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On Ozaki Close-to-Convex Functions with Bounded Boundary Rotation

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Abstract: In the present investigation, we introduce a new subclass of univalent functions $\mathcal{F}(u,\lambda)$ and a subclass of bi-univalent function $\mathcal{F}_{o,\Sigma}(u,\lambda)$ with bounded boundary and bounded radius rotation. Some examples of the functions belonging to the classes $\mathcal{F}(u,\lambda)$ are also derived. For these new classes, the authors derive many interesting relations between these classes and the existing familiar subclasses in the literature. Furthermore, the authors establish new coefficient estimates for these classes. Apart from the above, the first two initial coefficient bounds for the class $\mathcal{F}_{o,\Sigma}(u,\lambda)$ are established.

Keywords: analytic; Ozaki close-to-convex functions; convolution; bounded boundary rotations

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1. Introduction, Motivations and Definitions

Let $\mathcal{A}$ consist of all holomorphic functions $G : \mathbb{E} \longrightarrow \mathbb{C}$ normalized by $G(0) = 0$ and $G'(0) = 1$. Let $\mathcal{S} \subset \mathcal{A}$ be the class of all functions defined by

$$G(\vartheta) = \vartheta + \sum_{m=2}^{\infty} \xi_m \vartheta^m \tag{1}$$

which are analytic and univalent in $\mathbb{E} = \{\vartheta : |\vartheta| < 1\}$. A domain $D \subset \mathbb{C}$ is called as starlike with respect to a point $z_0 \in D$ if the line segment joining z_0 to every other point $z \in D$ lies entirely in D. A function $G \in \mathcal{A}$ is called starlike if $G(\mathbb{E})$ is a starlike domain with respect to the origin. The class of univalent starlike functions is denoted by $\mathcal{S}^*$. A domain $D \subset \mathbb{C}$ is called as convex if the line segment joining any two arbitrary points of D lies entirely in D, i.e., if it is starlike with respect to each point of D. A function $G \in \mathcal{A}$ is said to be convex in $\mathbb{D}$ if $G(\mathbb{E})$ is a convex domain. The class of all univalent convex functions is denoted by $\mathcal{K}$. Many times the analytic criteria of the above two functions provide a useful technique in analyzing the concepts and are as follows: a function $G \in \mathcal{S}$ is called as a starlike function of order λ $(0 \leq \lambda < 1)$ if and only if

$$\Re\left(\frac{\vartheta G'(\vartheta)}{G(\vartheta)}\right) > \lambda, \quad \vartheta \in \mathbb{E}.$$

The family of all starlike functions of order λ is denoted by $\mathcal{S}^*(\lambda)$. It is clear that for $\lambda = 0$, $\mathcal{S}^*(\lambda) \equiv \mathcal{S}^*$. Also, for $\lambda_1 \geq \lambda_2$, $\mathcal{S}^*(\lambda_1) \subseteq \mathcal{S}^*(\lambda_2)$. A function $G \in \mathcal{S}$ is called as a convex function of order λ $(0 \leq \lambda < 1)$ if and only if

$$\Re\left(1 + \frac{\vartheta G''(\vartheta)}{G'(\vartheta)}\right) > \lambda, \quad \vartheta \in \mathbb{E}.$$

The family of all convex functions of order λ is denoted by $\mathcal{K}(\lambda)$. It is clear that for $\lambda = 0$, $\mathcal{K}(\lambda) \equiv \mathcal{K}$. A function $G \in \mathcal{S}$ of the form (1) is called as a close-to-convex function if there exists a function $\phi \in \mathcal{K}$ such that

$$\Re\left(\frac{G'(\vartheta)}{\phi'(\vartheta)}\right) > 0.$$

The family of all close-to-convex functions is denoted by $\mathcal{C}$. Although the class of close-to-convex functions was introduced by Kaplan [1] in 1952, Ozaki [2] introduced the class $\mathcal{F}$ familiarly known as the Ozaki-close-to-convex function, which is defined as follows: a function G given in the form (1) belongs to the class $\mathcal{F}$ if G satisfies the condition

$$\Re\left(1 + \frac{\vartheta G''(\vartheta)}{G'(\vartheta)}\right) > -\frac{1}{2}, \quad \vartheta \in \mathbb{E}$$

or

$$\Re\left(1 + \frac{\vartheta G''(\vartheta)}{G'(\vartheta)}\right) < \frac{3}{2}, \quad \vartheta \in \mathbb{E}.$$

We observe that $\mathcal{F} \subset \mathcal{C}$ follows from the original definition of Kaplan [1], while Umezawa [3] proved that functions in $\mathcal{F}$ are not necessarily starlike but are convex in one direction. Singh and Singh [4] proved that functions in $\mathcal{F}$ are close-to-convex and bounded in $\mathbb{E}$. Let $\mathcal{F}(\eta)$ denote the class of locally univalent normalized analytic functions f in the unit disk, satisfying the condition

$$\Re\left(1 + \frac{\vartheta G''(\vartheta)}{G'(\vartheta)}\right) > \frac{1}{2} - \eta$$

for some $-\frac{1}{2} < \eta \leq 1$. If $-\frac{1}{2} \leq \eta \leq \frac{1}{2}$; then, $\mathcal{F}(\eta) \subset \mathcal{K} \subset \mathcal{S}^*$. Also, $\mathcal{F}(1/2) \equiv \mathcal{K}$. The functions of the class $\mathcal{F}(1)$ are known to be univalent and close-to-convex in $\mathbb{E}$. Note that the class $\mathcal{F}(\eta)$ plays an important role in determining the univalence criteria for sense-preserving harmonic mappings. The functions of the class $\mathcal{F}(1)$ are non-empty because it is very easy to see that the function $G : \mathbb{E} \longrightarrow \mathbb{C}$, defined by

$$G(\vartheta) = \frac{2}{3}\left[1 - \exp\left(-\frac{3}{2}\vartheta\right)\right], \quad \vartheta \in \mathbb{E},$$

belongs to the class $\mathcal{F}(1)$.

The family of all Ozaki-type close-to-convex functions denoted by $\mathcal{F}_o(\eta)$ is defined as:

Definition 1. *A function $G \in \mathcal{S}$ given by (1) is called Ozaki-type close-to-convex if and only if*

$$\Re\left[\frac{2\eta - 1}{2\eta + 1} + \frac{2}{2\eta + 1}\left(1 + \frac{\vartheta G''(\vartheta)}{G'(\vartheta)}\right)\right] > 0,$$

where $\frac{1}{2} \leq \eta \leq 1$.

For more details, one may see [5–7] and also [8,9]. Paatero [10] introduced the class $\mathcal{P}_u$ of functions with bounded turning. If a function $\psi \in \mathcal{P}_u$ satisfies $\psi(0) = 1$, then there exists a non-decreasing function ν with bounded variation in $[0, 2\pi]$ and satisfying

$$\int_0^{2\pi} d\nu(t) = 2 \quad and \quad \int_0^{2\pi} |d\nu(t)| \leq u$$

such that

$$\psi(\vartheta) = \frac{1}{2}\int_0^{2\pi} \frac{1 + \vartheta e^{-it}}{1 - \vartheta e^{-it}} d\nu(t).$$

Connection between the class $\mathcal{P}_u$ and the class $\mathcal{P}$ of functions with positive real part is stated in the following lemma that has been established earlier by Pinchuk [11].

