symmetry

MDPI

Article

Solutions for the Nonlinear Mixed Variational Inequality Problem in the System

Husain Gissy, Abdullah Ali H. Ahmadini and Salahuddin *

Department of Mathematics, College of Science, Jazan University, Jazan 45142, Saudi Arabia; hgissy@jazanu.edu.sa (H.G.); aahmadini@jazanu.edu.sa (A.A.H.A.)
* Correspondence: dr_salah12@yahoo.com or smohammad@jazanu.edu.sa

Abstract: Our paper proposes a system of nonlinear mixed variational inequality problems (SNMVIPs) on Banach spaces. Under suitable assumptions, using the K-Fan fixed point theorem and Minty techniques, we demonstrate that the solution set to the SNMVIP is nonempty, weakly compact, and unique. Additionally, we suggest a stability result for the SNMVIPs by perturbing the duality mappings. Furthermore, we present an optimal control problem that is governed by the SNMVIPs and show that it can be solved.

Keywords: system of nonlinear mixed variational inequality problem; inverse relaxed monotonicity; existence; uniqueness; stability; optimal control problem

MSC: 47J20; 49J30; 35A15; 49N45; 47H20; 49H52

check for updates

Citation: Gissy, H.; Ahmadini, A.A.H.; Salahuddin. Solutions for the Nonlinear Mixed Variational Inequality Problem in the System. *Symmetry* **2024**, *16*, 796. https://doi.org/10.3390/sym16070796

Academic Editor: Sergei D. Odintsov

Received: 16 April 2024
Revised: 10 May 2024
Accepted: 19 June 2024
Published: 25 June 2024

1. Introduction

Lin [1] introduced the system of generalized quasi-variational inclusion problems. This system includes the set of problems suggested on a product set. It includes several well-known problems such as variational inequalities, equilibrium problems, vector equilibrium problems, and variational inclusions/disclusion problems. The system of variational inequalities has concealed symmetries in both variational inequalities and fixed point theory. However, the appearance of scale symmetry in this system creates a void of symmetric hiddenness and has a correlation effect.

Undoubtedly, in the realm of engineering, sciences, technology, chemical processes, and economics, several challenging and complex problems frequently result in inequalities instead of straightforward equations. In this scenario, variational inequalities have become a formidable mathematical resource. Variational inequalities (VIs) essentially arise from applied models with an underlying convex foundation and have been the subject of extensive research since the 1960s, encompassing mathematical theories, numerical techniques, and practical applications (among other significant sources, see [2–7]).

It should be noted that the results mentioned earlier cannot be applied to coupled systems that consist of two elliptical mixed variational inequalities. System of VIs are a mathematical tool used to analyze mixed boundary value problems, control problems, and similar problems. More details can be found in [8–12].

In this paper, we suggest SNMVIPs and using the K-Fan fixed point theorem, the Minty techniques, and inverse relaxed monotonicity to establish the existence, convergence, uniqueness, stability, and optimal control of the problems.

Before we proceed, let us define the problem that will be discussed in this article. Consider two reflexive Banach spaces, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, with their dual spaces $(X^*, \|\cdot\|_{X^*})$ and $(Y^*, \|\cdot\|_{Y^*})$, *respectively*. We denote the duality pairing between X^* and X by $\langle\cdot,\cdot\rangle_X$ and between Y^* and Y by $\langle\cdot,\cdot\rangle_Y$. We use $\xrightarrow{w}$ and $\longrightarrow$ to denote the weak and the strong convergence in X, and X_w denotes X with weak topology. The limits, lower limits, and upper limits are considered as n approaches infinity, even if not explicitly stated.

We can formulate SNMVIPs on Banach spaces:

Problem 1. *Determine $(u,v) \in \Omega \times \mho$ such that*

$$\langle \mathscr{A}(v,u), y-u\rangle_X + \varphi(u,y) - \varphi(u,u) \geq \langle \gamma, y-u\rangle_X,\ \forall y \in \Omega, \gamma \in X^* \tag{1}$$

and

$$\langle \mathscr{B}(u,v), z-v\rangle_Y + \phi(v,z) - \phi(v,v) \geq \langle \zeta, z-v\rangle_Y,\ \forall z \in \mho, \zeta \in Y^*. \tag{2}$$

We note that if $\varphi(u,u) = \varphi(u)$ and $\phi(u,u) = \phi(u)$, then Problem 1 reduces to the problem of [13] for finding $(u,v) \in \Omega \times \mho$ such that

$$\begin{cases} \langle \mathscr{A}(v,u), y-u\rangle_X + \varphi(y) - \varphi(u) \geq \langle \gamma, y-u\rangle_X,\ \forall y \in \Omega \\ \langle \mathscr{B}(u,v), z-v\rangle_Y + \phi(z) - \phi(v) \geq \langle \zeta, z-v\rangle_Y,\ \forall z \in \mho. \end{cases} \tag{3}$$

Definition 1. *Let $\Omega \neq \emptyset$ be a subset of a Banach space X. Let $\varphi\colon \Omega \to \overline{\mathbb{R}}$ be a proper convex and lower semicontinuous function, and $\mathscr{Q}\colon \Omega \to X^*$. Then $\mathscr{Q}$ is called*

(i) *Monotone, if*

$$\langle \mathscr{Q}u - \mathscr{Q}v, u-v\rangle_X \geq 0,\ \forall u,v \in \Omega;$$

(ii) *Strictly monotone, if*

$$\langle \mathscr{Q}u - \mathscr{Q}v, u-v\rangle_X > 0,\ \forall u,v \in \Omega \ \textit{and}\ u \neq v;$$

(iii) *Inverse relaxed monotone with constant $\alpha_{\mathscr{Q}} > 0$, if*

$$\langle \mathscr{Q}u - \mathscr{Q}v, u-v\rangle_X \geq -\alpha_{\mathscr{Q}}\|Qu - Qv\|_X^2,\ \forall u,v \in \Omega;$$

(iv) *Lipschitz continuous with constant $\beta_{\mathscr{Q}} > 0$, if*

$$\|\mathscr{Q}u - \mathscr{Q}v\|_X \leq \beta_{\mathscr{Q}}\|u-v\|_X^2,\ \forall u,v \in \Omega;$$

(v) *φ-pseudomonotone, if*

$$\langle \mathscr{Q}u, v-u\rangle_X + \varphi(v) - \varphi(u) \geq 0, \forall u,v \in \Omega$$

then it implies that

$$\langle \mathscr{Q}v, v-u\rangle_X + \varphi(v) - \varphi(u) \geq 0;$$

(vi) *φ-stable-pseudomonotone with respect to the set $\mathscr{W} \subset X^*$, if $\mathscr{Q}$ and $u \rightarrowtail \mathscr{Q}u - w$ are φ-pseudomonotone for each $w \in \mathscr{W}$.*

Let Z and Y be topological spaces and $\emptyset \neq V \subset Z$. We use the notation 2^V to represent the set of all possible subsets of the set V. Let $\mathscr{B}\colon Z \to 2^Y$ be a set-valued map. $Gr(\mathscr{B})$ represents the graph of $\mathscr{B}$ and is defined as

$$Gr(\mathscr{B}) = \{(u,v) \in Z \times Y | v \in \mathscr{B}(u)\} \subset Z \times Y.$$

The graph of $\mathscr{B}$ is sequentially closed in $Z \times Y$ if any sequence $\{(u_n, v_n)\} \subset Gr(\mathscr{B})$ converging to $(u,v) \in Z \times Y$ as $n \to \infty$. Then,

$$(u,v) \in Gr(\mathscr{B}) \ \ (i.e.,\ \ v \in \mathscr{B}(u)).$$

Theorem 1. [14] *Let $D \neq \emptyset$ be a bounded, closed, and convex set of a subset of the reflexive Banach space Y. Let $\Lambda\colon D \to 2^D$ be a nonempty, closed, and convex set-values map whose graph is sequentially closed in the topology $Y_w \times Y_w$. Then, Λ has a fixed point.*

2. Main Results

In this section, we focus on the uniqueness of solutions and their existence to Problem 1. We use the Minty methodology, Theorem 1, and the K-Fan fixed point theorem to establish the existence theorem for the solutions to Problem 1 under given modest assumptions. Additionally, we deploy the inverse relaxed monotonicity and Lipschitz continuity to prove two uniqueness results for Problem 1.

Furthermore, we propose that $\mathscr{S}\colon \mho \to 2^{\Omega}$ and $\mathscr{T}:\Omega \to 2^{\mho}$ be the set-valued maps described by

$$\mathscr{S}(v) = \{u \in \Omega |\ u \text{ solves (1) which corresponds to } v\}, \forall v \in \mho,$$

and

$$\mathscr{T}(u) = \{v \in \mho |\ v \text{ solves (2) which corresponds to } u\}, \forall u \in \Omega,$$

respectively.

The following assumptions must be made to solve Problem 1:

(A): $\emptyset \neq \Omega \subset X$ and $\emptyset \neq \mho \subset Y$ are closed and convex.

(B): $\gamma \in X^*$ and $\zeta \in Y^*$.

