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Stability Analysis of Some Types of Singularly Perturbed Time-Delay Differential Systems: Symmetric Matrix Riccati Equation Approach

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Abstract: Several types of linear and nonlinear singularly perturbed time-delay differential systems are considered. Asymptotic stability of the linear systems and asymptotic stability of the trivial solution of the nonlinear systems, valid for any sufficiently small value of the parameter of singular perturbation, are analyzed. For the stability analysis in the linear case, a partial exact slow–fast decomposition of the original system and an application of the Symmetric Matrix Riccati Equation method are proposed. Such an analysis yields parameter-free conditions, providing the asymptotic stability of the considered linear singularly perturbed time-delay differential systems for any sufficiently small value of the parameter of singular perturbation. Using the asymptotic stability results for the considered linear systems and the method of asymptotic stability in the first approximation, parameter-free conditions, guaranteeing the asymptotic stability of the trivial solution to the considered nonlinear systems for any sufficiently small value of the parameter of singular perturbation, are derived. Illustrative examples are presented.

Keywords: time-delay differential system; singularly perturbed system; asymptotic stability; symmetric matrix Riccati equation method; partial slow–fast decomposition; parameter-free stability conditions

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1. Introduction

Differential systems with positive small multipliers for some of their highest-order derivatives, called singularly perturbed ones, have been of considerable interest in the literature for many years (see, e.g., [1–13] and references therein). The aforementioned small multipliers are called parameters of singular perturbations. An important class of singularly perturbed differential systems from the theoretical and practical viewpoints is the class of singularly perturbed time-delay systems. Various issues for singularly perturbed time-delay linear and nonlinear systems were studied in the literature. Among these issues are the following: (1) exact slow–fast decomposition; (2) asymptotic slow–fast decomposition; (3) solution stability; (4) stabilizability and stabilization; (5) controllability and observability; (6) asymptotic solution; and some others. Brief surveys on the topic of singularly perturbed time-delay systems can be found in [4,14,15].

One of the basic issues, studied in the theory of differential systems, is the stability of their solutions (see, e.g., [2,5,6,16–23]).

In this paper, we consider several types of linear and nonlinear singularly perturbed time-delay differential systems. We study the asymptotic stability of the linear systems and asymptotic stability of the trivial solution of the nonlinear systems. The stability of various types of time-delay differential systems has been extensively studied in the literature (see, e.g., [2,24–27] and references therein). One can immediately apply the results of these works to a singularly perturbed time-delay differential system for any specified value of the parameter of singular perturbation. However, a stiffness of the singularly perturbed system, as well as its high Euclidean space dimension, complicate such an application considerably. Also, this application depends on a pre-chosen value of the parameter of singular perturbation, while in various real-life problems the value of the singular perturbation's parameter is unknown, meaning that these problems are uncertain with respect to the parameter. Therefore, for singularly perturbed systems, another (other than in the aforementioned works) conditions of their stability should be derived. These conditions should be independent of the parameter of singular perturbation, while providing the stability for any sufficiently small value. Such conditions can be derived by the application of a slow–fast decomposition of a singularly perturbed system.

The slow–fast decomposition both, asymptotic and exact, was widely applied to the qualitative and quantitative analysis of various singularly perturbed systems without and with delays (see, e.g., [3,4,7,11,12,14,28–34] and references therein). In particular, the slow–fast decomposition approach was applied for the stability analysis of singularly perturbed differential systems. Thus, in [7,35–40], stability conditions for various singularly perturbed differential systems without delays were derived. For the stability analysis of singularly perturbed time-delay systems, several approaches were considered in the literature. Thus, in [31,41,42], the asymptotic/exponential stability of different types of singularly perturbed time-delay systems was analyzed by the block-wise estimate of their fundamental matrices using either the asymptotic slow–fast decomposition ([41,42]) or the exact slow–fast decomposition ([31]) of the original system. In [14,29,33,43–46], the spectrum analysis of various singularly perturbed time-delay systems was applied to establish their asymptotic/exponential stability. In [14,29,33,46], this analysis is based on the exact slow–fast decomposition of the considered system, while in [43–45] this analysis is based on the asymptotic slow–fast decomposition. Exponential/asymptotic stability analysis of some types of singularly perturbed time-delay systems using the Linear Matrix Inequality approach was carried out in [2,45,47–52].

In the present paper, we study the asymptotic stability of some types of linear singularly perturbed time-delay systems using the symmetric matrix Riccati equation method. This method is widely applied in the literature for the stability analysis of unperturbed time-delay systems (see, e.g., [25] and references therein). However, the extensive Google web-search does not show any work devoted to the direct application of the matrix Riccati equation method for the stability analysis of singularly perturbed time-delay systems. The matrix Riccati equation method consists of the analysis of the existence of a symmetric positive definite solution to a symmetric matrix Riccati algebraic equation associated in a proper way with the original time-delay system.

In the present paper, we propose the partial exact slow–fast decomposition of the considered singularly perturbed system. Namely, we decompose only either the un-delayed or the delayed part of the system. Using such a decomposition allows us to decompose the initially constructed Riccati equation into two much simpler and less dimensional Riccati equations. Asymptotic analysis of each of the latter yields parameter-free conditions guaranteeing the asymptotic stability of the original linear singularly perturbed time-delay system for any sufficiently small value of the parameter of singular perturbation. To the best of our knowledge, the partial slow–fast decomposition of a singularly perturbed time-delay system and the application of such a decomposition to the stability analysis of this system are proposed for the first time in the literature in the present paper. The application of the matrix Riccati algebraic equation method and the partial slow–fast decomposition of a singularly perturbed time-delay system to its stability analysis requires one to develop a significantly new approach to the asymptotic analysis of a symmetric matrix Riccati algebraic equation associated with the original singularly perturbed time-delay system.

Along with the linear singularly perturbed time-delay systems, we consider nonlinear singularly perturbed time-delay systems. Based on the aforementioned results for the considered linear systems and using the method of asymptotic stability in the first approximation, we derive parameter-free conditions guaranteeing the asymptotic stability of the trivial solution to the considered nonlinear singularly perturbed time-delay systems for any sufficiently small value of the parameter of singular perturbation.

It should be noted that the stability issue is of considerable importance, not only in the theory of differential equations but also in different real-life problems. Various examples of differential systems (without and with delays, unperturbed and perturbed), describing real-life problems, can be found, for instance, in [2,4,7,19,25,53]. Examples of stability analysis for real-life problems can be found in [2,19,44,53] and in references therein. However, not to overload the paper, we concentrate here only on the theoretical part of the stability issue. The consideration of the applied part of this issue requires a separate paper, or even more than one paper.

The motivations of the present, purely theoretical, paper are the following: (1) to propose the novel partial exact slow–fast decomposition of a singularly perturbed time-delay system and the direct application of the symmetric matrix Riccati equation method to the system's asymptotic stability analysis; (2) to realize this proposition for several types of linear and nonlinear singularly perturbed time-delay systems.

The structure of the paper as follows. The next section (Section 2) is devoted to the stability analysis of singularly perturbed systems (linear and nonlinear) with only delayed states in their right-hand sides. The type of the delay is a single point-wise delay proportional to the small parameter of singular perturbation. In Section 3, the stability analysis is carried out for singularly perturbed systems (linear and nonlinear) having undelayed and delayed states in their right-hand sides. The case of multiple point-wise delays is considered in Sections 3.1–3.4, while the case of a single point-wise delay is considered in Section 3.5. In both cases, the delays are independent of the small parameter of singular perturbation. Section 4 is devoted to conclusions and issues for future investigation.

The following main notations are applied in the paper:

1. E^n denotes the linear n-dimensional real vector space.
2. For an $n \times m$-matrix A, $(n \geq 1,\ m \geq 1)$, its norm is defined as $\|A\| \stackrel{\triangle}{=} \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|$, where a_{ij}, $(i = 1, \dots n;\ j = 1, \dots, m)$ are the entries of A.
3. The upper index $''T''$ denotes the transposition either of a vector x (x^T) or of a matrix A (A^T).
4. I_n denotes the identity matrix of dimension n.
5. $\mathrm{col}(x, y)$, where $x \in E^n$, $y \in E^m$, denotes the column block-vector of the dimension $n + m$ with the upper block x and the lower block y, i.e., $\mathrm{col}(x, y) = (x^T, y^T)^T$.
6. For a complex number λ, $\mathrm{Re}(\lambda)$ denotes its real part.
7. For a continuous vector-valued function $\varphi(\tau) : [\tau_1, \tau_2] \to E^n$, $\|\varphi(\cdot)\|_C$ denotes its uniform norm, i.e., $\|\varphi(\cdot)\|_C \stackrel{\triangle}{=} \max_{\tau \in [\tau_1, \tau_2]} \|\varphi(\tau)\|$.

2. First Type System

2.1. Formulation, Basic Definition, and Assertions

Consider the following differential system:

$$\frac{dx(t)}{dt} = H_1 x(t - \varepsilon h) + H_2 y(t - \varepsilon h), \quad t \geq 0,$$
$$\varepsilon \frac{dy(t)}{dt} = H_3 x(t - \varepsilon h) + H_4 y(t - \varepsilon h), \quad t \geq 0, \tag{1}$$

where $x(t) \in E^n$ and $y(t) \in E^m$; H_i, $(i = 1, \dots, 4)$ are given constant matrices of corresponding dimensions; $\varepsilon > 0$ is a small parameter; $h > 0$ is a given number independent of ε.

System (1) is a time-delay system with a point-wise form of the delay. Being a time-delay system, (1) is infinite-dimensional with the state variables $x(t + \tau)$, $y(t + \tau)$, $\tau \in [-\varepsilon h, 0]$. Moreover, this system is a singularly perturbed system with the parameter ε of singular perturbation. An additional feature of (1) is that the time delay is proportional to the small multiplier ε for the derivative $dy(t)/dt$ in the second equation.

For system (1), let us consider the initial conditions

$$x(\tau) = \varphi_x(\tau), \quad y(\tau) = \varphi_y(\tau), \quad \tau \in [-\varepsilon h, 0], \tag{2}$$

where $\varphi_x(\tau)$ is a given n-dimensional vector-valued function; $\varphi_y(\tau)$ is a given m-dimensional vector-valued function; both functions are continuous in the interval $[-\varepsilon h, 0]$.

Based on the results of [25], we introduce the definition.

Definition 1. *For a given $\varepsilon > 0$, the system (1) is called asymptotically stable if for any aforementioned functions $\varphi_x(\tau)$ and $\varphi_y(\tau)$, $\tau \in [-\varepsilon h, 0]$; the solution of the initial-value problem (1)–(2) $\mathrm{col}\Big(x\big(t;\varphi_x(\cdot),\varphi_y(\cdot)\big), y\big(t;\varphi_x(\cdot),\varphi_y(\cdot)\big)\Big)$ tends to zero as t tends to $+\infty$.*

Consider the block-form matrix

$$H \stackrel{\triangle}{=} \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}. \tag{3}$$

Also, we consider the following set of two matrix algebraic equations with respect to $m \times n$-matrix L and $n \times m$-matrix M:

$$\begin{aligned} \varepsilon L H_1 - H_4 L - \varepsilon L H_2 L + H_3 = 0, \\ -M(H_4 + \varepsilon L H_2) + \varepsilon (H_1 - H_2 L) M + H_2 = 0. \end{aligned} \tag{4}$$

In what follows in this section, we assume the following:

A1. The matrix H_4 is invertible, i.e., H_4^{-1} exists.

By virtue of the results of [7] (see Sections 2.2 and 2.4), we have the following assertion.

Proposition 1. *Let the assumption A1 be valid. Then, there exists a positive number ε_1 such that, for all $\varepsilon \in [0, \varepsilon_1]$, the set (4) has the solution $\{L(\varepsilon), M(\varepsilon)\}$ satisfying the inequalities*

$$\|L(\varepsilon) - H_4^{-1} H_3\| \le a\varepsilon, \quad \|M(\varepsilon) - H_2 H_4^{-1}\| \le a\varepsilon, \tag{5}$$

where $a > 0$ is some constant independent of ε

Let us introduce into consideration the block matrix

$$D(\varepsilon) \stackrel{\triangle}{=} \begin{pmatrix} I_n & \varepsilon M(\varepsilon) \\ -L(\varepsilon) & I_m - \varepsilon L(\varepsilon) M(\varepsilon) \end{pmatrix}, \quad \varepsilon \in [0, \varepsilon_1]. \tag{6}$$

This matrix is invertible and its inverse matrix is (for details, see [7], Section 2.4):

$$D^{-1}(\varepsilon) = \begin{pmatrix} I_n - \varepsilon M(\varepsilon) L(\varepsilon) & -\varepsilon M(\varepsilon) \\ L(\varepsilon) & I_m \end{pmatrix}, \quad \varepsilon \in [0, \varepsilon_1]. \tag{7}$$

Let us transform the state variables $x(t+\tau)$ and $y(t+\tau)$, $\tau \in [-\varepsilon h, 0]$ of system (1) as:

$$\begin{pmatrix} x(t+\tau) \\ y(t+\tau) \end{pmatrix} = D(\varepsilon) \begin{pmatrix} u(t+\tau) \\ v(t+\tau) \end{pmatrix}, \quad t \ge 0, \quad \tau \in [-\varepsilon h, 0], \quad \varepsilon \in (0, \varepsilon_1], \tag{8}$$

where $u(t) \in E^n$, $v(t) \in E^m$, $u(t+\tau)$, and $v(t+\tau)$ are new state variables.

Based on the results of [7] (see Section 2.4), as well as on Definition 1 and the existence of the matrix in (7), we directly obtain the following assertion.

Proposition 2. *Let the assumption A1 be valid. Then, for any given $\varepsilon \in (0, \varepsilon_1]$, the transformation (8) converts the initial-value problem (1)–(2) to the following equivalent initial-value problem:*

$$\frac{du(t)}{dt} = \big(H_1 - H_2L(\varepsilon)\big)u(t - \varepsilon h),$$
$$\varepsilon\frac{dv(t)}{dt} = \big(H_4 + \varepsilon L(\varepsilon)H_2\big)v(t - \varepsilon h), \quad (9)$$

$$u(\tau) = \big(I_n - \varepsilon M(\varepsilon)L(\varepsilon)\big)\varphi_x(\tau) - \varepsilon M(\varepsilon)\varphi_y(\tau), \quad \tau \in [-\varepsilon h, 0],$$
$$v(\tau) = L(\varepsilon)\varphi_x(\tau) + \varphi_y(\tau), \quad \tau \in [-\varepsilon h, 0]. \quad (10)$$

Moreover, system (1) is asymptotically stable if and only if each of the equations in system (9) is asymptotically stable.

Consider the following two symmetric Riccati algebraic equations with respect to $n \times n$-matrix P_1 and $m \times m$-matrix P_2:

$$\big(H_1 - H_2L(\varepsilon)\big)^T P_1 + P_1\big(H_1 - H_2L(\varepsilon)\big) + \varepsilon h P_1\big(H_1 - H_2L(\varepsilon)\big)R_1^{-1}\big(H_1 - H_2L(\varepsilon)\big)^T P_1$$
$$+\varepsilon h\big(H_1 - H_2L(\varepsilon)\big)^T R_1\big(H_1 - H_2L(\varepsilon)\big) = -Q_1, \quad (11)$$

$$\frac{1}{\varepsilon}\big(H_4 + \varepsilon L(\varepsilon)H_2\big)^T P_2 + \frac{1}{\varepsilon}P_2\big(H_4 + \varepsilon L(\varepsilon)H_2\big)$$
$$+\frac{1}{\varepsilon}hP_2\big(H_4 + \varepsilon L(\varepsilon)H_2\big)R_2^{-1}\big(H_4 + \varepsilon L(\varepsilon)H_2\big)^T P_2$$
$$+\frac{1}{\varepsilon}h\big(H_4 + \varepsilon L(\varepsilon)H_2\big)^T R_2\big(H_4 + \varepsilon L(\varepsilon)H_2\big) = -Q_2, \quad (12)$$

where R_1 and Q_1 are some symmetric positive definite matrices of the dimension $n \times n$, while R_2 and Q_2 are some symmetric positive definite matrices of the dimension $m \times m$.

By virtue of Proposition 1 and the results of [25] (see Chapter 7, Theorem 2.3), we directly obtain the following assertions.

Proposition 3. *Let the assumption A1 be valid. Let, for a given $\varepsilon \in (0, \varepsilon_1]$, there exist symmetric positive definite matrices R_1 and Q_1 such that Equation (11) has a symmetric positive definite solution $P_1 = P_1(\varepsilon)$. Then, for this ε, the first equation in system (9) is asymptotically stable.*

Proposition 4. *Let the assumption A1 be valid. Let, for a given $\varepsilon \in (0, \varepsilon_1]$, there exist symmetric positive definite matrices R_2 and Q_2 such that Equation (12) has a symmetric positive definite solution $P_2 = P_2(\varepsilon)$. Then, for this ε, the second equation in system (9) is asymptotically stable.*

2.2. Asymptotic Solution with Respect to ε of the Equation (11)

We look for the zero-order asymptotic solution P_{10} of (11). Using Proposition 1 and setting formally $\varepsilon = 0$ in Equation (11), we obtain the following equations for its zero-order asymptotic solution:

$$(H_1 - H_2H_4^{-1}H_3)^T P_{10} + P_{10}(H_1 - H_2H_4^{-1}H_3) = -Q_1. \quad (13)$$

In what follows of this section, we assume the following:

A2. There exists a symmetric positive definite matrix Q_1 such that the symmetric matrix Lyapunov algebraic Equation (13) has a symmetric positive definite solution P_{10}.

