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Optimal Harvesting Strategies for Timber and Non-Timber Forest Products with Nonlinear Harvesting Terms

Yaning Zhang , Lina Hao and Shan Zhang *

School of Mathematics and Statistics, Changchun University of Technology, Changchun 130012, China; zhangyaningjob@163.com (Y.Z.); haolina@ccut.edu.cn (L.H.)
* Correspondence: zhangshan9039@126.com

Abstract: Forest resources are renewable, and the rational exploitation and utilization of forest resources are not only conducive to sustainable development on a population scale, they can also lead to higher economic benefits. Based on the actual timber harvest problem, this paper establishes the joint harvest model of timber and non-timber with nonlinear harvest items. In the numerical simulation, by comparing the existing proportional harvest model, it is concluded that the optimal harvest strategy of nonlinear harvest items in this paper can obtain larger ecological benefits and be more conducive to the sustainable development of a population. Firstly, using the qualitative theory of ordinary differential equations, the dynamic behavior of the model is studied, and the existence and stability of the equilibrium point of the model are proven. Secondly, the optimal control solution is obtained by using the optimal control theory. Finally, the optimal harvesting strategy of timber and non-timber products is given based on the numerical simulation results, and a comparison of the effects of different parameters on the optimal harvest strategy, which provides a certain theoretical basis for the sustainable development of the ecological economy of forestry, is carried out.

Keywords: biomathematics model; nonlinear harvest; optimal control; non-timber forest products; sustainable timber harvest

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1. Introduction

Population dynamics is one of the branches of biomathematics, which focuses on the quantitative, spatial, and structural dynamics of populations, and can be used to describe the dynamic relationships between populations and their environment and between populations and other populations, as well as to explain, predict, regulate, and control the developmental processes and trends of species [1]. The updating and development of biological population dynamics models have effectively contributed to human knowledge of biological systems. At present, many scholars have established mathematical models to judge and predict the stable development of populations, such as forestry, agriculture, fisheries, the integrated control and management of pests, etc., which have given us practical production knowledge [2–4].

Symmetry in mathematics and its applications are very broad fields. Geometric symmetry plays an important role in design, architecture, and mathematical proofing. Algebraic symmetry helps us better understand and solve a wide range of equations and algebraic problems. In this paper, some concepts of symmetry are referred to, and related journal articles solve some problems in the processing of differential equations [5].

The problem of the optimal control of ecosystems has been one of the hot issues studied by scholars. The maximum sustainable yield policy obtained by studying the optimal harvesting of biological populations is an important ecological management tool to protect the survival of populations and to maximize economic benefits [6,7]. Faustmann's formula is a method used in forest economics to determine the optimal rotation age for a forest stand. While Faustmann's formula is a powerful tool in forest economics, it has limitations. One of the main challenges is accurately predicting future timber prices, costs, and growth rates over long periods, which can be highly uncertain [8]. The use of linear planning methods can use land, forest resources, equipment, labor, capital, etc., effectively, which is achieved by integrating numerous production and operational activities and related complex factors and various needs into the model integration, and these methods can accurately reflect the vertical and horizontal connection between the input and output of various activities in the process of change. This is difficult to achieve by other conventional methods, as it can not only reflect and feedback adjustment after a limited selection, so that the selected decision to complete is very convenient. Now, in the enterprise management of forestry production, many practical problems can be used in linear planning for optimal management [9,10].

Forests are important natural resources, which not only provide the necessary economic support for human production and life but also play a role in regulating the climate, water conservation, air purification, and biodiversity conservation [11,12]. In recent years, the increasing demand of the timber market and the intensification of commercial exploitation have led to the over-harvesting of forest resources, and the balance of timber and forest product supply has been greatly damaged. In order to ensure the sustainable development of forest resources, a series of logging policies have been established in various regions, and it is therefore necessary to develop reasonable timber harvesting strategies to obtain higher ecological benefits. However, because of such restrictions, it may not be economically viable to provide sufficient income for local residents and timber harvesters. Non-timber products such as fruits, seeds, leaves, bark, gum, etc., can also generate significant economic and ecological benefits and can even generate more economic income than timber harvesting or agricultural production. Therefore, the study of the ecological impacts of the utilization of non-timber forest products is also very important for the sustainable use of forest resources and biodiversity conservation [13,14]. Most previous studies have been devoted to timber harvesting, while non-timber harvesting has been studied less frequently and in a discontinuous manner. For example, in 2011, Isabel B et al. [15] modeled the ecological impacts of harvesting non-timber products using stage structure matrix models based on projection matrices. Although these models can quantify the effects of different life stages on plant population dynamics, they do not explicitly represent harvest intensity, which poses a challenge to quantify the effects of the harvest intensity on each stage of transition. In addition, matrix models have more parameters than classical logistic growth models and may be difficult to relate to well-established harvesting theories based on logistic growth models.