Lemma 1. *A function $\psi \in \mathcal{P}_u$ if and only if $\exists\ \psi_1, \psi_2 \in \mathcal{P}$ such that*

$$\psi(\vartheta) = \frac{u+2}{4}\psi_1(\vartheta) - \frac{u-2}{4}\psi_2(\vartheta).$$

A function $\psi \in \mathcal{P}_u(\lambda)$ if and only if there exists a function $\psi_3 \in \mathcal{P}_u$ such that

$$\psi(\vartheta) = \lambda + (1-\lambda)\psi_3(\vartheta). \tag{2}$$

The class $\mathcal{P}_u(\lambda)$ was introduced and investigated by Padmanabhan and Parvatham [12]. A function that is analytic and locally univalent in a given simply connected domain is said to be of bounded boundary rotation if its range has bounded boundary rotation, which is defined as the total variation in the direction angle of the tangent to the boundary curve under a complete circuit. Let $\mathcal{V}_u$ denote the family of functions G that map the unit disc $\mathbb{E}$ conformally onto an image domain $G(\mathbb{E})$ of bounded boundary rotation at most $u\pi$.

Umarani [13] introduced class $\mathcal{V}_u(b)$ functions of bounded boundary rotation of complex order b. We say a function $G \in \mathcal{V}_u(b)$, where b is a non-zero complex number if

$$\int_0^{2\pi} \left| \Re\left(1 + \frac{\vartheta G''(\vartheta)}{bG'(\vartheta)}\right) \right| d\nu \leq u\pi.$$

Umarani [13] showed that a function $G_b \in \mathcal{V}_u(b)$ if and only if there exists a function $G \in \mathcal{V}_u$ such that

$$G_b'(\vartheta) = \left[G'(\vartheta)\right]^b. \tag{3}$$

For the choice of b, the class $\mathcal{V}_u(b)$ reduces to the following important subclasses

(i) For $b = 1$, $\mathcal{V}_u(1) \equiv \mathcal{V}_u$, and the well-known class of functions of bounded boundary rotation at most $u\pi$ was introduced by Paatero [10].

(ii) For $b = 1-\lambda$, we have $\mathcal{V}_u(1-\lambda) \equiv \mathcal{V}_u(\lambda)$, introduced by Padmanabhan and Parvatham [12].

A function $G \in \mathcal{A}$ is said to be in the class $\mathcal{R}_u$, the class $\mathcal{R}_u$ of functions with bounded radius rotations (introduced by Tammi [14]), if for $u \geq 2$,

$$\int_0^{2\pi} \left| \Re\left(\frac{\vartheta G'(\vartheta)}{G(\vartheta)}\right) \right| d\nu \leq u\pi.$$

The integral representation for functions $G \in \mathcal{R}_u$ is given by

$$G(\vartheta) = \vartheta \exp\left[-\int_0^{2\pi} -\log(1-\vartheta e^{-it})d\nu\right],$$

where ν is a non-decreasing function with bounded variation in $[0, 2\pi]$ and satisfying

$$\int_0^{2\pi} d\nu(t) = 2 \quad \textit{and} \quad \int_0^{2\pi} |d\nu(t)| \leq u.$$

Pinchuk [11] showed that an Alexander-type relation between the classes $\mathcal{V}_u$ and $\mathcal{R}_u$ exists and is given by

$$G \in \mathcal{V}_u \text{ if and only if } \vartheta G' \in \mathcal{R}_u. \tag{4}$$

Lehto [15] (see [16]) proved that for the function $G \in \mathcal{V}_u$ given in the form (1)

$$|\xi_2| \leq \frac{u}{2} \quad \textit{and} \quad |\xi_3| \leq \frac{u^2+2}{6}. \tag{5}$$

Schiffer and Tammi [17] proved that for the function $G \in \mathcal{V}_u$ given in the form (1),

$$|\xi_4| \leq \frac{u^3 + 8u}{24}. \tag{6}$$

Following the work of Brannan [18], who proved that $\mathcal{V}_u$ is a subclass of the class $\mathcal{K}_u(\lambda)$ of the close-to-convex functions of order $\lambda = \frac{u}{2} - 1$, Koepf [19] showed that $\mathcal{V}_u(k)$ is a subclass of the class $\mathcal{K}_u(\lambda)$ of the k-fold symmetric close-to-convex functions of order $\lambda = \frac{u-2}{2k}$. This leads to the solution to the coefficient problem for k-fold symmetric functions of bounded boundary rotation when $u \geq 2k$. Moreover, for $k \in \mathbb{N}$, $\mathcal{V}_u(2u+2)$ consists of close-to-convex functions and, hence, are univalent functions. Further, Leach [20] investigated the concept of odd univalent functions with bounded boundary rotation, and this was extended to k-fold symmetric functions by the same author a little later in the same year, and for details, one may look at [21].

The Koebe one-quarter theorem [22] ensures that the image of $\mathbb{E}$ under every univalent function $G \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus, every univalent function G has an inverse G^{-1}, satisfying

$$\vartheta = G^{-1}(G(\vartheta)), \quad \vartheta \in \mathbb{E},$$

and

$$v = G(G^{-1}(v)), \quad \left(|v| \leq \rho_0(G);\ \rho_0(G) \geq \frac{1}{4}\right).$$

The inverse G^{-1} may have an analytic continuation to $\mathbb{D}$, where

$$G^{-1}(v) = \Phi(v) = v + \sum_{m=2}^{\infty} (-1)^{m-1} \Phi_m v^m. \tag{7}$$

where

$$\Phi_2 = \xi_2, \quad \Phi_3 = 2\xi_2^2 - \xi_3 \quad \textit{and} \quad \Phi_4 = 5\xi_2^3 - 5\xi_2\xi_3 + \xi_4.$$

A function $G \in \mathcal{S}$ is called as bi-univalent in $\mathbb{E}$ if both G and its inverse $G^{-1} = \Psi$ belong to the class $\mathcal{S}$. Indicate Σ to be the family of all bi-univalent functions in $\mathbb{E}$. The family Σ is non-empty as the functions $\frac{\vartheta}{1-\vartheta}$, $\frac{1}{2}\log\left(\frac{1+\vartheta}{1-\vartheta}\right)$ and $-\log(1-\vartheta)$ are in the family Σ. It is interesting that the famous Koebe function $\frac{\vartheta}{(1-\vartheta)^2}$ does not belong to the family Σ. The family of bi-univalent functions was investigated for the first time by Lewin [23], who obtained a non-sharp bound $|\xi_2| < 1.51$. This was followed by Brannan and Clunie [18] and Brannan and Taha [24], who worked on certain subclasses of the bi-univalent functions and obtained bounds for their initial coefficients. The study of bi-univalent functions gained concentration as well as thrust mainly due to the investigation of Srivastava et al. [25] and was followed by many authors.

In the current article, we introduce Ozaki-type close-to-convex functions with bounded boundary rotation denoted by $\mathcal{F}(u,\lambda)$. Examples showing that the class $\mathcal{F}(u,\lambda)$ is non-empty are discussed. The authors also derive many interesting connections between the class $\mathcal{F}(u,\lambda)$ and $\mathcal{S}^*$ and $\mathcal{K}$. Finally, a new subclass $\mathcal{F}_{o,\Sigma}(u,\lambda)$ of bi-univalent functions with bounded boundary and bounded radius rotation is introduced. For the class $\mathcal{F}_{o,\Sigma}(u,\lambda)$, the authors obtain interesting first two initial non-sharp coefficient bounds.