Remark 1. *Due to the well known properties of the symmetric matrix Lyapunov algebraic equation (see, e.g., [25,54]), we can conclude the following. If the assumption A2 is valid, then the matrix*

$$\mathcal{H} \overset{\triangle}{=} H_1 - H_2 H_4^{-1} H_3 \tag{14}$$

is a Hurwitz one, and the solution P_{10} is unique. Vice versa, if $\mathcal{H}$ is a Hurwitz matrix then, for any symmetric positive definite matrix Q_1, Equation (13) has the unique solution,

$$P_{10} = \int_0^{+\infty} \exp(\mathcal{H}^T \sigma) Q_1 \exp(\mathcal{H}\sigma) d\sigma, \tag{15}$$

and this solution is a symmetric positive definite matrix.

Theorem 1. *Let the assumptions A1 and A2 be valid. Then, there exists a number $0 < \varepsilon_1^* \leq \varepsilon_1$ such that, for all $\varepsilon \in (0, \varepsilon_1^*]$, Equation (11) has a symmetric positive definite solution $P_1(\varepsilon)$, satisfying the inequality*

$$\left\| P_1(\varepsilon) - P_{10} \right\| \leq a_1 \varepsilon, \quad \varepsilon \in (0, \varepsilon_1^*], \tag{16}$$

where $a_1 > 0$ is some constant independent of ε.

Proof. First of all, let us note the following. Using Equation (14) and the first inequality in (5), we can represent the matrix $H_1 - H_2 L(\varepsilon)$ in the form

$$H_1 - H_2 L(\varepsilon) = \mathcal{H} + \Gamma_1(\varepsilon), \quad \varepsilon \in [0, \varepsilon_1], \tag{17}$$

where the $n \times n$-matrix $\Gamma_1(\varepsilon)$ satisfies the inequality

$$\left\| \Gamma_1(\varepsilon) \right\| \leq c_1 \varepsilon, \quad c_1 = a \left\| H_2 \right\|, \quad \varepsilon \in (0, \varepsilon_1]. \tag{18}$$

Now, let us transform the unknown matrix in Equation (11) as:

$$P_1 = P_{10} + \Delta P_1, \tag{19}$$

where ΔP_1 is a new unknown matrix.

Substituting (19) into Equation (11) and using Equations (13), (14), (17), we obtain the following equation with respect to ΔP_1:

$$\begin{aligned} \big(\mathcal{H} + \Gamma_2(\varepsilon)\big)^T \Delta P_1 + \Delta P_1 \big(\mathcal{H} + \Gamma_2(\varepsilon)\big) &= -\Gamma_3(\varepsilon) \\ -\varepsilon h \Delta P_1 \big(\mathcal{H} + \Gamma_1(\varepsilon)\big) R_1^{-1} \big(\mathcal{H} &+ \Gamma_1(\varepsilon)\big)^T \Delta P_1, \end{aligned} \tag{20}$$

where

$$\begin{aligned} \Gamma_2(\varepsilon) &= \Gamma_1(\varepsilon) + \varepsilon h \big(\mathcal{H} + \Gamma_1(\varepsilon)\big) R_1^{-1} \big(\mathcal{H} + \Gamma_1(\varepsilon)\big)^T P_{10}, \\ \Gamma_3(\varepsilon) &= \Gamma_1^T(\varepsilon) P_{10} + P_{10} \Gamma_1(\varepsilon) + \varepsilon h P_{10} \big(\mathcal{H} + \Gamma_1(\varepsilon)\big) R_1^{-1} \big(\mathcal{H} + \Gamma_1(\varepsilon)\big)^T P_{10} \\ &\quad + \varepsilon h \big(\mathcal{H} + \Gamma_1(\varepsilon)\big)^T R_1 \big(\mathcal{H} + \Gamma_1(\varepsilon)\big). \end{aligned} \tag{21}$$

Due to inequality (18), we directly have

$$\left\| \Gamma_2(\varepsilon) \right\| \leq c_2 \varepsilon, \quad \left\| \Gamma_3(\varepsilon) \right\| \leq c_3 \varepsilon, \quad \varepsilon \in (0, \varepsilon_1], \tag{22}$$

where $c_2 > 0$ and $c_3 > 0$ are some constants independent of ε.

Furthermore, taking into account that $\mathcal{H}$ is a Hurwitz matrix (see Remark 1) and using the first inequality in (22) and the results of [55], we obtain the existence of a number

$0 < \varepsilon_2 \le \varepsilon_1$, such that, for all $\varepsilon \in (0, \varepsilon_2]$, the eigenvalues $\lambda_i(\varepsilon)$, $(i = 1, \dots, n)$ of the matrix $\mathcal{H} + \Gamma_2(\varepsilon)$ satisfy the inequality

$$\mathrm{Re}\big(\lambda_i(\varepsilon)\big) < -\gamma, \quad i = 1, \dots, n, \tag{23}$$

where $\gamma > 0$ is some constant independent of ε.

Due to inequality (23), we can rewrite Equation (20) in the following equivalent form for all $\varepsilon \in (0, \varepsilon_2]$:

$$\Delta P_1 = \int_0^{+\infty} \exp\left(\left(\mathcal{H} + \Gamma_2(\varepsilon)\right)^T \sigma\right) \Big[\Gamma_3(\varepsilon) + \varepsilon h \Delta P_1 \big(\mathcal{H} + \Gamma_1(\varepsilon)\big) R_1^{-1} \big(\mathcal{H} + \Gamma_1(\varepsilon)\big)^T \Delta P_1\Big] \exp\left(\left(\mathcal{H} + \Gamma_2(\varepsilon)\right)\sigma\right) d\sigma. \tag{24}$$

Consider the linear space Θ of all symmetric $n \times n$-matrix S. This space, endowed with the norm $\|S\|_\Theta = \|S\|$, is a Banach space.

For a sufficiently small number $\varepsilon > 0$, consider the ball in Θ,

$$\mathcal{B}(c_{\mathcal{B}}, \varepsilon) \stackrel{\triangle}{=} \Big\{S \in \Theta : \|S\|_\Theta \le c_{\mathcal{B}}\varepsilon\Big\}, \tag{25}$$

where $c_{\mathcal{B}} > 0$ is some constant independent of ε.

For the aforementioned $\varepsilon > 0$, consider the operator, given in the space Θ,

$$\mathcal{F}_\varepsilon(S) \stackrel{\triangle}{=} \int_0^{+\infty} \exp\left(\left(\mathcal{H} + \Gamma_2(\varepsilon)\right)^T \sigma\right) \Big[\Gamma_3(\varepsilon) + \varepsilon h S \big(\mathcal{H} + \Gamma_1(\varepsilon)\big) R_1^{-1} \big(\mathcal{H} + \Gamma_1(\varepsilon)\big)^T S\Big] \exp\left(\left(\mathcal{H} + \Gamma_2(\varepsilon)\right)\sigma\right) d\sigma, \quad S \in \Theta. \tag{26}$$

For any $c_{\mathcal{B}} > 0$ and any $\varepsilon > 0$, the operator $\mathcal{F}_\varepsilon(\cdot)$ maps the ball $\mathcal{B}(c_{\mathcal{B}}, \varepsilon)$ into the space Θ. Let us show that, for a proper choice of the numbers $c_{\mathcal{B}} > 0$ and $\widetilde{\varepsilon} \in (0, \varepsilon_2]$, this operator maps the ball $\mathcal{B}(c_{\mathcal{B}}, \varepsilon)$ into itself for any $\varepsilon \in (0, \widetilde{\varepsilon}]$.

Using inequalities (18), (22), (23), we can estimate the image of the operator $\mathcal{F}_\varepsilon(S)$ for any $S \in \mathcal{B}(c_{\mathcal{B}}, \varepsilon)$ as follows:

$$\begin{aligned}\|\mathcal{F}_\varepsilon(S)\|_\Theta &= \bigg\| \int_0^{+\infty} \exp\left(\left(\mathcal{H} + \Gamma_2(\varepsilon)\right)^T \sigma\right) \Big[\Gamma_3(\varepsilon) + \varepsilon h S \big(\mathcal{H} + \Gamma_1(\varepsilon)\big) R_1^{-1} \big(\mathcal{H} + \Gamma_1(\varepsilon)\big)^T S\Big] \exp\left(\left(\mathcal{H} + \Gamma_2(\varepsilon)\right)\sigma\right) d\sigma \bigg\|_\Theta \\ &\le \int_0^{+\infty} \|\exp\left(\left(\mathcal{H} + \Gamma_2(\varepsilon)\right)\sigma\right)\|_\Theta^2 \Big[\|\Gamma_3(\varepsilon)\|_\Theta + \varepsilon h \|S\|_\Theta^2 \|\mathcal{H} + \Gamma_2(\varepsilon)\|_\Theta^2 \|R_1^{-1}\|_\Theta\Big] d\sigma \\ &\le \frac{1}{2\gamma}\big[c_3 + h c_{\mathcal{B}}^2 c_4 \varepsilon^2\big]\varepsilon,\end{aligned} \tag{27}$$

where

$$c_4 \stackrel{\triangle}{=} \big(\|\mathcal{H}\|_\Theta + c_1\varepsilon_1\big)^2 \|R_1^{-1}\|_\Theta. \tag{28}$$

Using the definition of the ball $\mathcal{B}(c_{\mathcal{B}}, \varepsilon)$ (see the Equation (25)) and inequality (27), we can conclude the following. If, for some numbers $c_{\mathcal{B}} > 0$ and $\varepsilon > 0$,

$$\frac{1}{2\gamma}\big[c_3 + h c_{\mathcal{B}}^2 c_4 \varepsilon^2\big] \le c_{\mathcal{B}}, \tag{29}$$

then, for these $c_{\mathcal{B}}$ and ε, the operator $\mathcal{F}_\varepsilon(\cdot)$ maps the ball $\mathcal{B}(c_{\mathcal{B}}, \varepsilon)$ into itself.

Since $\lim_{\varepsilon \to +0} h c_{\mathcal{B}}^2 c_4 \varepsilon^2 = 0$, then, for any pre-chosen $\nu > 1$ and $c_{\mathcal{B}} = c_{\mathcal{B}}(\nu) = \nu \frac{c_3}{2\gamma}$, there exists $\widetilde{\varepsilon}(\nu) \in (0, \varepsilon_2]$, such that inequality (29) is valid for all $\varepsilon \in (0, \widetilde{\varepsilon}(\nu)]$. Thus, for the aforementioned $c_{\mathcal{B}} = c_{\mathcal{B}}(\nu)$ and ε, the operator $\mathcal{F}_\varepsilon(\cdot)$ maps the ball $\mathcal{B}(c_{\mathcal{B}}, \varepsilon)$ into itself.

Now, let us show that, for all sufficiently small $\varepsilon > 0$, the operator $\mathcal{F}_\varepsilon(\cdot)$ satisfies the Lipschitz condition in the ball $\mathcal{B}\big(c_{\mathcal{B}}(\nu),\varepsilon\big)$ with a constant $0 < q < 1$. For any $S_1 \in \mathcal{B}\big(c_{\mathcal{B}}(\nu),\varepsilon\big)$ and $S_2 \in \mathcal{B}\big(c_{\mathcal{B}}(\nu),\varepsilon\big)$, using inequalities (18) and (23) and Equations (25), (26), and (28), we obtain

$$\big\|\mathcal{F}_\varepsilon(S_1) - \mathcal{F}_\varepsilon(S_2)\big\|_\Theta \le \frac{hc_{\mathcal{B}}(\nu)c_4}{\gamma}\varepsilon^2\big\|S_1 - S_2\big\|_\Theta, \quad \varepsilon \in (0,\widetilde{\varepsilon}(\nu)]. \tag{30}$$

This inequality means that, for any pre-chosen number $0 < q < 1$, there exists $\widehat{\varepsilon}(q) \in (0,\widetilde{\varepsilon}(\nu)]$, such that the operator $\mathcal{F}_\varepsilon(\cdot)$ satisfies the Lipschitz condition in the ball $\mathcal{B}\big(c_{\mathcal{B}}(\nu),\varepsilon\big)$ with this constant for any $\varepsilon \in (0,\widehat{\varepsilon}(q)]$.

Summarizing the above presented analysis of the operator $\mathcal{F}_\varepsilon(\cdot)$, we have the following: For any $\varepsilon \in (0,\widehat{\varepsilon}(q)]$, the operator $\mathcal{F}_\varepsilon(\cdot)$ maps the ball $\mathcal{B}\big(c_{\mathcal{B}}(\nu),\varepsilon\big)$ into itself and satisfies the Lipschitz condition in this ball with the aforementioned constant $0 < q < 1$. Hence, by virtue of the results of [56,57], Equation (24) has the unique solution $\Delta P_1 = \Delta P_1(\varepsilon)$ in the ball $\mathcal{B}\big(c_{\mathcal{B}}(\nu),\varepsilon\big)$ for all $\varepsilon \in (0,\widehat{\varepsilon}(q)]$. Furthermore, since Equation (24) is equivalent to Equation (20), then $\Delta P_1(\varepsilon)$ is also the solution of this problem. Since $\Delta P_1(\varepsilon) \in \mathcal{B}\big(c_{\mathcal{B}}(\nu),\varepsilon\big)$ for all $\varepsilon \in (0,\widehat{\varepsilon}(q)]$, P_{10} is a symmetric matrix and $\|\cdot\|_\Theta = \|\cdot\|$, then, due to (19), Equation (11) has the symmetric solution $P_1 = P_1(\varepsilon)$, $\varepsilon \in (0,\widehat{\varepsilon}(q)]$, and this solution satisfies inequality (16) with $a_1 = c_{\mathcal{B}}(\nu)$ and $\varepsilon_1^* = \widehat{\varepsilon}(q)$. Moreover, since P_{10} is a positive definite matrix, then there exists a number $\bar{\varepsilon} \in \big(0,\widehat{\varepsilon}(q)\big]$, such that, for all $\varepsilon \in (0,\bar{\varepsilon}]$, the matrix $P_1(\varepsilon)$ is positive definite. Thus, the statements of the theorem are valid for $\varepsilon_1^* = \bar{\varepsilon}$ and $a_1 = c_{\mathcal{B}}(\nu)$. □

As a direct consequence of Proposition 3 and Theorem 1, we obtain the following assertion.

Corollary 1. *Let the assumptions A1 and A2 be valid. Then, for any given $\varepsilon \in (0,\varepsilon_1^*]$, the first equation in system (9) is asymptotically stable.*

2.3. Asymptotic Solution with Respect to ε of the Equation (12)

To solve asymptotically Equation (12), we choose the matrix Q_2 as follows:

$$Q_2 = Q_2(\varepsilon) = \frac{1}{\varepsilon}\widetilde{Q}, \quad \varepsilon \in (0,\varepsilon_1], \tag{31}$$

where $\widetilde{Q}$ is some symmetric positive definite matrix independent of ε.

Substituting (31) into (12), we directly obtain, after a simple rearrangement, the following equivalent symmetric Riccati algebraic equations with respect to the $m \times m$-matrix P_2:

$$\begin{aligned}\big(H_4 + \varepsilon L(\varepsilon)H_2\big)^T P_2 + P_2\big(H_4 + \varepsilon L(\varepsilon)H_2\big)\\ + hP_2\big(H_4 + \varepsilon L(\varepsilon)H_2\big)R_2^{-1}\big(H_4 + \varepsilon L(\varepsilon)H_2\big)^T P_2\\ + h\big(H_4 + \varepsilon L(\varepsilon)H_2\big)^T R_2\big(H_4 + \varepsilon L(\varepsilon)H_2\big) = -\widetilde{Q}.\end{aligned} \tag{32}$$

We look for the zero-order asymptotic solution P_{20} of (32). Using Proposition 1 and setting formally $\varepsilon = 0$ in Equation (32), we obtain the following equation for its zero-order asymptotic solution:

$$H_4^T P_{20} + P_{20}H_4 + hP_{20}H_4R_2^{-1}H_4^T P_{20} + hH_4^T R_2 H_4 = -\widetilde{Q}. \tag{33}$$

In what follows, we assume the following:
A3. There exist symmetric positive definite matrices R_2 and $\widetilde{Q}$, such that:
(a) the symmetric matrix Riccati algebraic Equation (33) has a symmetric positive definite

solution P_{20};
(b) all eigenvalues of the matrix

$$\mathcal{G} \stackrel{\triangle}{=} H_4 + hH_4R_2^{-1}H_4^TP_{20} \tag{34}$$

lie strictly inside either the left-hand half or the right-hand half of the complex plane.

Lemma 1. *Let the assumption A1 and the item (a) of the assumption A3 be valid. Then, H_4 is a Hurwitz matrix.*

Proof. Consider the Lyapunov algebraic equation with respect to the $m \times m$-matrix P,

$$H_4^TP + PH_4 = -hP_{20}H_4R_2^{-1}H_4^TP_{20} - hH_4^TR_2H_4 - \widetilde{Q}.$$

The matrix in the right-hand side of this equation is negative definite. Moreover, due to item (a) of assumption A3, this equation has the symmetric positive definite solution $P = P_{20}$. Hence, by the same arguments as in Remark 1, we conclude that the matrix H_4 is a Hurwitz one. □

Remark 2. *Based on Lemma 1 and the results of [25] (see Chapter 7, Theorem 2.3), we directly conclude the following. The validity of the assumption A3 (item (a)) guarantees that H_4 is a Hurwitz matrix and the equation*

$$\frac{dz(\xi)}{d\xi} = H_4z(\xi - h), \quad \xi \geq 0, \quad z(\xi) \in E^m \tag{35}$$

is asymptotically stable.

However, the only requirement that H_4 is a Hurwitz matrix does not, in general, guarantee the asymptotic stability of Equation (35) (see, e.g., Section 2.3 in [2]).

Theorem 2. *Let the assumptions A1 and A3 be valid. Then, there exists a number $0 < \varepsilon_2^* \leq \varepsilon_1$, such that, for all $\varepsilon \in (0, \varepsilon_2^*]$, Equation (32) has a symmetric positive definite solution $P_2(\varepsilon)$ satisfying the inequality*

$$\left\|P_2(\varepsilon) - P_{20}\right\| \leq a_2\varepsilon, \quad \varepsilon \in (0, \varepsilon_2^*], \tag{36}$$

where $a_2 > 0$ is some constant independent of ε.