In 2016, Gaoue et al. [16] developed a joint harvest model for timber and non-timber products based on a logistic timber growth equation with proportional harvest, which is the first continuous theoretical model for non-timber harvest. The model includes harvesting of timber and non-timber products and additional synergistic effects of harvesting on the growth rate of plant populations.

$$\begin{cases} \frac{dx}{dt} = rx(1-\frac{x}{K}) - h_L x \\ \tau\frac{dr}{dt} = r_e - r - \alpha h_N - \beta h_L, \end{cases} \tag{1}$$

It is also shown that the sustainability of timber and non-timber harvests depends on the effect of harvesting on species' growth rates for each type of population, and that there are two states in this differential equation: x denotes the population density, r represents the endogenous growth rate of the plant, and r_e is the maximum growth rate in the absence of harvest. The population x has logistic growth, and K is the environmental holding capacity. h_L is the timber harvest intensity, h_N is the non-timber harvest intensity, τ is the average population longevity, α is the population decay rate due to non-timber harvest, and β is the population decay rate due to timber harvest.

The harvest term of the first differential equation in this model uses proportional harvesting, but in practical ecological problems, the human harvest does not increase infinitely with the increase in harvest effort and biological resources. Therefore, more and more scholars have started to study predation systems with nonlinear harvest rates [17–19].

Therefore, this paper investigates the problem of joint harvesting of timber and non-timber products with nonlinear harvest items based on the current exploitation status of the relevant background of timber and non-timber products in forests, so as to derive the optimal harvesting strategy that is conducive to the conservation and recovery of populations.

2. Model Building

Agnew [20] considered the competition between fishing boats as an example of fishery harvesting, and the nonlinear harvest term was improved by taking into account the competition among fishing boats and the processing time of the captured fishery, and for a specific model that conforms to the logistic growth law, a nonlinear harvest function term of the following form was given after a simplified analysis.

$$H(x,E) = \frac{xE}{1 + a_1 E + a_2 x}, \tag{1a}$$

where E is the harvesting effort of timber products, a_1 is used to measure the multiple user groups with competing interests, and a_2 represents the proportionality constant of the handling time.

The harvesting and processing of timber products is more complicated and requires more processing time, while generally, for timber, permits need to be established and there are fixed contractors, so there is not much competition. Therefore, in this paper, we only consider the timber handling time without considering the competition among multiple user groups with competing interests and simplify the nonlinear harvesting term as follows:

$$H(x,E) = \frac{E_1 x}{1 + ax}, \tag{1b}$$

where x denotes the population density, E_1 is the harvesting effort of timber products, which represents the amount of harvesting over time, and a is the handling time of timber products, such as the transit time, etc. Non-timber products are easier to handle and are represented in the model as proportional harvest, i.e., the amount of harvesting effort for non-timber products E_2.Therefore, we introduced the nonlinear harvesting function into the model established by Gaoue et al. and established a harvesting model for timber and non-timber products with a nonlinear harvesting function:

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{K}) - \frac{E_1 x}{1+ax}, \\ \tau \frac{dr}{dt} = r_e - r - \alpha E_2 - \beta \frac{E_1}{1+ax}. \end{cases} \tag{1c}$$

3. Qualitative Analysis

3.1. Existence of Equilibrium Point

Theorem 1. *Equation (1c) has a trivial equilibrium point* $x_0 = 0$, *and when the condition* $r_e > \alpha E_2 + (1+\beta)E_1$ *is satisfied, the equation has a unique positive equilibrium point* x^*.

Proof. Considering the harvest of timber and non-timber, the dynamics of the long-term range is in quasi-steady state; therefore,

$$r = r_e - \alpha E_2 - \beta \frac{E_1}{1 + ax}, \tag{2a}$$

We use this to reduce Equation (2) into the following single equation:

$$\frac{dx}{dt} = (r_e - \alpha E_2 - \beta \frac{E_1}{1+ax}) x (1 - \frac{x}{K}) - \frac{E_1 x}{1+ax}. \tag{2b}$$

By discussing Equation (3), we have:

(1) The equation always has a trivial equilibrium point $x_0 = 0$.
(2) Otherwise, there is

$$(r_e - \alpha E_2 - \beta \frac{E_1}{1+ax})(1 - \frac{x}{K}) - \frac{E_1}{1+ax} = 0, \tag{2c}$$

From the basic theoretical knowledge of quadratic equations with one variable, the discriminant of the root is

$$\Delta = ((ka-1)(r_e - \alpha E_2) + \beta E_1)^2 + 4ka(r_e - \alpha E_2)(r_e - \alpha E_2 - (1+\beta)E_1) \tag{2d}$$

□

If $\Delta > 0$, the equation has two unequal real roots, and we know that $r_e > \alpha E_2 + (1+\beta)E_1$.