2. Ozaki Close-to-Convex Functions with Bounded Boundary Rotation

We start this section by introducing a new class of Ozaki-type close-to-convex functions with bounded boundary rotation and is defined as follows.

Definition 2. *Let $u \geq 2$ and $\frac{1}{2} \leq \eta \leq 1$. A function G given in the form (1) is called as Ozaki-type close-to-convex with bounded boundary rotation if G satisfies the following condition*

$$\frac{2\eta-1}{2\eta+1}+\frac{2}{2\eta+1}\left(1+\frac{\vartheta G''(\vartheta)}{G'(\vartheta)}\right) \in \mathcal{P}_u.$$

The family of all Ozaki-type close-to-convex functions with bounded boundary rotation is denoted by $\mathcal{F}(u,\eta)$.

Remark 1. *(i) If $\eta = \dfrac{1}{2}$, then $\mathcal{F}(u,\eta) \equiv \mathcal{F}\left(u, \frac{1}{2}\right) \equiv \mathcal{V}_u$, which consists of functions of bounded boundary rotation introduced in [11].*

(ii) If $u = 2$, then $\mathcal{F}(u,\eta) \equiv \mathcal{F}(2,\eta) \equiv \mathcal{F}_o(\eta)$ consists of Ozaki-type close-to-convex functions.

(iii) If $u = 2$ and $\eta = \dfrac{1}{2}$, then $\mathcal{F}(u,\eta) \equiv \mathcal{F}\left(2, \frac{1}{2}\right) \equiv \mathcal{K}$, which consists of convex functions introduced in [26].

Here, we show that the class $\mathcal{F}(u,\eta)$ is non-empty by providing a few examples. Examples of functions belonging to the class $\mathcal{F}(u,\eta)$.

Example 1. *The function $G_1 : \mathbb{E} \longrightarrow \mathbb{C}$ is defined by*

$$G_1(\vartheta) = \frac{2}{2\eta+1}\left[\exp\left(\frac{2\eta+1}{2}\vartheta\right)-1\right], \quad \vartheta \in \mathbb{E}.$$

Straightforward computations shows that

$$\frac{2\eta-1}{2\eta+1}+\frac{2}{2\eta+1}\left(1+\frac{\vartheta G''(\vartheta)}{G'(\vartheta)}\right) = 1+\vartheta.$$

Since

$$\int_0^{2\pi} \Re(1+\vartheta)dt = \int_0^{2\pi}(1+r\cos t)dt = 2\pi \leq u\pi, 0<r<1.$$

Therefore, $1+\vartheta \in \mathcal{P}_u$. Hence, the function $G_1 \in \mathcal{F}(u,\eta)$.

Example 2. *The function $G_2 : \mathbb{E} \longrightarrow \mathbb{C}$ is defined by*

$$G_2(\vartheta) = \frac{4}{2\eta+1}\left[\exp\left(\frac{2\eta+1}{4}\vartheta\right)-1\right], \quad \vartheta \in \mathbb{E}.$$

Straightforward computations shows that

$$\frac{2\eta-1}{2\eta+1}+\frac{2}{2\eta+1}\left(1+\frac{\vartheta G''(\vartheta)}{G'(\vartheta)}\right) = 1+\frac{\vartheta}{2}.$$

Since

$$\int_0^{2\pi} \Re\left(1+\frac{\vartheta}{2}\right)dt = \int_0^{2\pi}\left(1+\frac{r\cos t}{2}\right)dt = 2\pi \leq u\pi, 0<r<1.$$

Therefore, $1+\dfrac{\vartheta}{2} \in \mathcal{P}_u$. Hence, the function $G_2 \in \mathcal{F}(u,\eta)$.

Example 3. *The function $G_3 : \mathbb{E} \longrightarrow \mathbb{C}$ is defined by*

$$G_3(\vartheta) = \frac{4}{u(2\eta+1)}\left[\exp\left(\frac{u(2\eta+1)}{4}\vartheta\right)-1\right] \in \mathcal{F}(u,\eta).$$

Example 4. *The function* $G_4 : \mathbb{E} \longrightarrow \mathbb{C}$ *is defined by*

$$G_4(\vartheta) = \frac{8}{u(2\eta+1)}\left[\exp\left(\frac{u(2\eta+1)}{8}\vartheta\right) - 1\right] \in \mathcal{F}(u,\eta).$$

The images of the function G_1 and G_2 under unit disk $\mathbb{E}$ are shown as below in Figure 1.

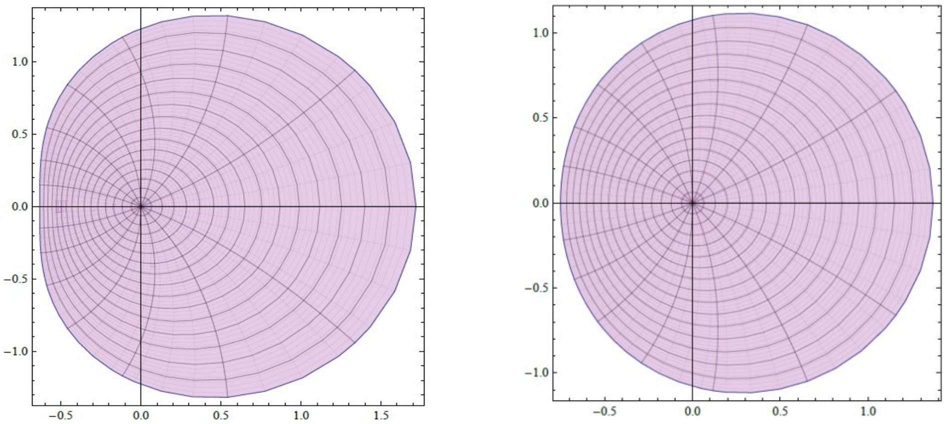

Figure 1. Image of G_1 under $\mathbb{E}$ for $\eta = 0.5$ is shown in the left picture and Image of G_2 under $\mathbb{E}$ for $\eta = 0.7$ is shown in the right picture.

Similarly, the images of the function G_3 and G_4 under unit disk $\mathbb{E}$ are shown as below in Figure 2.

