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Fixed Point of α-Modular Nonexpanive Mappings in Modular Vector Spaces $\ell_{p(\cdot)}$

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Abstract: Let C denote a convex subset within the vector space $\ell_{p(\cdot)}$, and let T represent a mapping from C onto itself. Assume $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index in $[0,1]^n$ such that $\sum_{i=1}^{n} \alpha_i = 1$, where $\alpha_1 > 0$ and $\alpha_n > 0$. We define $T_\alpha : C \to C$ as $T_\alpha = \sum_{i=1}^{n} \alpha_i T^i$, known as the mean average of the mapping T. While every fixed point of T remains fixed for T_α, the reverse is not always true. This paper examines necessary and sufficient conditions for the existence of fixed points for T, relating them to the existence of fixed points for T_α and the behavior of T-orbits of points in T's domain. The primary approach involves a detailed analysis of recurrent sequences in $\mathbb{R}$. Our focus then shifts to variable exponent modular vector spaces $\ell_{p(\cdot)}$, where we explore the essential conditions that guarantee the existence of fixed points for these mappings. This investigation marks the first instance of such results in this framework.

Keywords: electrorheological fluids; fixed point; modular mean–nonexpansive mapping; modular vector spaces; variable exponent spaces

MSC: 47H09; 46B20; 47H10; 47E10

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1. Introduction

The Banach Contraction Principle is a cornerstone in metric fixed-point theory, marking its inception with profound implications. This theorem asserts that any contraction mapping on a complete metric space must have a unique fixed point. Its significance lies in its elegant simplicity but also in its vast applicability across mathematics and other disciplines, serving as the foundational bedrock from which metric fixed-point theory blossoms. It offers essential tools for analyzing the stability and convergence of iterative processes, facilitating solutions to equations and systems in various scientific fields.

The fixed-point problem for nonexpansive mappings represents a natural and significant extension of the class of mappings beyond contraction mappings. Nonexpansive mappings, which do not contract distances between points, offer a broader and more complex challenge in identifying fixed points within a space. This extension is crucial because it encompasses a wider array of applications and theoretical scenarios, allowing for the exploration of fixed points in contexts where the strict contraction condition is relaxed. Studying nonexpansive mappings enriches our understanding of convergence, stability, and the structure of various mathematical and applied problems, highlighting the depth and diversity of metric fixed-point theory. Mean nonexpansive or α-nonexpansive mappings serve as a further extension in the hierarchy of nonexpansive mappings, offering a sophisticated approach to metric fixed-point theory.

We define $\alpha = (\alpha_1, \cdots, \alpha_p)$ for $p \geq 2$, within $[0,1]^p$, fulfilling $\sum_{i=1}^{p} \alpha_i = 1$ and ensuring $\alpha_1 \alpha_p \neq 0$, as a multi-index.

Definition 1 ([1])**.** *Let (M, d) be a metric space and $\alpha = (\alpha_1, \cdots, \alpha_p)$ be a multi-index. A function $T : M \to M$ is said to be an α-nonexpansive (or mean nonexpansive) mapping provided that*

$$\sum_{i=1}^{p} \alpha_i \, d\Big(T^i(x), T^i(y)\Big) \leq d(x, y), \quad \textit{for all} \;\; x, y \in M.$$

An element $x \in M$ is said to be a fixed point of T provided that $T(x) = x$. The set of the fixed points of T will be denoted by $Fix(T)$.

This class of mappings was introduced by the authors in [1]. The most surprising fixed-point result of these new class of mappings is the following:

Theorem 1 ([1])**.** *Let C be a convex subset of a Banach space. If C enjoys the fixed-point property for nonexpansive mappings, then any mean nonexpansive mapping $T : C \to C$, where the multi-index $\alpha = (\alpha_1, \cdots, \alpha_p)$ satisfies $\alpha_1 \sqrt[p-1]{2} \geq 1$, has a fixed point.*

Recall that a metric set is said to have the fixed-point property for nonexpansive mappings if every nonexpansive self-mapping within it has a fixed point. Given this elegant result, the equivalency between the fixed-point properties for nonexpansive mappings and mean nonexpansive mappings remains an unresolved question, particularly within the context of closed convex bounded subsets of a Banach space.