Because the quadratic coefficients of the equation $-a(r_e - \alpha E_2) < 0$, the parabola opens down.

When $r_e > \alpha E_2 + (1+\beta)E_1$,

according to Veda's theorem, $\frac{k(r_e - \alpha E_2 - (1+\beta)E_1}{-a(r_e - \alpha E_2)} < 0$. The equation has two different roots, in which case Equation (3) has a unique positive equilibrium point; thus,

$$\begin{aligned} x^* &= \frac{\sqrt{((ka-1)(r_e - \alpha E_2) + \beta E_1)^2 + 4ka(r_e - \alpha E_2)(r_e - \alpha E_2 - (1+\beta)E_1)}}{2a(r_e - \alpha E_2)} \\ &+ \frac{((ka-1)(r_e - \alpha E_2) + \beta E_1)}{2a(r_e - \alpha E_2)}. \end{aligned} \tag{2e}$$

3.2. Stability of Equilibrium Point

Theorem 2. *Equation (1c) has a locally asymptotically stable equilibrium point x_0 if $r_e < \alpha E_2 + (1+\beta)E_1$; Equation (1c) has a unique locally asymptotically stable positive equilibrium point x^** if *$r_e > \alpha E_2 + (1+\beta)E_1$ is satisfied.*

Proof. (1) The Jacobian matrix at equilibrium x_0 is given by

$$J(x) = [r_e - \alpha E_2 - (1+\beta)E_1], \tag{2f}$$

and it can be obtained that the eigenvalues

$$\lambda = r_e - \alpha E_2 - (1+\beta)E_1 \tag{2g}$$

Thus, where $r_e < \alpha E_2 + (1+\beta)E_1$, the equilibrium is locally asymptotically stable.

The Jacobian matrix x^* at the equilibrium is given by

$$J(x^*) = \left[\frac{x^*}{K} (\frac{KaE_1(1+\beta) + \beta E_1}{(1+ax)^2} + \alpha E_2 - r_e) \right], \tag{2h}$$

and the eigenvalues is

$$\lambda = \frac{x^*}{K} (\frac{KaE_1(1+\beta) + \beta E_1}{(1+ax)^2} + \alpha E_2 - r_e) \tag{2i}$$

where $\frac{x^*}{K} > 0$. We can obtain that

$$r_e > \frac{KaE_1(1+\beta) + \beta E_1}{(1+ax^*)^2} + \alpha E_2, \tag{2j}$$

and the condition can be strengthened by taking the maximum value at $x = 0$,

$$r_e > KaE_1(1+\beta) + \beta E_1 + \alpha E_2 \tag{2k}$$

□

When discussing the existence of the solution, there is a unique positive equilibrium point x^* of the equation if $r_e > \alpha E_2 + (1+\beta)E_1$. At this time, if the positive equilibrium is stable, the condition $r_e > KaE_1(1+\beta) + \beta E_1 + \alpha E_2$ will also be satisfied, and the above analysis holds.

In summary, when the condition $r_e < \alpha E_2 + (1+\beta)E_1$ is satisfied, Equation (1c) has a locally asymptotically stable trivial equilibrium point x_0. Harvesting timber and non-timber in this case leads to a high decay rate of the plant population, which will eventually lead to extinction if the plant population keeps growing at such a growth rate; when condition $r_e > \alpha E_2 + (1+\beta)E_1$ is satisfied, the equation has a unique positive equilibrium point x^*, and x^* is locally asymptotically stable. The plant growth rate is sustainable when combined timber and non-timber harvesting is carried out in this case.

4. Optimal Harvesting Strategy

In this section, we consider the issue of harvesting timber and non-timber products. The economic revenue benefits are maximized while minimizing the harvesting costs of timber and non-timber products, while ensuring the sustainability of system populations. Economic benefits are considered in the objective generalization to establish the optimal control problem.

$$\begin{aligned} \max_{E_1,E_2\in U} J(E_1,E_2) = \quad & A_T x(t) + \int_0^T e^{-\delta t}(Ax(t) + B_1\frac{E_1(t)}{1+ax(t)}x(t) \\ & + B_2E_2(t)x(t) - C_1(\frac{E_1(t)}{1+ax(t)})^2 - C_2E_2^2(t))dt \\ s.t. & \begin{cases} \frac{dx(t)}{dt} = r(t)x(t)(1-\frac{x(t)}{K}) - \frac{E_1(t)}{1+ax(t)}x(t). \\ \tau\frac{dr(t)}{dt} = r_e - r(t) - \alpha\frac{E_1(t)}{1+ax(t)} - \beta E_2(t). \\ 0 \le \mathrm{E}_1(t) \le 0.7, 0 \le \mathrm{E}_2(t) \le 1, \end{cases} \end{aligned} \tag{3a}$$