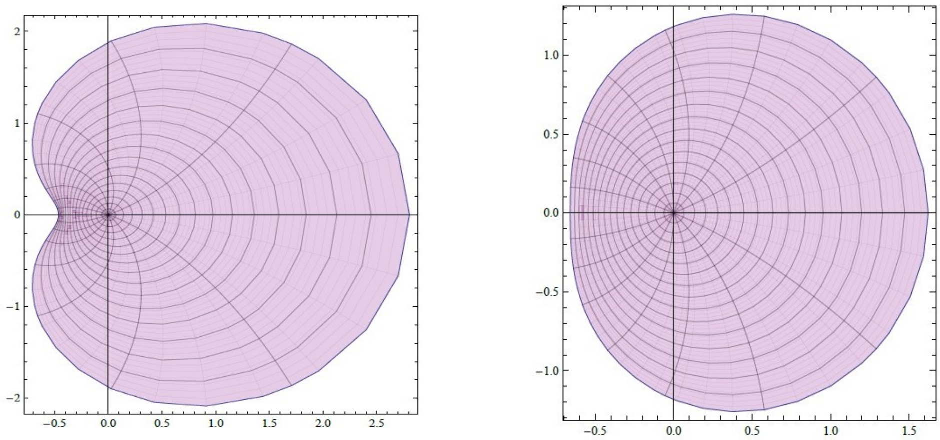

Figure 2. Image of G_3 under $\mathbb{E}$ for $\eta = 0.7,\ u = 0.3$ is shown in the left picture and Image of G_4 under $\mathbb{E}$ for $\eta = 0.7,\ u = 0.3$ is shown in the right picture.

Next, we prove new interesting properties of the class $\mathcal{F}(u,\eta)$, stated as Theorems 1–10. We start with proving an integral representation theorem.

2.1. Integral Representation of $\mathcal{F}(u,\eta)$

Theorem 1. *If* $G \in \mathcal{F}(u,\eta)$, *then*

$$G'(s) = \exp\left[-\frac{2\eta+1}{2}\int_0^{2\pi} \log(1 - se^{it})d\nu(t)\right], \tag{8}$$

where ν is a non-decreasing function with bounded variation in $[0, 2\pi]$ and satisfying

$$\int_0^{2\pi} d\nu(t) = 2 \quad and \quad \int_0^{2\pi} |d\nu(t)| \leq u.$$

Proof. Since $G \in \mathcal{F}(u,\eta)$, there exists an analytic function $\psi(\vartheta)$ belonging to the class $\mathcal{P}_u$ such that

$$\frac{2\eta - 1}{2\eta + 1} + \frac{2}{2\eta + 1}\left(1 + \frac{\vartheta G''(\vartheta)}{G'(\vartheta)}\right) = \psi(\vartheta). \tag{9}$$

Equation (9) can be written as

$$\frac{G''(\vartheta)}{G'(\vartheta)} = \frac{2\eta + 1}{2}\left(\frac{\psi(\vartheta) - 1}{\vartheta}\right). \tag{10}$$

Since $\psi \in \mathcal{P}_u$ by the representation theorem given by Paatero [10], there exists a non-decreasing function ν with bounded variation in $[0, 2\pi]$ and satisfying

$$\int_0^{2\pi} d\nu(t) = 2 \quad and \quad \int_0^{2\pi} |d\nu(t)| \leq u$$

such that

$$\psi(\vartheta) = \frac{1}{2}\int_0^{2\pi} \frac{1 + \vartheta e^{-it}}{1 - \vartheta e^{-it}} d\nu(t).$$

Therefore,

$$\int_0^{\vartheta} \frac{\psi(z) - 1}{z} dz = -\int_0^{2\pi} \log(1 - \vartheta e^{-it}) d\nu(t). \tag{11}$$

From (10) and (11), we obtain (8). The proof of Theorem 1 is hence completed. □

2.2. Relation between $\mathcal{F}(u,\eta)$ and $\mathcal{V}_u$

Theorem 2. *Let $u \geq 2$ and $\frac{1}{2} \leq \eta \leq 1$. A function $G \in \mathcal{F}(u,\eta)$ if $\exists$ a function $g \in \mathcal{V}_u$ such that*

$$G'(\vartheta) = [g'(\vartheta)]^{\frac{2\eta + 1}{2}}. \tag{12}$$

Proof. Since $G \in \mathcal{F}(u,\eta)$, there exists an analytic function $\psi(\vartheta)$ belonging to the class $\mathcal{P}_u$ such that

$$\frac{2\eta - 1}{2\eta + 1} + \frac{2}{2\eta + 1}\left(1 + \frac{\vartheta G''(\vartheta)}{G'(\vartheta)}\right) = \psi(\vartheta). \tag{13}$$

Since, $\psi \in \mathcal{P}_u$, there exists $g \in \mathcal{V}_u$ such that

$$\psi(\vartheta) = 1 + \frac{\vartheta g''(\vartheta)}{g'(\vartheta)}. \tag{14}$$

From (13) and (14), we obtain

$$\frac{\vartheta G''(\vartheta)}{G'(\vartheta)} = \frac{2\eta + 1}{2}\frac{g''(\vartheta)}{g'(\vartheta)}. \tag{15}$$

Upon integrating (15), we obtain (12). The proof of Theorem 2 is thus completed. □

We know that a function $g \in \mathcal{V}_u$ if and only if $\vartheta g' \in \mathcal{R}_u$. Hence, $g \in \mathcal{R}_u$ if and only if

$$h(\vartheta) = \int_0^{\vartheta} \frac{g(z)}{z} dz \in \mathcal{V}_u \iff h'(\vartheta) = \frac{g(\vartheta)}{\vartheta} \in \mathcal{V}_u. \tag{16}$$

From Theorem 2 and (16), we obtain the following result.

2.3. Relation between $\mathcal{F}(u,\eta)$ and $\mathcal{R}_u$

Theorem 3. *Let $u \geq 2$ and $\frac{1}{2} \leq \eta \leq 1$. A function $G \in \mathcal{F}(u,\eta)$ if $\exists$ a function $g \in \mathcal{R}_u$ such that*

$$G'(\vartheta) = \left[\frac{g(\vartheta)}{\vartheta}\right]^{\dfrac{2\eta+1}{2}}. \tag{17}$$

2.4. Relation between $\mathcal{F}(u,\eta)$ and $\mathcal{K}$

Theorem 4. *Let $u \geq 2$ and $\frac{1}{2} \leq \eta \leq 1$. A function $G \in \mathcal{F}(u,\eta)$ if $\exists$ functions $g, h \in \mathcal{K}$ such that*

$$G'(\vartheta) = \left\{\frac{[g'(\vartheta)]^{\frac{u+2}{4}}}{[h'(\vartheta)]^{\frac{u-2}{4}}}\right\}^{\dfrac{2\eta+1}{2}}. \tag{18}$$

Proof. Since $G \in \mathcal{F}(u,\eta)$,$\exists$ is an analytic function $\psi(\vartheta)$ that belongs to the class $\mathcal{P}_u$ such that

$$\frac{2\eta-1}{2\eta+1} + \frac{2}{2\eta+1}\left(1 + \frac{\vartheta G''(\vartheta)}{G'(\vartheta)}\right) = \psi(\vartheta). \tag{19}$$

Since $\psi \in \mathcal{P}_u$, from Lemma 1, then $\exists$ $\psi_1, \psi_2 \in \mathcal{P}$ such that

$$\psi(\vartheta) = \frac{u+2}{4}\psi_1(\vartheta) - \frac{u-2}{4}\psi_2(\vartheta). \tag{20}$$

Since $\psi_1, \psi_2 \in \mathcal{P}$, there exist functions $g, h \in \mathcal{K}$ such that

$$\psi_1(\vartheta) = 1 + \frac{\vartheta g''(\vartheta)}{g'(\vartheta)} \quad \textit{and} \quad \psi_2(\vartheta) = 1 + \frac{\vartheta h''(\vartheta)}{h'(\vartheta)}. \tag{21}$$

Hence, from (19)–(21), we obtain

$$\frac{G''(\vartheta)}{G'(\vartheta)} = \frac{2\eta+1}{2}\left[\frac{u+2}{4}\frac{g''(\vartheta)}{g'(\vartheta)} - \frac{u-2}{4}\frac{h''(\vartheta)}{h'(\vartheta)}\right]. \tag{22}$$

Upon Integrating (22), we obtain (18). This completes the proof of Theorem 4. □

Since we know that a function $h \in \mathcal{K}$ if and only if $\vartheta h' \in \mathcal{S}^*$ and $h \in \mathcal{S}^* \Leftrightarrow g = \frac{h(\vartheta)}{\vartheta} \in \mathcal{K}$, using Theorem 4, we have the following result.