(C): $\varphi\colon X \times X \to \overline{\mathbb{R}}$ is such that

(i) $\varphi(\tau, \cdot)\colon X \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function;

(ii) There exists $\varrho_\varphi \geq 0$ such that

$$\varphi(\tau_1, v_2) - \varphi(\tau_1, v_1) + \varphi(\tau_2, v_1) - \varphi(\tau_2, v_2) \leq \varrho_\varphi \|\tau_1 - \tau_2\|_X \|v_1 - v_2\|_X, \forall \tau_1, \tau_2, v_1, v_2 \in X;$$

(iii) For each $\tau \in X$, there exists $\varrho_\varphi(\tau) > 0$ such that [15]

$$\varphi(\tau, v_1) - \varphi(\tau, v_2) \leq \varrho_\varphi(\tau) \|v_1 - v_2\|_X, \forall v_1, v_2 \in X$$

(D): $\mathscr{A}: Y \times X \to X^*$ is such that

(i) $u \rightarrowtail \mathscr{A}(v, u)$ is φ-stable-pseudomonotone with $\{\gamma\}$ and fulfills

$$\limsup_{\lambda \to 0} \langle \mathscr{A}(v, \lambda y + (1-\lambda)u), y - u\rangle_X \leq \langle \mathscr{A}(v, u), y - u\rangle_X, \forall v \in Y \text{ and } u, y \in X.$$

(ii) It possesses

$$\limsup_{n \to \infty} \langle \mathscr{A}(v_n, y), y - u_n\rangle_X \leq \langle \mathscr{A}(v, y), y - u\rangle_X,$$

when $y \in X, (u, v) \in X \times Y, \{v_n\} \subset Y$ and $\{u_n\} \subset X$ are such that

$$v_n \xrightarrow{w} v \in Y \text{ and } u_n \xrightarrow{w} u \in X \text{ as } n \to \infty;$$

(iii) $\flat\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function such that

$$\langle \mathscr{A}(v, u), u\rangle_X \geq \flat(\|u\|_X, \|v\|_Y)\|u\|_X, \forall u \in X, v \in Y,$$

and

♠ every bounded set $\emptyset \neq D \subset \mathbb{R}^+$, we have

$$\flat(t, s) \to +\infty \text{ as } t \to +\infty, \forall s \in D,$$

♠ for any $\varrho_1, \varrho_2 \geq 0$, it holds that $\flat(t, \varrho_1 t + \varrho_2) \to +\infty$ as $t \to +\infty$.

(iv) There exists $\varrho_{\mathscr{A}} > 0$ such that

$$\|\mathscr{A}(v, u)\|_{X^*} \leq \varrho_{\mathscr{A}}(1 + \|u\|_X + \|v\|_Y), \forall (u, v) \in X \times Y.$$

(E): $\phi\colon Y \times Y \to \overline{\mathbb{R}}$ is such that

(i) $\phi(\tau,\cdot)\colon Y \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function;
(ii) There exists $\varrho_\phi \geq 0$ such that

$$\phi(\tau_1, v_2) - \phi(\tau_1, v_1) + \phi(\tau_2, v_1) - \phi(\tau_2, v_2) \leq \varrho_\phi \|\tau_1 - \tau_2\|_Y \|v_1 - v_2\|_Y, \forall \tau_1, \tau_2, v_1, v_2 \in Y;$$

(iii) For each $\tau \in X$, there exists $\varrho_\phi(\tau) > 0$ such that [15]

$$\phi(\tau, v_1) - \phi(\tau, v_2) \leq \varrho_\phi(\tau)\|v_1 - v_2\|_X, \forall v_1, v_2 \in X.$$

(F): $\mathscr{B}\colon X \times Y \to Y^*$ is such that
(i) $v \rightarrowtail \mathscr{B}(u, v)$ is ϕ-stable-pseudomonotone with $\{\zeta\}$ and fulfills

$$\lim_{\lambda \to 0} \sup \langle \mathscr{B}(u, \lambda z + (1-\lambda)v), z - v\rangle_Y \leq \langle \mathscr{B}(u, v), z - v\rangle_Y, \forall z, v \in Y, u \in X.$$

(ii) It possesses

$$\lim_{n\to\infty} \sup \langle \mathscr{B}(u_n, z), z - v_n\rangle_Y \leq \langle \mathscr{B}(u, z), z - v\rangle_Y,$$

when $z \in Y$, $(u, v) \in X \times Y$, $\{v_n\} \subset Y$ and $\{u_n\} \subset X$ are such that

$$v_n \xrightarrow{w} v \in Y \text{ and } u_n \xrightarrow{w} u \in X \text{ as } n \to \infty;$$

(iii) $\ell : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function such that

$$\langle \mathscr{B}(u, v), v\rangle_Y \geq \ell(\|v\|_Y, \|u\|_X)\|v\|_Y, \forall u \in X \text{ and } v \in Y,$$

and
♠ every bounded set $\emptyset \neq D \subset \mathbb{R}^+$, we have

$$\ell(t, s) \to +\infty \text{ as } t \to +\infty, \text{ for all } s \in D,$$

♠ any $\varrho_1, \varrho_2 \geq 0$, it holds $\ell(t, \varrho_1 t + \varrho_2) \to +\infty$ as $t \to +\infty$.
(iv) $\exists$ a constant $\varrho_{\mathscr{B}} > 0$ such that

$$\|\mathscr{B}(u, v)\|_{Y^*} \leq \varrho_{\mathscr{B}}(1 + \|u\|_X + \|v\|_Y), \forall (u, v) \in X \times Y.$$

Remark 1. *When $\flat$ appears in **(D)**(iii) (or ℓ appears in **(F)**(iii)) but has no effect on the second variable, the condition **(D)**(iii) (or **(F)**(iii)) becomes a subsequent uniformly coercive condition: **(D)**(iii)$'$: $\exists\, \flat\colon \mathbb{R}^+ \to \mathbb{R}$ with $\flat(s) \to +\infty$ as $s \to +\infty$ such that*

$$\langle \mathscr{A}(v, u), u\rangle_X \geq \flat(\|u\|_X)\|u\|_X, \forall u \in X \text{ and } v \in Y$$

*(respectively, **(F)**(iii)$'$: $\exists\, \ell\colon \mathbb{R}^+ \to \mathbb{R}$ with $\ell(s) \to +\infty$ as $s \to +\infty$ such that*

$$\langle \mathscr{B}(u, v), v\rangle_Y \geq \ell(\|v\|_Y)\|v\|_Y, \forall u \in X, \text{ and } v \in Y).$$

The accessibility of solutions for Problem 1 is the main theorem of this article.

Theorem 2. *Suppose that **(A)**, **(B)**, **(C)**, **(D)**, **(E)**, and **(F)** are held. Then, the solution set denoted by $\Gamma(\gamma, \zeta)$ of Problem 1 corresponding to $(\gamma, \zeta) \in X^* \times Y^*$ is nonempty and weakly compact in $X \times Y$.*

We require the following lemmas to prove this theorem:

Lemma 1. *Assume that **(A)**, **(B)**, **(C)**, and **(D)** are satisfied. Then, the following statements hold:*

(i) *For fixed $v \in Y$, $u \in \Omega$ solves the (1), if and only if u solves the Minty inequality for determining $u \in \Omega$ such that*

$$\langle \mathscr{A}(v,y), y-u\rangle_X + \varphi(u,y) - \varphi(u,u) \geq \langle \gamma, y-u\rangle_X, \forall y \in \Omega; \tag{4}$$

(ii) *For fixed $v \in Y$, the solution set $\mathscr{S}(v)$ of (1) is nonempty, bounded, closed and convex;*
(iii) *The graph of $\mathscr{S} : \mho \to 2^{\Omega}$ is sequentially closed in $Y_w \times X_w$, implying that $\mathscr{S}$ is sequentially closed from a waek topology Y into the subsets of a weak topology X;*
(iv) *If the map $u \rightarrowtail \mathscr{A}(v,u)$ is strictly monotone for a fixed $v \in Y$, then $\mathscr{S}$ is a weakly continuous point-to-point mapping.*

Proof. The assumptions (i) and (ii) are the straightforward consequences of ([16], Theorem 3.3). Now, we present the conclusion (iii).

Let $\{(v_n, u_n)\} \subset Gr(\mathscr{S})$ be such that

$$v_n \xrightarrow{w} v \in Y \text{ and } u_n \xrightarrow{w} u \in X \text{ as } n \to \infty \text{ for } (u,v) \in X \times Y. \tag{5}$$

Then, for each $n \in \mathbb{N}$, we have $u_n \in \mathscr{S}(v_n)$, *i.e.*,

$$\langle \mathscr{A}(v_n,u_n), y-u_n\rangle_X + \varphi(u_n,y) - \varphi(u_n,u_n) \geq \langle \gamma, y-u_n\rangle_X, \forall y \in \Omega.$$

The (i) asserts that

$$\langle \mathscr{A}(v_n,y), y-u_n\rangle_X + \varphi(u_n,y) - \varphi(u_n,u_n) \geq \langle \gamma, y-u_n\rangle_X, \forall y \in \Omega. \tag{6}$$

To establish the upper limit as $n \to \infty$, we use assumption **(D)**(ii) and the weak lower semicontinuity of φ (because φ is convex and lower semicontinuous) to determine

$$\begin{aligned}\langle \mathscr{A}(v,y), y-u\rangle_X + \varphi(u,y) - \varphi(u,u) &\geq \lim_{n\to\infty} \sup \langle \mathscr{A}(v_n,y), y-u_n\rangle_X \\ &\quad + \lim_{n\to\infty} \inf[\varphi(u_n,y)] - \lim_{n\to\infty} \inf[\varphi(u_n,u_n)] \\ &\geq \lim_{n\to\infty} \sup[\langle \mathscr{A}(v_n,y), y-u_n\rangle_X + \varphi(u_n,y) - \varphi(u_n,u_n)] \\ &\geq \lim_{n\to\infty} \sup \langle \gamma, y-u_n\rangle_X \\ &= \langle \gamma, y-u\rangle_X, \forall y \in \Omega.\end{aligned}$$

Using the assumption (i), we obtain

$$u \in \mathscr{S}(v).$$

Consequently, $(v,u) \in Gr(\mathscr{S})$, which is the the graph of the mapping $\mathscr{S} : \mho \to 2^{\Omega}$, is sequentially closure in $Y_w \times X_w$.