Exploring the fixed-point property for mean nonexpansive mappings significantly involves an auxiliary mapping T_α. Specifically, consider C as a convex subset of a Banach space and $T : C \to C$ a mean nonexpansive mapping, where $\alpha = (\alpha_1, \cdots, \alpha_p)$. Define $T_\alpha : C \to C$ by

$$T_\alpha(x) = \sum_{i=1}^{p} \alpha_i \, T^i(x), \quad \textit{for any} \;\; x \in C.$$

It is evident that $Fix(T) \subseteq Fix(T_\alpha)$. The core focus of the authors in [2] revolves around exploring the inverse scenario, specifically the implications when T_α possesses a fixed point. Some interesting and simple examples dealing with this question are found in the original work [1].

In this investigation, we explore the concept of α-nonexpansive mappings within modular vector spaces $\ell_{p(\cdot)}$ and assess the relevance of the main findings in [1] within this framework. Originating from Orlicz's seminal work in 1931 [3], these variable exponent spaces play a pivotal role in modeling non-Newtonian fluids, such as electrorheological fluids, where viscosity undergoes significant variations under electric or magnetic influences [4,5]. Understanding the variable integrability within these spaces is crucial for their potential applications. Nakano's seminal introduction of modular vector spaces in 1950 [6] greatly propelled the study of $\ell_{p(\cdot)}$ spaces [7].

The investigation of fixed-point theory within the vector spaces $\ell_{p(\cdot)}$ holds considerable interest among mathematicians due to the importance of these variable exponent spaces. The existence of fixed points often proves instrumental in tackling practical problems and elucidating their solutions. Maneuvering through the norms within these spaces can be intricate, whereas working with the modular is inherently more intuitive.

For the study of metric and modular fixed-point theory, we recommend the books [8,9].

2. Notes on Recurrent Sequences

Throughout, we have $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_p) \in [0,1]^p$, with $p \geq 2$, such that $\sum_{i=1}^{p} \alpha_i = 1$, and $\alpha_1 \alpha_p \neq 0$. Consider the vector space $\mathcal{S}_\alpha$ of recurrent sequences $\{x_n\}$ such that

$$x_n - \alpha_1\, x_{n+1} - \alpha_2\, x_{n+2} - \cdots - \alpha_p\, x_{n+p} = 0, \tag{1}$$

for all $n \in \mathbb{N}$. Note that $\mathcal{S}_\alpha$ is the set of the solutions to the higher-order linear difference equation whose characteristic polynomial is

$$P(x) = 1 - \alpha_1\, x - \alpha_2\, x^2 - \cdots - \alpha_p\, x^p.$$

It is well known [10,11] that if $r_1, r_2, \cdots, r_k$ are the roots of $P(x)$ with the respective multiplicities $m_1, m_2, \cdots, m_k$, then for any sequence $\{x_n\}$ in $\mathcal{S}_\alpha$, there exists unique p scalars $\{a_{i,j}\}$, with $j \in [1,k]$ and $i \in [0, m_j - 1]$, such that

$$x_n = \sum_{j=1}^{k} \sum_{i=0}^{m_j - 1} a_{i,j}\, n^i\, r_j^n, \tag{2}$$

for all $n \in \mathbb{N}$. The next technical result discusses the roots of $P(x)$.

Lemma 1. *Consider the polynomial function*

$$P(x) = 1 - \alpha_1\, x - \alpha_2\, x^2 - \cdots - \alpha_p\, x^p.$$

(1) $r_1 = 1$ *is a root of* $P(x)$ *with multiplicity* $m_1 = 1$*;*
(2) *if* r *is a root of* $P(x)$ *such that* $|r| = 1$*, then we must have* $r = 1$*;*
(3) *if* r *is a root of* $P(x)$ *such that* $r \neq 1$*, then we must have* $|r| > 1$*.*