where J is the objective function of the optimal control problem, and it is desired to find an optimal control pair (E_1^*, E_2^*) so that the objective function is maximized. In addition, the coefficient B_1, B_2 denotes the price of the two types of harvesting, and $B_1\frac{E_1(t)x(t)}{1+ax(t)} + B_2E_2(t)x(t)$ is the corresponding income; $e^{-\delta t}$ denotes the discount term, and a basic problem in the development of renewable resources is to determine the optimal balance between the current harvest and the future harvest. From the perspective of economics, a "time discount" can be used to solve the problem of inter-period economic benefits. Existing studies also show that the change in discount rate will affect the optimal solution of the harvest [21]; $C_1(\frac{E_1(t)}{1+ax(t)})^2 + C_2E_2^2(t)$ is the cost term, and in reference [15], Gaoue et al. used the nonlinear cost term of harvesting and the quadratic form of control. In order to facilitate the comparison of the difference between the harvesting strategies corresponding to nonlinear harvesting and proportional harvesting, the quadratic form of control is also used for the cost term in this paper.

The weighting factor A balances the importance of stock conservation, and $A_T x(t)$ indicates the conservation value of the stock at the end of the harvest. The total economic return is

$$\max_{E_1,E_2\in U} P(E_1,E_2) = \int_0^T e^{-\delta t}(B_1\frac{E_1(t)}{1+ax(t)}x(t) + B_2E_2(t)x(t) - C_1(\frac{E_1(t)}{1+ax(t)})^2 - C_2E_2^2(t))dt, \tag{3b}$$

The control set of the Lebesque measurable function boundary of the equation is

$$u = \left\{(E_1,E_2) \in (L^\infty(0,T))^2 : 0 \le E_1 \le M_1, 0 \le E_2 \le M_2, 0 \le t \le T\right\}, \tag{3c}$$

where M_1, M_2 is an upper bound on the control, for which the control set and the target generalization have appropriate tightness and convexity assumptions to guarantee the existence of optimal control pairs and corresponding states. The Hamiltonian function constructed here is

$$\begin{aligned} H = \; & \int_0^T e^{-\delta t}(Ax(t) + B_1(t)\frac{E_1(t)x(t)}{1+ax(t)} + B_2E_2(t)x(t) - C_1(\frac{E_1(t)}{1+ax(t)})^2 - C_2E_2^2(t)) \\ & +\lambda_x(r(t)x(t)(1-\frac{x}{K}) - \frac{E_1(t)}{1+ax(t)}) + \frac{\lambda_r}{\tau}(r_e - r(t) - (\alpha E_2(t) + \beta\frac{E_1(t)}{1+ax(t)})dt. \end{aligned} \tag{3d}$$

The concomitant function is derived from Pontryagin's principle of maximum value as

$$\begin{aligned} \lambda'_x = -\frac{\partial H}{\partial x} &= -e^{-\delta t}(A + \frac{B_1E_1(t)}{(1+ax(t))^2} + B_2E_2(t) + \frac{2aC_1E_1(t)}{(1+ax(t))^3}) \\ & -\lambda_x(r(t) - \frac{2x(t)r(t)}{K} - \frac{E_1}{(1+ax(t))^2}) + \frac{\lambda_r\beta E_1(t)a}{\tau(1+ax(t))^2}, \\ \lambda'_r = -\frac{\partial H}{\partial r} &= -\lambda_x x(t)(1-\frac{x(t)}{K}) + \frac{\lambda_r}{\tau}. \end{aligned} \tag{3e}$$

The transversality conditions are

$$\lambda_x(T) = A_T, \lambda_r(T) = 0. \tag{3f}$$

Also, from $\frac{\partial H_2}{\partial E_1} = 0$, $\frac{\partial H_2}{\partial E_2} = 0$, it can be derived that

$$\begin{cases} e^{-\delta t}(\frac{B_1x(t)}{1+ax(t)} - \frac{2C_1E_1(t)}{1+ax(t)}) - \frac{\lambda_x x(t)}{1+ax(t)} + \frac{\lambda_r\beta}{\tau(1+ax(t))} = 0, \\ e^{-\delta t}(B_2x(t) - 2C_2E_2(t) - \frac{\lambda_r\alpha}{\tau} = 0. \end{cases} \tag{3g}$$

We have the following:

$$\begin{cases} E_1^* = \frac{B_1x(t)(1+ax(t) - e^{\delta t}(1+ax(t)(\lambda_x x(t) - \lambda_r\beta)}{2C_1}, \\ E_2^* = \frac{B_2x(t) - \alpha\frac{\lambda_r}{\tau}e^{\delta t}}{2C_2}. \end{cases} \tag{3h}$$

For the timber harvesting problem with a nonlinear harvesting term, the objective function can be maximized when the optimal control pair (E_1^*, E_2^*) takes the value of J, so that the economic and ecological benefits are optimized.