2.5. Relation between $\mathcal{F}(u,\eta)$ and $\mathcal{S}^*$

Theorem 5. *For $u \geq 2$ and $\frac{1}{2} \leq \eta \leq 1$. A function $G \in \mathcal{F}(u,\eta)$ if $\exists$ functions $g, h \in \mathcal{S}^*$ such that*

$$G'(\vartheta) = \vartheta^{\frac{2\eta+1}{2}}\left\{\frac{[g(\vartheta)]^{\frac{u+2}{4}}}{[h(\vartheta)]^{\frac{u-2}{4}}}\right\}^{\dfrac{2\eta+1}{2}}. \tag{23}$$

Based on Equations (2)–(4), we have the following results, and they are stated by omitting the proof.

Theorem 6. *Let $u \geq 2$, $b \geq 1$ and $\frac{1}{2} \leq \eta \leq 1$. A function $G \in \mathcal{F}(u,\eta)$ if*
(i) $\exists$ a function $h \in \mathcal{V}_u(\lambda)$ such that

$$G'(\vartheta) = [h'(\vartheta)]^{\dfrac{2\eta+1}{2(1-\lambda)}}.$$

(ii) $\exists$ *a function* $g \in \mathcal{R}_u(\lambda)$ *such that*

$$G'(\vartheta) = \left[\frac{g(\vartheta)}{\vartheta}\right]^{\frac{2\eta+1}{2(1-\lambda)}}.$$

(iii) $\exists$ *a function* $g_b \in \mathcal{V}_u(b)$ *such that*

$$G'(\vartheta) = \left[g_b'(\vartheta)\right]^{\frac{2\eta+1}{2b}}.$$

Theorem 7. *Let* $u \geq 2$ *and* $\frac{1}{2} \leq \eta \leq 1$*. If a function* $G \in \mathcal{F}(u,\eta)$*, then for* $|\vartheta| = \rho < 1$

$$\left[\frac{(1-\rho)^{\frac{u-2}{2}}}{(1+\rho)^{\frac{u+2}{2}}}\right]^{\frac{2\eta+1}{2}} \leq |G'(\vartheta)| \leq \left[\frac{(1+\rho)^{\frac{u-2}{2}}}{(1-\rho)^{\frac{u+2}{2}}}\right]^{\frac{2\eta+1}{2}}. \tag{24}$$

Proof. Since $G \in \mathcal{F}(u,\eta)$, then from Theorem 1, we have

$$G'(\vartheta) = \exp\left[-\frac{2\eta+1}{2}\int_0^{2\pi}\log(1-\vartheta e^{-it})d\nu(t)\right]$$

where ν is a non-decreasing function with bounded variation in $[0,2\pi]$ and satisfying

$$\int_0^{2\pi} d\nu(t) = 2 \quad \textit{and} \quad \int_0^{2\pi}|d\nu(t)| \leq u.$$

Let us take $\vartheta = \rho e^{it}$. Then,

$$|G'(\rho e^{it})| \leq \exp\left[-\frac{2\eta+1}{2}\int_0^{2\pi}\log|1-\vartheta e^{-it}|d\nu(t)\right].$$

Since $\nu(t)$ is a non-decreasing functions with bounded variation in $[0,2\pi]$, we can write $\nu(t) = \nu_1(t) - \nu_2(t)$, where both $\nu_1(t)$ and $\nu_2(t)$ are non-decreasing functions with bounded variation in $[0,2\pi]$ and satisfying

$$\int_0^{2\pi} d\nu_1(t) \leq \frac{u+2}{2} \quad \textit{and} \quad \int_0^{2\pi} d\nu_2(t) \leq \frac{u-2}{2}.$$

Here, we can write

$$\begin{aligned}
&-\frac{2\eta+1}{2}\int_0^{2\pi}\log|1-\vartheta e^{-it}|d\nu(t) \\
&\qquad = \frac{2\eta+1}{2}\left[\int_0^{2\pi}\log|1-\vartheta e^{-it}|d\nu_2(t) - \int_0^{2\pi}\log|1-\vartheta e^{-it}|d\nu_1(t)\right] \\
&\qquad \leq \frac{u-2}{2}\log(1+\rho)^{\frac{2\eta+1}{2}} - \frac{u+2}{2}\log(1-\rho)^{\frac{2\eta+1}{2}} \\
&\qquad = \log\left[\frac{(1+\rho)^{\frac{u-2}{2}}}{(1-\rho)^{\frac{u+2}{2}}}\right]^{\frac{2\eta+1}{2}}.
\end{aligned}$$

To prove the lower bound, it is sufficient to show that

$$-\frac{2\eta+1}{2}\int_0^{2\pi}\log|1-se^{it}|d\nu(t) \geq \log\left[\frac{(1-\rho)^{\frac{u-2}{2}}}{(1+\rho)^{\frac{u+2}{2}}}\right]^{\frac{2\eta+1}{2}}.$$

If $\int_0^{2\pi} |d\nu(t)| = u$, then $\int_0^{2\pi} |d\nu_1(t)| \leq \frac{u+2}{2}$ and $\int_0^{2\pi} |d\nu_2(t)| \leq \frac{u-2}{2}$. Therefore, we have

$$\begin{aligned}-\frac{2\eta+1}{2}\int_0^{2\pi} \log|1-\vartheta e^{-it}|d\nu(t) &\geq \frac{u-2}{2}\log(1-\rho)^{\frac{2\eta+1}{2}} - \frac{u+2}{2}\log(1+\rho)^{\frac{2\eta+1}{2}} \\ &= \log\left[\frac{(1-\rho)^{\frac{u-2}{2}}}{(1+\rho)^{\frac{u+2}{2}}}\right]^{\dfrac{2\eta+1}{2}}.\end{aligned}$$

Hence, we obtain

$$|G'(\rho e^{it})| \geq \log\left[\frac{(1-\rho)^{\frac{u-2}{2}}}{(1+\rho)^{\frac{u+2}{2}}}\right]^{\dfrac{2\eta+1}{2}}.$$

The proof of Theorem 7 is thus finished. □

Theorem 8. *If a function $G(\vartheta)$ given in the form (1) belongs to the class $\mathcal{F}(u,\eta)$, then*

$$|\xi_2| \leq \frac{u(2\eta+1)}{4}, \tag{25}$$

$$|\xi_3| \leq \frac{(2\eta+1)[u^2(2\eta+1)+4]}{24} \tag{26}$$

and

$$|\xi_4| \leq \frac{(2\eta+1)[u^3(4\eta^2+16\eta+7)+u(48\eta+104)]}{768}. \tag{27}$$