Proof. Let us transform the unknown matrix in Equation (32) as:

$$P_2 = P_{20} + \Delta P_2, \tag{37}$$

where ΔP_2 is a new unknown matrix.

Substituting (37) into Equation (32) and using Equations (33) and (34), we obtain the following equation with respect to ΔP_2:

$$\begin{aligned}\big(\mathcal{G} + \Lambda_1(\varepsilon)\big)^T\Delta P_2 + \Delta P_2\big(\mathcal{G} + \Lambda_1(\varepsilon)\big) = -\Lambda_2(\varepsilon) \\ -h\Delta P_2\big(H_4 + \varepsilon L(\varepsilon)H_2\big)R_2^{-1}\big(H_4 + \varepsilon L(\varepsilon)H_2\big)^T\Delta P_2,\end{aligned} \tag{38}$$

where

$$\begin{aligned}\Lambda_1(\varepsilon) = \varepsilon\big(L(\varepsilon)H_2 + hH_4R_2^{-1}H_2^TL^T(\varepsilon)P_{20} + hL(\varepsilon)H_2R_2^{-1}H_4^TP_{20} \\ +\varepsilon hL(\varepsilon)H_2R_2^{-1}H_2^TL^T(\varepsilon)P_{20}\big), \\ \Lambda_2(\varepsilon) = \varepsilon\big(H_2^TL^T(\varepsilon)P_{20} + P_{20}L(\varepsilon)H_2 + hP_{20}L(\varepsilon)H_2R_2^{-1}H_4^TP_{20} \\ +hP_{20}H_4R_2^{-1}H_2^TL^T(\varepsilon)P_{20} + \varepsilon hP_{20}L(\varepsilon)H_2R_2^{-1}H_2^TL^T(\varepsilon)P_{20} + hH_4^TR_2L(\varepsilon)H_2 \\ +hH_2^TL^T(\varepsilon)R_2H_4 + \varepsilon hH_2^TL^T(\varepsilon)R_2L(\varepsilon)H_2\big).\end{aligned} \tag{39}$$

Due to Proposition 1, matrices $\Lambda_1(\varepsilon)$ and $\Lambda_2(\varepsilon)$ satisfy the inequalities

$$\|\Lambda_1(\varepsilon)\| \le b_1\varepsilon, \quad \|\Lambda_2(\varepsilon)\| \le b_2\varepsilon, \quad \varepsilon \in (0, \varepsilon_1], \tag{40}$$

where $b_1 > 0$ and $b_2 > 0$ are some constants independent of ε.

Moreover, since P_{20} and R_2 are symmetric matrices, then, for all $\varepsilon \in (0, \varepsilon_1]$, matrix $\Lambda_2(\varepsilon)$ is symmetric.

Furthermore, taking into account item (b) of assumption A3 and using the first inequality in (40) and the results of [55], we obtain the existence of a number $0 < \tilde{\varepsilon}_1 \le \varepsilon_1$, such that, for all $\varepsilon \in (0, \tilde{\varepsilon}_1]$, the eigenvalues $\mu_j(\varepsilon)$, $(j = 1, \ldots, m)$ of the matrix $\mathcal{G} + \Lambda_1(\varepsilon)$ satisfy either the inequality

$$\mathrm{Re}\big(\mu_j(\varepsilon)\big) < -\chi, \quad j = 1, \ldots, m, \tag{41}$$

or the inequality

$$\mathrm{Re}\big(\mu_j(\varepsilon)\big) > \chi, \quad j = 1, \ldots, m, \tag{42}$$

where $\chi > 0$ is some constant independent of ε.

If inequality (41) is valid, then Equation (38) can be rewritten in the equivalent form as:

$$\Delta P_2 = \int_0^{+\infty} \exp\big((\mathcal{G} + \Lambda_1(\varepsilon))^T\sigma\big)\big[\Lambda_2(\varepsilon) \\ + h\Delta P_2\big(H_4 + \varepsilon L(\varepsilon)H_2\big)R_2^{-1}\big(H_4 + \varepsilon L(\varepsilon)H_2\big)^T\Delta P_2\big]\exp\big((\mathcal{G} + \Lambda_1(\varepsilon))\sigma\big)d\sigma. \tag{43}$$

If inequality (42) is valid, then Equation (38) can be rewritten in the equivalent form as:

$$\Delta P_2 = -\int_0^{+\infty} \exp\big(-(\mathcal{G} + \Lambda_1(\varepsilon))^T\sigma\big)\big[\Lambda_2(\varepsilon) \\ + h\Delta P_2\big(H_4 + \varepsilon L(\varepsilon)H_2\big)R_2^{-1}\big(H_4 + \varepsilon L(\varepsilon)H_2\big)^T\Delta P_2\big]\exp\big(-(\mathcal{G} + \Lambda_1(\varepsilon))\sigma\big)d\sigma. \tag{44}$$

The rest of the proof is quite similar to the corresponding part of the proof of Theorem 1. This completes the proof of the present theorem. □

The following assertion is a direct consequence of Proposition 4 and Theorem 2.

Corollary 2. *Let the assumptions A1 and A3 be valid. Then, for any given $\varepsilon \in (0, \varepsilon_2^*]$, the second equation in system (9) is asymptotically stable.*

2.4. ε-Free Asymptotic Stability Conditions for System (1)

Let us denote

$$\varepsilon^* \stackrel{\triangle}{=} \min\{\varepsilon_1^*, \varepsilon_2^*\}. \tag{45}$$

Now, using Proposition 2, Corollaries 1 and 2, and Equation (45), we immediately have the following theorem.

Theorem 3. *Let the assumptions A1–A3 be valid. Then, for any given $\varepsilon \in (0, \varepsilon^*]$, system (1) is asymptotically stable.*

2.5. Nonlinear Singularly Perturbed System

Consider the following differential system:

$$\begin{aligned} \frac{dx(t)}{dt} &= f\big(x(t-\varepsilon h), y(t-\varepsilon h)\big), \quad t \geq 0, \\ \varepsilon\frac{dy(t)}{dt} &= g\big(x(t-\varepsilon h), y(t-\varepsilon h)\big), \quad t \geq 0, \end{aligned} \tag{46}$$

where $x(t) \in E^n$ and $y(t) \in E^m$; $f(\phi,\psi) : E^n \times E^m \to E^n$ and $g(\phi,\psi) : E^n \times E^m \to E^m$ are given functions; $f(0,0) = 0$ and $g(0,0) = 0$; $\varepsilon > 0$ is a small parameter; $h > 0$ is a given number independent of ε.

For this system (like for system (1)), we consider the initial conditions (2).

Based on the results of [25], we introduce the following definitions.

Definition 2. *For a given $\varepsilon > 0$, the trivial solution $\big(x(t) \equiv 0, y(t) \equiv 0\big)$ of system (46) is called stable if for any number $\alpha > 0$ there exists a number $\beta > 0$, dependent on α, such that the solution $\mathrm{col}\Big(x\big(t;\varphi_x(\cdot),\varphi_y(\cdot)\big), y\big(t;\varphi_x(\cdot),\varphi_y(\cdot)\big)\Big)$ of the initial-value problem (46), (2) satisfies the inequality $\Big\|\mathrm{col}\Big(x\big(t;\varphi_x(\cdot),\varphi_y(\cdot)\big), y\big(t;\varphi_x(\cdot),\varphi_y(\cdot)\big)\Big)\Big\| \leq \alpha$, $t \geq 0$, for any pair $\big(\varphi_x(\tau),\varphi_y(\tau)\big)$ satisfying the inequality*

$$\big\|\mathrm{col}\big(\varphi_x(\tau),\varphi_y(\tau)\big)\big\| \leq \beta, \quad \tau \in [-\varepsilon h, 0]. \tag{47}$$

Definition 3. *For a given $\varepsilon > 0$, the trivial solution $\big(x(t) \equiv 0, y(t) \equiv 0\big)$ of system (46) is called asymptotically stable if this solution is stable and there exists a number $\beta > 0$ such that, for any pair $\big(\varphi_x(\tau),\varphi_y(\tau)\big)$ satisfying (47), the solution $\mathrm{col}\Big(x\big(t;\varphi_x(\cdot),\varphi_y(\cdot)\big), y\big(t;\varphi_x(\cdot),\varphi_y(\cdot)\big)\Big)$ of the initial-value problem (46), (2) tends to zero as t tends to $+\infty$.*

In what follows of this section, we assume the following:

A4. The functions $f(\phi,\psi)$ and $g(\phi,\psi)$ are twice continuously differentiable for $(\phi,\psi) \in E^n \times E^m$.

The linearization of system (46) in a neighborhood of its trivial solution $\big(x(t) \equiv 0, y(t) \equiv 0, t \geq -\varepsilon h\big)$ yields the following system:

$$\begin{aligned} \frac{dx(t)}{dt} &= f_\phi(0,0)x(t-\varepsilon h) + f_\psi(0,0)y(t-\varepsilon h), \quad t \geq 0, \\ \varepsilon\frac{dy(t)}{dt} &= g_\phi(0,0)x(t-\varepsilon h) + g_\psi(0,0)y(t-\varepsilon h), \quad t \geq 0. \end{aligned} \tag{48}$$

As a direct consequence of Theorem 3, we have the following lemma.

Lemma 2. *Let the assumption A4 be valid. Let the assumptions A1–A3 be valid for $H_1 = f_\phi(0,0)$, $H_2 = f_\psi(0,0)$, $H_3 = g_\phi(0,0)$, and $H_4 = g_\psi(0,0)$. Then, there exists a number $\widetilde{\varepsilon}^* > 0$, such that, for any given $\varepsilon \in (0,\widetilde{\varepsilon}^*]$, system (48) is asymptotically stable.*

Theorem 4. *Let the assumption A4 be valid. Let the assumptions A1–A3 be valid for $H_1 = f_\phi(0,0)$, $H_2 = f_\psi(0,0)$, $H_3 = g_\phi(0,0)$, and $H_4 = g_\psi(0,0)$. Then, for any given $\varepsilon \in (0,\widetilde{\varepsilon}^*]$, the trivial solution of system (46) is asymptotically stable.*

Proof. First of all, let us note the following. Using the assumption A4 and applying the Taylor's formula to the functions $f(\phi,\psi)$ and $g(\phi,\psi)$, we obtain the following inequalities:

$$\begin{aligned} &\big\|f(\phi,\psi) - f_\phi(0,0) - f_\psi(0,0)\big\| \leq \omega_f(\phi,\psi)\big\|\mathrm{col}(\phi,\psi)\big\|^2, \quad (\phi,\psi) \in E^n \times E^m, \\ &\big\|g(\phi,\psi) - g_\phi(0,0) - g_\psi(0,0)\big\| \leq \omega_g(\phi,\psi)\big\|\mathrm{col}(\phi,\psi)\big\|^2, \quad (\phi,\psi) \in E^n \times E^m, \end{aligned} \tag{49}$$

where the positive functions $\omega_f(\phi,\psi)$ and $\omega_g(\phi,\psi)$, being estimates for the absolute values of all the second-order derivatives of all the entries of $f(\phi,\psi)$ and $g(\phi,\psi)$, respectively, at any given point $(\phi,\psi) \in E^n \times E^m$, are bounded in any bounded and closed set of the space $E^n \times E^m$.

Due to (49), we obtain the following inequalities for any given $\varepsilon \in (0,\widetilde{\varepsilon}^*]$ and any vector-valued functions $\varphi_x(\tau)$, $\varphi_y(\tau)$, $\tau \in [-\varepsilon h, 0]$, introduced in (2):

$$\begin{aligned}\left\|f\big(\varphi_x(\tau),\varphi_y(\tau)\big) - f_\phi(0,0) - f_\psi(0,0)\right\| &\le \omega_f\big(\varphi_x(\tau),\varphi_y(\tau)\big)\Big(\left\|\mathrm{col}(\varphi_x(\cdot),\varphi_y(\cdot))\right\|_C\Big)^2,\\ \left\|g\big(\varphi_x(\tau),\varphi_y(\tau)\big) - g_\phi(0,0) - g_\psi(0,0)\right\| &\le \omega_g\big(\varphi_x(\tau),\varphi_y(\tau)\big)\Big(\left\|\mathrm{col}(\varphi_x(\cdot),\varphi_y(\cdot))\right\|_C\Big)^2. \end{aligned} \quad (50)$$

Thus, for any given $\varepsilon \in (0,\widetilde{\varepsilon}^*]$ and $\tau \in [-\varepsilon h, 0]$,

$$\lim_{\|\mathrm{col}(\varphi_x(\cdot),\varphi_y(\cdot))\|_C \to 0} \omega_f\big(\varphi_x(\tau),\varphi_y(\tau)\big)\left\|\mathrm{col}(\varphi_x(\cdot),\varphi_y(\cdot))\right\|_C = 0, \quad (51)$$

$$\lim_{\|\mathrm{col}(\varphi_x(\cdot),\varphi_y(\cdot))\|_C \to 0} \frac{1}{\varepsilon}\omega_g\big(\varphi_x(\tau),\varphi_y(\tau)\big)\left\|\mathrm{col}(\varphi_x(\cdot),\varphi_y(\cdot))\right\|_C = 0. \quad (52)$$

Now, using the results of [2,21] on the asymptotic stability in the first approximation of time-delay equations, as well as the inequalities in (50), the limit equalities (51) and (52), and Lemma 2, we immediately obtain the statement of the theorem. □

2.6. Examples

2.6.1. Example 1

Consider a particular case of system (1) with the following data:

$$n = 2,\quad m = 2,\quad h = 1,\quad H_1 = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix},\quad H_2 = \begin{pmatrix} 4 & -3 \\ 1 & -1 \end{pmatrix},$$
$$H_3 = \begin{pmatrix} 2 & -1 \\ 4 & 8 \end{pmatrix},\quad H_4 = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.8 \end{pmatrix}. \quad (53)$$

Due to the form of the matrix H_4, assumption A1 is valid in this example.

For system (1), (53), the matrix $\mathcal{H}$, given by Equation (14), is

$$\mathcal{H} = \begin{pmatrix} 2 & -36 \\ 3 & -15 \end{pmatrix},$$

and the real part of both its eigenvalues equals -6.5. Thus, $\mathcal{H}$ is a Hurwitz matrix. Therefore, due to Remark 1, assumption A2 is valid in this example.

Proceed to the analysis and solution of the Riccati Equation (33) in this example. Let us choose the symmetric positive definite matrices R_2 and $\widetilde{Q}$ as:

$$R_2 = \begin{pmatrix} R_{21} & 0 \\ 0 & R_{22} \end{pmatrix},\quad \widetilde{Q} = \begin{pmatrix} \widetilde{Q}_1 & 0 \\ 0 & \widetilde{Q}_2 \end{pmatrix}. \quad (54)$$

Due to the diagonal form of R_2, $\widetilde{Q}$, and H_4, we look for the symmetric positive definite solution P_{20} of Equation (33) in the diagonal form, i.e.,

$$P_{20} = \begin{pmatrix} P_{20,1} & 0 \\ 0 & P_{20,2} \end{pmatrix}. \quad (55)$$

Substitution of the value h and the matrices H_4, R_2, $\widetilde{Q}$, P_{20} (see Equations (53), (54), and (55)) into Equation (33) yields, after a routine rearrangement, the following system of two disconnected scalar algebraic quadratic equations, the first of which is with respect to $P_{20,1}$, while the second is with respect to $P_{20,2}$:

$$\frac{1}{4R_{21}}P_{20,1}^2 - P_{20,1} + 0.25R_{21} + \widetilde{Q}_1 = 0,$$
$$\frac{16}{25R_{22}}P_{20,2}^2 - 1.6P_{20,2} + 0.64R_{22} + \widetilde{Q}_2 = 0. \quad (56)$$

Let us start with the analysis of the first equation in (56). This equation yields two solutions:

$$P_{20,1}^+ = 2R_{21}\left(1 + \sqrt{0.75 - \widetilde{Q}_1/R_{21}}\right),$$
$$P_{20,1}^- = 2R_{21}\left(1 - \sqrt{0.75 - \widetilde{Q}_1/R21}\right). \quad (57)$$

Each of these solutions is positive if and only if

$$0 < \widetilde{Q}_1 \le 0.75R_{21}. \quad (58)$$

Moreover, the expression

$$\mathcal{G}_1^+ \stackrel{\triangle}{=} H_4 + hH_4^2R_{21}^{-1}P_{20,1}^+ = 0.5\sqrt{0.75 - \widetilde{Q}_1/R_{21}} > 0, \quad (59)$$

while

$$\mathcal{G}_1^- \stackrel{\triangle}{=} H_4 + hH_4^2R_{21}^{-1}P_{20,1}^- = -0.5\sqrt{0.75 - \widetilde{Q}_1/R_{21}} < 0, \quad (60)$$

if and only if inequality (58) with the strict right-hand side is valid, i.e.,

$$0 < \widetilde{Q}_1 < 0.75R_{21}. \quad (61)$$

Proceed to the analysis of the second equation in (56). This equation yields two solutions:

$$P_{20,2}^+ = 1.25R_{22}\left(1 + \sqrt{0.36 - \widetilde{Q}_2/R_{22}}\right),$$
$$P_{20,2}^- = 1.25R_{22}\left(1 - \sqrt{0.36 - \widetilde{Q}_2/R22}\right). \quad (62)$$

Each of these solutions is positive if and only if

$$0 < \widetilde{Q}_2 \le 0.36R_{22}. \quad (63)$$

Moreover, the expression

$$\mathcal{G}_2^+ \stackrel{\triangle}{=} H_4 + hH_4^2R_{22}^{-1}P_{20,2}^+ = 0.8\sqrt{0.36 - \widetilde{Q}_2/R_{22}} > 0, \quad (64)$$

while

$$\mathcal{G}_2^- \stackrel{\triangle}{=} H_4 + hH_4^2R_{22}^{-1}P_{20,2}^- = -0.8\sqrt{0.36 - \widetilde{Q}_2/R_{22}} < 0, \quad (65)$$

if and only if inequality (63) with the strict right-hand side is valid, i.e.,

$$0 < \widetilde{Q}_2 < 0.36R_{22}. \quad (66)$$

In this example, for $P_{20,1} = P^+_{20,1}$ and $P_{20,2} = P^+_{20,2}$ the matrix $\mathcal{G}$, given by Equation (34), has the form

$$\mathcal{G} = \mathcal{G}^+ = \begin{pmatrix} 0.5\sqrt{0.75 - \widetilde{Q}_1/R_{21}} & 0 \\ 0 & 0.8\sqrt{0.36 - \widetilde{Q}_2/R_{22}} \end{pmatrix}. \tag{67}$$

For $P_{20,1} = P^-_{20,1}$, and $P_{20,2} = P^-_{20,2}$, the matrix $\mathcal{G}$, given by Equation (34), has the form

$$\mathcal{G} = \mathcal{G}^- = \begin{pmatrix} -0.5\sqrt{0.75 - \widetilde{Q}_1/R_{21}} & 0 \\ 0 & -0.8\sqrt{0.36 - \widetilde{Q}_2/R_{22}} \end{pmatrix}. \tag{68}$$

Taking into account Equations (54) and (55) and summarizing the above presented analysis of the system (56), we can conclude the following. For any values R_{21}, $\widetilde{Q}_1$ and R_{22}, $\widetilde{Q}_2$, satisfying inequalities (61) and (66), respectively, the assumption A3 is valid in this example.

