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Some Classes of Bazilevič-Type Close-to-Convex Functions Involving a New Derivative Operator

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Abstract: In the present paper, we are merging two interesting and well-known classes, namely those of Bazilevič and close-to-convex functions associated with a new derivative operator. We derive coefficient estimates for this broad category of analytic, univalent and bi-univalent functions and draw attention to the Fekete–Szegö inequalities relevant to functions defined within the open unit disk. Additionally, we identify several specific special cases of our results by specializing the parameters.

Keywords: analytic functions; univalent functions; starlike functions; Bazilevič functions; close-to-convex functions; Fekete–Szegö inequalities

MSC: 30C45; 30C50

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1. Introduction

The exploration of univalent functions traces its origins to the initial decades of the twentieth century and stands as a prominent focal point within the realm of complex analysis, commanding considerable interest among researchers. The collection of starlike and convex functions within univalent functions, considered as the most intuitive, has seen numerous fundamental properties established over the past century, yielding a collection of elegant theorems, yet also revealing a substantial number of remaining unresolved problems. Let $\mathcal{A}$ represent the collection of functions expressed in the following form:

$$h(z) = z + \sum_{k=2}^{\infty} c_k z^k, \tag{1}$$

where these functions are analytic within the open unit disk $\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$.

Additionally, define Q_γ $(0 \le \gamma < 1)$ as the collection of functions

$$q(z) = 1 + \sum_{k=1}^{\infty} \ell_k z^k, \tag{2}$$

which are also analytic within $\mathcal{U}$ and satisfy $\mathrm{Re}(q(z)) > \gamma$.

For any function $h \in \mathcal{A}$ given by (1), we define the derivative operator $D_{r,s} : \mathcal{A} \to \mathcal{A}$ as follows:

$$D^{\delta}_{r,s} h(z) = z + \sum_{k=2}^{\infty} \left(\frac{r+k}{r+1} s^{k-1} \right)^{\delta} c_k z^k, \tag{3}$$

where $s > 0$, $r \in \mathbb{N} := \{1, 2, 3, \cdots\}$, $\delta \in \mathbb{N} \cup \{0\} := \mathbb{N}_0$ and $z \in \mathcal{U}$.

Note that $D^0_{r,s}h(z) = h(z)$ and

$$\begin{aligned} D^1_{r,s}h(z) &= z + \sum_{k=2}^{\infty} \left(\frac{r+k}{r+1}\right) s^{k-1} c_k z^k \\ &= \frac{z^{1-r}}{(r+1)s}\left[(r+1)sz^r + \sum_{k=2}^{\infty}(r+k)s^k c_k z^{k+r-1}\right] \\ &= \frac{z^{1-r}}{(r+1)s}\left[sz^{r+1} + \sum_{k=2}^{\infty} s^k c_k z^{k+r}\right]' \\ &= \frac{z^{1-r}}{(r+1)s}\left[z^r\left(sz + \sum_{k=2}^{\infty} c_k s^k z^k\right)\right]' \\ &= \frac{z^{1-r}}{(r+1)s}[z^r h(sz)]'. \end{aligned}$$

A function $h \in \mathcal{A}$ is called starlike in $\mathcal{U}$ if the image of $\mathcal{U}$ under h forms a set that exhibits starlikeness with respect to the origin.

This set of functions is denoted as $\mathcal{M}^*$, and we commence by offering the widely acknowledged analytic characterization of starlike functions concerning functions possessing positive real parts, as given by Duren [1] (also referenced in Thomas et al. [2]). Let $h \in \mathcal{A}$. Then, $h \in \mathcal{M}^*$ if, and only if

$$\mathrm{Re}\left(\frac{zh'(z)}{h(z)}\right) > 0, \quad z \in \mathcal{U}.$$

This class has been expanded by defining a subclass $\mathcal{M}^*(\gamma)$ of starlike functions of order γ $(0 \le \gamma < 1)$ if

$$\mathrm{Re}\left(\frac{zh'(z)}{h(z)}\right) > \gamma, \quad z \in \mathcal{U},$$

and has been shown that the coefficient bounds for functions belonging to $\mathcal{M}^*(\gamma)$ are as follows (see [2] (Section 5.3), [3]):

$$|c_k| \le \frac{\prod_{n=2}^{k}(n - 2\gamma)}{(k-1)!}, \; k = 2, 3, 4, \cdots.$$

Also, for $g(z)$ starlike in $\mathcal{U}$, a function $h \in \mathcal{A}$ is considered close-to-convex if

$$\mathrm{Re}\left(\frac{zh'(z)}{g(z)}\right) > 0, \quad z \in \mathcal{U}.$$

This class of functions has been generalized to include close-to-convex of order γ $(0 \le \gamma < 1)$, denoted by $\mathcal{L}^*(\gamma)$, if

$$\mathrm{Re}\left(\frac{zh'(z)}{g(z)}\right) > \gamma, \quad z \in \mathcal{U}.$$

Both classes of functions, $\mathcal{L}^*(\gamma)$ and $\mathcal{M}^*(\gamma)$, consist only of univalent functions in $\mathcal{U}$.

A function h of a complex variable is considered univalent in $\mathcal{U}$ if it never assumes the same value twice. When h is univalent, we indicate a sub-collection of $\mathcal{A}$ by $\mathcal{S}$. The study of univalent functions is a longstanding and continuously evolving area of research, with roots reaching back more than a century. A significant portion of its historical development is intricately linked to the renowned Bieberbach conjecture [4], which postulates that for coefficients c_k, their absolute values are bounded by k for $k \ge 2$. De Branges' [5] resolution of the Bieberbach conjecture involved the application of sophisticated techniques from various branches of analysis, likely posing a significant open problem for researchers to comprehend.

Bazilevič [6] introduced the class $\mathcal{B}(\lambda, \nu, g)$ of functions, which is defined by the following integral:

$$h(z) = \left\{ \frac{\lambda}{1+\nu^2} \int_0^z (q(\gamma) - i\nu)\gamma^{-\left(1+\frac{i\lambda\nu}{1+\nu^2}\right)} \gamma^{\frac{\lambda}{1+\nu^2}} [g(\gamma)]^{\frac{\lambda}{1+\nu^2}} \, d\gamma \right\}^{\frac{1+i\nu}{\lambda}},$$

where $q \in Q(\equiv Q_0)$ and $g \in \mathcal{M}^*$. The numbers λ and ν are real, where $\lambda \geq 0$, and all powers are to be interpreted as principal values. Aside from the known property of being univalent, our understanding of the class of functions $\mathcal{B}(\lambda, \nu, g)$ is limited. However, in certain special cases, such as when $\nu = 0$ and $g(z) = z$, we arrive at the well-established class $\mathcal{B}(\lambda)$, which fulfills the following condition:

$$\mathrm{Re}\left(\frac{h(z)^{\lambda-1} h'(z)}{z^{\lambda-1}} \right) > 0, \quad z \in \mathcal{U}.$$

Singh [7] investigated a class of univalent functions characterized by functions that adhere to the geometric criterion below:

$$\mathrm{Re}\left(\frac{zh'(z)h(z)^{\lambda-1}}{g(z)^{\lambda}} \right) > 0, \quad z \in \mathcal{U},$$

where $\lambda \geq 0$. This particular class, denoted by $\mathcal{T}(\lambda)$, stands as a special case within the broadly recognized class of generally univalent Bazilevič functions. For $0 \leq \gamma < 1$, Babalola and Saka-Balogun [8] recently extended $\mathcal{T}(\lambda)$ to include a broader class of functions that satisfy the following condition:

$$\mathrm{Re}\left(\frac{zh'(z)h(z)^{\lambda-1}}{g(z)^{\lambda}} \right) > \gamma, \quad z \in \mathcal{U}.$$

Various subclasses of the well-known class of $\mathcal{A}$ and Bazilevič functions $\mathcal{B}(\lambda)$ have been studied in the past and more recently (see, for example, [1,9–15]).