Additionally, assume that $u \rightarrowtail \mathscr{A}(v,u)$ is strictly monotone. Let us consider $u_1, u_2 \in \Omega$ be two solutions to (1). Then, we have

$$\langle \mathscr{A}(v,u_1), y-u_1\rangle_X + \varphi(u_1,y) - \varphi(u_1,u_1) \geq \langle \gamma, y-u_1\rangle_X, \forall y \in \Omega, \gamma \in X^* \tag{7}$$

and

$$\langle \mathscr{A}(v,u_2), y-u_2\rangle_X + \varphi(u_2,y) - \varphi(u_2,u_2) \geq \langle \gamma, y-u_2\rangle_X, \forall y \in \Omega, \zeta \in Y^*. \tag{8}$$

In putting $y = u_2$ into (7) and $y = u_1$ in (8), we have

$$\langle \mathscr{A}(v,u_1), u_2-u_1\rangle_X + \varphi(u_1,u_2) - \varphi(u_1,u_1) \geq \langle \gamma, u_2-u_1\rangle_X, \tag{9}$$

and

$$\langle \mathscr{A}(v,u_2), u_1-u_2\rangle_X + \varphi(u_2,u_1) - \varphi(u_2,u_2) \geq \langle \gamma, u_1-u_2\rangle_X. \tag{10}$$

Adding (9) and (10), we have

$$\langle \mathscr{A}(v,u_1) - \mathscr{A}(v,u_2), u_1 - u_2\rangle_X - \varphi(u_1,u_2) + \varphi(u_1,u_1) - \varphi(u_2,u_1) + \varphi(u_2,u_2) \leq 0. \quad (11)$$

Given the assumption **(C)** and the strict monotonicity of $u \rightarrowtail \mathscr{A}(v,u)$, we obtain

$$u_1 = u_2.$$

Therefore, $\mathscr{S}$ is a point-to-point mapping. However, assumption (iii) shows that $\mathscr{S}$ is weakly continuous. □

Similarly, Problem (2) has the following lemma.

Lemma 2. *Assume that* **(A), (B), (E),** *and* **(F)** *are satisfied. Then, the following statements hold:*

(i) *For each fixed $u \in X$, $v \in \mho$ solves (2) if and only if v solves the Minty inequality to determine $v \in \mho$ such that*

$$\langle \mathscr{B}(u,z), z - v\rangle_Y + \phi(v,z) - \phi(v,v) \geq \langle \zeta, z - v\rangle_Y, \forall z \in \mho; \quad (12)$$

(ii) *For fixed $u \in X$, the solution set $\mathscr{T}(u)$ namely, of (2) is nonempty, bounded, closed, and convex;*

(iii) *The map $\mathscr{T} : \Omega \to 2^{\mho}$ has a sequentially closed graph in $X_w \times Y_w$;*

(iv) *If the map $v \rightarrowtail \mathscr{B}(u,v)$ is strictly monotone for a fixed $u \in X$, then $\mathscr{T}$ is a weakly continuous point-to-point mapping.*

Furthermore, we provide an a priori appraisal of the solution to Problem 1.

Lemma 3. *Let us assume that* **(A), (B), (C), (D), (E),** *and* **(F)** *have been fulfilled satisfactorily. If the solution set $\Gamma(\gamma,\zeta)$, namely, of Problem 1, is nonempty, then there exists $\mathscr{M} > 0$ such that*

$$\|u\|_X \leq \mathscr{M} \text{ and } \|v\|_Y \leq \mathscr{M}, \ \forall (u,v) \in \Gamma(\gamma,\zeta). \quad (13)$$

Proof. Suppose that $\Gamma(\gamma,\zeta) \neq \emptyset$. Let $(u,v) \in \Gamma(\gamma,\zeta)$ be arbitrary and $(u_0,v_0) \in (D(\varphi) \cap \Omega) \times (D(\phi) \cap \mho)$. By swapping $y = u_0$ and $z = v_0$ into (1) and (2), *respectively*, we obtain

$$\langle \mathscr{A}(v,u), u\rangle_X \leq \langle \mathscr{A}(v,u), u_0\rangle_X + \varphi(u,u_0) - \varphi(u,u) + \langle \gamma, u_0 - u\rangle_X \quad (14)$$

and

$$\langle \mathscr{B}(u,v), v\rangle_Y \leq \langle \mathscr{B}(u,v), v_0\rangle_Y + \phi(v,v_0) - \phi(v,v) + \langle \zeta, v_0 - v\rangle_Y. \quad (15)$$

Taking account of (14), we use hypotheses **(C)**(i), (iii), and **(D)**(iii)–(iv) to obtain

$$\begin{aligned}
\flat(\|u\|_X, \|v\|_Y)\|u\|_X &\leq \langle \mathscr{A}(v,u), u\rangle_X \\
&\leq \langle \mathscr{A}(v,u), u_0\rangle_X + \varphi(u,u_0) - \varphi(u,u) + \langle \gamma, u_0 - u\rangle_X \\
&\leq \|\mathscr{A}(v,u)\|_{X^*}\|u_0\|_X + \varphi(u,u_0) - \varphi(u,u) + \|\gamma\|_{X^*}(\|u_0\|_X + \|u\|_X) \\
&\leq \varrho_{\mathscr{A}}(1 + \|u\|_X + \|v\|_Y)\|u_0\|_X + \varrho_\varphi(u)(\|u_0\|_X + \|u\|_X) + \|\gamma\|_{X^*}(\|u_0\|_X + \|u\|_X).
\end{aligned}$$

This implies that

$$\flat(\|u\|_X, \|v\|_Y) \leq \frac{\varrho_{\mathscr{A}}(1 + \|u\|_X + \|v\|_Y)\|u_0\|_X}{\|u\|_X} + \frac{(\varrho_\varphi(u) + \|\gamma\|_{X^*})\|u_0\|_X}{\|u\|_X} + \varrho_\varphi(u) + \|\gamma\|_{X^*}. \quad (16)$$

Similarly, taking account of (15), we use hypotheses **(E)**(i), (iii) and **(F)**(iii)–(iv) to obtain

$$\ell(\|v\|_Y, \|u\|_X)\|v\|_Y \leq \varrho_{\mathscr{B}}(1 + \|u\|_X + \|v\|_Y)\|v_0\|_Y + \varrho_\phi(v)(\|v_0\|_Y + \|v\|_Y) + \|\zeta\|_{Y^*}(\|v_0\|_Y + \|v\|_Y)$$
$$\Longrightarrow$$
$$\ell(\|v\|_Y, \|u\|_X) \leq \frac{\varrho_{\mathscr{B}}(1 + \|u\|_X + \|v\|_Y)\|v_0\|_Y}{\|v\|_Y} + \frac{(\varrho_\phi(v) + \|\zeta\|_{Y^*})\|v_0\|_Y}{\|v\|_Y} + \varrho_\phi(v) + \|\zeta\|_{Y^*}. \tag{17}$$

Contrarily, suppose $\Gamma(\gamma, \zeta)$ is unbounded. Then, taking a subsequence, if necessary, it is possible to suggest a sequence $\{(u_n, v_n)\} \subset \Omega \times \mho$ so that

$$\|u_n\|_X \uparrow +\infty \text{ as } n \to \infty, \tag{18}$$

or

$$\|v_n\|_Y \uparrow +\infty \text{ as } n \to \infty. \tag{19}$$

Let us segregate the subsequent cases:

a Assume (18) is satisfied and the sequence $\{v_n\}$ is bounded in Y;
b Assume (19) is fulfilled and the sequence $\{u_n\}$ is bounded in X;
c Assuming that both (18) and (19) are fulfilled.