Proof. It is clear that $P(1) = 1 - \sum_{i=1}^{p} \alpha_i = 1 - 1 = 0$. Moreover, we have:

$$P'(1) = -\alpha_1 - 2\alpha_2 - \cdots - p\alpha_p \leq -\alpha_1 < 0.$$

So, $r = 1$ is a simple root of $P(x)$. This proves (1). As for (2), let r be a root of $P(x)$ such that $r \neq 1$ and $|r| = 1$. Let us write $r = e^{i\theta}$. We have

$$\begin{aligned} P(r) &= 1 - \alpha_1\, e^{i\theta} - \alpha_2\, e^{2i\theta} - \cdots - \alpha_p\, e^{pi\theta} \\ &= \Big(1 - \sum_{j=1}^{p} \alpha_j\, \cos(j\theta)\Big) - i \sum_{j=1}^{p} \alpha_j\, \sin(j\theta) = 0. \end{aligned}$$

This will imply

$$1 - \sum_{j=1}^{p} \alpha_j\, \cos(j\theta) = 0 \;\; \textit{and} \;\; \sum_{j=1}^{p} \alpha_j\, \sin(j\theta) = 0.$$

Since

$$1 - \sum_{j=1}^{p} \alpha_j\, \cos(j\theta) = \sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{p} \alpha_j\, \cos(j\theta) = \sum_{j=1}^{p} \alpha_j\, (1 - \cos(j\theta)) = 0,$$

and the numbers are all positive, we deduce that $\cos(\theta) = 1$, which forces $\sin(\theta) = 0$, i.e., $r = 1$ as claimed. As for (3), let r be a root of $P(x)$ such that $r \neq 1$. From (2), we know that $|r| \neq 1$. Assume $|r| < 1$. Since r is a root of $P(x)$, we obtain

$$\alpha_1\, r + \alpha_2\, r^2 - \cdots + \alpha_p\, r^p = 1.$$

Hence,

$$1 = |\alpha_1 \, r + \alpha_2 \, r^2 - \cdots + \alpha_p \, r^p| \leq \sum_{j=1}^{p} \alpha_j \, |r|^j < \sum_{j=1}^{p} \alpha_j = 1.$$

This contradiction forces $|r| > 1$, as claimed in (3). □

Before we jump into some fundamental properties of the recurrent sequences (RSs), let us explain some simple facts about the characteristic polynomial $P(x)$. Indeed, we have

$$-P(x) = -1 + \sum_{j=1}^{p} \alpha_j \, x^j = -\sum_{j=1}^{p} \alpha_j + \sum_{j=1}^{p} \alpha_j \, x^j = \sum_{j=1}^{p} \alpha_j \, (x^j - 1).$$

Using the fact that $x^n - 1 = (x-1)(1 + x + x^2 + \cdots + x^{n-1})$, we obtain

$$-P(x) = \sum_{j=1}^{p} \alpha_j \, (x-1)(1 + x + x^2 + \cdots + x^{j-1}).$$

which implies

$$P(x) = -(x-1) \, \sum_{j=1}^{p} \alpha_j \, (1 + x + x^2 + \cdots + x^{j-1}).$$

A straightforward calculation implies

$$P(x) = -(x-1) \, \sum_{j=1}^{p-1} (\alpha_{j+1} + \alpha_{j+2} + \cdots + \alpha_p) x^j.$$

Set $\beta_j = \alpha_{j+1} + \alpha_{j+2} + \cdots + \alpha_p = 1 - \alpha_1 - \alpha_2 - \cdots - \alpha_j$, for $j \in [1, p-1]$. We have

$$0 < \alpha_p = \beta_{p-1} \leq \beta_{p-2} \leq \cdots \leq \beta_1 = 1 - \alpha_1 < 1.$$

Hence, we have $P(x) = -(x-1)Q(x)$, where $Q(x) = 1 + \sum\limits_{j=1}^{p-1} \beta_j \, x^j$.

Proposition 1. *Under the above notations, we conclude that the roots of $P(x)$ not equal to 1 are the roots of $Q(x)$ and vice versa.*

Let us now consider the vector space $\mathcal{S}_\beta$ of recurrent sequences $\{x_n\}$ such that

$$x_n + \beta_1 \, x_{n+1} + \cdots + \beta_{p-1} \, x_{n+p-1} = 0, \tag{3}$$

for all $n \in \mathbb{N}$.