It is noteworthy that the Koebe 1/4-theorem [1] ensures that any function h belonging to the class of univalent functions has an inverse $H := h^{-1}$ such that

$$H(h(z)) = z, \quad z \in \mathcal{D},$$

and

$$h(H(w)) = w, \quad (|w| < r_0(f);\ r_0(f) \geq 1/4).$$

Moreover, $H(w)$ has the Taylor–Maclaurin series of the form

$$H(w) = w + \sum_{k=2}^{\infty} A_k w^k,$$

and for initial values of k, we have

$$A_2 = -c_2, \quad A_3 = 2c_2^2 - c_3 \quad \text{and} \quad A_4 = 5c_2^3 - 5c_2c_3 + c_4. \tag{4}$$

The function $h \in \mathcal{A}$ given by (1) is bi-univalent, denoted by σ, if both h and H are univalent in $\mathcal{U}$. The earliest reference of bi-univalent functions seems to have originated in Lewin's verbal exchange (refer to [16]), where he posed the fundamental inquiry about the sharp upper bounds for $|c_k|$ in the Taylor series expansion of h. This inquiry has since proved to be notably challenging, with relatively scant progress made even after half a century.

Lewin demonstrated that $|c_2|$ is less than 1.51..., while Brannan and Clunie conjectured in [17] that $|c_2| \leq \sqrt{2}$. However, Netanyahu proved in [18] that the best upper bound for $|c_2|$ is $4/3$, for the subclass of σ consisting of all functions that are bi-univalent where their ranges contain the unit disk $\mathcal{U}$. Little significant progress appears to have been made on these early results. The recent groundbreaking work by Srivastava et al. [19], which has been extensively cited, promoted the analysis of bi-univalent functions to a specialized level and led to further studies about the class (see, for example, [20–30]).

Brannan and Taha [31] introduced two interesting subclasses of the function class σ, in analogy to the subclass of starlike functions of order γ of the class $\mathcal{S}$. The function $h(z)$, defined by (1), is said to be in the class $\mathcal{M}^*_\sigma(\gamma)$, the class of bi-starlike functions of order γ $(0 \leq \gamma < 1)$ if each of the following conditions are satisfied:

$$h \in \sigma \quad \text{and} \quad \operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) > \gamma, \quad z \in \mathcal{U},$$

and

$$\operatorname{Re}\left(\frac{wH'(w)}{H(w)}\right) > \gamma, \quad w \in \mathcal{U}.$$

Furthermore, it has been proven that for $h(z) \in \mathcal{M}^*_\sigma(\gamma)$ defined by (1), the following bounds hold:

$$|c_2| \leq \sqrt{2(1-\gamma)}$$

and

$$|c_3| \leq 2(1-\gamma).$$

Motivated by the aforementioned works and the significance of starlikeness and convexity in geometric function theory within a complex analysis, we unify the Bazilevič and close-to-convex functions. We highlight non-sharp bounds and Fekete–Szegö inequalities for analytic functions within the defined classes. The merging of mathematical classes and the derivation operator are common practices in complex analysis and geometric function theory, as well as in the study of special functions like Bessel functions and hypergeometric functions. These techniques are used to analyze the properties of functions under various transformations and operations, providing insights into their behavior and convergence properties. Additionally, Fekete–Szegö inequalities find widespread application in areas such as potential theory, approximation theory, and numerical analysis, offering valuable information about the distribution of zeros or critical points of analytic functions.

Now, employing the derivative operator provided in (3), we introduce the following classes of Bazilevič close-to-convex functions and bi-Bazilevič close-to-convex functions.

Definition 1. *For $h \in \mathcal{A}$ defined as in (1) and $g \in \mathcal{M}^*(\equiv \mathcal{M}^*(0))$, we consider h to be a member of the class*

$$\mathcal{MT}_\lambda(r,\ s,\ \delta;\ \gamma)\ (s > 0,\ r \in \mathbb{N},\ \delta \in \mathbb{N}_0,\ \lambda \geq 1,\ 0 \leq \gamma < 1)$$

if it satisfies the following condition:

$$\operatorname{Re}\left(\frac{z\left(D^\delta_{r,s}h(z)\right)'}{D^\delta_{r,s}h(z)}\left(\frac{z\left(D^\delta_{r,s}h(z)\right)'}{D^\delta_{r,s}g(z)}\right)^{\lambda-1}\right) > \gamma, \quad z \in \mathcal{U}. \tag{5}$$

Remark 1. *Throughout this work, all powers are defined as principal values.*

Remark 2. *It is noteworthy that the class $\mathcal{MT}_\lambda(r,s,\delta;\gamma)$ represents an extension of various previously investigated classes. Here are some examples:*

1. If $\delta = 0$ and $g(z) = z$, then the class $\mathcal{MT}_\lambda(r,s,\delta;\gamma)$ reduces to the class $\mathcal{L}(\lambda,\gamma)$ of λ-pseudo-starlike functions of order γ [32].
2. If $\delta = 0$ and $\gamma = 0$, then the class $\mathcal{MT}_\lambda(r,s,\delta;\gamma)$ reduces to the class $\mathcal{MT}_\lambda$ of λ-pseudo-close-to-convex functions [8].
3. If $\delta = 0$ and $\lambda = 1$, then the class $\mathcal{MT}_\lambda(r,s,\delta;\gamma)$ reduces to the well-known class $\mathcal{M}^*(\gamma)$ of starlike functions of order γ (cf. [1–3,33]).
4. If $\delta = 0$, $\lambda = 1$ and $\gamma = 0$, then the class $\mathcal{MT}_\lambda(r,s,\delta;\gamma)$ reduces to the well-known class $\mathcal{M}^*$ of starlike functions (cf. [1,2,34]).

Definition 2. *A function $h(z)$ given by (1) is said to be in the class*

$$\mathcal{MT}_\lambda^\sigma(r,\ s,\ \delta;\ \gamma)\ (s>0,\ r\in\mathbb{N},\ \delta\in\mathbb{N}_0,\ \lambda\geq 1,\ 0\leq\gamma<1),$$

if it satisfies the following conditions:

$$h\in\mathcal{MT}_\lambda(r,\ s,\ \delta;\ \gamma)\quad\text{and}\quad H\in\mathcal{MT}_\lambda(r,\ s,\ \delta;\ \gamma), \tag{6}$$

where H is the inverse function of h.

Remark 3. *The class $\mathcal{MT}_\lambda^\sigma(r,s,\delta;\gamma)$ extends the scope of various previously studied classes. Let us present some examples [31]:*

1. If $\delta = 0$ and $\lambda = 1$, then the class reduces to the class of bi-starlike functions of order γ.
2. If $\delta = 0 = \gamma$ and $\lambda = 1$, then the class $\mathcal{MT}_\lambda^\sigma(r,s,\delta;\gamma)$ reduces to the class of bi-starlike functions.