Assuming a is valid, substitute $u = u_n$ and $v = v_n$ into (16) to obtain

$$\flat(\|u_n\|_X, \|v_n\|_Y) \leq \frac{\varrho_{\mathscr{A}}(1 + \|u_n\|_X + \|v_n\|_Y)\|u_0\|_X}{\|u_n\|_X} + \frac{(\varrho_\varphi(u) + \|\gamma\|_{X^*})\|u_0\|_X}{\|u_n\|_X} + \varrho_\varphi(u) + \|\gamma\|_{X^*}. \tag{20}$$

When we let n approach infinity in the inequality (20) and make use of (18) along with property **(D)**(iii), we obtain the following:

$$\begin{aligned} +\infty &= \lim_{n\to\infty} \flat(\|u_n\|_X, \|v_n\|_Y) \\ &\leq \lim_{n\to\infty} \left[\frac{\varrho_{\mathscr{A}}(1 + \|u_n\|_X + \|v_n\|_Y)\|u_0\|_X}{\|u_n\|_X} + \frac{(\varrho_\varphi(u) + \|\gamma\|_{X^*})\|u_0\|_X}{\|u_n\|_X} + \varrho_\varphi(u) + \|\gamma\|_{X^*} \right] \\ &= \varrho_{\mathscr{A}}\|u_0\|_X + \varrho_\varphi(u) + \|\gamma\|_{X^*}. \end{aligned} \tag{21}$$

Consequently, (21) produces a contradiction. Similarly, for b, we could use (17) to obtain a contradiction. However, we assume that c holds, and we will proceed to discuss two additional situations:

(1) $\dfrac{\|v_n\|_Y}{\|u_n\|_X} \to +\infty \text{ as } n \to \infty;$
(2) There exist $n_0 \in \mathbb{N}$ and $\widehat{\varrho_0} > 0$ such that

$$\frac{\|v_n\|_Y}{\|u_n\|_X} \leq \widehat{\varrho_0}, \ \forall n \geq n_0.$$

If item (1) is true, we enter $u = u_n$ and $v = v_n$ into (17) to yield

$$\ell(\|v_n\|_Y, \|u_n\|_X) \leq \frac{\varrho_{\mathscr{B}}(1 + \|u_n\|_X + \|v_n\|_Y)\|v_0\|_Y}{\|v_n\|_Y} + \frac{(\varrho_\phi(v) + \|\zeta\|_{Y^*})\|v_0\|_Y}{\|v_n\|_Y} + \varrho_\phi(v) + \|\zeta\|_{Y^*}.$$

Taking the limit as n approaches infinity for the inequality mentioned above yields

$$\begin{aligned} +\infty &= \lim_{n\to\infty} \ell(\|v_n\|_Y, \|u_n\|_X) \\ &\leq \lim_{n\to\infty} \left[\frac{\varrho_{\mathscr{B}}(1 + \|u_n\|_X + \|v_n\|_Y)\|v_0\|_Y}{\|v_n\|_Y} + \frac{(\varrho_\phi(v) + \|\zeta\|_{Y^*})\|v_0\|_Y}{\|v_n\|_Y} + \varrho_\phi(v) + \|\zeta\|_{Y^*} \right] \\ &= \varrho_{\mathscr{B}}\|v_0\|_Y + \varrho_\phi(v) + \|\zeta\|_{Y^*}. \end{aligned} \tag{22}$$

It is obviously impossible; however, for a situation (2), we can deduce from (16) that

$$\begin{aligned} +\infty \leftarrow & \flat(\|u_n\|_X, \|v_n\|_Y) \quad (\text{as } n \to \infty) \\ \leq & \frac{\varrho_{\mathscr{A}}(1+\|u_n\|_X+\|v_n\|_Y)\|u_0\|_X}{\|u_n\|_X} + \frac{(\varrho_\varphi(u)+\|\gamma\|_{X^*})\|u_0\|_X}{\|u_n\|_X} + \varrho_\varphi(u) + \|\gamma\|_{X^*} \\ = & \varrho_{\mathscr{A}}(2+\widehat{\varrho}_0)\|u_0\|_X + \varrho_\varphi(u) + \|\gamma\|_{X^*}\|u_0\|_X + \varrho_\varphi(u) + \|\gamma\|_{X^*}, \text{ for } n \geq n_1, \end{aligned} \tag{23}$$

where $n_1 \geq n_0$ is such that

$$\|u_{n_1}\|_X > 1.$$

This leads to a contradiction. Thus, $\Gamma(\gamma, \zeta)$ is bounded in $X \times Y$, allowing us to determine $\mathscr{M} > 0$ satisfying (13). □

Consider the set-valued mapping $\Lambda : \Omega \times \mho \to 2^{\Omega \times \mho}$ defined by

$$\Lambda(u,v) = (\mathscr{S}(v), \mathscr{T}(u)), \forall (u,v) \in \Omega \times \mho. \tag{24}$$

By invoking Lemma 1 and Lemma 2, it can be seen that Λ is well defined. In addition, $\exists$ a bounded, closed, and convex set $\mathfrak{D}$ in $\Omega \times \mho$ such that Λ maps $\mathfrak{D}$ into itself.

Lemma 4. *Suppose* **(A), (B), (C), (D), (E),** *and* **(F)** *are met. Then,* $\exists$ *a constant* $\widehat{\mathscr{M}} > 0$ *satisfy*

$$\Lambda(\overline{\mathscr{B}(0,\widehat{\mathscr{M}})}) \subset \overline{\mathscr{B}(0,\widehat{\mathscr{M}})},$$

where $\overline{\mathscr{B}(0,\widehat{\mathscr{M}})} \subset X \times Y$ *is defined as*

$$\overline{\mathscr{B}(0,\widehat{\mathscr{M}})} = \{(u,v) \in \Omega \times \mho \mid \ \|u\|_X \leq \widehat{\mathscr{M}} \text{ and } \|v\|_Y \leq \widehat{\mathscr{M}}\}.$$

Proof. Our proof will be based on contradiction. Assume that

$$\Gamma(\overline{\mathscr{B}(0,n)}) \not\subset \overline{\mathscr{B}(0,n)}, \text{ for } n \in \mathbb{N}.$$

Then, for each $n \in \mathbb{N}$, we may determine $(u_n, v_n) \in \overline{\mathscr{B}(0,n)}$ and $(w_n, z_n) \in \Gamma(u_n, v_n)$ (i.e., $w_n \in \mathscr{S}(v_n)$ and $z_n \in \mathscr{T}(u_n)$) so that

$$\|w_n\|_X > n \quad \text{or} \quad \|z_n\|_Y > n. \tag{25}$$

Thus, assuming $\|w_n\|_X > n$ for each $n \in \mathbb{N}$ (similarly for $\|z_n\|_Y > n$ for each $n \in \mathbb{N}$). We employ (16); one has

$$\flat(\|w_n\|_X, \|v_n\|_Y) \leq \frac{\varrho_{\mathscr{A}}(1+\|w_n\|_X+\|v_n\|_Y)\|u_0\|_X}{\|w_n\|_X} + \frac{(\varrho_\varphi(u)+\|\gamma\|_{X^*})\|u_0\|_X}{\|w_n\|_X} + \varrho_\varphi(u) + \|\gamma\|_{X^*}.$$

Since

$$\|v_n\|_Y \leq n < \|w_n\|_X.$$

Therefore, passing to the limit as $n \to \infty$ for the inequality above, we have

$$\begin{aligned} +\infty = & \lim_{n\to\infty} \flat(\|w_n\|_X, \|v_n\|_Y) \\ \leq & \lim_{n\to\infty} \left[\frac{\varrho_{\mathscr{A}}(1+\|w_n\|_X+\|v_n\|_Y)\|u_0\|_X}{\|w_n\|_X} + \frac{(\varrho_\varphi(u)+\|\gamma\|_{X^*})\|u_0\|_X}{\|w_n\|_X} + \varrho_\varphi(u) + \|\gamma\|_{X^*} \right] \\ \leq & 2\varrho_{\mathscr{A}}\|u_0\|_X + \varrho_\varphi(u) + \|\gamma\|_{X^*}. \end{aligned}$$

This leads to a contradiction. Thus, $\exists$ a constant $\widehat{\mathscr{M}} > 0$ satisfying

$$\Lambda(\overline{\mathscr{B}(0,\widehat{\mathscr{M}})}) \subset \overline{\mathscr{B}(0,\widehat{\mathscr{M}})}.$$

□

Proof. **(Proof of Theorem 2)** Let us see that if Λ has a fixed point (u^*, v^*), then

$$u^* \in \mathscr{S}(v^*) \text{ and } v^* \in \mathscr{T}(u^*).$$

By employing the concepts of $\mathscr{S}$ and $\mathscr{T}$, it offers

$$\langle \mathscr{A}(v^*, u^*), y - u^* \rangle_X + \varphi(u^*, y) - \varphi(u^*, u^*) \geq \langle \gamma, y - u^* \rangle_X, \forall y \in \Omega,$$

and

$$\langle \mathscr{B}(u^*, v^*), z - v^* \rangle_Y + \phi(v^*, z) - \phi(v^*, v^*) \geq \langle \zeta, z - v^* \rangle_Y, \forall z \in \mho.$$

Thus, it is clear that (u^*, v^*) solves Problem 1. We will apply Theorem 1, the K-Fan fixed point theorem, to determine the existence of a fixed point for Λ.