Proposition 2. *Let $\{x_n\}$ be in $\mathcal{S}_\alpha$. The general form of this sequence is given by (GF). One of the roots is $r = 1$, which is simple. In this case, we have*

$$x_n = a + y_n,$$

for all $n \in \mathbb{N}$, where a is a scalar and $\{y_n\}$ is in $\mathcal{S}_\beta$, i.e., $\mathcal{S}_\beta$ is a linear subspace of $\mathcal{S}_\alpha$ with co-dimension equal to 1.

Theorem 2. *Let $\{x_n\}$ be any sequence in $\mathcal{S}_\beta$. Assume $\{x_n\}$ is bounded. Then, $\{x_n\}$ is constant equal to the zero sequence, i.e., $x_n = 0$, for all $n \in \mathbb{N}$.*

Proof. Let $\{x_n\}$ be in $\mathcal{S}_\beta$. We have $x_n = -\sum_{j=1}^{p-1} \beta_j \, x_{n+j}$ for all $n \in \mathbb{N}$. Set

$$X_n = \begin{pmatrix} x_n \\ x_{n+1} \\ \cdot \\ \cdot \\ x_{n+p-2} \end{pmatrix} \quad and \quad B = \begin{pmatrix} -\beta_1 & -\beta_2 & \cdots & -\beta_{p-2} & -\beta_{p-1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Clearly, we have $X_n = B\, X_{n+1}$ for all $n \in \mathbb{N}$. Next, we compute the characteristic polynomial of the matrix B, $p_B(\lambda) = \det(\lambda I - B)$, which will give the eigenvalues of B. We have

$$p_B(\lambda) = \begin{vmatrix} \lambda + \beta_1 & \beta_2 & \cdots & \beta_{p-2} & \beta_{p-1} \\ -1 & \lambda & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & -1 & \lambda \end{vmatrix}.$$

A straightforward calculation gives

$$p_B(\lambda) = \beta_{p-1} + \beta_{p-2}\, \lambda + \beta_{p-3}\, \lambda^2 + \cdots + \beta_1\, \lambda^{p-2} + \lambda^{p-1},$$

which implies

$$p_B(\lambda) = \lambda^{p-1}\, Q\left(\frac{1}{\lambda}\right).$$

Note that $\lambda = 0$ is not a root of $p_B(\lambda)$. Using the properties of the roots of $Q(x)$, we conclude that all eigenvalues of B are strictly less than 1. Therefore, the sequences of matrices $\{B^n\}$ will converge to the zero-matrix. But we have $X_0 = B^n X_n$ for all $n \in \mathbb{N}$. Since $\{X_n\}$ is an abounded sequence, we conclude that $X_0 = 0$, which forces the sequence $\{x_n\}$ to be equal to 0. □

Remark 1. *Theorem 2 suggests that the conclusion applies solely to sequences of scalars. However, it is possible to modify the proof elegantly for application in normed vector spaces. Indeed, let* $(X, \|.\|)$ *be a normed vector. Consider the normed vector space* $(X^{p-1}, \|.\|_{p-1})$ *where the norm is defined by*

$$\left\| \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{p-1} \end{pmatrix} \right\|_{p-1} = \sum_{i=1}^{p-1} \|x_i\|.$$

Let $\{x_n\}$ *be a sequence of vectors in X such that* $x_n = -\sum_{i=1}^{p} \beta_i x_{n+i}$ *for all* $n \in \mathbb{N}$*, where* (β_i) *satisfies the general assumptions assumed above. Then, the following holds:*

$$X_n = B\, X_{n+1},$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n+1} \\ \cdot \\ \cdot \\ x_{n+p-2} \end{pmatrix} \quad and \quad B = \begin{pmatrix} -\beta_1 & -\beta_2 & \cdots & -\beta_{p-2} & -\beta_{p-1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

which implies $X_0 = B^n X_n$ for all $n \in \mathbb{N}$. If $\{x_n\}$ is bounded in X, and using the properties of matrix B (as described above), we conclude that $X_0 = 0$. Hence, $x_n = 0$ for all $n \in \mathbb{N}$.

As a direct consequence of Theorem 2, we have the following result:

Theorem 3. *Let $\{x_n\}$ be any sequence in $\mathcal{S}_\alpha$. Assume $\{x_n\}$ is bounded. Then, $\{x_n\}$ is a constant sequence.*

Using the approach described in Remark 1, we obtain a vector version of Corollary 3 as well.