2. Auxiliary Lemmas

The establishment of our main findings will be enriched by the inclusion of the following lemmas.

Lemma 1 ([1])**.** *Let $p(z) = 1+\sum_{k=1}^{\infty} d_k z^k \in Q$. Then*

$$|d_k|\leq 2\quad\text{for all}\quad k\in\mathbb{N}.$$

Equality is attained by the Möbius function $\mathcal{L}_0(z) = (1+z)/(1-z)$.

Lemma 2 ([8])**.** *Let $p(z) = 1+\sum_{k=1}^{\infty} d_k z^k \in Q$. Then, we have the sharp inequality*

$$\left|d_2-\varepsilon\frac{d_1^2}{2}\right|\leq 2\max\{1,|\varepsilon-1|\}.$$

Note that, if $q(z) = 1+\ell_1 z+\ell_2 z^2+\cdots\in Q_\gamma$, then

$$q(z) = \gamma+(1-\gamma)p(z) = 1+(1-\gamma)d_1 z+(1-\gamma)d_2 z^2+\cdots. \tag{7}$$

Thus, Lemmas 1 and 2 have been rewritten by Babalola and Saka-Balogun [8], as showcased below:

Lemma 3. *Let $q(z) = 1+\sum_{k=1}^{\infty} \ell_k z^k \in Q_\gamma$. Then*

$$|\ell_k|\leq 2(1-\gamma)\quad\text{for all}\quad k\in\mathbb{N}.$$

Equality is attained by $\mathcal{L}_{0,\lambda}(z) = [1+(1-2\lambda)z]/(1-z)$.

Lemma 4. *Let $q(z) = 1+\sum_{k=1}^{\infty} \ell_k z^k \in Q_\gamma$. Then, we have the sharp inequality*

$$\left|\ell_2 - \varepsilon\frac{\ell_1^2}{2}\right| \le 2(1-\gamma)\max\{1, |(1-\gamma)\varepsilon - 1|\}.$$

Lemma 5 ([35]). *Let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{M}^*$. Then*

$$|b_k| \le k$$

and

$$\left|b_3 - \mu b_2^2\right| \le 2\max\{1, |4\mu - 3|\}.$$

3. Coefficient Properties of Functions in $\mathcal{MT}_\lambda(r, s, \delta; \gamma)$

Theorem 1. *Let the function $h(z)$ be analytic in $\mathcal{U}$ and defined by (1). If $h(z) \in \mathcal{MT}_\lambda(r, s, \delta; \gamma)$, then*

$$|c_2| \le \frac{2\left[(1-\gamma)(1+r)^\delta + (\lambda-1)(2+r)^\delta s^\delta\right]}{(2\lambda-1)(2+r)^\delta s^\delta}$$

and

$$|c_3| \le \begin{cases} \frac{2\left[(1-\gamma)\left[4\left(\gamma+2\left(\frac{2+r}{1+r}\right)^\delta s^\delta\right)\lambda^2+2\left(2-4\gamma-5\left(\frac{2+r}{1+r}\right)^\delta s^\delta\right)\lambda+2\gamma+2\left(\frac{2+r}{1+r}\right)^\delta s^\delta-1\right]+E\right]}{F}, & \lambda \in \left[1, 1+1/\sqrt{2}\right] \\ \frac{2\left[(1-\gamma)\left[4\left(1+2\left(\frac{2+r}{1+r}\right)^\delta s^\delta\right)\lambda^2-2\left(2+5\left(\frac{2+r}{1+r}\right)^\delta s^\delta\right)\lambda+2\left(\frac{2+r}{1+r}\right)^\delta s^\delta+1\right]+E\right]}{F}, & \lambda \in [1+1/\sqrt{2}, \infty) \end{cases}$$

where

$$E := (\lambda-1)\left[12\lambda^2 - \left(12 + 2\left(\frac{2+r}{1+r}\right)^{2\delta} s^{2\delta}\right)\lambda + 3\right], \tag{8}$$

$$F := (3\lambda-1)(2\lambda-1)^2\left(\frac{3+r}{1+r}\right)^\delta s^{2\delta}. \tag{9}$$

Proof. For $h(z) \in \mathcal{MT}_\lambda(r, s, \delta; \gamma)$, there exists $q(z) = 1 + \sum_{k=1}^{\infty} \ell_k z^k \in Q_\gamma$ such that

$$\frac{z\left(D_{r,s}^\delta h(z)\right)'}{D_{r,s}^\delta h(z)}\left(\frac{z\left(D_{r,s}^\delta h(z)\right)'}{D_{r,s}^\delta g(z)}\right)^{\lambda-1} = q(z). \tag{10}$$

By careful computation, the left side of (10) expands as follows:

$$\begin{aligned} \frac{z\left(D_{r,s}^\delta h(z)\right)'}{D_{r,s}^\delta h(z)}\left(\frac{z\left(D_{r,s}^\delta h(z)\right)'}{D_{r,s}^\delta g(z)}\right)^{\lambda-1} &= 1 + \left[(2\lambda-1)\left(\frac{2+r}{1+r}\right)^\delta s^\delta c_2 - (\lambda-1)\left(\frac{2+r}{1+r}\right)^\delta s^\delta b_2\right] z \\ &+ \left[(3\lambda-1)\left(\frac{3+r}{1+r}\right)^\delta s^{2\delta} c_3 + [2\lambda(\lambda-1) - (2\lambda-1)]\left(\frac{2+r}{1+r}\right)^{2\delta} s^{2\delta} c_2^2 - (\lambda-1)\left(\frac{3+r}{1+r}\right)^\delta s^{2\delta} b_3\right. \\ &\left. + \frac{\lambda^2-\lambda}{2}\left(\frac{2+r}{1+r}\right)^{2\delta} s^{2\delta} b_2^2 - \left(2\lambda^2 - 3\lambda + 1\right)\left(\frac{2+r}{1+r}\right)^{2\delta} s^{2\delta} c_2 b_2\right] z^2 + \cdots. \end{aligned}$$

Upon comparing the coefficients of both sides of (10), we find

$$2\lambda(2+r)^{\delta}s^{\delta}c_2 = \ell_1(1+r)^{\delta} + (2+r)^{\delta}s^{\delta}c_2 + (\lambda-1)(2+r)^{\delta}s^{\delta}b_2,$$

and thus, we obtain

$$c_2 = \frac{\ell_1(1+r)^{\delta} + (\lambda-1)(2+r)^{\delta}s^{\delta}\mathrm{b}_2}{(2\lambda-1)(2+r)^{\delta}s^{\delta}}. \tag{11}$$