Moreover, Lemmas 1, 2, and 4, in fact, infer that $\Lambda\colon \overline{\mathscr{B}(0, \widehat{\mathscr{M}})} \to 2^{\overline{\mathscr{B}(0, \widehat{\mathscr{M}})}}$ has nonempty, closed, and convex values; the graph of Λ is sequentially closed in $(X \times Y)_w \times (X \times Y)_w$. The conditions stated in Theorem 1 have been verified. From this theorem, It can be shown that Problem 1 has a solution $(u^*, v^*) \in \Omega \times \mho$, such that

$$(u^*, v^*) \in \Lambda(u^*, v^*).$$

Hence,

$$\Gamma(\gamma, \zeta) \neq \varnothing.$$

Lemma 3 clearly shows that $\Gamma(\gamma, \zeta)$ is bounded in $X \times Y$. Therefore, we will demonstrate that $\Gamma(\gamma, \zeta)$ is weakly closed. Consider $\{(u_n, v_n)\} \subset \Gamma(\gamma, \zeta)$ such that

$$(u_n, v_n) \xrightarrow{w} (u, v) \in X \times Y \text{ as } n \to \infty, \text{ for some } (u, v) \in \Omega \times \mho. \tag{26}$$

It is clear that for each natural number n, the pair $(u_n, v_n) \in \Lambda(u_n, v_n)$. Since Λ is sequentially closed from $(X \times Y)_w$ to $(X \times Y)_w$ (*see* Lemma 1 and Lemma 2), we can conclude that

$$(u, v) \in \Lambda(u, v).$$

This signifies that

$$(u, v) \in \Gamma(\gamma, \zeta).$$

Thus, due to the boundedness of $\Gamma(\gamma, \zeta)$, we can conclude that $\Gamma(\gamma, \zeta)$ is weakly compact. □

Theorem 2 shows that the solution set of Problem 1 is both nonempty and weakly compact. However, it raises the question of whether it is possible to prove the uniqueness of the solution under certain assumptions. Fortunately, the theorems below provide a positive solution to this problem.

Theorem 3. *Assume that* **(A), (B), (C), (D), (E),** *and* **(F)** *are fulfilled. In addition, if the inequality below holds,*

$$\begin{aligned}\langle \mathscr{A}(v_1, u_1) - \mathscr{A}(v_2, u_2), u_1 - u_2 \rangle_X + \langle \mathscr{B}(u_1, v_1) - \mathscr{B}(u_2, v_2), v_1 - v_2 \rangle_Y + \varrho_\varphi \|u_1 - u_2\|_X^2 \\ + \varrho_\phi \|v_1 - v_2\|_Y^2 > 0, \forall (u_1, v_1), (u_2, v_2) \in X \times Y \text{ with } (u_1, v_1) \neq (u_2, v_2).\end{aligned} \tag{27}$$

Then Problem 1 has a unique solution.

Proof. Theorem 2 assures that

$$\Gamma(\gamma, \zeta) \neq \varnothing.$$

We now demonstrate that Problem 1 has unique solution. Assume $(u_1, v_1), (u_2, v_2) \in \Gamma(\gamma, \zeta)$ are the two different solutions. Then,

$$\langle \mathscr{A}(v_i, u_i), y - u_i \rangle_X + \varphi(u_i, y) - \varphi(u_i, u_i) \geq \langle \gamma, y - u_i \rangle_X, \forall y \in \Omega, \tag{28}$$

and

$$\langle \mathscr{B}(u_i, v_i), z - v_i \rangle_Y + \phi(v_i, z) - \phi(v_i, v_i) \geq \langle \zeta, z - v_i \rangle_Y, \forall z \in \mho. \tag{29}$$

After setting $i = 1$ to correspond to $y = u_2$ and $i = 2$ to correspond to $y = u_1$ in Equation (28), we add the two equations to obtain

$$\langle \mathscr{A}(v_1, u_1) - \mathscr{A}(v_2, u_2), u_1 - u_2 \rangle_X - \varphi(u_1, u_2) + \varphi(u_1, u_1) - \varphi(u_2, u_1) + \varphi(u_2, u_2) \leq 0. \tag{30}$$

Similarly, assigning $i = 1$ to correspond to $z = v_2$ and $i = 2$ to correspond to $z = v_1$ in Equation (29), we add the two equations to obtain

$$\langle \mathscr{B}(u_1, v_1) - \mathscr{B}(u_2, v_2), v_1 - v_2 \rangle_Y - \phi(v_1, v_2) + \phi(v_1, v_1) - \phi(v_2, v_1) + \phi(v_2, v_2) \leq 0. \tag{31}$$

By using (30), (31), and assertions **(C)** and **(E)**, we have

$$\langle \mathscr{A}(v_1, u_1) - \mathscr{A}(v_2, u_2), u_1 - u_2 \rangle_X + \langle \mathscr{B}(u_1, v_1) - \mathscr{B}(u_2, v_2), v_1 - v_2 \rangle_Y + \varrho_\varphi \|u_1 - u_2\|_X^2 + \varrho_\phi \|v_1 - v_2\|_Y^2 \leq 0.$$

This, combined with the condition (27), implies that $u_1 = u_2$ and $v_1 = v_2$. Thus, Problem 1 has a unique solution. □

By adding an additional condition to (27), the resulting theorem establishes a unique solution for Problem 1.

Theorem 4. *Assume that* **(A), (B), (C), (D), (E),** *and* **(F)** *are fulfilled. If the following assumptions are met:*

① *The function $u \rightarrowtail \mathscr{A}(v, u)$ is inversely relaxed monotone and Lipschitz continuous for $v \in Y$, with constants $\alpha_{\mathscr{A}} > 0$ and $\beta_{\mathscr{A}} > 0$. Moreover, for each $u \in X$ the function $v \rightarrowtail \mathscr{A}(v, u)$ is Lipschitz continuous with $\mathscr{L}_{\mathscr{A}} > 0$;*
② *The function $v \rightarrowtail \mathscr{B}(u, v)$ is inversely relaxed monotone and Lipschitz continuous for $u \in X$, with $\alpha_{\mathscr{B}} > 0$ and $\beta_{\mathscr{B}} > 0$. Moreover, for every $v \in Y$ the function $u \rightarrowtail \mathscr{B}(u, v)$ is Lipschitz continuous with $\mathscr{L}_{\mathscr{B}} > 0$;*
③ $\dfrac{\mathscr{L}_{\mathscr{A}} \mathscr{L}_{\mathscr{B}}}{(\alpha_{\mathscr{A}} \beta_{\mathscr{A}} + \varrho_\varphi)(\alpha_{\mathscr{B}} \beta_{\mathscr{B}} + \varrho_\phi)} < 1.$
Then Problem 1 has a unique solution.

Proof. Let (u_1, v_1) and (u_2, v_2) be two solutions to Problem 1. Then, it has

$$\langle \mathscr{A}(v_1, u_1) - \mathscr{A}(v_2, u_2), u_1 - u_2 \rangle_X - \varrho_\varphi \|u_1 - u_2\|_X^2 \leq 0, \tag{32}$$

and

$$\langle \mathscr{B}(u_1, v_1) - \mathscr{A}(u_2, v_2), v_1 - v_2 \rangle_Y - \varrho_\phi \|v_1 - v_2\|_Y^2 \leq 0. \tag{33}$$

Again, from the inverse relaxed monotonicity and Lipschitz continuity of $\mathscr{A}$, we have

$$\begin{aligned} -\alpha_{\mathscr{A}} \beta_{\mathscr{A}} \|u_1 - u_2\|_X^2 &\leq \langle \mathscr{A}(v_1, u_1 - \mathscr{A}(v_1, u_2), u_1 - u_2 \rangle_X \\ &\leq \mathscr{L}_{\mathscr{A}} \|v_1 - v_2\|_Y \|u_1 - u_2\|_X. \end{aligned} \tag{34}$$

Thus, from (32) and (34), we have

$$(-\alpha_{\mathscr{A}} \beta_{\mathscr{A}} - \varrho_\varphi) \|u_1 - u_2\|_X^2 \leq \mathscr{L}_{\mathscr{A}} \|v_1 - v_2\|_Y \|u_1 - u_2\|_X. \tag{35}$$

Similarly, we obtain

$$\begin{aligned} -\alpha_{\mathscr{B}}\beta_{\mathscr{B}}\|v_1 - v_2\|_Y^2 &\leq \langle \mathscr{B}(u_1, v_1) - \mathscr{B}(u_1, v_2), v_1 - v_2 \rangle_Y \\ &\leq \mathscr{L}_{\mathscr{B}}\|v_1 - v_2\|_Y \|u_1 - u_2\|_X. \end{aligned} \tag{36}$$

Again, from (33) and (36), we have

$$(-\alpha_{\mathscr{B}}\beta_{\mathscr{B}} - \varrho_{\phi})\|v_1 - v_2\|_Y^2 \leq \mathscr{L}_{\mathscr{B}}\|v_1 - v_2\|_Y \|u_1 - u_2\|_X. \tag{37}$$

Combining equations (35) and (37) yields

$$\|u_1 - u_2\|_X \leq \frac{\mathscr{L}_{\mathscr{A}}\mathscr{L}_{\mathscr{B}}}{(\alpha_{\mathscr{A}}\beta_{\mathscr{A}} + \varrho_{\varphi})(\alpha_{\mathscr{B}}\beta_{\mathscr{B}} + \varrho_{\phi})}\|u_1 - u_2\|_X. \tag{38}$$

However, the inequality $\frac{\mathscr{L}_{\mathscr{A}}\mathscr{L}_{\mathscr{B}}}{(\alpha_{\mathscr{A}}\beta_{\mathscr{A}} + \varrho_{\varphi})(\alpha_{\mathscr{B}}\beta_{\mathscr{B}} + \varrho_{\phi})} < 1$ implies that $u_1 = u_2$ and $v_1 = v_2$. Therefore, Problem 1 has a unique solution. □

3. Stability Results

In this section, we delve into examining the stability of the system of nonlinear mixed variational inequality problems. Firstly, we present a set of regularized problems perturbed by duality mappings that correspond to Problem 1. Secondly, we arrive at a stability conclusion that demonstrates that every solution sequence to a regularized problem contains at least one subsequence that solves Problem 1.