Remark 2. *Note that if $\{x_n\}$ is in $\mathcal{S}_\alpha$, then $\{x_n - x_{n+1}\}$ is in $\mathcal{S}_\beta$. Hence, if we assume that $\{x_n - x_{n+1}\}$ is bounded, then we will know that $\{x_n - x_{n+1}\}$ is the zero sequence, i.e., $\{x_n\}$ is a constant sequence.*

In the next section, we use the obtained results to investigate the fixed-point problem for α-modular nonexpansive mappings defined within the $\ell_{p(\cdot)}$ spaces.

3. Variable Exponent Sequence Spaces $\ell_{p(\cdot)}$

In this work, we will investigate the theory of mean nonexpansive mappings or α-nonexpansive mappings defined within the $\ell_{p(\cdot)}$ spaces. This attempt has never been carried out before. We will mainly deal with the main conclusions of [2] from the setting of linear normed vector spaces to the case of the modular structure of $\ell_{p(\cdot)}$.

Our work requires tools from the field of modular fixed-point theory, for which the reader is referred to the book [9].

We initiate this section by outlining basic facts regarding $\ell_{p(\cdot)}$ linear spaces.

Definition 2 ([3,12])**.** *Consider $p : \mathbb{N} \to [1, \infty)$ and define the linear spaces known as $\ell_{p(\cdot)}$ as*

$$\ell_{p(\cdot)} = \left\{ \{x_n\} \subset \mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{\infty} \frac{1}{p(n)} \left| \frac{x_n}{\beta} \right|^{p(n)} < +\infty,\ \textit{for some } \beta > 0 \right\}.$$

Orlicz, in [3], originally introduced these spaces with slightly different terminology and notation. Nakano drew inspiration from these spaces and went on to develop a more comprehensive theory, which is now recognized as modular vector spaces [13,14].

Proposition 3 ([13–16])**.** *For $\ell_{p(\cdot)}$, we define the functional $\varrho : \ell_{p(\cdot)} \to [0, \infty]$ as*

$$\varrho(x) = \varrho\Big(\{x_n\}\Big) = \sum_{n=0}^{\infty} \frac{1}{p(n)} |x_n|^{p(n)}.$$

The function ϱ possesses the following properties:

(1) $\varrho(x) = 0$ if and only if $x = 0$;
(2) $\varrho(\pm x) = \varrho(x)$;
(3) $\varrho(\alpha\, x + \beta\, y) \le \alpha\, \varrho(x) + \beta\, \varrho(y)$ for all $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$

for any $x, y \in \ell_{p(\cdot)}$.

Moreover, ϱ exhibits left-continuity, meaning that for any $x \in \ell_{p(\cdot)}$, $\lim\limits_{\alpha \to 1-} \varrho(\alpha x) = \varrho(x)$. Subsequently, we extend several concepts from the metric setting to the modular case.

Definition 3 ([9])**.**

(1) A sequence $\{x_n\} \subset \ell_{p(\cdot)}$ converges with respect to ϱ to $x \in \ell_{p(\cdot)}$ if and only if $\varrho(x_n - x) \to 0$. It is evident that if a ϱ-limit exists, it is necessarily unique.

(2) *A sequence* $\{x_n\} \subset \ell_{p(\cdot)}$ *is said to be* ϱ*-Cauchy if* $\varrho(x_n - x_m) \to 0$ *as n and m tend to infinity.*
(3) *A subset* $C \subset \ell_{p(\cdot)}$ *is* ϱ*-closed if, for any sequence* $\{x_n\} \subset C$ *that* ϱ*-converges to* x*, it follows that* $x \in C$*.*
(4) *A subset* $C \subset \ell_{p(\cdot)}$ *is said to be* ϱ*-bounded if the supremum of* $\varrho(x - y)$ *for all pairs* $x, y \in C$*, denoted by* $\delta_\varrho(C)$*, is finite, i.e.,* $\delta_\varrho(C) < \infty$*.*