Therefore, by employing Lemmas 3 and 5, we achieve the desired bound on $|c_2|$. Additionally, upon further comparison of coefficients, we have

$$3\lambda(3+r)^{\delta}s^{2\delta}c_3 + 2\lambda(\lambda-1)\frac{(2+r)^{2\delta}}{(1+r)^{\delta}}s^{2\delta}c_2^2 = (3+r)^{\delta}s^{2\delta}c_3 + \ell_2(1+r)^{\delta} + (2+r)^{\delta}s^{\delta}\ell_1c_2$$
$$+(\lambda-1)\left((3+r)^{\delta}s^{2\delta}b_3 + \frac{(2+r)^{2\delta}}{(1+r)^{\delta}}s^{2\delta}c_2b_2 + (2+r)^{\delta}s^{\delta}\ell_1b_2 + \frac{(\lambda-2)(2+r)^{2\delta}}{2(1+r)^{\delta}}s^{2\delta}b_2^2\right),$$

and this leads to

$$(3\lambda-1)(3+r)^{\delta}s^{2\delta}c_3 = \ell_2(1+r)^{\delta} + (2+r)^{\delta}s^{\delta}\ell_1c_2 + (\lambda-1)\Big((3+r)^{\delta}s^{2\delta}b_3$$
$$+\frac{(2+r)^{2\delta}}{(1+r)^{\delta}}s^{2\delta}c_2b_2 + (2+r)^{\delta}s^{\delta}\ell_1b_2 + \frac{(\lambda-2)(2+r)^{2\delta}}{2(1+r)^{\delta}}s^{2\delta}b_2^2\Big) - 2\lambda(\lambda-1)\frac{(2+r)^{2\delta}}{(1+r)^{\delta}}s^{2\delta}c_2^2. \tag{12}$$

Substituting the value of c_2 from (11) in (12) and arranging the equation, we have

$$(3\lambda-1)(3+r)^{\delta}s^{2\delta}c_3 = (1+r)^{\delta}\left(\ell_2 - \frac{2\lambda^2-4\lambda+1}{(2\lambda-1)^2}\ell_1^2\right) + (\lambda-1)\left((3+r)^{\delta}s^{2\delta}b_3 - \frac{\lambda(2+r)^{2\delta}}{2(2\lambda-1)^2(1+r)^{\delta}}s^{2\delta}b_2^2\right)$$
$$+\frac{4\lambda^2-5\lambda+1}{(2\lambda-1)^2}(2+r)^{\delta}s^{\delta}\ell_1b_2. \tag{13}$$

Noting that $4\lambda^2-5\lambda+1 \geqq 0$, hence, for $\lambda \geq 1$, we obtain

$$(3\lambda-1)(3+r)^{\delta}s^{2\delta}|c_3| \leq (1+r)^{\delta}\left|\ell_2 - \frac{2\lambda^2-4\lambda+1}{(2\lambda-1)^2}\ell_1^2\right| + (\lambda-1)\left|(3+r)^{\delta}s^{2\delta}b_3 - \frac{\lambda(2+r)^{2\delta}}{2(2\lambda-1)^2(1+r)^{\delta}}s^{2\delta}b_2^2\right|$$
$$+\frac{4\lambda^2-5\lambda+1}{(2\lambda-1)^2}(2+r)^{\delta}s^{\delta}|\ell_1||b_2|. \tag{14}$$

Now, applying Lemmas 3, 4, and 5 on (14), we obtain

$$(3\lambda-1)(3+r)^{\delta}s^{2\delta}|c_3| \leq 4(1-\gamma)(2+r)^{\delta}s^{\delta}\frac{4\lambda^2-5\lambda+1}{(2\lambda-1)^2} + 2(1+r)^{\delta}(1-\gamma)\max\left\{1, \left|\frac{4\gamma\lambda^2+(4-8\gamma)\lambda+2\gamma-1}{(2\lambda-1)^2}\right|\right\}$$
$$+2(1+r)^{\delta}(\lambda-1)\max\left\{1, \left|\frac{12\lambda^2 - \left(12+2\left(\frac{2+r}{1+r}\right)^{2\delta}s^{2\delta}\right)\lambda+3}{(2\lambda-1)^2}\right|\right\}. \tag{15}$$

Furthermore, it is easy to see that

$$\max\left\{1, \left|\frac{12\lambda^2 - \left(12 + 2\left(\frac{2+r}{1+r}\right)^{2\delta} s^{2\delta}\right)\lambda + 3}{(2\lambda-1)^2}\right|\right\} = \frac{12\lambda^2 - \left(12 + 2\left(\frac{2+r}{1+r}\right)^{2\delta} s^{2\delta}\right)\lambda + 3}{(2\lambda-1)^2} \geq 1, \quad \lambda \geq 1,$$

and

$$\max\left\{1, \left|\frac{4\gamma\lambda^2 + (4-8\gamma)\lambda + 2\gamma - 1}{(2\lambda-1)^2}\right|\right\} = \begin{cases} \frac{4\gamma\lambda^2+(4-8\gamma)\lambda+2\gamma-1}{(2\lambda-1)^2}, & \lambda \in \left[1, 1+1/\sqrt{2}\right] \\ 1, & \lambda \in [1+1/\sqrt{2}, \infty) \end{cases}.$$

Thus, when $\lambda \in \left[1, 1+1/\sqrt{2}\right]$, Equation (15) reduces to

$$(3\lambda-1)(3+r)^\delta s^{2\delta}|c_3| \leq 4(1-\gamma)(2+r)^\delta s^\delta \frac{4\lambda^2 - 5\lambda + 1}{(2\lambda-1)^2} + 2(1+r)^\delta(1-\gamma)\frac{4\gamma\lambda^2 + (4-8\gamma)\lambda + 2\gamma - 1}{(2\lambda-1)^2}$$
$$+ 2(1+r)^\delta(\lambda-1)\frac{12\lambda^2 - \left(12 + 2\left(\frac{2+r}{1+r}\right)^{2\delta} s^{2\delta}\right)\lambda + 3}{(2\lambda-1)^2}.$$

Upon simplification, this results in the first part of the bound on $|c_3|$. In addition, if $\lambda \in [1+1/\sqrt{2}, \infty)$, then

$$(3\lambda-1)(3+r)^\delta s^{2\delta}|c_3| \leq 2(1+r)^\delta(1-\gamma) + 4(1-\gamma)(2+r)^\delta s^\delta \frac{4\lambda^2 - 5\lambda + 1}{(2\lambda-1)^2}$$
$$+ 2(1+r)^\delta(\lambda-1)\frac{12\lambda^2 - \left(12 + 2\left(\frac{2+r}{1+r}\right)^{2\delta} s^{2\delta}\right)\lambda + 3}{(2\lambda-1)^2}.$$

This leads to the intended second part of the bound on $|c_3|$. □

By letting $g(z) = z$ and $\delta = 0$ in Theorem 1, we conclude the following consequence.

Corollary 1. *Let $h(z) \in \mathcal{L}(\lambda, \gamma)$. Then*

$$|c_2| \leq \frac{2(1-\gamma)}{2\lambda - 1}$$

and

$$|c_3| \leq \begin{cases} \frac{2(4\lambda-1)(1-\gamma)}{(3\lambda-1)(2\lambda-1)^2}, & \lambda \in \left[1, 1+1/\sqrt{2}\right] \\ \frac{2(1-\gamma)}{3\lambda-1}, & \lambda \in [1+1/\sqrt{2}, \infty) \end{cases}.$$

By setting $\delta = 0$ and $\gamma = 0$ in Theorem 1, we conclude the following result.