Let X and Y be two reflexive Banach spaces, and let X^* and Y^* be their dual spaces. We assume that X and Y are strictly convex without losing the generality. Let $J_X\colon X \to X^*$ and $J_Y\colon Y \to Y^*$ be the duality mappings, so that

$$J_X(u) = \{u^* \in X^* | \langle u^*, u \rangle_X = \|u\|_X^2 = \|u^*\|_{X^*}^2\},$$

$$J_Y(v) = \{v^* \in Y^* | \langle v^*, v \rangle_Y = \|v\|_Y^2 = \|v^*\|_{Y^*}^2\}.$$

Let $\{\delta_n\}$ and $\{\varepsilon_n\}$ be real sequences such that

$$\varepsilon_n > 0, \quad \delta_n > 0, \quad \varepsilon_n \to 0 \text{ and } \delta_n \to 0. \tag{39}$$

Consider the following perturbated problem for every $n \in \mathbb{N}$, which corresponds to Problem 1.

Problem 2. *Determine* $(u_n, v_n) \in \Omega \times \mho$ *so that*

$$\langle \mathscr{A}(v_n, u_n) + \varepsilon_n J_X(u_n), y - u_n \rangle_X + \varphi(u_n, y) - \varphi(u_n, u_n) \geq \langle \gamma, y - u_n \rangle_X, \forall y \in \Omega, \gamma \in X^*, \tag{40}$$

and

$$\langle \mathscr{B}(u_n, v_n) + \delta_n J_Y(v_n), z - v_n \rangle_Y + \phi(v_n, z) - \phi(v_n, v_n) \geq \langle \zeta, z - v_n \rangle_Y, \forall z \in \mho, \zeta \in Y^*. \tag{41}$$

We assume the following.

(G): $u \rightarrowtail \mathscr{A}(v, u)$ and $v \rightarrowtail \mathscr{B}(u, v)$ are monotone, and meet

$$\limsup_{\lambda \to 0} \langle \mathscr{A}(v, \lambda y + (1-\lambda)u), y - u \rangle_X \leq \langle \mathscr{A}(v, u), y - u \rangle_X,$$

$$\limsup_{\lambda \to 0} \langle \mathscr{B}(u, \lambda z + (1-\lambda)v), z - v \rangle_Y \leq \langle \mathscr{B}(u, v), z - v \rangle_Y, \forall z, v \in Y, \forall y, u \in X.$$

(H): $u \rightarrowtail \mathscr{A}(v,u)$ is inverse relaxed monotone with $\alpha_{\mathscr{A}} > 0$ and Lipschitz continuous with $\beta_{\mathscr{A}} > 0$; similarly, $v \rightarrowtail \mathscr{B}(u,v)$ is inverse relaxed monotone with $\alpha_{\mathscr{B}} > 0$ and Lipschitz continuous with $\beta_{\mathscr{B}} > 0$, and fulfil

$$\limsup_{\lambda \to 0} \langle \mathscr{A}(v, \lambda y + (1-\lambda)u), y - u \rangle_X \leq \langle \mathscr{A}(v,u), y-u \rangle_X,$$

$$\limsup_{\lambda \to 0} \langle \mathscr{B}(u, \lambda z + (1-\lambda)v), z - v \rangle_Y \leq \langle \mathscr{B}(u,v), z-v \rangle_Y, \forall z, v \in Y, \forall y, u \in X.$$

The theory described below ensures that solutions to Problem 2 exist and converge.

Theorem 5. *Assume that* **(A), (B), (C), (D)***(ii)–(iv),* **(E),** *and* **(F)***(ii)–(iv) are satisfied. Then the following assertions hold:*

(i) *If, in addition to assumption* **(G)***, Problem 2 has at least one solution* $(u_n, v_n) \in \Omega \times \mho$ *for every* $n \in \mathbb{N}$*;*

(ii) *Furthermore, if* **(G)** *holds, there is a subsequence* $\{(u_n, v_n)\}$ *for every solution of the sequence* $\{(u_n, v_n)\}$ *to Problem 2, such that*

$$(u_n, v_n) \xrightarrow{w} (u, v) \in X \times Y \;\; as \;\; n \to \infty, \tag{42}$$

where $(u, v) \in \Omega \times \mho$ *solves the Problem 2;*

(iii) *Under the conditions of* **(H)***, any sequence of solutions* $\{(u_n, v_n)\}$ *of Problem 2 has a subsequence* $\{(u_n, v_n)\}$ *such that*

$$(u_n, v_n) \to (u, v) \in X \times Y \;\; as \;\; n \to \infty, \tag{43}$$

where $(x, y) \in \Omega \times \mho$ *solves Problem 1.*

Proof. (i) Assign

$$\mathscr{A}_n(v,u) = \mathscr{A}(v,u) + \varepsilon_n J_X(u)$$

and

$$\mathscr{B}_n(u,v) = \mathscr{B}(u,v) + \delta_n J_Y(v), \; \forall \; (u,v) \in X \times Y.$$

We shall confirm that $\mathscr{A}_n$ and $\mathscr{B}_n$ satisfy, *respectively*, **(D)** and **(F)**. Observe that J_X is demicontinuous and

$$0 \leq (\|u\|_X - \|y\|_X)^2 \leq \langle J_X(u) - J_X(y), u - y \rangle_X, \forall u, y \in X. \tag{44}$$

Using hypotheses **(G)**, we determine that **(D)**(i) is satisfied for each $v \in Y$, $u \rightarrowtail \mathscr{A}_n(v,u)$. Utilizing the information that $\|J_X(u)\|_X = \|u\|_X$ and

$$\langle J_X(u), u \rangle_X = \|u\|_X^2, \forall u \in X.$$

It is easy to show that $\mathscr{A}_n$ satisfies **(D)**(ii)–(iv). Similarly, $\mathscr{B}_n$ satisfies **(F)**. Consequently, by using Theorem 2, we can argue that Problem 2 has a solution.

(ii) Let $\{(u_n, v_n)\}$ be any arbitrary sequence that solves Problem 2. Next, a meticulous calculation yields

$$\begin{aligned}
\flat(\|u_n\|_X, \|v_n\|_Y) &\leq \flat(\|u_n\|_X, \|v_n\|_Y) + \frac{\varepsilon_n \langle J_X(u_n), u_n\rangle_X}{\|u_n\|_X} \\
&\leq \frac{\varrho_{\mathscr{A}}(1 + \|u_n\|_X + \|v_n\|_Y)\|u_0\|_X}{\|u_n\|_X} + \frac{(\varepsilon_n \|J_X(u_n)\|_{X^*} + \|\gamma\|_{X^*})\|u_0\|_X + \varrho_\varphi(u)}{\|u_n\|_X} \\
&\quad + \varrho_\varphi(u) + \|\gamma\|_{X^*} \\
&= \frac{\varrho_{\mathscr{A}}(1 + \|u_n\|_X + \|v_n\|_Y)\|u_0\|_X}{\|u_n\|_X} + \frac{(\varepsilon_n \|u_n\|_X + \|\gamma\|_{X^*})\|u_0\|_X + \varrho_\varphi(u)}{\|u_n\|_X} \\
&\quad + \varrho_\varphi(u) + \|\gamma\|_{X^*},
\end{aligned} \tag{45}$$

and

$$\begin{aligned}
\ell(\|v_n\|_Y, \|u_n\|_X) &\leq \frac{\varrho_{\mathscr{B}}(1 + \|u_n\|_X + \|v_n\|_Y)\|v_0\|_Y}{\|v_n\|_Y} + \frac{(\delta_n \|v_n\|_Y + \|\zeta\|_{Y^*})\|v_0\|_Y + \varrho_\phi(v)}{\|v_n\|_Y} \\
&\quad + \varrho_\phi(v) + \|\zeta\|_{Y^*}.
\end{aligned} \tag{46}$$

The same argument that was employed in Lemma 3's proof that $\{(u_n, v_n)\}$ is bounded in $X \times Y$.
If necessitated we can go to a relabeled subsequence and assume that

$$(u_n, v_n) \xrightarrow{w} (u, v) \in X \times Y \text{ as } n \to \infty, \text{ for some } (u, v) \in \Omega \times \mho. \tag{47}$$

By using the monotonicity of $u \rightarrowtail \mathscr{A}(v, u)$ and $v \rightarrowtail \mathscr{B}(u, v)$, we can make the following deduction:

$$\langle \mathscr{A}(v_n, y) + \varepsilon_n J_X(u_n), y - u_n\rangle_X + \varphi(u_n, y) - \varphi(u_n, u_n) \geq \langle \gamma, y - u_n\rangle_X, \forall y \in \Omega, \tag{48}$$

and

$$\langle \mathscr{B}(u_n, z) + \delta_n J_Y(v_n), z - v_n\rangle_Y + \phi(v_n, z) - \phi(v_n, v_n) \geq \langle \zeta, z - v_n\rangle_Y, \forall z \in \mho. \tag{49}$$