Proof. Since $G \in \mathcal{F}(u,\eta)$, then from Theorem 2 $\exists\, g \in \mathcal{V}_u$ such that

$$G'(\vartheta) = [g'(\vartheta)]^{\dfrac{2\eta+1}{2}} \tag{28}$$

where

$$g(\vartheta) = \vartheta + a_2\vartheta^2 + a_3\vartheta^3 + a_4\vartheta^4 + \cdots. \tag{29}$$

Hence, from (28) and (29), we obtain

$$2\xi_2 = (2\eta+1)a_2, \tag{30}$$

$$3\xi_3 = \frac{3(2\eta+1)}{2}a_3 + \frac{(4\eta^2-1)}{2}a_2^2 \tag{31}$$

and

$$4\xi_4 = 2(2\eta+1)a_4 + \frac{3(4\eta^2-1)}{4}a_2a_3 + \frac{(8\eta^3-12\eta^2-2\eta+3)}{6}a_2^3. \tag{32}$$

Using (5) in (30), (31) and (6) in (32), we obtain (25)–(27), respectively, which essentially completes the proof of Theorem 8. □

Remark 2. *(i) For $\eta = \dfrac{1}{2}$, Theorem 8 verifies the bounds of $|\xi_2|$ and $|\xi_3|$ obtained by Letho [15].*

(ii) For $\eta = \dfrac{1}{2}$, Theorem 8 verifies the bound of $|\xi_4|$ obtained by Schiffer and Tammi [17].

Theorem 9. *If $G \in \mathcal{F}(u,\eta)$ then the function $G_d(\vartheta)$ defined by*

$$G'_\beta(\vartheta) = \frac{G'\left(\dfrac{\vartheta + d}{1 + \bar{d}\vartheta}\right)}{[G'(d)]^{\frac{2\eta+1}{2}}(1 + \bar{d}s)^{2\eta+1}} \tag{33}$$

also belongs to $\mathcal{F}(u,\eta)$.

Proof. Since $G \in \mathcal{F}(u,\eta)$, then from Theorem 2, $\exists\, h \in \mathcal{V}_u$ such that

$$G'(\vartheta) = [h'(\vartheta)]^{\dfrac{2\eta+1}{2}}. \tag{34}$$

Robertson [27] showed that if $h \in \mathcal{V}_u$, then $H(\vartheta)$, defined by

$$H(\vartheta) = \frac{h\left(\dfrac{\vartheta + d}{1 + \bar{d}\vartheta}\right) - h(d)}{h'(d)(1 - |d|^2)}, d \in \mathbb{E}$$

which also belongs to $\mathcal{V}_u$. Therefore, we obtain

$$G'_d(\vartheta) = \frac{\left[h\left(\dfrac{\vartheta + d}{1 + \bar{d}\vartheta}\right)\right]^{\frac{2\eta+1}{2}}}{[h'(d)]^{\frac{2\eta+1}{2}}(1 + \bar{d}\vartheta)^{2\eta+1}}. \tag{35}$$

The proof of Theorem 9 is thus completed. □

Theorem 10. *If $G \in \mathcal{F}(u,\eta)$ and $\dfrac{u(2\eta+1)}{2} < 1$, then G is univalent in $\mathbb{E}$ and*

$$\left|\frac{\vartheta G''(\vartheta)}{G'(\vartheta)} - \frac{|\vartheta|^2(2\eta+1)}{1 - |\vartheta|^2}\right| \le \frac{u(2\eta+1)|\vartheta|}{2(1 - |\vartheta|^2)}. \tag{36}$$

Proof. If $G \in \mathcal{F}(u,\eta)$, then $G_d(\vartheta)$ given in (33) belongs to $\mathcal{F}(u,\eta)$. By differentiating (33) with respect to ϑ and substituting $\vartheta = 0$, we obtain

$$G''_d(0) = \frac{G''(d)}{G'(d)}(1 - |d|^2) - \bar{d}(2\eta+1).$$

Therefore,

$$\xi_2 = \frac{G''_d(0)}{2} = \frac{1}{2}\left[\frac{G''(d)}{G'(d)}(1 - |d|^2) - \bar{d}(2\eta+1)\right].$$

By using the bound of ξ_2 given in (25) and replacing d by ϑ, we have

$$\left|\frac{G''(\vartheta)}{G'(\vartheta)}(1 - |\vartheta|^2) - \bar{\vartheta}(2\eta+1)\right| \le |\xi_2| \le \frac{u(2\eta+1)}{2}. \tag{37}$$

Equation (37) can be rewritten as

$$\left|\frac{\vartheta G''(\vartheta)}{G'(\vartheta)}(1 - |\vartheta|^2) - |\vartheta|^2(2\eta+1)\right| \le \frac{u|\vartheta|(2\eta+1)}{2}. \tag{38}$$

If $\dfrac{u(2\eta+1)}{2} < 1$, then according to Ahlfors [28] univalence criterion, G is univalent in $\mathbb{E}$. Equation (38) gives (36). The proof of Theorem 10 is hence finished. □

Theorem 11. *If $G \in \mathcal{F}(u,\eta)$, then for $0 \le \nu_1 < \nu_2 \le 2\pi$, $\vartheta = \rho e^{it}$ we have*

$$\int_{\nu_1}^{\nu_2} \Re\left(1 + \frac{\vartheta G''(\vartheta)}{G'(\vartheta)}\right) d\nu > -\frac{2\eta+1}{2}\left(\frac{u}{2} - 1\right)\pi. \tag{39}$$

Proof. Since $G \in \mathcal{F}(u,\eta)$, then from Theorem 2, $\exists\, h \in \mathcal{V}_u$ such that

$$G'(\vartheta) = [h'(\vartheta)]^{\frac{2\eta+1}{2}}. \tag{40}$$

Brannan [29] showed that a function if $h \in \mathcal{V}_u$ and $0 \le \nu_1 < \nu_2 \le 2\pi$, then

$$\int_{\nu_1}^{\nu_2} \Re\left(1 + \frac{\vartheta h''(\vartheta)}{h'(\vartheta)}\right) d\nu > -\left(\frac{u}{2} - 1\right)\pi. \tag{41}$$

Equations (40) and (41) gives (39). □

3. Ozaki-Type Bi-Close-to-Convex Functions with Bounded Boundary Rotation

Definition 3. *Suppose $0 \le \lambda < 1$, $\frac{1}{2} \le \eta \le 1$, and $u \ge 2$. A function G given by (1) is called as Ozaki-type bi-close-to-convex functions with bounded boundary rotation of order λ if the conditions*

$$\frac{2\eta-1}{2\eta+1} + \frac{2}{2\eta+1}\left(1 + \frac{\vartheta G''(\vartheta)}{G'(\vartheta)}\right) \in \mathcal{P}_u(\lambda)$$

and

$$\frac{2\eta-1}{2\eta+1} + \frac{2}{2\eta+1}\left(1 + \frac{\upsilon \Phi''(\upsilon)}{\Phi'(\upsilon)}\right) \in \mathcal{P}_u(\lambda)$$

are satisfied. The family of all Ozaki-type bi-close-to-convex functions with bounded boundary rotation of order λ is denoted by $\mathcal{F}_\Sigma^\lambda(u,\eta)$.