Thus, all assumptions A1,A2, and A3 are valid in this example. Therefore, due to Theorem 3, there exists a number $\varepsilon^* > 0$, such that, for any given $\varepsilon \in (0, \varepsilon^*]$, system (1), (53) is asymptotically stable.

2.6.2. Example 2

In this example, we are going to show that the time delay in system (1) of the order of the small parameter ε, i.e., of the form εh, is considerable.

Let us consider the scalar differential equation

$$\varepsilon\frac{dy(t)}{dt} = Hy\big(t - h(\varepsilon)\big), \quad t \geq 0, \tag{69}$$

where H is a given number; $\varepsilon > 0$ is a small parameter; $h(\varepsilon) > 0$ is a given function of ε, such that $h(\varepsilon)/\varepsilon \to +\infty$ for $\varepsilon \to +0$. As a particular case, $h(\varepsilon)$ can be a positive constant.

Consider the scalar quadratic equation with respect to unknown P,

$$\frac{h(\varepsilon)}{\varepsilon^2}\frac{H^2}{R}P^2 + \frac{2}{\varepsilon}HP + \frac{h(\varepsilon)}{\varepsilon^2}H^2R + Q = 0, \tag{70}$$

where $R > 0$ and $Q > 0$ are some numbers.

Due to the results of [25] (see Chapter 7, Theorem 2.3), if for a given $\varepsilon > 0$ there exist positive numbers R and Q, such that Equation (70) has a real positive solution, then the differential Equation (69) is asymptotically stable.

Let us show that Equation (70) does not have a positive solution for any sufficiently small $\varepsilon > 0$ and any positive numbers R and Q. Indeed, since $\frac{h(\varepsilon)}{\varepsilon^2}\frac{H^2}{R} > 0$ and $\frac{h(\varepsilon)}{\varepsilon^2}H^2R + Q > 0$, then, due to the Vieta formulas, the necessary condition for the existence of the aforementioned solution to (70) is the fulfilment of the inequality $H < 0$. Using this inequality and solving Equation (70), we obtain two of its solutions:

$$P_1 = \frac{1 + \sqrt{1 - \big(h(\varepsilon)/\varepsilon\big)^2H^2 - \big(h(\varepsilon)/R\big)Q}}{\big(h(\varepsilon)/\varepsilon\big)\big(|H|/R\big)},$$

$$P_1 = \frac{1 - \sqrt{1 - \big(h(\varepsilon)/\varepsilon\big)^2H^2 - \big(h(\varepsilon)/R\big)Q}}{\big(h(\varepsilon)/\varepsilon\big)\big(|H|/R\big)}.$$

Since $h(\varepsilon)/\varepsilon \to +\infty$ for $\varepsilon \to +0$, then the expression $1 - \big(h(\varepsilon)/\varepsilon\big)^2H^2 - \big(h(\varepsilon)/R\big)Q$ is negative for all sufficiently small $\varepsilon > 0$ and any positive numbers R and Q. The latter means that, for such ε, R and Q, both solutions of Equation (70) are not real. Hence, the aforementioned asymptotic stability condition is not applicable to differential Equation (69) for any constant coefficient H and all sufficiently small $\varepsilon > 0$. Moreover, due to the results of [2] (see Section 2.3), differential Equation (69) is not asymptotically stable for any constant coefficient H and all sufficiently small $\varepsilon > 0$.

2.6.3. Example 3

Consider a particular case of system (46) with the following data:

$$\begin{aligned} n = 1, \quad m = 1, \quad h = 1, \\ f\big(x(t-\varepsilon h), y(t-\varepsilon h)\big) = \sin\big(y(t-\varepsilon h) - 4x(t-\varepsilon h)\big), \\ g\big(x(t-\varepsilon h), y(t-\varepsilon h)\big) = \big(x(t-\varepsilon h) - 0.4y(t-\varepsilon h) + 1\big)^2 - 1. \end{aligned} \tag{71}$$

It is seen that $f(0,0) = 0$, $g(0,0) = 0$. Moreover, assumption A4 is valid in this example.

The linearization of the system (46), (71) in a neighborhood of its trivial solution $\big(x(t) \equiv 0, y(t) \equiv 0, t \geq -\varepsilon h\big)$, yields the system

$$\begin{aligned} \frac{dx(t)}{dt} &= -4x(t-\varepsilon h) + y(t-\varepsilon h), \quad t \geq 0, \\ \varepsilon\frac{dy(t)}{dt} &= 2x(t-\varepsilon h) - 0.8y(t-\varepsilon h), \quad t \geq 0. \end{aligned} \tag{72}$$

System (72) is a particular case of system (1) with the scalar coefficients

$$H_1 = -4, \quad H_2 = 1, \quad H_3 = 2, \quad H_4 = -0.8. \tag{73}$$

Thus, assumption A1 is valid for (72). Moreover, since $\mathcal{H}$, given by (14), is negative ($\mathcal{H} = -1.5$) then, due to Remark 1, assumption A2 is also valid.

Equation (33) for system (72) has the form

$$\frac{16}{25R_2}P_{20}^2 - 1.6P_{20} + 0.64R_2 + \widetilde{Q} = 0,$$

i.e., it is the same as the second equation in (56). Therefore, for any scalars R_2 and $\widetilde{Q}$, satisfying the inequality $0 < \widetilde{Q} < 0.36R_2$, assumption A3 is valid for system (72). Hence, by virtue of Lemma 2, this system is asymptotically stable for any given sufficiently small $\varepsilon > 0$, and by virtue of Theorem 4, the trivial solution of the nonlinear system (46), (71) is asymptotically stable for such ε.

3. Second Type System

3.1. Formulation, Basic Definition, and Assertions

Consider the following differential system:

$$\begin{aligned} \frac{dx(t)}{dt} &= A_1x(t) + A_2y(t) + \sum_{j=1}^{N}\big[H_{1j}x(t-h_j) + H_{2j}y(t-h_j)\big], \quad t \geq 0, \\ \varepsilon\frac{dy(t)}{dt} &= A_3x(t) + A_4y(t) + \sum_{j=1}^{N}\big[H_{3j}x(t-h_j) + H_{4j}y(t-h_j)\big], \quad t \geq 0, \end{aligned} \tag{74}$$

where $x(t) \in E^n$ and $y(t) \in E^m$; A_i and H_{ij}, $(i = 1,\dots,4;\ j = 1,\dots,N)$ are given constant matrices of corresponding dimensions; $\varepsilon > 0$ is a small parameter; $0 < h_1 < \dots < h_N$ are given time delays.

Like system (1), system (74) is a time-delay system. However, in contrast with (1), system (74) has in its right-hand side both delayed and undelayed, $x(\cdot)$ and $y(\cdot)$. Moreover, the delayed $x(\cdot)$ and $y(\cdot)$ are with multiple point-wise delays. Being a time-delay system, (74) is infinite-dimensional with the state variables $x(t+\tau)$, $y(t+\tau)$, $\tau \in [-h_N, 0]$.

Like (1), system (74) is a singularly perturbed system with the parameter ε of singular perturbation. However, in contrast with (1), the time delays in (74) are independent of ε.

For system (74), we consider the initial conditions

$$x(\tau) = \varphi_x(\tau), \quad y(\tau) = \varphi_y(\tau), \quad \tau \in [-h_N, 0], \tag{75}$$

where $\varphi_x(\tau)$ is a given n-dimensional vector-valued function; $\varphi_y(\tau)$ is a given m-dimensional vector-valued function; and both functions are continuous in the interval $[-h_N, 0]$.

Similarly to Definition 1, we introduce the following definition.

Definition 4. *For a given $\varepsilon > 0$, system (74) is called asymptotically stable if, for any aforementioned functions $\varphi_x(\tau)$ and $\varphi_y(\tau)$, $\tau \in [-h_N, 0]$, the solution of the initial-value problem (74)–(75) $\mathrm{col}\Big(x\big(t;\varphi_x(\cdot),\varphi_y(\cdot)\big), y\big(t;\varphi_x(\cdot),\varphi_y(\cdot)\big)\Big)$ tends to zero as t tends to $+\infty$.*

Consider the block-form matrix

$$A \triangleq \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}. \tag{76}$$

Also, we consider the following set of two matrix algebraic equations with respect to $m \times n$-matrix $\mathcal{L}$ and $n \times m$-matrix $\mathcal{M}$:

$$\begin{gathered} \varepsilon\mathcal{L}A_1 - A_4\mathcal{L} - \varepsilon\mathcal{L}A_2\mathcal{L} + A_3 = 0, \\ -\mathcal{M}(A_4 + \varepsilon\mathcal{L}A_2) + \varepsilon(A_1 - A_2\mathcal{L})\mathcal{M} + A_2 = 0. \end{gathered} \tag{77}$$

In what follows of this section, we assume the following:

A5. Matrix A_4 is invertible, i.e., A_4^{-1} exists.

Similarly to Proposition 1, we have the following assertion.

Proposition 5. *Let assumption A5 be valid. Then, there exists a positive number $\bar{\varepsilon}_1$, such that, for all $\varepsilon \in [0, \bar{\varepsilon}_1]$, the set (77) has the solution $\{\mathcal{L}(\varepsilon), \mathcal{M}(\varepsilon)\}$ satisfying the inequalities*

$$\|\mathcal{L}(\varepsilon) - A_4^{-1}A_3\| \le \bar{a}\varepsilon, \quad \|\mathcal{M}(\varepsilon) - A_2A_4^{-1}\| \le \bar{a}\varepsilon, \tag{78}$$

where $\bar{a} > 0$ is some constant independent of ε.

Consider the following block matrix:

$$\mathcal{D}(\varepsilon) \triangleq \begin{pmatrix} I_n & \varepsilon\mathcal{M}(\varepsilon) \\ -\mathcal{L}(\varepsilon) & I_m - \varepsilon\mathcal{L}(\varepsilon)\mathcal{M}(\varepsilon) \end{pmatrix}, \quad \varepsilon \in [0, \bar{\varepsilon}_1]. \tag{79}$$

Similarly to the matrix $D(\varepsilon)$, $\mathcal{D}(\varepsilon)$ is an invertible matrix and its inverse matrix is:

$$\mathcal{D}^{-1}(\varepsilon) = \begin{pmatrix} I_n - \varepsilon\mathcal{M}(\varepsilon)\mathcal{L}(\varepsilon) & -\varepsilon\mathcal{M}(\varepsilon) \\ \mathcal{L}(\varepsilon) & I_m \end{pmatrix}, \quad \varepsilon \in [0, \bar{\varepsilon}_1]. \tag{80}$$

Let us transform the state variables $x(t+\tau)$ and $y(t+\tau)$, $\tau \in [-h_N, 0]$ of the system (74) as:

$$\begin{pmatrix} x(t+\tau) \\ y(t+\tau) \end{pmatrix} = \mathcal{D}(\varepsilon)\begin{pmatrix} u(t+\tau) \\ v(t+\tau) \end{pmatrix}, \quad t \ge 0, \ \ \tau \in [-h_N, 0], \ \ \varepsilon \in (0, \bar{\varepsilon}_1], \tag{81}$$

where $u(t) \in E^n$, $v(t) \in E^m$; $u(t+\tau)$; and $v(t+\tau)$ are new state variables.

Similarly to Proposition 2, we directly have the following assertion:

Proposition 6. *Let assumption A5 be valid. Then, for any given $\varepsilon \in (0, \bar{\varepsilon}_1]$, transformation (81) converts the initial-value problem (74)–(75) to the following equivalent initial-value problem:*

$$\frac{du(t)}{dt} = \big(A_1 - A_2\mathcal{L}(\varepsilon)\big)u(t) + \sum_{j=1}^{N}\big[\mathcal{H}_{1j}(\varepsilon)u(t-h_j) + \mathcal{H}_{2j}(\varepsilon)v(t-h_j)\big],$$
$$\varepsilon\frac{dv(t)}{dt} = \big(A_4 + \varepsilon\mathcal{L}(\varepsilon)A_2\big)v(t) + \sum_{j=1}^{N}\big[\mathcal{H}_{3j}(\varepsilon)u(t-h_j) + \mathcal{H}_{4j}(\varepsilon)v(t-h_j)\big], \quad (82)$$

$$u(\tau) = \big(I_n - \varepsilon\mathcal{M}(\varepsilon)\mathcal{L}(\varepsilon)\big)\varphi_x(\tau) - \varepsilon\mathcal{M}(\varepsilon)\varphi_y(\tau), \quad \tau \in [-h_N, 0],$$
$$v(\tau) = \mathcal{L}(\varepsilon)\varphi_x(\tau) + \varphi_y(\tau), \quad \tau \in [-h_N, 0], \quad (83)$$

where

$$\mathcal{H}_j(\varepsilon) \triangleq \begin{pmatrix} \mathcal{H}_{1j}(\varepsilon) & \mathcal{H}_{2j}(\varepsilon) \\ \mathcal{H}_{3j}(\varepsilon) & \mathcal{H}_{4j}(\varepsilon) \end{pmatrix} = \mathcal{D}^{-1}(\varepsilon)\begin{pmatrix} H_{1j} & H_{2j} \\ H_{3j} & H_{4j} \end{pmatrix}\mathcal{D}(\varepsilon), \quad j = 1, \ldots, N. \quad (84)$$

Moreover, system (74) is asymptotically stable if and only if system (82) is asymptotically stable.

Consider the following symmetric Riccati algebraic equation with respect to $(n+m) \times (n+m)$-matrix P:

$$\mathcal{A}^T(\varepsilon)\mathcal{E}^{-1}(\varepsilon)P + P\mathcal{E}^{-1}(\varepsilon)\mathcal{A}(\varepsilon) + \sum_{j=1}^{N}\big[P\mathcal{E}^{-1}(\varepsilon)\mathcal{H}_j(\varepsilon)R_j^{-1}\mathcal{H}_j^T(\varepsilon)\mathcal{E}^{-1}(\varepsilon)P + R_j\big] = -Q, \quad (85)$$

where R_j, $(j = 1, \ldots, N)$ and Q are some symmetric positive definite matrices of the dimension $(n+m) \times (n+m)$;

$$\mathcal{E}(\varepsilon) \triangleq \begin{pmatrix} I_n & 0 \\ 0 & \varepsilon I_m \end{pmatrix}, \quad \mathcal{A}(\varepsilon) \triangleq \begin{pmatrix} A_1 - A_2\mathcal{L}(\varepsilon) & 0 \\ 0 & A_4 + \varepsilon\mathcal{L}(\varepsilon)A_2 \end{pmatrix}. \quad (86)$$

By virtue of Proposition 5 and the results of [25] (see Chapter 7, Theorem 3.1), we directly obtain the following assertion.

Proposition 7. *Let assumption A5 be valid. Let, for a given $\varepsilon \in (0, \bar{\varepsilon}_1]$, there exist symmetric positive definite matrices R_j, $(j = 1, \ldots, N)$ and Q such that the Equation (85) has a symmetric positive definite solution $P = P(\varepsilon)$. Then, for this ε, system (82) is asymptotically stable.*

We choose the matrices R_j, $(j = 1, \ldots, N)$ and Q in the block form as:

$$R_j = \begin{pmatrix} R_{1j} & 0 \\ 0 & R_{2j} \end{pmatrix}, \quad j = 1, \ldots, N;$$
$$Q = Q(\varepsilon) = \begin{pmatrix} Q_1 & Q_2(\varepsilon) \\ Q_2^T(\varepsilon) & Q_3 \end{pmatrix}, \quad (87)$$

where the matrices R_{1j}, $(j = 1, \ldots, N)$ and Q_1 are of the dimension $n \times n$; the matrices R_{2j}, $(j = 1, \ldots, N)$, and Q_3 are of the dimension $m \times m$.

Based on the block form of the matrices $\mathcal{E}(\varepsilon)$, $\mathcal{A}(\varepsilon)$, and R_j, $(j = 1, \ldots, N)$, we look for the symmetric positive definite solution $P = P(\varepsilon)$ of Riccati Equation (85) in the block form

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & \varepsilon P_2 \end{pmatrix}, \quad (88)$$

where the matrices P_1 and P_2 are of the dimensions $n \times n$ and $m \times m$, respectively.