By taking the upper limit as $n \to \infty$ and applying conditions **(D)**(ii) and **(F)**(ii), we infer that

$$\langle \mathscr{A}(v, y), y - u\rangle_X + \varphi(u, y) - \varphi(u, u) \geq \langle \gamma, y - u\rangle_X, \forall y \in \Omega,$$

and

$$\langle \mathscr{B}(u, z), z - v\rangle_Y + \phi(v, z) - \phi(v, v) \geq \langle \zeta, z - v\rangle_Y, \forall z \in \mho,$$

Here, we utilized the boundedness of $\{(u_n, v_n)\} \in X \times Y$. Using the Minty approach, we find $(u, v) \in \Omega \times \mho$ to solve Problem 1, *i.e.*,

$$(u, v) \in \Gamma(\gamma, \zeta).$$

(iii) It can be deduced from (ii) that if we have a sequence of solutions denoted by $\{(u_n, v_n)\}$ for Problem 2, there will always exist a subsequence of $\{(u_n, v_n)\}$ that satisfies (42). We assert that the sequence $\{(u_n, v_n)\}$ has a strong convergence to (u, v). It is simple to demonstrate that

$$\begin{aligned}
-\alpha_{\mathscr{A}}\beta_{\mathscr{A}}\|u_n - u\|_X^2 &\leq \langle \mathscr{A}(v_n, u_n) - \mathscr{A}(v_n, u), u_n - u\rangle_X \\
&\leq \langle \mathscr{A}(v, u) - \mathscr{A}(v_n, u), u_n - u\rangle_X + \varepsilon_n \langle J_X(u_n), u - u_n\rangle_X \\
&\Longrightarrow \\
\alpha_{\mathscr{A}}\beta_{\mathscr{A}}\|u_n - u\|_X^2 &\leq \langle \mathscr{A}(v, u) - \mathscr{A}(v_n, u), u - u_n\rangle_X + \varepsilon_n \langle J_X(u_n), u_n - u\rangle_X.
\end{aligned} \tag{50}$$

By using hypothesis **(D)**(ii) and taking the upper limit as n approaches infinity on the above inequality, we obtain

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \inf \alpha_{\mathscr{A}} \beta_{\mathscr{A}} \|u_n - u\|_X^2 \\ &\leq \lim_{n\to\infty} \sup \|u_n - u\|_X^2 \\ &\leq \langle \mathscr{A}(v,u) - \mathscr{A}(v_n,u), u - u_n \rangle_X + \lim_{n\to\infty} \sup \varepsilon_n \|u_n\|_X \|u_n - u\|_X \\ &\leq 0. \end{aligned}$$

This implies that

$$u_n \to u \in X \text{ as } n \to \infty.$$

On the other hand, it has

$$v_n \to v \in Y \text{ as } n \to \infty.$$

□

4. Optimal Control

In this section, we explore optimal control for the SNMVIPs. Additionally, we examine and demonstrate the solveability of an optimal control problem that is influenced by the nonlinear mixed variational inequality system.

Consider two Banach spaces Z_1 and Z_2 with continuous embeddings from X to Z_1 and from Y to Z_2. Let $u_0 \in Z_1$ and $v_0 \in Z_2$ be two target profiles. We define subspaces $U \subset X^*$ and $V \subset Y^*$ such that the embeddings from U to X^* and V to Y^* are compact. We now examine the ensuing optimal control problem:

Problem 3. *Find $(\gamma^*, \zeta^*) \in U \times V$ such that*

$$\urcorner(\gamma^*, \zeta^*) = \inf_{(\gamma,\zeta)\in U\times V} \urcorner(\gamma, \zeta), \tag{51}$$

in which $\urcorner \colon U \times V \to \mathbb{R}$ is defined as

$$\urcorner(\gamma, \zeta) = \inf_{(u,v)\in\Gamma(\gamma,\zeta)} \left(\frac{\rho}{2} \|u - u_0\|_{Z_1}^2 + \frac{\theta}{2} \|v - v_0\|_{Z_2}^2 \right) + Y(\gamma, \zeta). \tag{52}$$

Here, $\Gamma(\gamma, \zeta)$ denotes the set of solution to Problem 1 for $(\gamma, \zeta) \in X^* \times Y^*$, where the regularized parameters are $\rho > 0$ and $\theta > 0$.

We assume that the function Y satisfies the following conditions:
(K): $Y : U \times V \to \mathbb{R}$ is such that

(i) Y is bounded from beneath;
(ii) Y is coercive on $U \times V$, that is, it maintains

$$\lim_{\substack{(\gamma,\zeta)\in U\times V \\ \|\gamma\|_U + \|\zeta\|_V \to \infty}} Y(\gamma, \zeta) \to +\infty;$$

(iii) Y is weakly lower semicontinuous on $U \times V$, *i.e.*,

$$\lim_{n\to\infty} \inf Y(\gamma_n, \zeta_n) \geq Y(\gamma, \zeta),$$

whenever $\{(\gamma_n, \zeta_n)\} \subset U \times V$ and $(\gamma, \zeta) \in U \times V$ are such that

$$(\gamma_n, \zeta_n) \xrightarrow{w} (\gamma, \zeta) \in U \times V \text{ as } n \to \infty.$$

In this context, we are exploring the existence result for Problem 3.

Theorem 6. *Assume that* **(A), (B), (C), (D)***(ii)–(iv),* **(E),** *and* **(F)***(ii)–(iv) hold. If* **(K)** *and* **(G)** *are also satisfied, then Problem 3 has an optimal control pair.*

Proof. For each fix $(\gamma,\zeta) \in U \times V$, the closedness of $\Gamma(\gamma,\zeta)$ ensures that $(\hat{u},\hat{v}) \in \Gamma(\gamma,\zeta)$ such that

$$\frac{\rho}{2}\|\hat{u}-u_0\|_{Z_1}^2 + \frac{\theta}{2}\|\hat{v}-v_0\|_{Z_2}^2 = \inf_{(u,v)\in\Gamma(\gamma,\zeta)}\left(\frac{\rho}{2}\|u-u_0\|_{Z_1}^2 + \frac{\theta}{2}\|v-v_0\|_{Z_2}^2\right), \tag{53}$$

is attainable.

According to $\daleth$ and **(K)**(i), there exists a minimizing sequence $\{(\gamma_n,\zeta_n)\} \subset U \times V$ such that

$$\lim_{n\to\infty} \daleth(\gamma_n,\zeta_n) = \inf_{(\gamma,\zeta)\in U\times V} \daleth(\gamma,\zeta). \tag{54}$$

We assume that the sequence $\{(\gamma_n,\zeta_n)\}$ is bounded in $U \times V$. To arrive at a contradiction, we suppose that

$$\|\gamma_n\|_U + \|\zeta_n\|_V \to +\infty \ \text{ as } \ n \to \infty.$$

Using the latter with **(K)**(ii), we can conclude that

$$\begin{aligned}\inf_{(\gamma,\zeta)\in U\times V} \daleth(\gamma,\zeta) &= \lim_{n\to\infty} \daleth(\gamma_n,\zeta_n)\\ &\geq \lim_{n\to\infty} \Upsilon(\gamma_n,\zeta_n)\\ &= +\infty.\end{aligned} \tag{55}$$

The result is a contradiction, which means that $\{(\gamma_n,\zeta_n)\}$ is bounded in $U \to V$. Passing to a relabeled subsequence if necessary, we may assume that

$$(\gamma_n,\zeta_n) \xrightarrow{w} (\gamma^*,\zeta^*) \in U \times V \ \text{ as } \ n \to \infty, \ \text{ for some } \ (\gamma^*,\zeta^*) \in U \times V. \tag{56}$$

Let $\{(u_n,v_n)\} \subset \Omega \times \mho$ satisfy (53) by letting $\hat{u} = u_n$, $\hat{v} = v_n$, and $(\gamma,\zeta) = (\gamma_n,\zeta_n)$. We will now prove that $\{(u_n,v_n)\} \subset \Omega \times \mho$ is uniformly bounded in $X \times Y$. A simple computation reveals that

$$\begin{aligned}\flat(\|u_n\|_X,\|v_n\|_Y) \leq{}& \frac{\varrho_{\mathscr{A}}(1+\|u_n\|_X+\|v_n\|_Y)\|u_0\|_X}{\|u_n\|_X} + \frac{(\varrho_\varphi(u)+\|\gamma_n\|_{X^*})\|u_0\|_X}{\|u_n\|_X}\\ &+ \varrho_\varphi(u) + \|\gamma_n\|_{X^*},\end{aligned} \tag{57}$$

and

$$\begin{aligned}\ell(\|v_n\|_Y,\|u_n\|_X) \leq{}& \frac{\varrho_{\mathscr{B}}(1+\|u_n\|_X+\|v_n\|_Y)\|v_0\|_Y}{\|v_n\|_Y} + \frac{(\varrho_\phi(v)+\|\zeta_n\|_{Y^*}\|v_0\|_Y}{\|v_n\|_Y}\\ &+ \varrho_\phi(v) + \|\zeta_n\|_{Y^*}.\end{aligned} \tag{58}$$