To prove the main results of this section, we need the following lemma.

Lemma 2. *[30] Let $0 \le \lambda < 1$. If a function $\psi(\vartheta) = 1 + \sum_{m=1}^{\infty} \psi_m \vartheta^m$ belongs to the class $\mathcal{P}_u$, then*

$$|\psi_m| \le u(1-\lambda).$$

Theorem 12. *Suppose $0 \le \lambda < 1$, $\frac{1}{2} \le \eta \le 1$, and $u \ge 2$. If a function G given by (1) is in the class $\mathcal{F}_\Sigma^\lambda(u,\eta)$, then*

$$|\xi_2| \le \sqrt{\frac{u(1-\lambda)(2\eta+1)}{4}}, \tag{42}$$

$$|\xi_3| \le \frac{u(1-\lambda)(2\eta+1)}{4} \tag{43}$$

and

$$|\xi_3 - \Lambda\xi_2^2| \le \begin{cases} \dfrac{u(1-\lambda)(2\eta+1)(1-\Lambda)}{4} & \text{for } \Lambda < \dfrac{2}{3}, \\ \dfrac{u(1-\lambda)(2\eta+1)}{12} & \text{for } \dfrac{2}{3} \le \mu \le \dfrac{4}{3}, \\ \dfrac{u(1-\lambda)(2\eta+1)(\Lambda-1)}{4} & \text{for } \Lambda > \dfrac{4}{3} \end{cases} \tag{44}$$

where Λ is a real number.

Proof. Let the function $G \in \mathcal{F}_{\Sigma}^{\lambda}(u,\eta)$. Then, there exist $y(\vartheta)$ and $x(v)$ in $\mathcal{P}_u(\lambda)$ such that

$$\frac{2\eta-1}{2\eta+1}+\frac{2}{2\eta+1}\left(1+\frac{\vartheta G''(\vartheta)}{G'(\vartheta)}\right)=y(\vartheta) \tag{45}$$

and

$$\frac{2\eta-1}{2\eta+1}+\frac{2}{2\eta+1}\left(1+\frac{v\Phi''(v)}{\Phi'(v)}\right)=x(v) \tag{46}$$

where

$$y(\vartheta)=1+y_1\vartheta+y_2\vartheta^2+y_3\vartheta^3+\cdots \tag{47}$$

and

$$x(v)=1+x_1v+x_2v^2+x_3v^3+\cdots. \tag{48}$$

From Equations (45)–(48), we obtain

$$\frac{4\xi_2}{2\eta+1}=y_1, \tag{49}$$

$$\frac{12\xi_3}{2\eta+1}-\frac{8\xi_2^2}{2\eta+1}=y_2, \tag{50}$$

$$-\frac{4\xi_2}{2\eta+1}=x_1, \tag{51}$$

and

$$\frac{16\xi_2^2}{2\eta+1}-\frac{12\xi_3}{2\eta+1}=x_2. \tag{52}$$

Hence from (49) and (51), we obtain $y_1+x_1=0$. Again, from (50) and (52), we obtain

$$\frac{8\xi_2^2}{2\eta+1}=y_2+x_2. \tag{53}$$

An application of Lemma 2 in (53) gives

$$|\xi_2|^2\le\frac{u(1-\lambda)(2\eta+1)}{4}. \tag{54}$$

Hence, (54) gives (42). Here, again from (50) and (52) and by applying (53), we obtain

$$\frac{12\xi_3}{2\eta+1}=2y_2+x_2. \tag{55}$$

An application of Lemma 2 in (55) gives (43). Hence, for any $\Lambda\in\mathbb{R}$ and by using (53) and (55), we have

$$\xi_3-\lambda\xi_2^2=\frac{2\eta+1}{24}[(4-3\Lambda)y_2+(2-3\Lambda)x_2]. \tag{56}$$

Here, using triangle inequality and an application of Lemma 2 in (56) implies

$$|\xi_3-\Lambda\xi_2^2|\le\frac{u(1-\lambda)(2\eta+1)}{24}[|4-3\Lambda|+|2-3\Lambda|]. \tag{57}$$

Hence, (57) gives (44). The proof of Theorem 12 is thus finished. □

For the choice $u=2$, the class $\mathcal{F}_{\Sigma}^{\lambda}(2,\eta)$ reduces to the class $\mathcal{F}_{o,\Sigma}^{\eta}(\lambda)$. Hence, we obtain the following result for the functions belonging to the class $\mathcal{F}_{o,\Sigma}^{\eta}(\lambda)$, which verifies the bound of $|\xi_2|$ and $|\xi_3|$ obtained by Tezelcı and Sümer Eker [6].

Corollary 1. *Suppose* $0 \leq \lambda < 1$ *and* $\frac{1}{2} \leq \eta \leq 1$. *If a function G given by* (1) *is in the class* $\mathcal{F}^{\eta}_{o,\Sigma}(\lambda)$, *then*

$$|\xi_2| \leq \sqrt{\frac{(1-\lambda)(2\eta+1)}{2}}, \tag{58}$$

$$|\xi_3| \leq \frac{(1-\lambda)(2\eta+1)}{2} \tag{59}$$

and

$$|\xi_3 - \Lambda\xi_2^2| \leq \begin{cases} \frac{(1-\lambda)(2\eta+1)(1-\Lambda)}{2} & \text{for } \Lambda < \frac{2}{3}, \\ \frac{(1-\lambda)(2\eta+1)}{6} & \text{for } \frac{2}{3} \leq \Lambda \leq \frac{4}{3}, \\ \frac{(1-\lambda)(2\eta+1)(\Lambda-1)}{2} & \text{for } \Lambda > \frac{4}{3}. \end{cases} \tag{60}$$

Let $\eta = \frac{1}{2}$. Then, the class $\mathcal{F}^{\lambda}_{\Sigma}\left(u, \frac{1}{2}\right)$ reduces to the class $\mathcal{C}_{\Sigma}[u,\lambda]$, which denotes the class of all bi-convex functions with the bounded boundary rotation of order λ. Hence, we obtain the following result for the functions belonging to the class $\mathcal{C}_{\Sigma}[u,\lambda]$, which verifies the bound obtained by (Li et al. [31], Corollary 3.2).

Corollary 2. *Suppose* $0 \leq \lambda < 1$ *and* $u \geq 2$. *If a function G given by* (1) *belongs to the class* $\mathcal{C}_{\Sigma}[u,\lambda]$, *then*

$$|\xi_2| \leq \sqrt{\frac{u(1-\lambda)}{2}}, \tag{61}$$

$$|\xi_3| \leq \frac{u(1-\lambda)}{2} \tag{62}$$

and

$$|\xi_3 - \Lambda\xi_2^2| \leq \begin{cases} \frac{u(1-\lambda)(1-\Lambda)}{2} & \text{for } \Lambda < \frac{2}{3}, \\ \frac{u(1-\lambda)}{6} & \text{for } \frac{2}{3} \leq \Lambda \leq \frac{4}{3}, \\ \frac{u(1-\lambda)(\Lambda-1)}{2} & \text{for } \Lambda > \frac{4}{3}. \end{cases} \tag{63}$$

Remark 3. *Corollary 2 verifies the results obtained by (Sharma et al. [32] Corollary 7).*

4. Concluding Remarks and Observations

In the current article, we introduced a new subclass of univalent functions $\mathcal{F}(u,\eta)$ with bounded boundary rotation. Many interesting examples are constructed for the class $\mathcal{F}(u,\eta)$. Interesting connections between the class $\mathcal{F}(u,\eta)$ and the familiar classes of starlike, convex and convex functions of bounded boundary rotation and starlike functions of bounded radius rotation are obtained. Following the interesting connection, the first three coefficient bounds for the new subclass $\mathcal{F}(u,\eta)$ were derived. Furthermore, the authors also introduced a new subclass of bi-univalent functions $\mathcal{F}_{o,\Sigma}(u,\lambda)$ associated with bounded boundary rotation. For the new class $\mathcal{F}_{o,\Sigma}(u,\lambda)$, the authors obtained new initial two coefficient estimates. Apart from the new interesting coefficient estimates, the established coefficient estimates also generalize the earlier existing results.