Substituting the block-form representations of the matrices $\mathcal{H}_j(\varepsilon)$, $(j = 1, \ldots, N)$, $\mathcal{E}(\varepsilon)$, $\mathcal{A}(\varepsilon)$, R_j, $(j = 1, \ldots, N)$, $Q(\varepsilon)$ and P (see Equations (84), (86), (87), and (88)) into Equation (85) yields, after a routine matrix algebra, the following equivalent set of three matrix equations with respect to P_1 and P_2:

$$\big(A_1 - A_2\mathcal{L}(\varepsilon)\big)^T(\varepsilon)P_1 + P_1\big(A_1 - A_2\mathcal{L}(\varepsilon)\big)(\varepsilon) + \sum_{j=1}^{N}\left[P_1\big(\mathcal{H}_{1j}(\varepsilon)R_{1j}^{-1}\mathcal{H}_{1j}^T(\varepsilon) + \mathcal{H}_{2j}(\varepsilon)R_{2j}^{-1}\mathcal{H}_{2j}^T(\varepsilon)\big)P_1 + R_{1j}\right] = -Q_1, \tag{89}$$

$$\big(A_4 + \varepsilon\mathcal{L}(\varepsilon)A_2\big)^T(\varepsilon)P_2 + P_2\big(A_4 + \varepsilon\mathcal{L}(\varepsilon)A_2\big) + \sum_{j=1}^{N}\left[P_2\big(\mathcal{H}_{3j}(\varepsilon)R_{1j}^{-1}\mathcal{H}_{3j}^T(\varepsilon) + \mathcal{H}_{4j}(\varepsilon)R_{2j}^{-1}\mathcal{H}_{4j}^T(\varepsilon)\big)P_2 + R_{2j}\right] = -Q_3, \tag{90}$$

$$P_1\Big(\sum_{j}^{N}\big(\mathcal{H}_{1j}(\varepsilon)R_{1j}^{-1}\mathcal{H}_{3j}^T(\varepsilon) + \mathcal{H}_{2j}(\varepsilon)R_{2j}^{-1}\mathcal{H}_{4j}^T(\varepsilon)\big)\Big)P_2 = -Q_2(\varepsilon). \tag{91}$$

Corollary 3. *Let assumption A5 be valid. Let, for a given $\varepsilon \in (0, \bar{\varepsilon}_1]$, there exist symmetric positive definite matrices R_{1j}, R_{2j}, $(j = 1, \ldots, N)$, Q_1, Q_3 such that Equations (89) and (90) have symmetric positive definite solutions $P_1 = P_1(\varepsilon)$ and $P_2 = P_2(\varepsilon)$. Moreover, let the matrix $Q(\varepsilon)$, given in (87) with*

$$Q_2(\varepsilon) = -P_1(\varepsilon)\Big(\sum_{j=1}^{N}\big(\mathcal{H}_{1j}(\varepsilon)R_{1j}^{-1}\mathcal{H}_{3j}^T(\varepsilon) + \mathcal{H}_{2j}(\varepsilon)R_{2j}^{-1}\mathcal{H}_{4j}^T(\varepsilon)\big)\Big)P_2(\varepsilon), \tag{92}$$

be positive definite. Then, for this ε, system (82) is asymptotically stable.

Proof. Due to the assumptions of the corollary, we obtain the following. For any given $\varepsilon \in (0, \bar{\varepsilon}_1]$ and for the matrices R_j and Q, given by (87), Equation (85) has the symmetric positive definite solution $P = P(\varepsilon)$ of the form (88), where $P_1 = P_1(\varepsilon)$ and $P_2 = P_2(\varepsilon)$ are the symmetric positive definite solutions of Equations (89) and (90). Therefore, by virtue of Proposition 7, system (82) is asymptotically stable for any given $\varepsilon \in (0, \bar{\varepsilon}_1]$. □

3.2. Asymptotic Solution with Respect to ε of Equations (89) and (90)

We look for the zero-order asymptotic solutions P_{10} and P_{20} of (89) and (90), respectively. Setting formally $\varepsilon = 0$ in Equations (89) and (90), and using Proposition 5 and Equation (84), we obtain the following equations for their zero-order asymptotic solutions:

$$(A_1 - A_2A_4^{-1}A_3)^T P_{10} + P_{10}(A_1 - A_2A_4^{-1}A_3) + \sum_{j=1}^{N}\left[P_{10}\big(\mathcal{H}_{1j}(0)R_{1j}^{-1}\mathcal{H}_{1j}^T(0) + \mathcal{H}_{2j}(0)R_{2j}^{-1}\mathcal{H}_{2j}^T(0)\big)P_{10} + R_{1j}\right] = -Q_1, \tag{93}$$

$$A_4^T P_{20} + P_{20}A_4 + \sum_{j=1}^{N}\left[P_{20}\big(\mathcal{H}_{3j}(0)R_{1j}^{-1}\mathcal{H}_{3j}^T(0) + \mathcal{H}_{4j}(0)R_{2j}^{-1}\mathcal{H}_{4j}^T(0)\big)P_{20} + R_{2j}\right] = -Q_3, \tag{94}$$

where

$$\mathcal{H}_{1j}(0) = H_{1j} - H_{2j}A_4^{-1}A_3, \quad \mathcal{H}_{2j}(0) = H_{2j},$$
$$\mathcal{H}_{3j}(0) = A_4^{-1}A_3H_{1j} + H_{3j} - A_4^{-1}A_3H_{2j}A_4^{-1}A_3 - H_{4j}A_4^{-1}A_3,$$
$$\mathcal{H}_{4j}(0) = A_4^{-1}A_3H_{2j} + H_{4j}. \tag{95}$$

In what follows, we assume the following:
A6. There exist symmetric positive definite matrices R_{1j}, R_{2j}, $(j = 1, \dots, N)$, Q_1, Q_3, such that:
(a) Equations (93) and (94) have symmetric positive definite solutions P_{10} and P_{20}, respectively;
(b) all eigenvalues of each of the matrices

$$\mathcal{G}_{10} \triangleq A_1 - A_2A_4^{-1}A_3 + \sum_{j=1}^{N}\left[\mathcal{H}_{1j}(0)R_{1j}^{-1}\mathcal{H}_{1j}^T(0) + \mathcal{H}_{2j}(0)R_{2j}^{-1}\mathcal{H}_{2j}^T(0)\right]P_{10},$$
$$\mathcal{G}_{20} \triangleq A_4 + \sum_{j=1}^{N}\left[\mathcal{H}_{3j}(0)R_{1j}^{-1}\mathcal{H}_{3j}^T(0) + \mathcal{H}_{4j}(0)R_{2j}^{-1}\mathcal{H}_{4j}^T(0)\right]P_{20} \tag{96}$$

lie strictly inside either the left-hand half or the right-hand half of the complex plane;
(c) the matrix

$$Q_0 = \begin{pmatrix} Q_1 & Q_{20} \\ Q_{20}^T & Q_3 \end{pmatrix}, \tag{97}$$

with

$$Q_{20} = -P_{10}\Big(\sum_{j=1}^{N}\left(\mathcal{H}_{1j}(0)R_{1j}^{-1}\mathcal{H}_{3j}^T(0) + \mathcal{H}_{2j}(0)R_{2j}^{-1}\mathcal{H}_{4j}^T(0)\right)\Big)P_{20}, \tag{98}$$

is positive definite.

Remark 3. *Based on the results of [25] (see Chapter 7, Theorem 3.1), we obtain (similarly to Lemma 1 and Remark 2) the following conclusions. The validity of assumption A6 (items (a) and (b)) guarantees that $A_1 - A_2A_4^{-1}A_3$ and A_4 are Hurwitz matrices. Moreover, the equations*

$$\frac{dz_1(t)}{dt} = (A_1 - A_2A_4^{-1}A_3)z_1(t) + \sum_{j=1}^{N}\mathcal{H}_{1j}(0)z_1(t - h_j), \quad t \geq 0, \quad z_1(t) \in E^n \tag{99}$$

and

$$\frac{dz_2(t)}{dt} = A_4z_2(t) + \sum_{j=1}^{N}\mathcal{H}_{4j}(0)z_2(t - h_j), \quad t \geq 0, \quad z_2(t) \in E^m \tag{100}$$

are asymptotically stable.

However, the only requirement that $A_1 - A_2A_4^{-1}A_3$ is a Hurwitz matrix does not, in general, guarantee the asymptotic stability of Equation (99) (see, e.g., Section 2.3 in [2]). Similarly, the only requirement that A_4 is a Hurwitz matrix does not, in general, guarantee the asymptotic stability of Equation (100).

Theorem 5. *Let assumptions A5 and A6 be valid. Then, there exists a number $0 < \bar{\varepsilon}_2 \leq \bar{\varepsilon}_1$, such that, for all $\varepsilon \in (0, \bar{\varepsilon}_2]$, Equations (89) and (90) have symmetric positive definite solutions $P_1 = P_1(\varepsilon)$ and $P_2 = P_2(\varepsilon)$, satisfying the inequalities*

$$\left\|P_1(\varepsilon) - P_{10}\right\| \leq \bar{a}_1\varepsilon, \quad \left\|P_2(\varepsilon) - P_{20}\right\| \leq \bar{a}_2\varepsilon, \quad \varepsilon \in (0, \bar{\varepsilon}_2], \tag{101}$$

where $\bar{a}_1 > 0$ and $\bar{a}_2 > 0$ are some constants independent of ε.

Moreover, there exists a number $0 < \bar{\varepsilon}_3 \leq \bar{\varepsilon}_2$, such that, for all $\varepsilon \in (0, \bar{\varepsilon}_3]$, the matrix $Q(\varepsilon)$, given in (87) with the block $Q_2(\varepsilon)$ given by (92), is positive definite.

Proof. The first statement of the theorem is proven quite similarly to Theorems 1 and 2. Proceed to the proof of the second statement of the theorem. Using Proposition 5; Equations (79), (80), (84), (92), and (98); and the inequalities in (101), we directly obtain the inequality $\|Q_2(\varepsilon) - Q_{20}\| \leq \bar{a}_3 \varepsilon \;\; \forall \varepsilon \in (0, \bar{\varepsilon}_2]$, where $\bar{a}_3 > 0$ is some constant independent of ε. This inequality and item (c) of assumption A6 directly yield the validity of the second statement of the theorem. □

3.3. ε-Free Asymptotic Stability Conditions for System (74)

Let us denote $\bar{\varepsilon}^* \stackrel{\triangle}{=} \bar{\varepsilon}_3$.

Now, using Proposition 6, Corollary 3, and Theorem 5, we obtain the following assertion.

Theorem 6. *Let assumptions A5 and A6 be valid. Then, for any given $\varepsilon \in (0, \bar{\varepsilon}^*]$, system (74) is asymptotically stable.*

3.4. Nonlinear Singularly Perturbed System

Consider the following differential system:

$$\begin{aligned} \frac{dx(t)}{dt} &= f\big(x(t), y(t), x(t-h_1), \dots, x(t-h_N), y(t-h_1), \dots, y(t-h_N)\big), \;\; t \geq 0, \\ \varepsilon \frac{dy(t)}{dt} &= g\big(x(t), y(t), x(t-h_1), \dots, x(t-h_N), y(t-h_1), \dots, y(t-h_N)\big), \;\; t \geq 0, \end{aligned} \tag{102}$$

where $x(t) \in E^n$ and $y(t) \in E^m$; $f(\phi, \psi, \phi_1, \dots, \phi_N, \psi_1, \dots, \psi_N) : E^n \times E^m \times E^n \cdots \times E^n \times E^m \cdots \times E^m \to E^n$ and $g(\phi, \psi, \phi_1, \dots, \phi_N, \psi_1, \dots, \psi_N) : E^n \times E^m \times E^n \cdots \times E^n \times E^m \cdots \times E^m \to E^m$ are given functions; $f(0,0,0,\dots,0) = 0$ and $g(0,0,0,\dots,0) = 0$; $0 < h_1 < \cdots < h_N$ are given time delays.

For this system (like for system (74)), we consider the initial conditions (75).

Similarly to Definitions 2–3, we introduce the following definitions.

Definition 5. *For a given $\varepsilon > 0$, the trivial solution $\big(x(t) \equiv 0, y(t) \equiv 0\big)$ of system (102) is called stable if for any number $\alpha > 0$ there exists a number $\beta > 0$, dependent on α, such that the solution $\mathrm{col}\Big(x\big(t; \varphi_x(\cdot), \varphi_y(\cdot)\big), y\big(t; \varphi_x(\cdot), \varphi_y(\cdot)\big)\Big)$ of the initial-value problem (102), (75) satisfies the inequality $\Big\|\mathrm{col}\Big(x\big(t; \varphi_x(\cdot), \varphi_y(\cdot)\big), y\big(t; \varphi_x(\cdot), \varphi_y(\cdot)\big)\Big)\Big\| \leq \alpha$, $t \geq 0$, for any pair $\big(\varphi_x(\tau), \varphi_y(\tau)\big)$ satisfying the inequality*

$$\big\|\mathrm{col}\big(\varphi_x(\tau), \varphi_y(\tau)\big)\big\| \leq \beta, \quad \tau \in [-h_N, 0]. \tag{103}$$

Definition 6. *For a given $\varepsilon > 0$, the trivial solution $\big(x(t) \equiv 0, y(t) \equiv 0\big)$ of system (102) is called asymptotically stable if this solution is stable and there exists a number $\beta > 0$, such that, for any pair $\big(\varphi_x(\tau), \varphi_y(\tau)\big)$ satisfying (103), the solution $\mathrm{col}\Big(x\big(t; \varphi_x(\cdot), \varphi_y(\cdot)\big), y\big(t; \varphi_x(\cdot), \varphi_y(\cdot)\big)\Big)$ of the initial-value problem (102), (75) tends to zero as t tends to $+\infty$.*

In what follows of this section, we assume the following:

A7. The functions $f(\phi, \psi, \phi_1, \dots, \phi_N, \psi_1, \dots, \psi_N)$ and $g(\phi, \psi, \phi_1, \dots, \phi_N, \psi_1, \dots, \psi_N)$ are twice continuously differentiable for $(\phi, \psi, \phi_1, \dots, \phi_N, \psi_1, \dots, \psi_N) \in E^n \times E^m \times E^n \cdots \times E^n \times E^m \cdots \times E^m$.

The linearization of system (102) in a neighborhood of its trivial solution $(x(t) \equiv 0, y(t) \equiv 0, t \geq -h_N)$ yields the system

$$\frac{dx(t)}{dt} = f_\phi(0,0,0,\ldots,0)x(t) + f_\psi(0,0,0,\ldots,0)y(t)$$
$$+ \sum_{J=1}^{N} \left[f_{\phi_j}(0,0,0,\ldots,0)x(t-h_j) + f_{\psi_j}(0,0,0,\ldots,0)y(t-h_j) \right], \quad t \geq 0,$$
$$\varepsilon\frac{dy(t)}{dt} = g_\phi(0,0,0,\ldots,0)x(t) + g_\psi(0,0,0,\ldots,0)y(t)$$
$$+ \sum_{J=1}^{N} \left[g_{\phi_j}(0,0,0,\ldots,0)x(t-h_j) + g_{\psi_j}(0,0,0,\ldots,0)y(t-h_j) \right], \quad t \geq 0. \tag{104}$$

The following lemma is a direct consequence of Theorem 6.

Lemma 3. *Let assumption A7 be valid. Let assumptions A5 and A6 be valid for* $A_1 = f_\phi(0,0,0,\ldots,0)$, $A_2 = f_\psi(0,0,0,\ldots,0)$, $A_3 = g_\phi(0,0,0,\ldots,0)$, $A_4 = g_\psi(0,0,0,\ldots,0)$, $H_{1j} = f_{\phi_j}(0,0,0,\ldots,0)$, $H_{2j} = f_{\psi_j}(0,0,0,\ldots,0)$, $H_{3j} = g_{\phi_j}(0,0,0,\ldots,0)$ *and* $H_{4j} = g_{\psi_j}(0,0,0,\ldots,0)$, $(j = 1,\ldots,N)$. *Then, there exists a number* $\widehat{\varepsilon}^* > 0$, *such that, for any given* $\varepsilon \in (0, \widehat{\varepsilon}^*]$, *system (104) is asymptotically stable.*

Theorem 7. *Let assumption A7 be valid. Let assumptions A5 and A6 be valid for* $A_1 = f_\phi(0,0,0,\ldots,0)$, $A_2 = f_\psi(0,0,0,\ldots,0)$, $A_3 = g_\phi(0,0,0,\ldots,0)$, $A_4 = g_\psi(0,0,0,\ldots,0)$, $H_{1j} = f_{\phi_j}(0,0,0,\ldots,0)$, $H_{2j} = f_{\psi_j}(0,0,0,\ldots,0)$, $H_{3j} = g_{\phi_j}(0,0,0,\ldots,0)$ *and* $H_{4j} = g_{\psi_j}(0,0,0,\ldots,0)$, $(j = 1,\ldots,N)$. *Then, for any given* $\varepsilon \in (0, \widehat{\varepsilon}^*]$, *the trivial solution of system (102) is asymptotically stable.*

Proof. Using Lemma 3, the theorem is proven quite similarly to Theorem 4. □

3.5. Case of a Single Delay in System (74): Alternative Approach to Asymptotic Stability Analysis

Consider the following differential system:

$$\frac{dx(t)}{dt} = A_1x(t) + A_2y(t) + H_1x(t-h) + H_2y(t-h), \quad t \geq 0,$$
$$\varepsilon\frac{dy(t)}{dt} = A_3x(t) + A_4y(t) + H_3x(t-h) + H_4y(t-h), \quad t \geq 0, \tag{105}$$

where $x(t) \in E^n$ and $y(t) \in E^m$; A_i and H_i, $(i = 1,\ldots,4)$ are given constant matrices of corresponding dimensions; $\varepsilon > 0$ is a small parameter; $h > 0$ is a given time delay.

System (105) is a particular case of system (74), with a single point-wise time delay.