Additionally, ϱ exhibits the Fatou property; in other words, if a sequence $\{x_n\} \subseteq \ell_{p(\cdot)}$ ϱ-converges to x, then for any $y \in \ell_{p(\cdot)}$, the following holds:

$$\varrho(x - y) \leq \liminf_{n \to \infty} \varrho(x_n - y).$$

The Luxemburg norm induced by the modular ϱ on $\ell_{p(\cdot)}$ is expressed as

$$\|x\|_\varrho = \inf\left\{\lambda > 0; \varrho\left(\frac{1}{\lambda}x\right) \leq 1\right\}.$$

Set

$$p^- = \inf_{n \in \mathbb{N}} p(n) \quad and \quad p^+ = \sup_{n \in \mathbb{N}} p(n).$$

Equipped with the Luxemburg norm, $(\ell_{p(\cdot)}, \|.\|_\varrho)$ forms a Banach space. Many of the geometric characteristics typical of Banach spaces remain valid as long as both p^- and p^+ are not equal to 1 or $+\infty$. Specifically, $(\ell_{p(\cdot)}, \|.\|_\varrho)$ is uniform convexity if and only if $1 < p^- \leq p^+ < +\infty$ [16].

Subsequently, we introduce the category of mappings for which we will explore the existence of fixed points.

Definition 4. *Suppose C is a nonempty subset of* $\ell_{p(\cdot)}$*. A mapping* $T : C \to \ell_{p(\cdot)}$ *is*

(1) *Ref. [9]* ϱ*-nonexpansive if*

$$\varrho(T(x) - T(x)) \leq \varrho(x - y)$$

(2) *Ref. [9]* α*-*ϱ*-nonexpansive, where* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_p) \in [0,1]^p$*, with* $p \geq 2$*, such that* $\sum_{i=1}^{p} \alpha_i = 1$*, and* $\alpha_1 \alpha_p \neq 0$*, provided the following holds:*

$$\sum_{i=1}^{p} \alpha_i \, \varrho\left(T^i(x) - T^i(y)\right) \leq \varrho(x - y)$$

for all $x, y \in C$*. A* $x \in C$ *point is a fixed point of T if* $T(x) = x$*. In the sequel, Fix*(T) *will denote the set of all fixed points of T. The sequence* $\{T^n(x)\}$ *is known as the orbit of T at x.*

Clearly, ϱ-nonexpansive mappings are obviously α-ϱ-nonexpansive with respect to any index α. But the converse is not true [2]. Therefore, the class of α-ϱ-nonexpansive is larger, which explains the interest of mathematicians working in metric fixed-point theory investigating these mappings.

4. Main Results

This section discusses the modular version of the main result of [2], which has not been investigated yet. The setting for our investigation is $\ell_{p(\cdot)}$ modular vector spaces. The first result is a simple fact in any abstract vector space.

Proposition 4. *Let C be a nonempty convex subset of $\ell_{p(\cdot)}$, $T : C \to C$ be a map and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_p) \in [0,1]^p$, with $p \geq 2$, such that $\sum_{i=1}^{p} \alpha_i = 1$, and $\alpha_1 \alpha_p \neq 0$. Define $T_\alpha : C \to C$ by*

$$T_\alpha(x) = \sum_{i=1}^{p} \alpha_i\, T^i(x).$$

Let $x \in Fix(T)$; then, we have $T^n(x) = x \in Fix(T_\alpha)$ for all $n \in \mathbb{N}$, i.e., the orbit $\{T^n(x)\}$ is in $Fix(T_\alpha)$.

Remark 3. *In general, it is not clear that an orbit of T is in $Fix(T_\alpha)$, provided this fixed-point set is nonempty. But if T is affine, then we have*

$$T(T_\alpha(x)) = T\Big(\sum_{i=1}^{p} \alpha_i\, T^i(x)\Big) = \sum_{i=1}^{p} \alpha_i\, T(T^i(x)) = \sum_{i=1}^{p} \alpha_i\, T^{i+1}(x) = T_\alpha(T(x)),$$

for any $x \in C$. Hence if $x \in Fix(T_\alpha)$, then the orbit $\{T^n(x)\}$ is in $Fix(T_\alpha)$. This is the main motivation for the authors to consider affine mappings in [2].