5. Numerical Simulation

In this section, the optimal harvesting strategy is simulated numerically, and the numerical results of the optimal harvesting strategy with a nonlinear harvesting term are compared with proportional harvesting, and parameter sensitivity analysis is performed. The values of the parameters are shown in Table 1.

Table 1. Notation for the model and optimal control and value.

	Values	Definition	References
x_0	80	Initial population density at $t = 0$(each−1)	[22]
K	100	Carrying capacity for the plant (each−1)	[22]
r_e	0.03, 0.25	Maximum growth rate without harvest (years−1)	[23,24]
τ	1, 5, 10, 20	Average lifespan of the plant in years (years−1)	Estimated
α	0.4	Growth decay rate for non-timber harvest	[25]
β	0.23	Growth decay rate due to timber harvest	[26]
A	0, 0.1	Weight for the value of conservation	[22]
A_T	0, 0.1	Weight for the value of conservation at the end of harvest	[22]
δ	0.05	Discount rate	[22]
B_1	0.3	Benefit from timber harvest (1 × 10−2 dollar)	[22]
B_2	0.15	Benefit from non-timber harvest (1 × 10−2 dollar)	[22]
C_1	15	Cost coefficient of timber harvesting (dollar−1)	[22]
C_2	15	Cost coefficient of non-timber harvest (dollar−1)	[22]
a	0.01, 0.1	processing time for timber harvesting (year−1)	Estimated
M_1	0.7	Upper bound for timber harvest rate	[22]
M_2	1	Upper bound for non-timber harvest rate	[22]

5.1. The Effect of Proportional Harvesting and Nonlinear Harvesting on the Optimal Harvesting Strategy

1. Some parameters of proportional harvesting and nonlinear harvesting are taken as $T = 5, \tau = 5, r_e = 0.25, a = 0.1, A = 0.1, A_T = 0.1, \delta = 0.05$. See Table 1 for other parameters.

In the actual harvest problem, there are many influencing factors, and it cannot be a simple linear harvest, so this paper adopts the nonlinear harvest to build the model and uses the optimal control theory to study the optimal capture problem. Figure 1 shows the harvesting strategy with proportional harvesting, and it can be seen that the timber harvesting starts around the fourth year after the non-timber harvesting starts, and at the end of harvesting, the timber population decreases by about 10%, and the objective function of the optimal control problem has a maximum value of $J_{\max} = 60.779, p_{\max} = 19.783$. Figure 2 shows the nonlinear harvesting strategy, and it can be seen that when nonlinear harvesting is used, the timber harvesting starts around year 3.7, and at the end of harvesting, the population size is the maximum value of the objective function of the optimal control problem is $J_{\max} = 60.2475, p_{\max} = 16.8275$. Compared with proportional harvesting, the economic benefit is reduced by 14.87%, but the economic benefit plus ecological benefit is only reduced by 0.8%. This shows that compared with the existing results of proportional harvest, the nonlinear harvest method has greatly improved the ecological benefits, which is more conducive to the recovery of the population.

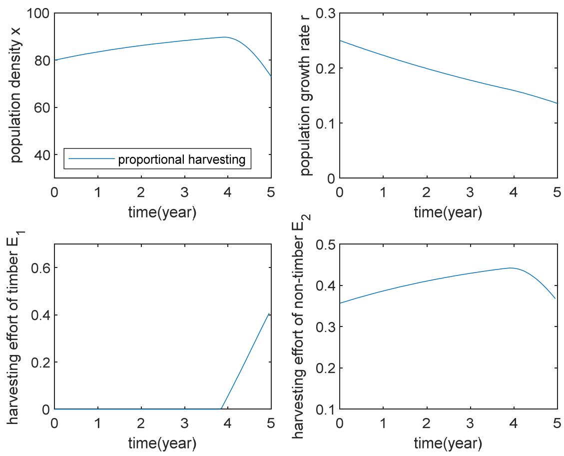


Figure 1. Optimal solution with proportional harvesting.

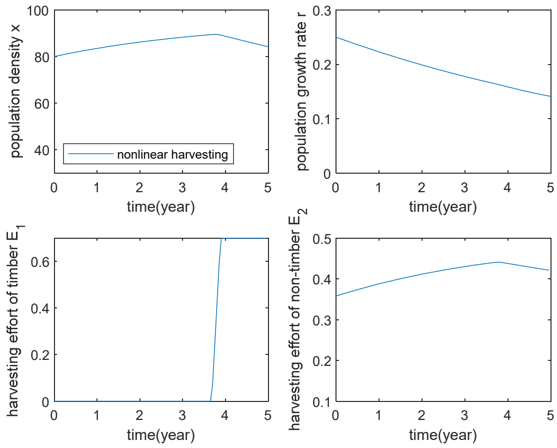


Figure 2. Optimal solution with nonlinear harvesting.