Corollary 2. *Let $h(z) \in \mathcal{MT}_\lambda$. Then*

$$|c_2| \leq \frac{2\lambda}{2\lambda - 1}$$

and

$$|c_3| \le \begin{cases} \frac{2\left(12\lambda^3-18\lambda^2+11\lambda-2\right)}{(2\lambda-1)^2(3\lambda-1)}, & \lambda \in \left[1, 1+1/\sqrt{2}\right] \\ \frac{2\lambda\left(12\lambda^2-14\lambda+3\right)}{(2\lambda-1)^2(3\lambda-1)}, & \lambda \in [1+1/\sqrt{2}, \infty) \end{cases}.$$

By setting $\delta = 1$, $r = 1$, and $\gamma = 0$ in Theorem 1, we conclude the following result.

Corollary 3. *Let $h(z) \in \mathcal{MT}_\lambda(s,1,1;0)$. Then*

$$|c_2| \le \frac{4(1+s(\lambda-1))}{3s(2\lambda-1)}$$

and

$$|c_3| \le \begin{cases} \frac{24\lambda^3-3\left(3s^2-8s+16\right)\lambda^2+\left(9s^2-30s+38\right)\lambda+6s-8}{2s^2(12\lambda^3-16\lambda^2+7\lambda-1)}, & \lambda \in \left[1, 1+1/\sqrt{2}\right] \\ \frac{24\lambda^3-\left(9s^2-24s+40\right)\lambda^2+\left(9s^2-30s+22\right)\lambda+6s-4}{2s^2(12\lambda^3-16\lambda^2+7\lambda-2)}, & \lambda \in [1+1/\sqrt{2}, \infty) \end{cases}.$$

Theorem 2. *Let the function $h(z)$ be analytic in $\mathcal{U}$ and defined by (1). If $h(z) \in \mathcal{MT}_\lambda(r,s,\delta;\gamma)$ and*

$$\vartheta \le \frac{(4\lambda^2-5\lambda+1)(1+r)^\delta}{6(3+r)^\delta s^{2\delta}\lambda^2 - 8(3+r)^\delta s^{2\delta}\lambda + 2(3+r)^\delta s^{2\delta}},$$

then

$$\left|c_3 - \vartheta c_2^2\right| \le \frac{(1+r)^\delta}{(3\lambda-1)(3+r)^\delta s^{2\delta}}\left(2(1-\gamma)I + (\lambda-1)J + \frac{4(1-\gamma)(2+r)^\delta s^\delta}{(2\lambda-1)^2(1+r)^{2\delta}} \times \right.$$
$$\left.\left[\left(4(1+r)^\delta - 6\vartheta(3+r)^\delta s^{2\delta}\right)\lambda^2 - \left(5(1+r)^\delta - 8\vartheta(3+r)^\delta s^{2\delta}\right)\lambda - 2\vartheta(3+r)^\delta s^{2\delta} + (1+r)^\delta\right]\right),$$

where

$$I = \max\left\{1, \frac{2(2\lambda^2-4\lambda+1)(2+r)^\delta(1-\gamma)s^\delta - 2\vartheta(1-\gamma)(3\lambda-1)(3+r)^\delta s^{2\delta} - (2\lambda-1)^2(2+r)^\delta s^\delta}{(2\lambda-1)^2(2+r)^\delta s^\delta}\right\},$$

and

$$J = \max\left\{1, \frac{4\left(\frac{2+r}{1+r}\right)^{2\delta}s^{2\delta}\lambda - 8\vartheta(\lambda-1)(3\lambda-1)\left(\frac{3+r}{1+r}\right)^\delta s^{2\delta} - 3(2\lambda-1)^2}{(2\lambda-1)^2}\right\}.$$

Proof. Using Equations (11) and (13), and arranging, we find

$$c_3-\vartheta c_2^2=\frac{(\lambda-1)}{(3\lambda-1)}\left[b_3+\frac{6\vartheta\lambda^2-\left(8\vartheta+\frac{(2+r)^{2\delta}}{(1+r)^{\delta}(3+r)^{\delta}}\right)\lambda+2\vartheta}{2(2\lambda-1)^2}b_2^2\right]$$
$$+\frac{(2+r)^{\delta}s^{\delta}}{(3\lambda-1)(2\lambda-1)^2(3+r)^{2\delta}}\left[\left(4\lambda^2-5\lambda+1\right)(1+r)^{\delta}-\vartheta\left(6\lambda^2-8\lambda+2\right)(3+r)^{\delta}\right]\ell_1 b_2$$
$$+\frac{(1+r)^{\delta}}{(3\lambda-1)(3+r)^{\delta}s^{2\delta}}\left[\ell_2-\frac{2(2+r)^{\delta}s^{\delta}\lambda^2-\left(4(2+r)^{\delta}s^{\delta}-3\vartheta(3+r)^{\delta}s^{2\delta}\right)\lambda+\vartheta(3+r)^{\delta}s^{2\delta}+(2+r)^{\delta}s^{\delta}}{(2\lambda-1)^2(2+r)^{\delta}s^{\delta}}\ell_1^2\right].$$

Therefore, the inequality follows by applying Lemmas 3, 4, and 5, along with the condition on ϑ. □

Remark 4. *For $\lambda=1$ and $\delta=0$, the Fekete–Szegö inequalities for the special cases $\mathcal{MT}_\lambda(r,s,\delta;\gamma)\equiv\mathcal{M}^*(\gamma)$ are well known.*

4. Coefficient Properties of Functions in $\mathcal{MT}_\lambda^\sigma(r,s,\delta;\gamma)$

Theorem 3. *If $h\in\sigma$ of the form (1) is in the class $\mathcal{MT}_\lambda^\sigma(r,\ s,\ \delta;\ \gamma)$, then*

$$|c_2|\le\sqrt{\frac{2\left((1-\gamma)(2\lambda-1)(1+r)^{2\delta}+2(1-\gamma)(2\lambda^2-3\lambda+1)(1+r)^{\delta}(2+r)^{\delta}s^{\delta}+(2\lambda^3-7\lambda^2+7\lambda-2)(2+r)^{2\delta}s^{2\delta}\right)}{\left((2\lambda^3-7\lambda^2+5\lambda-1)(2+r)^{2\delta}+(6\lambda^2-5\lambda+1)(1+r)^{\delta}(3+r)^{\delta}\right)s^{2\delta}}},$$

and

$$|c_3|\le\frac{1}{(3\lambda-1)(2\lambda-1)^3(2+r)^{2\delta}(3+r)^{\delta}s^{2\delta}}\times\Big[2(1-\gamma)(2\lambda-1)^3(1+r)^{\delta}(2+r)^{2\delta}+3(\lambda-1)(2\lambda-1)^3(2+r)^{2\delta}(3+r)^{\delta}s^{2\delta}$$
$$+4(1-\gamma)^2(1+r)^{\delta}\left|\left(6\lambda^2-5\lambda+1\right)(1+r)^{\delta}(3+r)^{\delta}-\lambda\left(2\lambda^2-3\lambda+1\right)(2+r)^{2\delta}\right|$$
$$+8(1-\gamma)(2+r)^{\delta}s^{\delta}\left|\left(6\lambda^3-11\lambda^2+6\lambda-1\right)(1+r)^{\delta}(3+r)^{\delta}-\lambda\left(2\lambda^3-5\lambda^2+4\lambda-1\right)(2+r)^{2\delta}\right|$$
$$+4(2+r)^{2\delta}s^{2\delta}\left|\left(6\lambda^4-25\lambda^3-5\lambda^2-13\lambda+2\right)(3+r)^{\delta}-\lambda\left(2\lambda^4-7\lambda^3+9\lambda^2-5\lambda+1\right)\frac{(2+r)^{2\delta}}{(1+r)^{\delta}}\right|\Big].$$