Since there is continuity in the embeddings from U to X^* and from V to Y^*, we use the same approach as in the proof of Lemma 3 to show that the sequence $\{(u_n,v_n)\} \subset \Omega \times \mho$ is uniformly bounded in $X \times Y$. Without loss of generality, we can assume that

$$(u_n,v_n) \xrightarrow{w} (u^*,v^*) \in X \times Y, \ \text{ and } \ Z_1 \times Z_2 \ \text{ as } \ n \to \infty, \ \text{ for some } \ (u^*,v^*) \in \Omega \times \mho. \tag{59}$$

Using the Minty approach yields

$$\langle \mathscr{A}(v_n,y), y-u_n\rangle_X + \varphi(u_n,y) - \varphi(u_n,u_n) \geq \langle \gamma_n, y-u_n\rangle_X, \forall y \in \Omega, \tag{60}$$

and

$$\langle \mathscr{B}(u_n, z), z - v_n \rangle_Y + \phi(v_n, z) - \phi(v_n, v_n) \geq \langle \zeta_n, z - v_n \rangle_Y, \forall z \in \mho. \quad (61)$$

The embedding from (U, V) into $(X^*, Y^*)$is compact, and (56) implies that

$$(\gamma_n, \zeta_n) \to (\gamma^*, \zeta^*) \in X^* \times Y^* \text{ as } n \to \infty.$$

Taking the upper limit as $n \to \infty$ for (60) and (61), we obtain

$$\langle \mathscr{A}(v^*, y), y - u^* \rangle_X + \varphi(u^*, y) - \varphi(u^*, u^*) \geq \langle \gamma^*, y - u^* \rangle_X, \forall y \in \Omega,$$

and

$$\langle \mathscr{B}(u^*, z), z - v^* \rangle_Y + \phi(v^*, z) - \phi(v^*, v^*) \geq \langle \zeta^*, z - v^* \rangle_Y, \forall z \in \mho,$$

where we used **(F)**(ii) and **(D)**(ii). Using the Minty trick once more, we accomplish

$$(u^*, v^*) \in \Gamma(\gamma^*, \zeta^*).$$

The weaker lower semicontinuity of $\| \cdot \|_{Z_1}$ and $\| \cdot \|_{Z_2}$, however, suggests that

$$\frac{\rho}{2}\|u^* - u_0\|_{Z_1}^2 + \frac{\theta}{2}\|v^* - v_0\|_{Z_2}^2 \leq \liminf_{n\to\infty}\left[\frac{\rho}{2}\|u_n - u_0\|_{Z_1}^2 + \frac{\theta}{2}\|v_n - v_0\|_{Z_2}^2\right]. \quad (62)$$

Note that Υ is weakly lower semicontinuous on $U \times V$; it implies

$$\Upsilon(\gamma^*, \zeta^*) \leq \liminf_{n\to\infty} \Upsilon(\gamma_n, \zeta_n). \quad (63)$$

Referring to Equations (62) and (63), we have

$$\begin{aligned}
\liminf_{n\to\infty} \daleth(\gamma_n, \zeta_n) &\geq \liminf_{n\to\infty} \inf_{(u,v)\in\Gamma(\gamma_n,\zeta_n)} \left(\frac{\rho}{2}\|u - u_0\|_{Z_1}^2 + \frac{\theta}{2}\|v - v_0\|_{Z_2}^2\right) + \liminf_{n\to\infty} \Upsilon(\gamma_n, \zeta_n) \\
&= \liminf_{n\to\infty}\left(\frac{\rho}{2}\|u_n - u_0\|_X^2 + \frac{\theta}{2}\|v_n - v_0\|_Y^2\right) + \liminf_{n\to\infty} \Upsilon(\gamma_n, \zeta_n) \\
&\geq \frac{\rho}{2}\|u^* - u_0\|_{Z_1}^2 + \frac{\theta}{2}\|v^* - v_0\|_{Z_2}^2 + \Upsilon(\gamma^*, \zeta^*), \ \ (\text{where } (u^*, v^*) \in \Gamma(\gamma^*, \zeta^*)) \\
&\geq \inf_{(u,v)\in\Gamma(\gamma^*,\zeta^*)} \left(\frac{\rho}{2}\|u - u_0\|_{Z_1}^2 + \frac{\theta}{2}\|v - v_0\|_{Z_2}^2\right) + \Upsilon(\gamma^*, \zeta^*) \\
&= \daleth(\gamma^*, \zeta^*).
\end{aligned} \quad (64)$$

We can use Equation (64) along with (54) to arrive at the following conclusion:

$$\daleth(\gamma^*, \zeta^*) \leq \inf_{(\gamma,\zeta)\in U\times V} \daleth(\gamma, \zeta),$$

namely, (γ^*, ζ^*) is an optimal control pair of Problem 3. □

Author Contributions: Conceptualization, H.G. and A.A.H.A.; methodology, H.G.; software, A.A.H.A.; validation, H.G., A.A.H.A. and S.; formal analysis, H.G.; investigation, H.G.; resources, Salahuddin; data curation, H.G.; writing—original draft preparation, S.; writing—review and editing, S.; visualization, S.; supervision, H.G.; project administration, H.G.; funding acquisition, S. All authors have read and agreed to the published version of the manuscript.

Funding: Deputyship for Research & Innovation, Ministry of Education in Saudi Arabia funded this research work through Project Number: ISP-2024.

Data Availability Statement: The datasets used and/or analyzed during the current study are available from the corresponding author upon reasonable request.

Acknowledgments: The authors are grateful to the anonymous referee for their useful comments and suggestions, which have improved the quality of the paper.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Lin, L.J. Systems of generalized quasi-variational inclusions problems with applications with applications to variational analysis and optimization problems. *J. Glob. Optim.* **2007**, *38*, 21–39. [CrossRef]
2. Chang, S.; Salahuddin, S.; Wang, L.; Ma, Z. The convergence results of differential variational inequality problems. *Symmetry* **2022**, *14*, 760. doi.org/10.3390/sym 14040760. [CrossRef]
3. Han, W.; Sofonea, M. *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*; American Mathematical Society, International Press: Providence, RI, USA, 2002.
4. Hlaváček, I.; Haslinger, J.; Nečas, J.; Lovíšek, J. *Solution of Variational Inequalities in Mechanics*; Applied Mathematical Sciences; Springer: New York, NY, USA, 1988; Volume 66.
5. Ceng, L.C.; Yuan, Q. Systems of variational inequalities with nonlinear operators. *Mathematics* **2019**, *7*, 338. doi.org/10.3390/math7040338. [CrossRef]
6. Kheawborisut, A.; Kangtunyakarn, A. Modified subgradient extragradient method for system of variational inclusion problem and finite family of variational inequalities problem in real Hilbert space. *J. Inequal. Appl.* **2021**, *2021*, 53. [CrossRef]
7. Kinderlehrer, D.; Stampacchia, G. *An Introduction to Variational Inequalities and Their Application*; Academic Press: New York, NY, USA, 1980.
8. Abd-Allah, A.A.; Ahmadini, A.A.H.; Salahuddin. Hybrid iterative algorithms for finding common solutions of a system of generalized mixed quasi-equilibrium problems and fixed point problems of nonexpansive semigroups. *Int. J. Nonlinear Anal. Appl.* **2023**, *14*, 2271–2788.
9. Denkowski, Z.; Migórski, S.; Papageorgiou, N.S. *An Introduction to Nonlinear Analysis: Theory*; Kluwer Academic/Plenum Publishers: Boston, MA, USA; Dordrecht, The Netherlands; London, UK; New York, NY, USA, 2003.
10. Gasinski, L.; Papageorgiou, N.S. Nonlinear analysis. In *Mathematical Analysis and Applications*; Chapman & Hall/CRC: Boca Raton, FL, USA, 2006.
11. Liu, Y.; Xu, H.; Lin, G.H. Stability analysis of two-stage stochastic mathematical programs with complementarity constraints via NLP regularization. *SIAM J. Optim.* **2011**, *21*, 669–705. [CrossRef]
12. Zeidler, E. *Nonlinear Functional Analysis and Its Applications:II/B: Nonlinear Monotone Operators*; Springer: Berlin, Germany, 2007.
13. Liu, J.; Yang, X.; Zeng, S.; Zhao, Y. Coupled variational inequalities: Existence, Stability and Optimal Control. *J. Optim. Theo. Appl.* **2022**, *193*, 877–909. [CrossRef]
14. Papageorgiou, N.S.; Kyritsi, S. *Handbook of Applied Analysis*; Springer: New York, NY, USA, 2009.
15. Cai, D.L.; Sofonea, M.; Xiao, Y.B. Convergence results for elliptic variational-hemivariational inequalities. *Adv. Nonlinear Anal.* **2021**, *10*, 2–23. [CrossRef]
16. Liu, Z.H.; Motreanu, D.; Zeng, S.D. Nonlinear evolutionary systems are driven by mixed variational inequalities and their applications. *Nonlinear Anal.* **2018**, *42*, 409–421. [CrossRef]