5.2. Sensitivity Analysis of Parameters

Sensitivity analysis is a test of the sensitivity to changes in the parameters of the model results and can help evaluate the stability and reliability of the model. In this section, the sensitivity of the objective function with respect to each parameter is calculated using the partial rank correlation coefficient (PRCC) method for the optimal control problem.

Figure 3 shows the parameter sensitivity analysis of J, and it can be seen that the importance of population conservation at the end of harvest A_T has a greater impact on the objective function; at the same time, the timber treatment time a also has an impact on the objective function, indicating that it is necessary for us to consider the timber treatment time; the price and cost of non-timber B_2, C_2 also have a greater impact on the sensitivity of J, so a combined harvest of timber and non−timber can yield increased benefits.

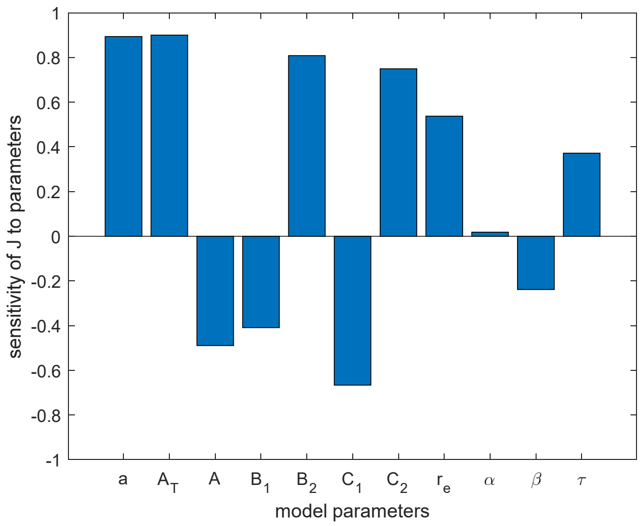


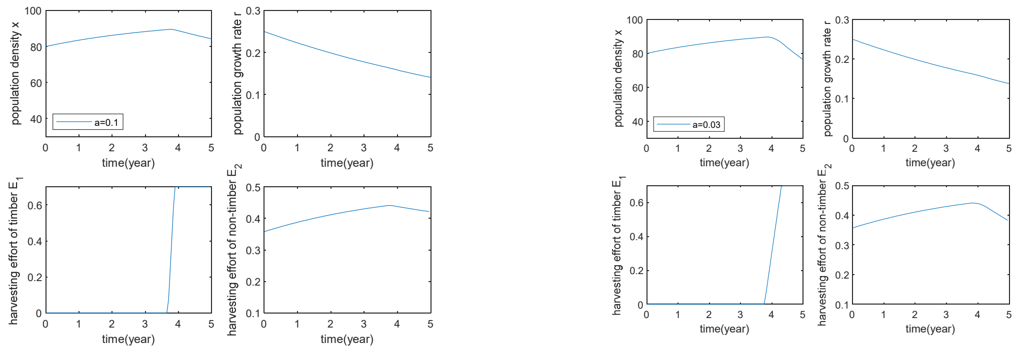
Figure 3. Sensitivity analysis of the objective value J.

5.3. Effect of Different Parameters on the Optimal Harvesting Strategy

Based on the conclusion in Section 5.2., this section specifically examines the effects of the timber treatment time a, population growth rate r_e, species lifespan τ, importance of species conservation during harvest A, and species conservation at the end of harvest A_T, which affects the optimal harvesting strategy. The price of timber B_1, the cost of timber harvesting C_1, the price of non-timber B_2, and the cost of non-timber harvesting C_2 are influenced by the market and are not specifically analyzed here.

1. To determine the effect of the timber processing time on the optimal control strategy, some parameters are taken as $T = 5, \tau = 5, r_e = 0.25, A = 0.1, A_T = 0.1$. See Table 1 for other parameters.

The objective functions of the optimal harvesting strategies for different timber treatment times are $J_{\max} = 60.2475, p_{\max} = 16.8275$ $J_{\max} = 60.6665, p_{\max} = 19.7374$ As shown in Figure 4 when the nonlinear harvest is used, the lower the processing time for the timber is, the higher the economic benefit is, the lower the ecological benefit is, and the smaller the population is. Therefore, if you want to harvest with high economic benefits, try to choose species with short processing times, and if you want to have high ecological benefits, try to choose species with long processing times.