Finally, the study considered in this article can be extended by taking different types of convolution operators existing in the literature. Also, similar types of results can be investigated for interesting special functions.

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References

1. Kaplan, W. Close-to-convex schlicht functions. *Mich. Math. J.* **1952**, *1*, 169–185. [CrossRef]
2. Ozaki, S. On the theory of multivalent functions. II. *Sci. Rep. Tokyo Bunrika Daigaku. Sect. A* **1941**, *4*, 45–87.
3. Umezawa, T. Analytic functions convex in one direction. *J. Math. Soc. Jpn.* **1952**, *4*, 194–202. [CrossRef]
4. Singh, R.; Singh, S. Some sufficient conditions for univalence and starlikeness. *Colloq. Math.* **1982**, *47*, 309–314. [CrossRef]
5. Srivastava, H.M.; Rath, B.; Sanjay Kumar, K.; Vamshee Krishna, D. Some sharp bounds of the third-order Hankel determinant for the inverses of the Ozaki type close-to-convex functions. *Bull. Sci. Math.* **2023**, *191* , 103381. [CrossRef]
6. Tezelcı, M.; Sümer Eker, S. On strongly Ozaki bi-close-to-convex functions. *Turk. J. Math.* **2019**, *43*, 862–870. [CrossRef]
7. Vijaya, K.; Murugusundaramoorthy, G.; Breaz, D.; Oros, G.I.; El-Deeb, S.M. Ozaki-Type Bi-Close-to-Convex and Bi-Concave Functions Involving a Modified Caputo's Fractional Operator Linked with a Three-Leaf Function. *Fractal Fract.* **2024**, *8*, 220. [CrossRef]
8. Thomas, D.K.; Tuneski, N.; Vasudevarao, A. *Univalent Functions*; De Gruyter Studies in Mathematics; De Gruyter: Berlin, Germany, 2018; Volume 69.
9. Vasudevarao, A.; Thomas, D.K.; Tuneski, N. On Ozaki close-to-convex functions. *Bull. Aust. Math. Soc.* **2019**, *99*, 89–100.
10. Paatero, V. Über die konforme Abbildung mehrblä ttriger Gebiete von beschränkter Randdrehung. *Ann. Acad. Sci. Fenn. Ser. AI Math.-Phys.* **1933**, *37*, 9p.
11. Pinchuk, B. Functions of bounded boundary rotation. *Isr. J. Math.* **1971**, *10*, 6–16. [CrossRef]
12. Padmanabhan, K.S.; Parvatham, R. Properties of a class of functions with bounded boundary rotation. *Ann. Polon. Math.* **1976**, *31*, 311–323. [CrossRef]
13. Umarani, P.G. Functions of bounded boundary rotation of complex order. *Math. Balk.* **1989**, *3*, 34–43.
14. Tammi, O. On the maximalization of the coefficients of schlicht and related functions. *Ann. Acad. Sci. Fenn. Ser. AI Math.-Phys.* **1952**, *114*, 51p.
15. Lehto, O.E. On the distortion of conformal mappings with bounded boundary rotation. *Ann. Acad. Sci. Fenn. Ser. A I Math.-Phys.* **1952**, *124*, 14.
16. Thomas, D.K. On the coefficients of functions with bounded boundary rotation. *Proc. Am. Math. Soc.* **1972**, *36*, 123–129. [CrossRef]
17. Schiffer, M.M.; Tammi, O. On the fourth coefficient of univalent functions with bounded boundary rotation. *Ann. Acad. Sci. Fenn. Ser. AI* **1967**, *396*, 1–26. [CrossRef]
18. Brannan, D.A.; Clunie, J.G. *Aspects of Contemporary Complex Analysis*; Academic Press, Inc.: London, UK, 1980.
19. Koepf, W.A. Coefficients of symmetric functions of bounded boundary rotation. *Proc. Am. Math. Soc.* **1989**, *105*, 324–329. [CrossRef]
20. Leach, R. On Odd Functions of bounded boundary rotation. *Can. J. Math.* **1974**, *26*, 551–564. [CrossRef]
21. Leach, R. Coefficients of symmetric functions of bounded boundary rotation. *Can. J. Math.* **1974**, *26*, 1351–1355. [CrossRef]
22. Duren, P.L. Univalent functions. In *Grundlehren der Mathematischen Wissenschaften*; Springer: New York, NY, USA, 1983; Volume 259.
23. Lewin, M. On a coefficient problem for bi-univalent functions. *Proc. Am. Math. Soc.* **1967**, *18*, 63–68. [CrossRef]
24. Brannan, D.A.; Taha, T.S. *On Some Classes of Bi-Univalent Functions, in Mathematical Analysis and Its Applications KUWAIT*; KFAS Proceedings Series; Pergamon: Oxford, UK, 1985; Volume 3, pp. 53–60.
25. Srivastava, H.M.; Mishra, A.K.; Gochhayat, P. Certain subclasses of analytic and bi-univalent functions. *Appl. Math. Lett.* **2010** *23*, 1188–1192. [CrossRef]
26. Robertson, M.S. On the theory of univalent functions. *Ann. Math.* **1936**, *37*, 374–408. [CrossRef]
27. Robertson, M.S. Coefficients of functions with bounded boundary rotation. *Can. J. Math.* **1969**, *21*, 1477–1482. [CrossRef]
28. Ahlfors, L.V. Sufficient Conditions for Quasiconformal Extension. *Ann. Math. Stud.* **1973**, *79*, 23–29.
29. Brannan, D.A. On functions of bounded boundary rotation I. *Proc. Edinb. Math. Soc.* **1969**, *16*, 339–347. [CrossRef]
30. Alkahtani, B.S.T.; Goswami, P.; Bulboacă, T. Estimate for initial MacLaurin coefficients of certain subclasses of bi-univalent functions. *Miskolc Math. Notes* **2016**, *17*, 739–748. [CrossRef]

31. Li, Y.; Vijaya, K.; Murugusundaramoorthy, G.; Tang, H. On new subclasses of bi-starlike functions with bounded boundary rotation. *AIMS Math.* **2020**, *5*, 3346–3356. [CrossRef]
32. Sharma, P.; Sivasubramanian, S.; Cho, N.E. Initial coefficient bounds for certain new subclasses of bi-univalent functions with bounded boundary rotation. *AIMS Math.* **2023**, *8*, 29535–29554. [CrossRef]