For system (105), we consider the initial conditions

$$x(\tau) = \varphi_x(\tau), \quad y(\tau) = \varphi_y(\tau), \quad \tau \in [-h, 0], \tag{106}$$

where $\varphi_x(\tau)$ is a given n-dimensional vector-valued function; $\varphi_y(\tau)$ is a given m-dimensional vector-valued function; both functions are continuous in the interval $[-h, 0]$.

The asymptotic stability of system (105) is defined quite similarly to the asymptotic stability of system (74) (see Definition 4).

3.5.1. ε-Dependent Asymptotic Stability Conditions

Let assumption A1 be valid. Based on this assumption, let us transform the state variables $x(t+\tau)$ and $y(t+\tau)$, $\tau \in [-h, 0]$ of the system (105) as:

$$\begin{pmatrix} x(t+\tau) \\ y(t+\tau) \end{pmatrix} = D(\varepsilon)\begin{pmatrix} u(t+\tau) \\ v(t+\tau) \end{pmatrix}, \quad t \geq 0, \ \ \tau \in [-h, 0], \ \ \varepsilon \in (0, \varepsilon_1], \tag{107}$$

where $u(t) \in E^n$, $v(t) \in E^m$; $u(t+\tau)$, and $v(t+\tau)$ are new state variables; the positive number ε_1 is defined in Proposition 1; the $(n+m) \times (n+m)$-matrix $D(\varepsilon)$ is defined by Equation (6) and its inverse matrix $D^{-1}(\varepsilon)$ is defined by Equation (7).

Similarly to Propositions 2 and 6, we directly obtain the following assertion.

Proposition 8. *Let assumption A1 be valid. Then, for any given $\varepsilon \in (0, \varepsilon_1]$, transformation (107) converts the initial-value problem (105)–(106) to the following equivalent initial-value problem:*

$$\frac{du(t)}{dt} = \widetilde{\mathcal{A}}_1(\varepsilon)u(t) + \widetilde{\mathcal{A}}_2(\varepsilon)v(t) + \big(H_1 - H_2L(\varepsilon)\big)u(t-h),$$
$$\varepsilon\frac{dv(t)}{dt} = \widetilde{\mathcal{A}}_3(\varepsilon)u(t) + \widetilde{\mathcal{A}}_4(\varepsilon)v(t) + \big(H_4 + \varepsilon L(\varepsilon)H_2\big)v(t-h), \tag{108}$$

$$u(\tau) = \big(I_n - \varepsilon M(\varepsilon)L(\varepsilon)\big)\varphi_x(\tau) - \varepsilon M(\varepsilon)\varphi_y(\tau), \quad \tau \in [-h, 0],$$
$$v(\tau) = L(\varepsilon)\varphi_x(\tau) + \varphi_y(\tau), \quad \tau \in [-h, 0]. \tag{109}$$

where

$$\widetilde{\mathcal{A}}(\varepsilon) \triangleq \begin{pmatrix} \widetilde{\mathcal{A}}_1(\varepsilon) & \widetilde{\mathcal{A}}_2(\varepsilon) \\ \widetilde{\mathcal{A}}_3(\varepsilon) & \widetilde{\mathcal{A}}_4(\varepsilon) \end{pmatrix} = D^{-1}(\varepsilon)\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}D(\varepsilon). \tag{110}$$

Moreover, system (105) is asymptotically stable if and only if system (108) is asymptotically stable.

Consider the following symmetric Riccati algebraic equation with respect to $(n+m) \times (n+m)$-matrix $\widetilde{P}$:

$$\widetilde{\mathcal{A}}^T(\varepsilon)\mathcal{E}^{-1}(\varepsilon)\widetilde{P} + \widetilde{P}\mathcal{E}^{-1}(\varepsilon)\widetilde{\mathcal{A}}(\varepsilon) + \widetilde{P}\mathcal{E}^{-1}(\varepsilon)\mathcal{H}(\varepsilon)\widetilde{R}^{-1}\mathcal{H}^T(\varepsilon)\mathcal{E}^{-1}(\varepsilon)\widetilde{P} + \widetilde{R} = -\widetilde{Q}, \tag{111}$$

where $\widetilde{R}$ and $\widetilde{Q}$ are some symmetric positive definite matrices of the dimension $(n+m) \times (n+m)$; the $(n+m) \times (n+m)$-matrix $\mathcal{E}(\varepsilon)$ is given in (86);

$$\widetilde{\mathcal{H}}(\varepsilon) \triangleq \begin{pmatrix} H_1 - H_2L(\varepsilon) & 0 \\ 0 & H_4 + \varepsilon L(\varepsilon)H_2 \end{pmatrix}. \tag{112}$$

Based on Proposition 1 and the results of [25] (see Chapter 7, Theorem 3.1), we directly obtain the following assertion.

Proposition 9. *Let assumption A1 be valid. Let, for a given $\varepsilon \in (0, \varepsilon_1]$, there exist symmetric positive definite matrices $\widetilde{R}$ and $\widetilde{Q}$, such that Equation (111) has a symmetric positive definite solution $\widetilde{P} = \widetilde{P}(\varepsilon)$. Then, for this ε, system (108) is asymptotically stable.*

We choose the matrices $\widetilde{R}$ and $\widetilde{Q}$ in the block form as:

$$\widetilde{R} = \begin{pmatrix} \widetilde{R}_1 & 0 \\ 0 & \widetilde{R}_2 \end{pmatrix}, \quad \widetilde{Q} = \widetilde{Q}(\varepsilon) = \begin{pmatrix} \widetilde{Q}_1 & \widetilde{Q}_2(\varepsilon) \\ \widetilde{Q}_2^T(\varepsilon) & \widetilde{Q}_3 \end{pmatrix}, \tag{113}$$

where the matrices $\widetilde{R}_1$ and $\widetilde{Q}_1$ are of the dimension $n \times n$; the matrices $\widetilde{R}_2$ and $\widetilde{Q}_3$ are of the dimension $m \times m$.

Due to the block form of the matrices $\mathcal{E}(\varepsilon)$, $\widetilde{\mathcal{H}}(\varepsilon)$, and $\widetilde{R}$, we look for the symmetric positive definite solution $\widetilde{P} = \widetilde{P}(\varepsilon)$ of the Riccati Equation (111) in the block form

$$\widetilde{P} = \begin{pmatrix} \widetilde{P}_1 & 0 \\ 0 & \varepsilon\widetilde{P}_2 \end{pmatrix}, \tag{114}$$

where the matrices $\widetilde{P}_1$ and $\widetilde{P}_2$ are of the dimensions $n \times n$ and $m \times m$, respectively.

Substituting the block-form representations of the matrices $\mathcal{E}(\varepsilon)$, $\widetilde{\mathcal{A}}(\varepsilon)$, $\widetilde{\mathcal{H}}(\varepsilon)$, $\widetilde{R}$, $\widetilde{Q}(\varepsilon)$, and $\widetilde{P}$ (see Equations (86), (110), (112), (113), and (114)) into the Equation (111), we obtain, after a routine matrix algebra, the following equivalent set of three matrix equations with respect to $\widetilde{P}_1$ and $\widetilde{P}_2$:

$$\widetilde{\mathcal{A}}_1^T(\varepsilon)\widetilde{P}_1 + \widetilde{P}_1\widetilde{\mathcal{A}}_1(\varepsilon) + \widetilde{P}_1\big(H_1 - H_2L(\varepsilon)\big)\widetilde{R_1}^{-1}\big(H_1 - H_2L(\varepsilon)\big)^T\widetilde{P}_1 + \widetilde{R}_1 = -\widetilde{Q}_1, \tag{115}$$

$$\widetilde{\mathcal{A}}_4^T(\varepsilon)\widetilde{P}_2 + \widetilde{P}_2\widetilde{\mathcal{A}}_4(\varepsilon) + \widetilde{P}_2\big(H_4 + \varepsilon L(\varepsilon)H_2\big)\widetilde{R_2}^{-1}\big(H_4 + \varepsilon L(\varepsilon)H_2\big)^T\widetilde{P}_2 + \widetilde{R}_2 = -\widetilde{Q}_3 \tag{116}$$

$$\widetilde{\mathcal{A}}_3^T(\varepsilon)\widetilde{P}_2 + \widetilde{P}_1\widetilde{\mathcal{A}}_2(\varepsilon) = -Q_2(\varepsilon). \tag{117}$$

From Proposition 9, we directly obtain the following assertion.

Corollary 4. *Let assumption A1 be valid. Let, for a given $\varepsilon \in (0, \varepsilon_1]$, there exist symmetric positive definite matrices $\widetilde{R}_1$, $\widetilde{R}_2$, $\widetilde{Q}_1$, $\widetilde{Q}_3$, such that Equations (115) and (116) have symmetric positive definite solutions $\widetilde{P}_1 = \widetilde{P}_1(\varepsilon)$ and $\widetilde{P}_2 = \widetilde{P}_2(\varepsilon)$. Moreover, let the matrix $Q(\varepsilon)$, given in (113) with*

$$\widetilde{Q}_2(\varepsilon) = -\big(\widetilde{\mathcal{A}}_3^T(\varepsilon)\widetilde{P}_2(\varepsilon) + \widetilde{P}_1(\varepsilon)\widetilde{\mathcal{A}}_2(\varepsilon)\big), \tag{118}$$

be positive definite. Then, for this ε, system (108) is asymptotically stable.

3.5.2. Asymptotic Solution with Respect to ε of Equations (115) and (116)

We look for the zero-order asymptotic solutions $\widetilde{P}_{10}$ and $\widetilde{P}_{20}$ of (115) and (116), respectively. Setting formally $\varepsilon = 0$ in Equations (115) and (116), and using Proposition 1 and Equation (110), we obtain the following equations for their zero-order asymptotic solutions:

$$\begin{aligned} &\widetilde{\mathcal{A}}_1^T(0)\widetilde{P}_{10} + \widetilde{P}_{10}\widetilde{\mathcal{A}}_1(0) \\ &+\widetilde{P}_{10}(H_1 - H_2H_4^{-1}H_3)\widetilde{R_1}^{-1}(H_1 - H_2H_4^{-1}H_3)^T\widetilde{P}_{10} + \widetilde{R}_1 = -\widetilde{Q}_1, \end{aligned} \tag{119}$$

$$\widetilde{\mathcal{A}}_4^T(0)\widetilde{P}_{20} + \widetilde{P}_{20}\widetilde{\mathcal{A}}_4(0) + \widetilde{P}_{20}H_4\widetilde{R_2}^{-1}H_4^T\widetilde{P}_{20} + \widetilde{R}_2 = -\widetilde{Q}_3, \tag{120}$$

where

$$\widetilde{\mathcal{A}}_1(0) = A_1 - A_2H_4^{-1}H_3, \quad \widetilde{\mathcal{A}}_4(0) = H_4^{-1}H_3A_2 + A_4. \tag{121}$$

In what follows, we assume the following:

A8. There exist symmetric positive definite matrices $\widetilde{R}_1$, $\widetilde{R}_2$, $\widetilde{Q}_1$, $\widetilde{Q}_3$, such that:

(a) Equations (119) and (120) have symmetric positive definite solutions $\widetilde{P}_{10}$ and $\widetilde{P}_{20}$, respectively;

(b) all eigenvalues of each of the matrices

$$\begin{aligned} &\widetilde{\mathcal{G}}_{10} \triangleq \widetilde{\mathcal{A}}_1(0) + (H_1 - H_2H_4^{-1}H_3)\widetilde{R}_1^{-1}(H_1 - H_2H_4^{-1}H_3)^T\widetilde{P}_{10}, \\ &\widetilde{\mathcal{G}}_{20} \triangleq \widetilde{\mathcal{A}}_4(0) + H_4\widetilde{R}_2^{-1}H_4^T\widetilde{P}_{20} \end{aligned} \tag{122}$$

lie strictly inside either the left-hand half or the right-hand half of the complex plane;

(c) the matrix

$$\widetilde{Q}_0 = \begin{pmatrix} \widetilde{Q}_1 & \widetilde{Q}_{20} \\ \widetilde{Q}_{20}^T & \widetilde{Q}_3 \end{pmatrix}, \tag{123}$$

with

$$\widetilde{Q}_{20} = -\big(\widetilde{\mathcal{A}}_3^T(0)\widetilde{P}_{20} + \widetilde{P}_{10}\widetilde{\mathcal{A}}_2(0)\big) \tag{124}$$

and

$$\widetilde{\mathcal{A}}_2(0) = A_2, \quad \widetilde{\mathcal{A}}_3(0) = H_4^{-1}H_3A_1 + A_3 - H_4^{-1}H_3A_2H_4^{-1}H_3 - A_4H_4^{-1}H_3, \tag{125}$$

is positive definite.

Remark 4. *Using the same arguments as in Remark 3, we can conclude the following. The validity of assumption A8 (items (a) and (b)) guarantees that $\widetilde{\mathcal{A}}_1(0)$ and $\widetilde{\mathcal{A}}_4(0)$ are Hurwitz matrices. Moreover, the equations*

$$\frac{d\widetilde{z}_1(t)}{dt} = \widetilde{\mathcal{A}}_1(0)\widetilde{z}_1(t) + (H_1 - H_2H_4^{-1}H_3)\widetilde{z}_1(t-h), \quad t \geq 0, \quad \widetilde{z}_1(t) \in E^n \tag{126}$$

and

$$\frac{d\widetilde{z}_2(t)}{dt} = \widetilde{\mathcal{A}}_4(0)\widetilde{z}_2(t) + H_4\widetilde{z}_2(t-h), \quad t \geq 0, \quad \widetilde{z}_2(t) \in E^m \tag{127}$$

are asymptotically stable.

However, the only requirement that $\widetilde{\mathcal{A}}_1(0)$ is a Hurwitz matrix does not, in general, guarantee the asymptotic stability of Equation (126) (see, e.g., Section 2.3 in [2]). Similarly, the only requirement that $\widetilde{\mathcal{A}}_4(0)$ is a Hurwitz matrix does not, in general, guarantee the asymptotic stability of Equation (127).

Quite similarly to Theorem 5, we obtain the following assertion.

Theorem 8. *Let assumptions A1 and A8 be valid. Then, there exists a number $0 < \widetilde{\varepsilon}_1 \leq \varepsilon_1$, such that, for all $\varepsilon \in (0, \widetilde{\varepsilon}_1]$, Equations (119) and (120) have symmetric positive definite solutions $\widetilde{P}_1 = \widetilde{P}_1(\varepsilon)$ and $\widetilde{P}_2 = \widetilde{P}_2(\varepsilon)$, satisfying the inequalities*

$$\left\|\widetilde{P}_1(\varepsilon) - \widetilde{P}_{10}\right\| \leq \widetilde{a}_1\varepsilon, \quad \left\|\widetilde{P}_2(\varepsilon) - \widetilde{P}_{20}\right\| \leq \widetilde{a}_2\varepsilon, \quad \varepsilon \in (0, \widetilde{\varepsilon}_1], \tag{128}$$

where $\widetilde{a}_1 > 0$ and $\widetilde{a}_2 > 0$ are some constants independent of ε.

Moreover, there exists a number $0 < \widetilde{\varepsilon}_2 \leq \widetilde{\varepsilon}_1$, such that, for all $\varepsilon \in (0, \widetilde{\varepsilon}_2]$, the matrix $\widetilde{Q}(\varepsilon)$, given in (113) with the block $\widetilde{Q}_2(\varepsilon)$ given by (118), is positive definite.

3.5.3. ε-Free Asymptotic Stability Conditions for System (105)

Denoting $\widetilde{\varepsilon}^* \stackrel{\triangle}{=} \widetilde{\varepsilon}_2$ and using Proposition 8, Corollary 4, and Theorem 8, we obtain the following assertion.

Theorem 9. *Let assumptions A1 and A8 be valid. Then, for any given $\varepsilon \in (0, \widetilde{\varepsilon}^*]$, the system (105) is asymptotically stable.*

3.5.4. Particular Case of the Nonlinear Singularly Perturbed System (102)

Consider the following differential system:

$$\begin{aligned} \frac{dx(t)}{dt} &= f\big(x(t), y(t), x(t-h), y(t-h)\big), \quad t \geq 0, \\ \varepsilon\frac{dy(t)}{dt} &= g\big(x(t), y(t), x(t-h), y(t-h)\big), \quad t \geq 0, \end{aligned} \tag{129}$$

where $x(t) \in E^n$ and $y(t) \in E^m$; $f(\phi, \psi, \phi_1, \psi_1) : E^n \times E^m \times E^n \times E^m \to E^n$ and $g(\phi, \psi, \phi_1, \psi_1) : E^n \times E^m \times E^n \times E^m \to E^m$ are given functions; $f(0,0,0,0) = 0$ and $g(0,0,0,0) = 0$; $h > 0$ is a given time delay.

System (129) is a particular case of system (102), with a single point-wise time delay.

For this system (like for system (105)), we consider the initial conditions (106). The stability and asymptotic stability of the trivial solution $\big(x(t) \equiv 0, y(t) \equiv 0\big)$ to system (129) are defined quite similarly to the stability and asymptotic stability of the trivial solution $\big(x(t) \equiv 0, y(t) \equiv 0\big)$ to system (102) (see Definitions 5 and 6).

In what follows, we assume the following:

A9. The functions $f(\phi, \psi, \phi_1, \psi_1)$ and $g(\phi, \psi, \phi_1, \psi_1)$ are twice continuously differentiable for $(\phi, \psi, \phi_1, \psi_1) \in E^n \times E^m \times E^n \times E^m$.