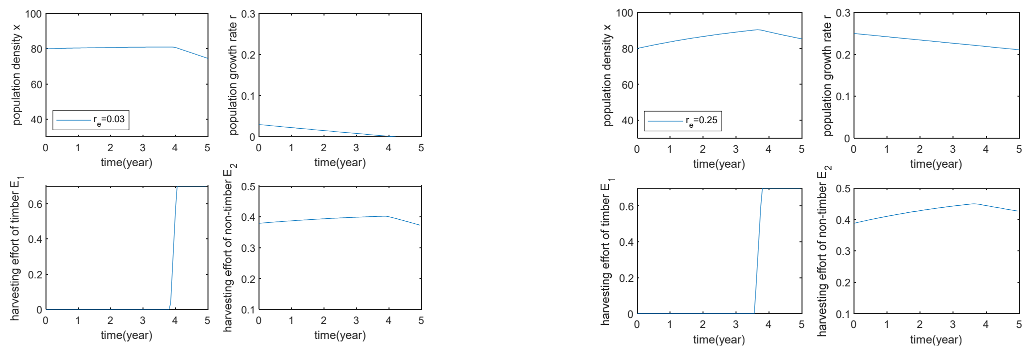


(**a**) a=0.1 (**b**) $a = 0.03$.

Figure 4. Effect of optimal harvest for different timber treatment times.

2. To determine the effect of the plant growth rate on the optimal control strategy, some parameters are taken as $T = 5, \tau = 20, a = 0.1, A = 0.1, A_T = 0.1$. See Table 1 for other parameters.

The objective functions for slow-growing and fast-growing trees are as follows: $J_{\max} = 54.9543, p_{\max} = 14.8346$, $J_{\max} = 60.918 p_{\max} = 17.1486$. As shown in Figure 5 the plants with a fast growth rate can obtain more economic and ecological benefits. Slow-growing plants can start harvesting timber earlier and do not reduce the biological population, or they decrease less, while enabling harvesting of more non−timber products. Therefore, selecting species with fast plant growth as far as possible can obtain higher economic and ecological benefits.



(**a**) $r_e = 0.03$ (**b**) $r_e = 0.25$

Figure 5. Effect of optimal harvest for fast-growing species and slow-growing species.

3. To determine the effect of different species' lifespan on the optimal control strategy, some parameters are taken as $T = 5, r_e = 0.25, a = 0.1, A = 0.1, A_T = 0.1$. See Table 1 for other parameters.

The species objective function for the short and long harvest lifespans are $J_{\max} = 60.2457, p_{\max} = 16.8275$; $J_{\max} = 60.9181, p_{\max} = 17.1486$; at the same time, it can be analyzed through Figure 6 the length of life has little effect on the optimal harvest strategy. Species with different lifespans also approximate populations at the end of the harvest. However, the population growth rate at the last moment of harvest is much lower than the long-lifespan species, indicating that timber harvest has a greater impact on short-lived timber.

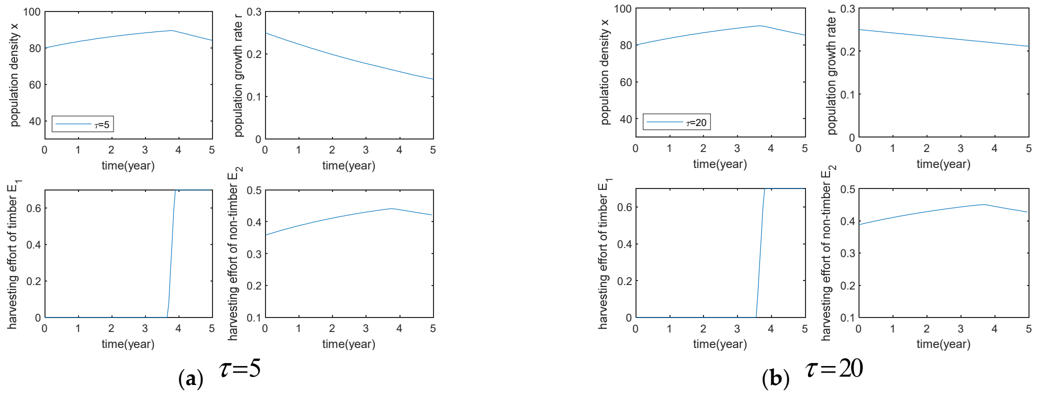


Figure 6. Effect of optimal harvest for short- and long-lived species.

4. To determine the importance of species conservation and the influence of the importance of species conservation at the end of harvest on the optimal control strategy, some parameters are taken as $T = 5, \tau = 5, r_e = 0.25, a = 0.01$. See Table 1 for other parameters.

By comparing (b) with (c) in Figure 7, it is concluded that species with lower importance for species conservation are harvested from the beginning and maintain the harvest behavior until the end of the harvest, when the population decreases more. By comparing (c) with (d), it can be concluded that species with higher conservation importance begin wood harvesting several years later, and that fewer populations decrease at the end of the harvest. Therefore, species with low conservation importance impact the harvest.