Proof. It follows from (6) that there exists

$$q_1(z)=1+\ell_1 z+\ell_2 z^2+\cdots\in Q_\gamma$$

and

$$q_2(w)=1+\jmath_1 w+\jmath_2 w^2+\cdots\in Q_\gamma$$

such that

$$q_1(z)=1+\left[(2\lambda-1)\left(\frac{2+r}{1+r}\right)^{\delta}s^{\delta}c_2-(\lambda-1)\left(\frac{2+r}{1+r}\right)^{\delta}s^{\delta}b_2\right]z$$
$$+\Bigg[(3\lambda-1)\left(\frac{3+r}{1+r}\right)^{\delta}s^{2\delta}c_3+[2\lambda(\lambda-1)-(2\lambda-1)]\left(\frac{2+r}{1+r}\right)^{2\delta}s^{2\delta}c_2^2-(\lambda-1)\left(\frac{3+r}{1+r}\right)^{\delta}s^{2\delta}b_3$$
$$+\frac{\lambda^2-\lambda}{2}\left(\frac{2+r}{1+r}\right)^{2\delta}s^{2\delta}b_2^2-\left(2\lambda^2-3\lambda+1\right)\left(\frac{2+r}{1+r}\right)^{2\delta}s^{2\delta}c_2b_2\Bigg]z^2+\cdots,\quad(16)$$

and

$$
\begin{aligned}
q_2(w) = 1 &+ \left[(\lambda-1)\left(\frac{2+r}{1+r}\right)^{\delta} s^{\delta} b_2 - (2\lambda-1)\left(\frac{2+r}{1+r}\right)^{\delta} s^{\delta} c_2\right] w \\
&+ \left[(3\lambda-1)\left(2c_2^2 - c_3\right)\left(\frac{3+r}{1+r}\right)^{\delta} s^{2\delta} - (\lambda-1)\left(2b_2^2 - b_3\right)\left(\frac{3+r}{1+r}\right)^{\delta} s^{2\delta} + \frac{\lambda^2-\lambda}{2}\left(\frac{2+r}{1+r}\right)^{2\delta} s^{2\delta} b_2^2 \right. \\
&\left. - \left(2\lambda^2 - 3\lambda + 1\right)\left(\frac{2+r}{1+r}\right)^{2\delta} s^{2\delta} c_2 b_2 - (2\lambda-1)\left(\frac{2+r}{1+r}\right)^{2\delta} s^{2\delta} c_2^2\right] w^2 + \cdots. \qquad (17)
\end{aligned}
$$

Comparing coefficients in (16) and (17) yields

$$
[(2\lambda-1)c_2 - (\lambda-1)b_2](2+r)^{\delta} s^{\delta} = (1+r)^{\delta} \ell_1, \qquad (18)
$$

$$
\begin{aligned}
&(3\lambda-1)(1+r)^{\delta}(3+r)^{\delta} s^{2\delta} c_3 + \left(2\lambda^2 - 4\lambda + 1\right)(2+r)^{2\delta} s^{2\delta} c_2^2 - (\lambda-1)(1+r)^{\delta}(3+r)^{\delta} s^{2\delta} b_3 + \frac{\lambda^2-\lambda}{2}(2+r)^{2\delta} s^{2\delta} b_2^2 \\
&\quad - (2\lambda^2 - 3\lambda + 1)(2+r)^{2\delta} s^{2\delta} c_2 b_2 = (1+r)^{2\delta} \ell_2, \qquad (19)
\end{aligned}
$$

$$
[(\lambda-1)b_2 - (2\lambda-1)c_2](2+r)^{\delta} s^{\delta} = (1+r)^{\delta} \jmath_1, \qquad (20)
$$

and

$$
\begin{aligned}
&\left[(3\lambda-1)\left(2c_2^2 - c_3\right)(3+r)^{\delta} - (\lambda-1)\left(2b_2^2 - b_3\right)(3+r)^{\delta}\right](1+r)^{\delta} s^{2\delta} \\
&\quad + \left[\frac{\lambda^2-\lambda}{2} b_2^2 - \left(2\lambda^2 - 3\lambda + 1\right) c_2 b_2 - (2\lambda-1) c_2^2\right](2+r)^{2\delta} s^{2\delta} = (1+r)^{2\delta} \jmath_2. \qquad (21)
\end{aligned}
$$

From (18) and (20), we obtain

$$
\ell_1 = -\jmath_1, \qquad (22)
$$

and

$$
2(2\lambda-1)^2(2+r)^{2\delta} s^{2\delta} c_2^2 = (1+r)^{2\delta}\left(\ell_1^2 + \jmath_1^2\right) + \left[4\left(2\lambda^2 - 3\lambda + 1\right) b_2 - 2(\lambda-1)^2 b_2^2\right](2+r)^{2\delta} s^{2\delta}. \qquad (23)
$$

Adding (19) to (21) and substituting the value of $\ell_1^2 + \jmath_1^2$ from (23), we obtain

$$
\begin{aligned}
&(1+r)^{2\delta}(\ell_2 + \jmath_2) + \frac{(2+r)^{\delta} s^{\delta}}{2\lambda-1}\left[2\left(2\lambda^2 - 3\lambda + 1\right)(1+r)^{\delta} \ell_1 b_2 + \left(2\lambda^3 - 7\lambda^2 + 7\lambda - 2\right)(2+r)^{\delta} s^{\delta} b_2^2\right] \\
&\quad = \left[2\left(\lambda^2 - 3\lambda + 1\right)(2+r)^{2\delta} + 2(3\lambda-1)(1+r)^{\delta}(3+r)^{\delta}\right] s^{2\delta} c_2^2. \qquad (24)
\end{aligned}
$$

Further computations using (24) yield

$$
c_2^2 = \frac{(\ell_2 + \jmath_2)(2\lambda-1)(1+r)^{2\delta} + 2(2\lambda^2 - 3\lambda + 1)(1+r)^{\delta}(2+r)^{\delta} s^{\delta} \ell_1 b_2 + (2\lambda^3 - 7\lambda^2 + 7\lambda - 2)(2+r)^{2\delta} s^{2\delta} b_2^2}{2\left[(2\lambda^3 - 7\lambda^2 + 5\lambda - 1)(2+r)^{2\delta} + (6\lambda^2 - 5\lambda + 1)(1+r)^{\delta}(3+r)^{\delta}\right] s^{2\delta}}. \qquad (25)
$$

Taking the absolute value of (25) and applying Lemmas 3 and 4 for the unknown coefficients, we deduce the desired bound on $|c_2|$.