The linearization of system (129) in a neighborhood of its trivial solution $\big(x(t) \equiv 0, y(t) \equiv 0, t \geq -h\big)$ yields the system

$$\begin{aligned} \frac{dx(t)}{dt} &= f_{\phi}(0,0,0,0)x(t) + f_{\psi}(0,0,0,0)y(t) \\ &+ f_{\phi_1}(0,0,0,0)x(t-h) + f_{\psi_1}(0,0,0,0)y(t-h), \quad t \geq 0, \\ \varepsilon\frac{dy(t)}{dt} &= g_{\phi}(0,0,0,0)x(t) + g_{\psi}(0,0,0,0)y(t) \\ &+ g_{\phi_1}(0,0,0,0)x(t-h) + g_{\psi_1}(0,0,0,0)y(t-h), \quad t \geq 0. \end{aligned} \tag{130}$$

Using Theorem 9 and the results of [2,21] on the asymptotic stability in the first approximation of time-delay equations, we obtain (quite similarly to Lemma 3 and Theorem 7) the following assertion.

Theorem 10. *Let assumption A9 be valid. Let assumptions A1 and A8 be valid for* $A_1 = f_{\phi}(0,0,0,0)$, $A_2 = f_{\psi}(0,0,0,0)$, $A_3 = g_{\phi}(0,0,0,0)$, $A_4 = g_{\psi}(0,0,0,0)$, $H_1 = f_{\phi_1}(0,0,0,0)$, $H_2 = f_{\psi_1}(0,0,0,0)$, $H_3 = g_{\phi_1}(0,0,0,0)$, *and* $H_4 = g_{\psi_1}(0,0,0,0)$. *Then, there exists a number* $\check{\varepsilon}^* > 0$ *such that, for any given* $\varepsilon \in (0, \check{\varepsilon}^*]$, *system (130) is asymptotically stable. Moreover, for such* ε, *the trivial solution of system (129) is asymptotically stable.*

3.6. Examples

3.6.1. Example 1

Consider a particular case of system (74) with the following data:

$$\begin{aligned} n = 1, \quad m = 1, \quad N = 2, \quad 0 < h_1 < h_2, \\ A_1 = 1, \quad A_2 = -7, \quad A_3 = 9, \quad A_4 = -9, \\ H_{11} = -1, \quad H_{12} = -1.5, \quad H_{21} = 2, \quad H_{22} = -0.5, \\ H_{31} = 0.5, \quad H_{32} = -1.5, \quad H_{41} = 2.5, \quad H_{42} = 0.5. \end{aligned} \tag{131}$$

To analyze the asymptotic stability of system (74), (131), we use Theorem 6.

Since $A_4 \neq 0$, assumption A5 is valid in this example.

Using Equation (95) and the data (131) of this example, we obtain

$$\begin{aligned} \mathcal{H}_{11}(0) = 1, \quad \mathcal{H}_{12}(0) = -2, \quad \mathcal{H}_{21}(0) = 2, \quad \mathcal{H}_{22}(0) = -0.5, \\ \mathcal{H}_{31}(0) = 2, \quad \mathcal{H}_{32}(0) = 1, \quad \mathcal{H}_{41}(0) = 0.5, \quad \mathcal{H}_{42}(0) = 1. \end{aligned} \tag{132}$$

Furthermore, using Equations (93), (94), (131), and (132) and choosing

$$R_{11} = R_{12} = R_{21} = R_{22} = R > 0,$$

we can write down the scalar quadratic equations with respect to the unknown P_{10} and P_{20} as:

$$\begin{aligned} 9.25R^{-1}P_{10}^2 - 12P_{10} + 2R + Q_1 = 0, \\ 6.25R^{-1}P_{20}^2 - 18P_{20} + 2R + Q_3 = 0, \end{aligned} \tag{133}$$

where $Q_1 > 0$ and $Q_3 > 0$.

Solving the first equation in (133), we obtain two its solutions:

$$P_{10}^{+}=\frac{R\Big(6+\sqrt{36-9.25\big(2+Q_1/R\big)}\Big)}{9.25},$$
$$P_{10}^{-}=\frac{R\Big(6-\sqrt{36-9.25\big(2+Q_1/R\big)}\Big)}{9.25}. \tag{134}$$

Each of these solutions is positive if and only if

$$0<Q_1\le\frac{17.5}{9.25}R\approx 1.89189R. \tag{135}$$

Moreover, the particular case of $\mathcal{G}_{10}$ in (96) for $P_{10}=P_{10}^{+}$ satisfies the inequality

$$\mathcal{G}_{10}^{+}=\sqrt{36-9.25\big(2+Q_1/R\big)}>0, \tag{136}$$

while the particular case of $\mathcal{G}_{10}$ in (96) for $P_{10}=P_{10}^{-}$ satisfies the inequality

$$\mathcal{G}_{10}^{-}=-\sqrt{36-9.25\big(2+Q_1/R\big)}<0, \tag{137}$$

if and only if

$$0<Q_1<\frac{17.5}{9.25}R. \tag{138}$$

Similarly, the second equation in (133) has two solutions:

$$P_{20}^{+}=\frac{R\Big(9+\sqrt{81-6.25\big(2+Q_3/R\big)}\Big)}{6.25},$$
$$P_{20}^{-}=\frac{R\Big(9-\sqrt{81-6.25\big(2+Q_3/R\big)}\Big)}{6.25}. \tag{139}$$

Each of these solutions is positive if and only if

$$0<Q_3\le 10.96R. \tag{140}$$

Moreover, similarly to (136) and (137), the particular case of $\mathcal{G}_{20}$ in (96) for $P_{20}=P_{20}^{+}$ satisfies the inequality

$$\mathcal{G}_{20}^{+}=\sqrt{81-6.25\big(2+Q_3/R\big)}>0, \tag{141}$$

while the particular case of $\mathcal{G}_{20}$ in (96) for $P_{20}=P_{20}^{-}$ satisfies the inequality

$$\mathcal{G}_{10}=-\sqrt{81-6.25\big(2+Q_3/R\big)}<0, \tag{142}$$

if and only if

$$0<Q_3<10.96R. \tag{143}$$

Thus, items (a) and (b) of assumption A6 are valid. Let us show the validity of item (c) of this assumption.

Using Equations (98) and (132), we obtain Q_{20} as:

$$Q_{20}=-\frac{0.5P_{10}P_{20}}{R}. \tag{144}$$

Let us choose

$$Q_1 = 1.8R, \quad Q_3 = 10R. \tag{145}$$

Such chosen Q_1 and Q_3 satisfy inequalities (138) and (143), respectively. For these Q_1 and Q_3, due to Equations (134) and (139), we have

$$P_{10}^{+} = 0.74832R, \quad P_{10}^{-} = 0.54898R, \quad P_{20}^{+} = 1.83192R, \quad P_{20}^{-} = 1.04808R,$$

which, along with (144), yields

$$\begin{aligned} Q_{20}^{++} &= -\frac{0.5P_{10}^{+}P_{20}^{+}}{R} = -0.68543R, \quad Q_{20}^{+-} = -\frac{0.5P_{10}^{+}P_{20}^{-}}{R} = -0.39215R, \\ Q_{20}^{-+} &= -\frac{0.5P_{10}^{-}P_{20}^{+}}{R} = -0.50284R, \quad Q_{20}^{--} = -\frac{0.5P_{10}^{-}P_{20}^{-}}{R} = -0.28769R. \end{aligned} \tag{146}$$

Now, using Equations (97), (145), and (146), we can construct the following matrices:

$$Q_0^{++} = \begin{pmatrix} 1.8R & -0.68543R \\ -0.68543R & 10R \end{pmatrix}, \quad Q_0^{+-} = \begin{pmatrix} 1.8R & -0.39215R \\ -0.39215R & 10R \end{pmatrix},$$
$$Q_0^{-+} = \begin{pmatrix} 1.8R & -0.50284R \\ -0.50284R & 10R \end{pmatrix}, \quad Q_0^{--} = \begin{pmatrix} 1.8R & -0.28769R \\ -0.28769R & 10R \end{pmatrix}.$$

All these matrices are positive definite, meaning that item (c) of assumption A6 is valid for all the pairs (P_{10}^{+}, P_{20}^{+}), (P_{10}^{+}, P_{20}^{-}), (P_{10}^{-}, P_{20}^{+}), (P_{10}^{-}, P_{20}^{-}). Thus, assumptions A5 and A6 are valid for system (74), (131). Therefore, by virtue of Theorem 6, there exists a positive number $\bar{\varepsilon}^*$, such that, for any given $\varepsilon \in (0, \bar{\varepsilon}^*]$, system (74), (131) is asymptotically stable.

3.6.2. Example 2

Consider a particular case of system (102) with the following data:

$$\begin{aligned} n = 1, \quad m = 1, \quad N = 1, \quad h_1 > 0, \\ f\big(x(t), y(t), x(t-h_1), y(t-h_1)\big) = \exp\big(x(t) - 4y(t) - x(t-h_1) + y(t-h_1)\big) - 1, \\ g\big(x(t), y(t), x(t-h_1), y(t-h_1)\big) = \sin(5x(t) - 5y(t) + 2x(t-h_1)) \\ +1 - \cos\big(y(t-h_1)\big). \end{aligned} \tag{147}$$

It is seen that $f(0,0,0,0) = 0$ and $g(0,0,0,0) = 0$. Moreover, it is seen that system (102), (147) is also a particular case of system (129), and both assumptions A7 and A9 are valid for this system.

Let us find out which theorem (either Theorem 7 or Theorem 10) is applicable for the asymptotic stability analysis of system (102), (147). For this purpose, we linearize (102), (147) in a neighborhood of its trivial solution $\big(x(t) \equiv 0, y(t) \equiv 0, t \geq -h_1\big)$, which yields the system

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t) - 4y(t) - x(t-h_1) + y(t-h_1), \quad t \geq 0, \\ \varepsilon\frac{dy(t)}{dt} &= 5x(t) - 5y(t) + 2x(t-h_1), \quad t \geq 0. \end{aligned} \tag{148}$$

This system can be considered as a particular case of each of the systems (74) and (105) with $n = 1$, $m = 1$, $N = 1$ and

$$\begin{aligned} A_1 = 1, \quad A_2 = -4, \quad A_3 = 5, \quad A_4 = -5, \\ H_{11} = H_1 = -1, \quad H_{21} = H_2 = 1, \quad H_{31} = H_3 = 2, \quad H_{41} = H_4 = 0. \end{aligned} \tag{149}$$

Since $H_4 = 0$, then assumption A1 is not valid for (148). Consequently, Theorem 10 is not applicable for the asymptotic stability analysis of system (102), (147). Let us try to apply

Theorem 7 for such an analysis. We start with the verification of the validity of assumptions A5 and A6 for system (148). Since $A_4 \neq 0$, assumption A5 is valid.

Using Equations (95) and (149), we obtain

$$\mathcal{H}_{11}(0) = 0, \quad \mathcal{H}_{21}(0) = 1, \quad \mathcal{H}_{31}(0) = 2, \quad \mathcal{H}_{41}(0) = -1. \tag{150}$$

Furthermore, using Equations (93), (94), (150) and choosing

$$R_{11} = R_{21} = R > 0,$$

we obtain the scalar quadratic equations with respect to the unknown P_{10} and P_{20} as:

$$\begin{aligned} R^{-1}P_{10}^2 - 6P_{10} + R + Q_1 = 0, \\ 5R^{-1}P_{20}^2 - 10P_{20} + R + Q_3 = 0, \end{aligned} \tag{151}$$

where $Q_1 > 0$ and $Q_3 > 0$.

Solving the first equation in (151), we obtain its two solutions:

$$\begin{aligned} P_{10}^+ &= R\Big(3 + \sqrt{9 - (1 + Q_1/R)}\Big), \\ P_{10}^- &= R\Big(3 - \sqrt{9 - (1 + Q_1/R)}\Big). \end{aligned} \tag{152}$$

Each of these solutions is positive if and only if

$$0 < Q_1 \leq 8R. \tag{153}$$

The particular case of $\mathcal{G}_{10}$ in (96) for $P_{10} = P_{10}^+$ satisfies the inequality

$$\mathcal{G}_{10}^+ = \sqrt{9 - (1 + Q_1/R)} > 0, \tag{154}$$

while the particular case of $\mathcal{G}_{10}$ in (96) for $P_{10} = P_{10}^-$ satisfies the inequality

$$\mathcal{G}_{10}^- = -\sqrt{9 - (1 + Q_1/R)} < 0, \tag{155}$$

if and only if

$$0 < Q_1 < 8R. \tag{156}$$

Similarly, we obtain two solutions of the second equation in (151):

$$\begin{aligned} P_{20}^+ &= \frac{R\Big(5 + \sqrt{25 - 5(1 + Q_3/R)}\Big)}{5}, \\ P_{20}^- &= \frac{R\Big(5 - \sqrt{25 - 5(1 + Q_3/R)}\Big)}{5}, \end{aligned} \tag{157}$$

and each of these solutions is positive if and only if

$$0 < Q_3 \leq 4R. \tag{158}$$

Moreover, similarly to (154) and (155), the particular case of $\mathcal{G}_{20}$ in (96) for $P_{20} = P_{20}^+$ satisfies the inequality

$$\mathcal{G}_{20}^+ = \sqrt{25 - 5(1 + Q_3/R)} > 0, \tag{159}$$

while the particular case of $\mathcal{G}_{20}$ in (96) for $P_{20} = P_{20}^{-}$ satisfies the inequality

$$\mathcal{G}_{10} = -\sqrt{25 - 5(1 + Q_3/R)} < 0, \tag{160}$$

if and only if

$$0 < Q_3 < 4R. \tag{161}$$

Hence, items (a) and (b) of assumption A6 are valid. Let us show that item (c) of this assumption is also valid.

Using Equations (98) and (150), we obtain Q_{20} as:

$$Q_{20} = \frac{P_{10}P_{20}}{R}. \tag{162}$$

Let us choose

$$Q_1 = 7R, \quad Q_3 = 3R. \tag{163}$$

These Q_1 and Q_3 satisfy inequalities (156) and (161), respectively. For Q_1 and Q_3 from (163), due to Equations (152) and (157), we have

$$P_{10}^{+} = 4R, \quad P_{10}^{-} = 2R, \quad P_{20}^{+} = 1.44721R, \quad P_{20}^{-} = 0.55279R,$$

which, along with (162), yields

$$Q_{20}^{++} = \frac{P_{10}^{+}P_{20}^{+}}{R} = 5.78884R, \quad Q_{20}^{+-} = \frac{P_{10}^{+}P_{20}^{-}}{R} = 2.21116R,$$
$$Q_{20}^{-+} = \frac{P_{10}^{-}P_{20}^{+}}{R} = 2.89442R, \quad Q_{20}^{--} = \frac{P_{10}^{-}P_{20}^{-}}{R} = 1.10558R. \tag{164}$$

Now, based on Equations (97), (163), and (164), we can construct the following matrices:

$$Q_0^{++} = \begin{pmatrix} 7R & 5.78884R \\ 5.78884R & 3R \end{pmatrix}, \quad Q_0^{+-} = \begin{pmatrix} 7R & 2.21116R \\ 2.21116R & 3R \end{pmatrix},$$
$$Q_0^{-+} = \begin{pmatrix} 7R & 2.89442R \\ 2.89442R & 3R \end{pmatrix}, \quad Q_0^{--} = \begin{pmatrix} 7R & 1.10558R \\ 1.10558R & 3R \end{pmatrix}.$$

It is verified directly that the matrix Q_0^{++} is not positive definite, while the other matrices Q_0^{+-}, Q_0^{-+}, and Q_0^{--} are positive definite. This means that item (c) of assumption A6 is valid for the pairs (P_{10}^{+}, P_{20}^{-}), (P_{10}^{-}, P_{20}^{+}), and (P_{10}^{-}, P_{20}^{-}). Thus, assumptions A5 and A6 are valid for system (148). Therefore, by virtue of Lemma 3, there exists a positive number $\widehat{\varepsilon}^{*}$ such that, for any given $\varepsilon \in (0, \widehat{\varepsilon}^{*}]$, system (148) is asymptotically stable. Moreover, due to Theorem 7, the trivial solution of system (102), (147) is asymptotically stable for any given $\varepsilon \in (0, \widehat{\varepsilon}^{*}]$.

3.6.3. Example 3

Consider a particular case of system (105) with the following data:

$$n = 1, \quad m = 1, \quad A_1 = -7, \quad A_2 = 3, \quad A_3 = -6, \quad A_4 = 0,$$
$$H_1 = 1, \quad H_2 = -3, \quad H_3 = 2, \quad H_4 = -2. \tag{165}$$

It should be noted that system (105), (165) is also a particular case of system (74). However, since $A_4 = 0$, Theorem 6 is not applicable for the asymptotic stability analysis of (105), (165). Taking into account that $H_4 \neq 0$, i.e., assumption A1 is valid, we try to apply Theorem 9 for the asymptotic stability analysis of this system.

Using Equations (121) and (125) and the data (165) of this example, we obtain

$$\widetilde{\mathcal{A}}_1(0) = -4, \quad \widetilde{\mathcal{A}}_2(0) = 3, \quad \widetilde{\mathcal{A}}_3(0) = -2, \quad \widetilde{\mathcal{A}}_4(0) = -3. \tag{166}$$

Now, using Equations (119), (120), (165), (166) and choosing

$$\widetilde{R}_1 = 2\widetilde{R}, \quad \widetilde{R}_2 = 4\widetilde{R}, \quad \widetilde{R} > 0,$$

we can write down the scalar quadratic equations with respect to the unknown $\widetilde{P}_{10}$ and $\widetilde{P}_{20}$ as:

$$\begin{aligned} 2\widetilde{R}^{-1}\widetilde{P}_{10}^2 - 8\widetilde{P}_{10} + 2\widetilde{R} + \widetilde{Q}_1 &= 0, \\ \widetilde{R}^{-1}\widetilde{P}_{20}^2 - 6\widetilde{P}_{20} + 4\widetilde{R} + \widetilde{Q}_3 &= 0, \end{aligned} \tag{167}$$

where $\widetilde{Q}_1 > 0$ and $\widetilde{Q}_3 > 0$.