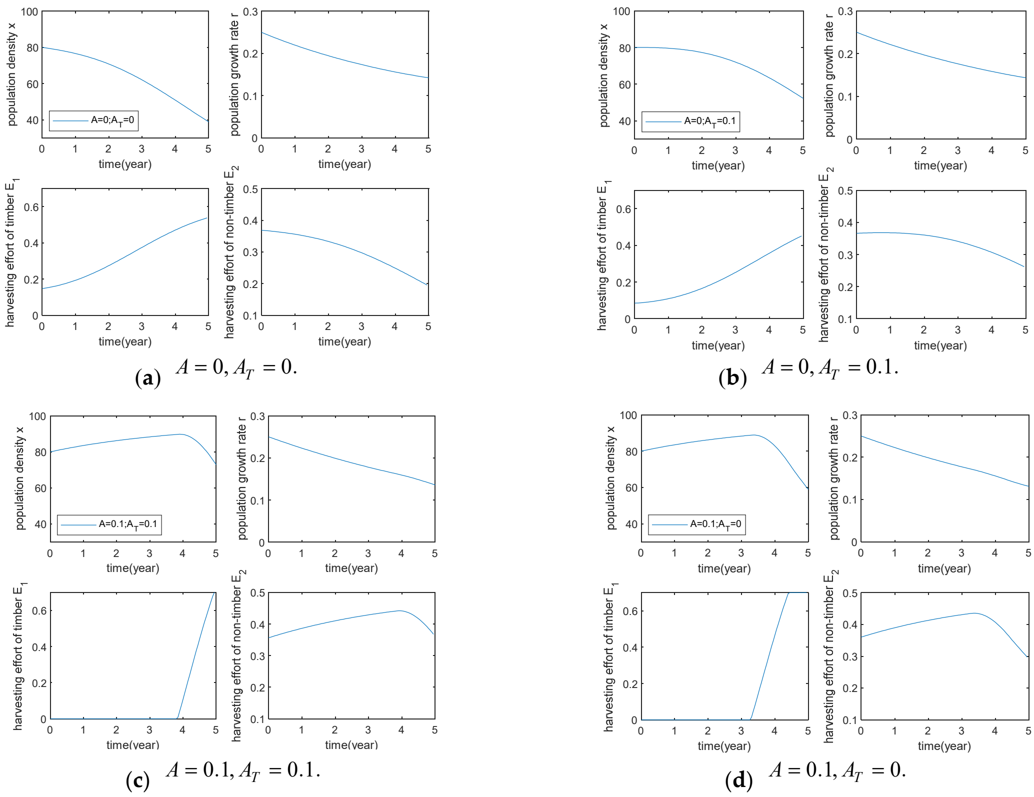


Figure 7. Effect of optimal harvest for conservation weight coefficient A and conservation weight coefficient at the end of the harvest A_T.

6. Conclusions

Most of the current harvest studies are based on proportional harvesting. In this paper, we assess timber and non−timber products in nonlinear harvest projects and present the optimal harvest strategy. Compared with the existing proportional harvest study results, the optimal harvest strategy given under this nonlinear harvest model can lead to higher ecological benefits and be more conducive to the sustainable development of the population. The combination of the nonlinear cutting period of timber and the nonlinear cutting strategy can ensure higher ecological benefits. The numerical simulation results provide theoretical suggestions for the harvest strategy and provide a theoretical basis for the sustainable development of forestry's ecological economy.

We studied the dynamic behavior of the model. It was concluded that when combined harvesting was carried out, there was a unique locally asymptotically stable positive equilibrium point in Equation (1c) when the condition $r_e > \alpha E_2 + (1+\beta)E_1$ was satisfied, and the plant growth rate was sustainable at that time for both timber and non−timber harvesting. When $r_e < \alpha E_2 + (1+\beta)E_1$, there was only one trivial equilibrium point in Equation (1c), and harvesting timber and non−timber resulted in a large decay rate of the plant population, which eventually led to extinction if the plant population kept growing at such a growth rate.

For the optimal control problem of combined timber and non−timber harvesting with nonlinear harvesting, expression (3h) of the optimal control strategy was obtained using the Hamilitonian function and Pontryagin's optimality principle, and numerical simulations were carried out using MATLAB. Compared with the existing proportional harvest results, the optimal harvesting strategy with a nonlinear harvesting term can lead to higher ecological benefits and more sustainable development of the population.

Through the parameter sensitivity analysis, it is concluded that the price and cost of timber and non−timber, B_2 and C_2, have a great impact on the harvest benefit, so more economic and ecological benefits can be obtained from the combined harvest of timber and non−timber. Considering the specific effects of parameters on the harvest strategy, the analysis concluded that higher economic and ecological benefits can be obtained for species with fast growth rates and low importance to species conservation. However, the timber's livespan τ had little effect on the optimal harvesting strategy. The numerical simulation provided theoretical suggestions for the harvest strategy.

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