The primary focus of researchers studying these kinds of mappings has been to determine whether the nonemptiness of $Fix(T_\alpha)$ will force the existence of a fixed point for T. Proposition 4 makes it evident that to demonstrate $Fix(T)$ is nonempty, one can assume that an orbit of T within $Fix(T_\alpha)$ is not at an extreme assumption.

Theorem 4. *Let C be a nonempty convex ϱ-bounded subset of $\ell_{p(\cdot)}$, $T : C \to C$ be a map and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_p) \in [0,1]^p$, with $p \geq 2$, such that $\sum_{i=1}^{p} \alpha_i = 1$, and $\alpha_1 \alpha_p \neq 0$. Define $T_\alpha : C \to C$ by*

$$T_\alpha(x) = \sum_{i=1}^{p} \alpha_i\, T^i(x).$$

Assume that there exists $z \in C$ such that its orbit $\{T^n(z)\}_{n\in\mathbb{N}}$ is in $Fix(T_\alpha)$. Then, z is a fixed point of T, i.e., $z \in Fix(T)$.

Proof. Let $z \in Fix(T_\alpha)$ such that its orbit $\{T^n(z)\}_{n\in\mathbb{N}}$ is in $Fix(T_\alpha)$. Since C is ϱ-bounded, then $\{T^n(z)\}_{n\in\mathbb{N}}$ is ϱ-bounded. The fact that the orbit is in $Fix(T_\alpha)$ will imply

$$T_\alpha(T^n(z)) = T^n(z), \quad n = 0, 1, \cdots$$

In other words, we have

$$\alpha_1\, T^{n+1}(z) + \alpha_2\, T^{n+2}(z) + \cdots + \alpha_p\, T^{n+p}(z) = T^n(z), \quad n = 0, 1, \cdots \tag{4}$$

Fix $k \in \mathbb{N}$. Define the functional $e_k^* : \ell_{p(\cdot)} \to \mathbb{R}$ by

$$e_k^*(x) = e_k^*\Big(\{x_n\}\Big) = x_k.$$

The functional e_k^* is linear and satisfies

$$|e_k^*(x)| \leq \Big(p(k)\, \varrho(x)\Big)^{1/p(k)},$$

for any $x \in \ell_{p(\cdot)}$. Since $\{T^n(z)\}_{n\in\mathbb{N}}$ is ϱ-bounded, we deduce that $\left\{e_k^*\left(T^n(z)\right)\right\}_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. The equation (RS) implies

$$\alpha_1\, e_k^*\left(T^{n+1}(z)\right) + \alpha_2\, e_k^*\left(T^{n+2}(z)\right) + \cdots + \alpha_p\, e_k^*\left(T^{n+p}(z)\right) = e_k^*\left(T^n(z)\right), \quad n = 0, 1, \cdots$$

i.e., the sequence $\left\{e_k^*\left(T^n(z)\right)\right\}_{n\in\mathbb{N}}$ is in $\mathcal{S}_\alpha$. All the assumptions of Theorem 3 are satisfied to conclude that $\left\{e_k^*\left(T^n(z)\right)\right\}_{n\in\mathbb{N}}$ is a constant sequence. Hence, we have

$$e_k^*\left(T(z)\right) = e_k^*\left(T^0(z)\right) = e_k^*(z).$$

Since k was taken arbitrarily in $\mathbb{N}$, we conclude that $T(z) = z$, i.e., $z \in Fix(T)$. □

5. Conclusions

The assumption of boundedness in Theorem 4 can be made less strict by considering the set $\{^*_k(x); x \in C\}$ to be a bounded subset of $\mathbb{R}$ for every $k \in \mathbb{N}$. It should be noted that the authors did not define the boundedness condition in [2] for Theorem 2 due to the underlying space being a topological vector space.

The existence of a fixed point for T_α can be derived from a comprehensive range of studies [9]. Specifically, when $p^- > 1$ and C is both a ϱ-closed convex and ϱ-bounded nonempty subset of $\ell_{p(\cdot)}$ and $T : C \to C$ is α-ϱ-nonexpansive, it follows that T_α possesses a fixed point [17]. However, it is unclear if possessing any modular geometric property is sufficient to ensure that an orbit of T belongs to $Fix(T_\alpha)$, given that T_α has a fixed point.

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