Next, in order to determine the bound on $|c_3|$, we subtract (21) from (19) and utilize (22), deducing

$$2(3\lambda-1)(3+r)^{\delta}s^{2\delta}c_3+\left[2\lambda(\lambda-1)\frac{(2+r)^{2\delta}}{(1+r)^{\delta}}-2(3\lambda-1)(3+r)^{\delta}\right]s^{2\delta}c_2^2$$
$$=(1+r)^{\delta}(\ell_2-\jmath_2)+2(\lambda-1)(3+r)^{\delta}s^{2\delta}b_3-2(\lambda-1)(3+r)^{\delta}s^{2\delta}b_2^2. \quad (26)$$

Now, substituting the value of c_2^2 from (23), we obtain

$$\begin{aligned}
&2(3\lambda-1)(3+r)^{\delta}s^{2\delta}c_3=(1+r)^{\delta}(\ell_2-\jmath_2)+2(\lambda-1)(3+r)^{\delta}s^{2\delta}b_3\\
&+\frac{1}{(2\lambda-1)^3(2+r)^{2\delta}}\bigg[\Big(\ell_1^2+\jmath_1^2\Big)(1+r)^{\delta}\Big(\Big(6\lambda^2-5\lambda+1\Big)(1+r)^{\delta}(3+r)^{\delta}\\
&-\lambda\Big(2\lambda^2-3\lambda+1\Big)(2+r)^{2\delta}\Big)+4\ell_1b_2(2+r)^{\delta}s^{\delta}\Big(\Big(6\lambda^3-11\lambda^2+6\lambda-1\Big)(1+r)^{\delta}(3+r)^{\delta}\\
&-\lambda\Big(2\lambda^3-5\lambda^2+4\lambda-1\Big)(2+r)^{2\delta}\Big)+2b_2^2(2+r)^{2\delta}s^{2\delta}\Big(\Big(6\lambda^4-25\lambda^3-5\lambda^2-13\lambda+2\Big)(3+r)^{\delta}\\
&-\lambda\Big(2\lambda^4-7\lambda^3+9\lambda^2-5\lambda+1\Big)\frac{(2+r)^{2\delta}}{(1+r)^{\delta}}\Big)\bigg]. \qquad (27)
\end{aligned}$$

With further computation of (27), we deduce that

$$\begin{aligned}
&c_3=\frac{1}{2(3\lambda-1)(2\lambda-1)^3(2+r)^{2\delta}(3+r)^{\delta}s^{2\delta}}\\
&\times\bigg[(1+r)^{\delta}(2\lambda-1)^3(2+r)^{2\delta}(\ell_2-\jmath_2)+2(\lambda-1)(2\lambda-1)^3(2+r)^{2\delta}(3+r)^{\delta}s^{2\delta}b_3\\
&+\Big(\ell_1^2+\jmath_1^2\Big)(1+r)^{\delta}\Big(\Big(6\lambda^2-5\lambda+1\Big)(1+r)^{\delta}(3+r)^{\delta}-\lambda\Big(2\lambda^2-3\lambda+1\Big)(2+r)^{2\delta}\Big)\\
&+4\ell_1b_2(2+r)^{\delta}s^{\delta}\Big(\Big(6\lambda^3-11\lambda^2+6\lambda-1\Big)(1+r)^{\delta}(3+r)^{\delta}-\lambda\Big(2\lambda^3-5\lambda^2+4\lambda-1\Big)(2+r)^{2\delta}\Big)\\
&+2b_2^2(2+r)^{2\delta}s^{2\delta}\Big(\Big(6\lambda^4-25\lambda^3-5\lambda^2-13\lambda+2\Big)(3+r)^{\delta}\\
&-\lambda\Big(2\lambda^4-7\lambda^3+9\lambda^2-5\lambda+1\Big)\frac{(2+r)^{2\delta}}{(1+r)^{\delta}}\Big)\bigg]. \qquad (28)
\end{aligned}$$

Taking the absolute value of (28) and applying Lemmas 3 and 5, we obtain the desired estimate on $|c_3|$. □

By setting $\delta=0$ in Theorem 3, we conclude the following result.

Corollary 4. *Let* $h(z)\in\mathcal{MT}_{\lambda}^{\sigma}(\gamma)(\equiv\mathcal{MT}_{\lambda}^{\sigma}(r,\ s,\ 0;\ \gamma))$. *Then,*

$$|c_2|\le\sqrt{\frac{2(2\lambda-1)(\gamma-\lambda-2\gamma\lambda+\lambda^2+1)}{\lambda^2(2\lambda-1)}}$$

and

$$|c_3|\le\frac{1}{(3\lambda-1)(2\lambda-1)^3}\Big[4\Big(2\lambda^5-13\lambda^4+34\lambda^3+14\lambda-2\Big)+(2\lambda-1)^3(3\lambda-2\gamma-1)$$
$$+4(1-\gamma)^2\Big|2\lambda^3-9\lambda^2+6\lambda-1\Big|+8(1-\gamma)\Big(2\lambda^4-11\lambda^3+15\lambda^2-7\lambda+1\Big)\Big].$$

Next, we determine the bounds for the first two terms of the function $H(w)$. Given that $A_2 = -c_2$ (by Equation (4)), any upper bound obtained for $|c_2|$ also applies to $|A_2|$. Additionally, to derive the upper bound for $|A_3| = \left|2c_2^2 - c_3\right|$, we need to perform calculations based on the equation provided in (4), which we elaborate on in the subsequent theorem.

Theorem 4. *If $h \in \sigma$ of the form (1) is in the class $\mathcal{MT}_\lambda^\sigma(r,\ s,\ \delta;\ \gamma)$, then*

$$\left|2c_2^2 - c_3\right| \leq \frac{4(1-\gamma)(2\lambda-1)(1+r)^{2\delta}}{N_0 s^{2\delta}} + \frac{2}{(3\lambda-1)(2\lambda-1)^3(3+r)^\delta}\left[\frac{2(1-\gamma)(N_3 + |N_4 - N_5|) + N_6 + N_7 + N_8}{N_2} + \frac{2(1-\gamma)(2\lambda-1)^3(1+r)^\delta + 3(\lambda-1)(2\lambda-1)^3(3+r)^\delta s^{2\delta} + 4(1-\gamma)^2|N_1|}{s^{2\delta}}\right],$$

where

$$N_0 := \left(2\lambda^3 - 7\lambda^2 + 5\lambda - 1\right)(2+r)^{2\delta} + \left(6\lambda^2 - 5\lambda + 1\right)(1+r)^\delta(3+r)^\delta;$$

$$N_1 := \lambda\left(2\lambda^2 - 3\lambda + 1\right)(1+r)^\delta - \left(6\lambda^2 - 5\lambda + 1\right)\left(\frac{1+r}{2+r}\right)^{2\delta}(3+r)^\delta;$$

$$N_2 := \left(2\lambda^3 - 7\lambda^2 + 5\lambda - 1\right)(2+r)^{2\delta} + \left(6\lambda^2 - 5\lambda + 1\right)(1+r)^\delta(3+r)^\delta;$$

$$N_3 := 2\lambda\left(24\lambda^5 - 69\lambda^4 + 74\lambda^3 - 39\lambda^2 + 58\lambda - 1\right)(3+r)^\delta(2+r)^{2\delta};$$

$$N_4 := \lambda\left(2\lambda^3 - 7\lambda^2 + 5\lambda - 1\right)\left(2\lambda^3 - 5\lambda^2 + 4\lambda - 1\right)\frac{(2+r)^{4\delta}}{(1+r)^\delta};$$

$$N_5 := \left(6\lambda^2 - 5\lambda + 1\right)\left(6\lambda^3 - 11\lambda^2 + 6\lambda - 1\right)(1+r)^\delta(3+r)^{2\delta};$$

$$N_6 := \left(48\lambda^7 - 216\lambda^6 + 432\lambda^5 - 474\lambda^4 + 490\lambda^3 - 66\lambda^2 + 3\lambda - 2\right)(3+r)^\delta(2+r)^{2\delta};$$

$$N_7 := \lambda\left(2\lambda^3 - 7\lambda^2 + 5\lambda - 1\right)\left(2\lambda^4 - 7\lambda^3 + 9\lambda^2 - 5\lambda + 1\right)\frac{(2+r)^{4\delta}}{(1+r)^\delta};$$

$$N_8 := \left(6\lambda^2 - 5\lambda + 1\right)\left(6\lambda^4 - 25\lambda^3 - 5\lambda^2 - 13\lambda + 2\right)(1+r)^\delta(3+r)^{2\delta}.$$