The first equation in (167) yields the following two solutions:

$$\begin{aligned} \widetilde{P}_{10}^{+} &= \frac{\widetilde{R}}{2}\left(4 + \sqrt{16 - 2\big(2 + \widetilde{Q}_1/\widetilde{R}\big)}\right), \\ \widetilde{P}_{10}^{-} &= \frac{\widetilde{R}}{2}\left(4 - \sqrt{16 - 2\big(2 + \widetilde{Q}_1/\widetilde{R}\big)}\right). \end{aligned} \tag{168}$$

Each of these solutions is positive if and only if

$$0 < \widetilde{Q}_1 \le 6\widetilde{R}. \tag{169}$$

The particular case of $\widetilde{\mathcal{G}}_{10}$ in (122) for $\widetilde{P}_{10} = \widetilde{P}_{10}^{+}$ satisfies the inequality

$$\widetilde{\mathcal{G}}_{10}^{+} = \sqrt{16 - 2\big(2 + \widetilde{Q}_1/\widetilde{R}\big)} > 0, \tag{170}$$

while the particular case of $\widetilde{\mathcal{G}}_{10}$ in (122) for $\widetilde{P}_{10} = \widetilde{P}_{10}^{-}$ satisfies the inequality

$$\widetilde{\mathcal{G}}_{10}^{-} = -\sqrt{16 - 2\big(2 + \widetilde{Q}_1/\widetilde{R}\big)} < 0, \tag{171}$$

if and only if

$$0 < Q_1 < 6\widetilde{R}. \tag{172}$$

The second equation in (167) has the following two solutions:

$$\begin{aligned} \widetilde{P}_{20}^{+} &= \widetilde{R}\left(3 + \sqrt{9 - \big(4 + \widetilde{Q}_3/\widetilde{R}\big)}\right), \\ \widetilde{P}_{20}^{-} &= \widetilde{R}\left(3 - \sqrt{9 - \big(4 + \widetilde{Q}_3/\widetilde{R}\big)}\right), \end{aligned} \tag{173}$$

and each of these solutions is positive if and only if

$$0 < \widetilde{Q}_3 \le 5\widetilde{R}. \tag{174}$$

Moreover, similarly to (170) and (171), the particular case of $\widetilde{\mathcal{G}}_{20}$ in (122) for $\widetilde{P}_{20} = \widetilde{P}_{20}^{+}$ satisfies the inequality

$$\widetilde{\mathcal{G}}_{20}^{+} = \sqrt{9 - \big(4 + \widetilde{Q}_3/\widetilde{R}\big)} > 0, \tag{175}$$

while the particular case of $\widetilde{\mathcal{G}}_{20}$ in (122) for $\widetilde{P}_{20} = \widetilde{P}_{20}^{-}$ satisfies the inequality

$$\widetilde{\mathcal{G}}_{20}^{-} = -\sqrt{9 - \big(4 + \widetilde{Q}_3/\widetilde{R}\big)} < 0, \tag{176}$$

if and only if

$$0 < \widetilde{Q}_3 < 5\widetilde{R}. \tag{177}$$

Thus, items (a) and (b) of assumption A8 are valid. Let us show that item (c) of this assumption is also valid.

Using Equations (124) and (166), we obtain $\widetilde{Q}_{20}$ as:

$$\widetilde{Q}_{20} = 2\widetilde{P}_{20} - 3\widetilde{P}_{10}. \tag{178}$$

We choose $\widetilde{Q}_1$ and $\widetilde{Q}_3$ as:

$$\widetilde{Q}_1 = 5\widetilde{R}, \quad \widetilde{Q}_3 = 4\widetilde{R}. \tag{179}$$

These $\widetilde{Q}_1$ and $\widetilde{Q}_3$ satisfy inequalities (172) and (177), respectively. For $\widetilde{Q}_1$ and $\widetilde{Q}_3$ chosen in (179), due to Equations (168) and (173), we have

$$\widetilde{P}_{10}^{+} = 2.70711\widetilde{R}, \quad \widetilde{P}_{10}^{-} = 1.29289\widetilde{R}, \quad \widetilde{P}_{20}^{+} = 4\widetilde{R}, \quad \widetilde{P}_{20}^{-} = 2\widetilde{R},$$

which, along with (178), yields

$$\begin{aligned} \widetilde{Q}_{20}^{++} &= 2\widetilde{P}_{20}^{+} - 3\widetilde{P}_{10}^{+} = -0.12133\widetilde{R}, \quad \widetilde{Q}_{20}^{+-} = 2\widetilde{P}_{20}^{-} - 3\widetilde{P}_{10}^{+} = -4.12133\widetilde{R}, \\ \widetilde{Q}_{20}^{-+} &= 2\widetilde{P}_{20}^{+} - 3\widetilde{P}_{10}^{-} = 4.12133\widetilde{R}, \quad \widetilde{Q}_{20}^{--} = 2\widetilde{P}_{20}^{-} - 3\widetilde{P}_{10}^{-} = 0.12133\widetilde{R}. \end{aligned} \tag{180}$$

Now, using Equations (123), (179), and (180), we can construct the following matrices:

$$\widetilde{Q}_0^{++} = \begin{pmatrix} 5\widetilde{R} & -0.12133\widetilde{R} \\ -0.12133\widetilde{R} & 4\widetilde{R} \end{pmatrix}, \quad \widetilde{Q}_0^{+-} = \begin{pmatrix} 5\widetilde{R} & -4.12133\widetilde{R} \\ -4.12133\widetilde{R} & 4\widetilde{R} \end{pmatrix},$$
$$\widetilde{Q}_0^{-+} = \begin{pmatrix} 5\widetilde{R} & 4.12133\widetilde{R} \\ 4.12133\widetilde{R} & 4\widetilde{R} \end{pmatrix}, \quad \widetilde{Q}_0^{--} = \begin{pmatrix} 5\widetilde{R} & 0.12133\widetilde{R} \\ 0.12133\widetilde{R} & 4\widetilde{R} \end{pmatrix}.$$

It is verified directly that all the matrices $\widetilde{Q}_0^{++}$, $\widetilde{Q}_0^{+-}$, $\widetilde{Q}_0^{-+}$, and $\widetilde{Q}_0^{--}$ are positive definite. This means that item (c) of assumption A8 is valid for all the pairs $(\widetilde{P}_{10}^{+}, \widetilde{P}_{20}^{+})$, $(\widetilde{P}_{10}^{+}, \widetilde{P}_{20}^{-})$, $(\widetilde{P}_{10}^{-}, \widetilde{P}_{20}^{+})$, and $(\widetilde{P}_{10}^{-}, \widetilde{P}_{20}^{-})$. Thus, assumptions A1 and A8 are valid for system (105), (165). Therefore, by virtue of Theorem 9, there exists a positive number $\widetilde{\varepsilon}^*$ such that, for any given $\varepsilon \in (0, \widetilde{\varepsilon}^*]$, system (105), (165) is asymptotically stable.

3.6.4. Example 4

Consider a particular case of system (129) with the following data:

$$\begin{aligned} n = 1, \quad m = 1, \quad h > 0, \\ f\big(x(t), y(t), x(t-h), y(t-h)\big) = \sin\big(-6x(t) + 2y(t) + 2x(t-h) - y(t-h)\big), \\ g\big(x(t), y(t), x(t-h), y(t-h)\big) = \exp\big(-3x(t) - x(t-h) + y(t-h)\big) - \cos\big(y(t)\big). \end{aligned} \tag{181}$$

It is seen that $f(0,0,0,0) = 0$ and $g(0,0,0,0) = 0$. Moreover, it is seen that system (129), (181) is also a particular case of system (102) with $N = 1$, and both assumptions A7 and A9 are valid for this system. As in Example 2 (Section 3.6.2), let us find out which theorem (either Theorem 10 or Theorem 7) is applicable for the asymptotic stability analysis of system (129), (181). For this purpose, we carry out the linearization of this system in a neighborhood of its trivial solution $\big(x(t) \equiv 0, y(t) \equiv 0, t \geq -h\big)$. This linearization yields the system

$$\begin{aligned} \frac{dx(t)}{dt} &= -6x(t) + 2y(t) + 2x(t-h) - y(t-h), \quad t \geq 0, \\ \varepsilon\frac{dy(t)}{dt} &= -3x(t) - x(t-h) + y(t-h), \quad t \geq 0. \end{aligned} \tag{182}$$

This system can be considered as a particular case of each of the systems (105) and (74) with $n = 1$, $m = 1$, $N = 1$ and

$$\begin{aligned} A_1 = -6, \quad A_2 = 2, \quad A_3 = -3, \quad A_4 = 0, \\ H_1 = H_{11} = 2, \quad H_2 = H_{21} = -1, \quad H_3 = H_{31} = -1, \quad H_4 = H_{41} = 1. \end{aligned} \tag{183}$$

Since $A_4 = 0$, then assumption A5 is not valid for (182). Consequently, Theorem 7 is not applicable for the asymptotic stability analysis of system (129), (181). Let us try to apply Theorem 10 for such an analysis. We start with the verification of the validity of assumptions A1 and A8 for system (182). Since $H_4 \neq 0$, assumption A1 is valid.

Using Equations (121) and (125) and the coefficients (183) of system (182), we obtain

$$\widetilde{\mathcal{A}}_1(0) = -4, \quad \widetilde{\mathcal{A}}_2(0) = 2, \quad \widetilde{\mathcal{A}}_3(0) = 1, \quad \widetilde{\mathcal{A}}_4(0) = -2. \tag{184}$$

Now, using Equations (119), (120), (183), and (184) and choosing

$$\widetilde{R}_1 = \widetilde{R}_2 = \widetilde{R} > 0, \quad \widetilde{Q}_1 = 6\widetilde{R}, \quad \widetilde{Q}_3 = 2\widetilde{R}, \tag{185}$$

we can write down the scalar quadratic equations with respect to the unknown $\widetilde{P}_{10}$ and $\widetilde{P}_{20}$ as:

$$\begin{aligned} \widetilde{R}^{-1}\widetilde{P}_{10}^2 - 8\widetilde{P}_{10} + 7\widetilde{R} = 0, \\ \widetilde{R}^{-1}\widetilde{P}_{20}^2 - 4\widetilde{P}_{20} + 3\widetilde{R} = 0. \end{aligned} \tag{186}$$

The first equation in (186) yields the following two solutions:

$$\widetilde{P}_{10}^{+} = 7\widetilde{R} > 0, \quad \widetilde{P}_{10}^{-} = \widetilde{R} > 0, \tag{187}$$

while the second equation in (186) yields the following two solutions:

$$\widetilde{P}_{20}^{+} = 3\widetilde{R} > 0, \quad \widetilde{P}_{20}^{-} = \widetilde{R} > 0. \tag{188}$$

The particular cases of $\widetilde{\mathcal{G}}_{10}$ in (122) for $\widetilde{P}_{10} = \widetilde{P}_{10}^{+}$ and $\widetilde{P}_{10} = \widetilde{P}_{10}^{-}$ are $\widetilde{\mathcal{G}}_{10}^{+} = 3 > 0$ and $\widetilde{\mathcal{G}}_{10}^{+} = -3 < 0$, respectively. The particular cases of $\widetilde{\mathcal{G}}_{20}$ in (122) for $\widetilde{P}_{20} = \widetilde{P}_{20}^{+}$ and $\widetilde{P}_{20} = \widetilde{P}_{20}^{-}$ are $\widetilde{\mathcal{G}}_{20}^{+} = 1 > 0$ and $\widetilde{\mathcal{G}}_{10}^{+} = -1 < 0$, respectively.

Thus, items (a) and (b) of assumption A8 are valid for system (182). Let us show that item (c) of this assumption is also valid for (182).

Using Equations (124) and (184), we obtain $\widetilde{Q}_{20}$ as:

$$\widetilde{Q}_{20} = -(2\widetilde{P}_{10} + \widetilde{P}_{20}), \tag{189}$$

which, along with (187) and (188), yields

$$\begin{aligned} \widetilde{Q}_{20}^{++} = -(2\widetilde{P}_{10}^{+} + \widetilde{P}_{20}^{+}) = -17\widetilde{R}, \quad \widetilde{Q}_{20}^{+-} = -(2\widetilde{P}_{10}^{+} + \widetilde{P}_{20}^{-}) = -15\widetilde{R}, \\ \widetilde{Q}_{20}^{-+} = -(2\widetilde{P}_{10}^{-} + \widetilde{P}_{20}^{+}) = -5\widetilde{R}, \quad \widetilde{Q}_{20}^{--} = -(2\widetilde{P}_{10}^{-} + \widetilde{P}_{20}^{-}) = -3\widetilde{R}. \end{aligned} \tag{190}$$

Now, using Equations (123), (185), and (190), we can construct the following matrices:

$$\begin{aligned} \widetilde{Q}_0^{++} = \begin{pmatrix} 6\widetilde{R} & -17\widetilde{R} \\ -17\widetilde{R} & 2\widetilde{R} \end{pmatrix}, \quad \widetilde{Q}_0^{+-} = \begin{pmatrix} 6\widetilde{R} & -15\widetilde{R} \\ -15\widetilde{R} & 2\widetilde{R} \end{pmatrix}, \\ \widetilde{Q}_0^{-+} = \begin{pmatrix} 6\widetilde{R} & -5\widetilde{R} \\ -5\widetilde{R} & 2\widetilde{R} \end{pmatrix}, \quad \widetilde{Q}_0^{--} = \begin{pmatrix} 6\widetilde{R} & -3\widetilde{R} \\ -3\widetilde{R} & 2\widetilde{R} \end{pmatrix}. \end{aligned}$$

It is verified directly that the matrices $\widetilde{Q}_0^{++}$, $\widetilde{Q}_0^{+-}$, and $\widetilde{Q}_0^{-+}$ are not positive definite, while the matrix $\widetilde{Q}_0^{--}$ is positive definite. This means that item (c) of assumption A8 is valid only for the pair $\left(\widetilde{P}_{10}^{-}, \widetilde{P}_{20}^{-}\right)$. However, along with the validity of items (a) and (b) for $\left(\widetilde{P}_{10}^{-}, \widetilde{P}_{20}^{-}\right)$, it is enough for the validity of assumption A8 for system (182).

Thus, assumptions A1 and A8 are valid for system (182), meaning, by virtue of Theorem 10, the existence of a positive number $\check{\varepsilon}^*$ such that, for any given $\varepsilon \in (0, \check{\varepsilon}^*]$, linear system (182) and the trivial solution of nonlinear system (129), (181) are asymptotically stable.

4. Conclusions

In this paper, three types of linear singularly perturbed time-delay differential systems were considered. The first type presents the system right-hand side, which depends only on the delayed unknown functions. The delay is a single point-wise one, and it is proportional to the small parameter $\varepsilon > 0$ of the singular perturbation. The second type of the considered systems is the system containing in its right-hand side both, un-delayed and delayed, unknown functions. The delays are multiple point-wise ones, and they are independent of ε. The third type is the particular case of the second type with the single delay. For each of these systems, its asymptotic stability was studied. To carry out this study, the partial exact slow–fast decomposition of the original system and the application of the symmetric matrix Riccati equation method were proposed and realized. For the first type system, the partial slow–fast decomposition becomes the complete decomposition because this system contains in its right-hand side only the delayed terms with the single delay. This decomposition essentially depends on the assumption that the matrix-valued coefficient for the "fast" state in the "fast" equation of the original singularly perturbed system is an invertible matrix. The slow–fast decompositions of the second and third types systems are partial. Namely, in the second type system only the undelayed part is decomposed, while in the third type system only the delayed part is decomposed. Similarly to the decomposition of the first type of system, the decompositions of the second and third types of system essentially depend on the invertibility of the corresponding matrix-valued coefficients for the "fast" state (undelayed or delayed) in the "fast" equation. It should be noted that the aforementioned assumptions on the invertibility of the corresponding matrices are unavoidable in the exact slow–fast partial/complete decomposition. Using the aforementioned decomposition allowed us to decompose the matrix Riccati algebraic equation, associated with the original linear singularly perturbed time-delay differential system, into two much simpler and less dimensional Riccati equations, which are not connected with each other. Then, the asymptotic analysis of each of these equations was carried out separately, yielding the ε-free conditions guaranteeing the asymptotic stability of the original linear singularly perturbed time-delay system for any sufficiently small value of ε. Based on the obtained results for the considered linear systems and using the method of asymptotic stability in the first approximation, the ε-free conditions guaranteeing the asymptotic stability of the trivial solution to the corresponding nonlinear singularly perturbed time-delay systems for any sufficiently small value of the parameter of singular perturbation were derived.

In the completion of this section, we would like to mention several issues connected with the topic of the paper, which are interesting ones for future investigations. These issues are the following: (a) to establish (maybe subject to some additional conditions) the uniformity with respect to ε of the asymptotic stability for the considered systems; (b) to establish (maybe subject to some additional conditions) the exponential stability, uniform in ε, for the considered systems; (c) to obtain an estimate of the small positive number ε^* appearing in Theorem 3, as well as estimates of the numbers $\bar{\varepsilon}^*$ and $\widetilde{\varepsilon}^*$ appearing in Theorems 6 and 9, using either an analytical approach or an extensive computer simulation; (d) to extend the approach proposed in this paper to the asymptotic stability analysis of another types of singularly perturbed time-delay systems, for instance: (1) the systems with undelayed states, point-wise delayed states, and distributed delayed states; for such systems, the partial exact slow–fast decomposition can be applied separately, either for the undelayed part in the right-hand side, or for the point-wise delayed part, or for the distributed delayed part; in the latter case, the corresponding matrix-valued coefficients should be constant; (2) the systems having simultaneously small (of order of ε) and non-small (of order of 1) time delays; (e) to compare by an extensive computer simulation the method proposed in this paper with existing literature methods of stability analysis for singularly perturbed time-delay systems; (f) based on the theoretical results of this paper and their possible extensions (see item (d)), to carry out the stability analysis of various real-life problems modeled by singularly perturbed systems with delays.

In order not to overload the present paper (thus keeping its readability), the aforementioned issues are not considered here. They will be investigated in separate forthcoming papers.

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