Proof. By using Equations (25) and (28) , we can find

$$\begin{aligned}
2c_2^2 - c_3 = {} & \frac{(\ell_2 + \jmath_2)(2\lambda-1)(1+r)^{2\delta}}{(2\lambda^3 - 7\lambda^2 + 5\lambda - 1)(2+r)^{2\delta}\delta^{2\delta} + (6\lambda^2 - 5\lambda + 1)(1+r)^\delta(3+r)^\delta s^{2\delta}} \\
& - \frac{1}{2(3\lambda-1)(2\lambda-1)^3(3+r)^\delta s^{2\delta}}\left[(\ell_2 - \jmath_2)(2\lambda-1)^3(1+r)^\delta + 2(\lambda-1)(2\lambda-1)^3(3+r)^\delta s^{2\delta} b_3\right. \\
& \left. + \left(\ell_1^2 + \jmath_1^2\right)\left(\left(6\lambda^2 - 5\lambda + 1\right)\left(\frac{1+r}{2+r}\right)^{2\delta}(3+r)^\delta - \lambda\left(2\lambda^2 - 3\lambda + 1\right)(1+r)^\delta\right)\right] \\
& + \frac{1}{((2\lambda^3 - 7\lambda^2 + 5\lambda - 1)(2+r)^{2\delta} + (6\lambda^2 - 5\lambda + 1)(1+r)^\delta(3+r)^\delta)(3\lambda-1)(2\lambda-1)^3(3+r)^\delta}
\end{aligned}$$

$$\times\Bigg[2\ell_1 b_2\Bigg(2\lambda\Big(24\lambda^5-69\lambda^4+74\lambda^3-39\lambda^2+58\lambda-1\Big)(3+r)^{\delta}(2+r)^{2\delta}$$

$$+\lambda\Big(2\lambda^3-7\lambda^2+5\lambda-1\Big)\Big(2\lambda^3-5\lambda^2+4\lambda-1\Big)\frac{(2+r)^{4\delta}}{(1+r)^{\delta}}$$

$$-(6\lambda^2-5\lambda+1)\Big(6\lambda^3-11\lambda^2+6\lambda-1\Big)(1+r)^{\delta}(3+r)^{2\delta}\Bigg)$$

$$+b_2^2\Bigg(\Big(48\lambda^7-216\lambda^6+432\lambda^5-474\lambda^4+490\lambda^3-66\lambda^2+3\lambda-2\Big)(3+r)^{\delta}(2+r)^{2\delta}$$

$$+\lambda\Big(2\lambda^3-7\lambda^2+5\lambda-1\Big)\Big(2\lambda^4-7\lambda^3+9\lambda^2-5\lambda+1\Big)\frac{(2+r)^{4\delta}}{(1+r)^{\delta}}$$

$$+(6\lambda^2-5\lambda+1)\Big(6\lambda^4-25\lambda^3-5\lambda^2-13\lambda+2\Big)(1+r)^{\delta}(3+r)^{2\delta}\Bigg)\Bigg].$$

Finally, by applying Lemmas 3 and 5, we achieve the desired estimate for $\left|2c_2^2-c_3\right|$. □

By setting $\delta=0$ in Theorem 4, we conclude the following result.

Corollary 5. *Let $h(z)\in\mathcal{MT}_{\lambda}^{\sigma}(\gamma)$. Then,*

$$\left|2c_2^2-c_3\right|\leq\frac{(2\lambda-1)^3 3(\lambda-)-2(1-\gamma)(2\lambda-1)^3+4|N_1|(\gamma-1)^2}{(2\lambda-1)^3(3\lambda-1)}+\frac{2(N_4+2(1-\gamma)(N_2+|N_3|))}{\lambda^2(2\lambda-1)^4(3\lambda-1)}+\frac{4(1-\gamma)}{\lambda^2}$$

where

$$\begin{aligned}
N_1 &:= 2\lambda^3-9\lambda^2+6\lambda-1;\\
N_2 &:= 2\lambda(24\lambda^5-69\lambda^4+74\lambda^3-39\lambda^2+58\lambda-1);\\
N_3 &:= (2\lambda-1)^2(\lambda^5-5\lambda^4-\lambda^3+10\lambda^2-6\lambda+1);\\
N_4 &:= \lambda\Big(4\lambda^7+20\lambda^6-103\lambda^5+142\lambda^4-284\lambda^3+371\lambda^2+16\lambda-21\Big).
\end{aligned}$$

Finally, we provide nontrivial examples of Definitions 1 and 2 for $\delta=0$, $\lambda=2$, and $\gamma=0$.

Example 1. *If we take*

$$h(z)=z-\frac{2}{3}z^{3/2}+\frac{1}{9}z^2+\frac{2}{5}z^{5/2}-\frac{2}{15}z^3-\frac{2}{21}z^{7/2}+\cdots,$$

and

$$g(z)=z-z^2+z^3-z^4+\cdots,$$

then, by a simple computation, the left side of (5) is as follows:

$$1-z+z^2-z^3+\cdots\in Q,$$

which has a positive real part. So, $h\in\mathcal{MT}_{\lambda}(r,\ s,\ \delta;\ \gamma)$.

Example 2. *If we consider*

$$h(z)=z-\frac{1}{2}z^2+\frac{1}{3}z^3-\frac{1}{4}z^4+\cdots,$$

and

$$g(z) = z + z^2 + z^3 + z^4 + \cdots,$$

it becomes easy to see that $h \in \mathcal{MT}_{\lambda}^{\sigma}(r,\ s,\ \delta;\ \gamma)$, *as defined in Definition* 2.

5. Conclusions

In this paper, we have integrated two significant and established classes, namely those introduced by Bazilevič and close-to-convex functions, under a novel derivative operator. Our analysis has yielded coefficient estimates for these extensive subclasses of analytic, univalent, and bi-univalent functions, shedding light on the pertinent Fekete–Szegö inequalities within the open unit disk. Furthermore, by parameter specialization, we have identified several noteworthy special cases within our findings.

As an open problem, we aim for this study to inspire researchers to explore additional coefficient inequalities using various polynomials and the subordination method within these specified subclasses. Also, we recommend researchers to define and study strongly bi-Bazilevič close-to-convex functions and to find coefficient estimates, Fekete–Szegö inequalities, and the distortion and growth theorems of this effective class. Furthermore, we encourage researchers to find the second and third Hankel and Teoplitz determinants of logarithmic coefficients for functions in the Bazilevič close-to-convex class defined in the Introductory section.

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