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Convolution Theorem for (p, q)-Gamma Integral Transforms and Their Application to Some Special Functions

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Abstract: This article introduces (p, q)-analogs of the gamma integral operator and discusses their expansion to power functions, (p, q)-exponential functions, and (p, q)-trigonometric functions. Additionally, it validates other findings concerning (p, q)-analogs of the gamma integrals to unit step functions as well as first- and second-order (p, q)-differential operators. In addition, it presents a pair of (p, q)-convolution products for the specified (p, q)-analogs and establishes two (p, q)-convolution theorems.

Keywords: q-derivative; (p, q)-convolution theorem; (p, q)-trigonometric functions; (p, q)-exponential functions; (p, q)-differential operators

MSC: 05A30; 26D10; 26D15; 26A33

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1. Introduction

Quantum calculus, also referred to as q-calculus, is a branch of calculus that focuses on derivatives without limits [1]. It attracts a lot of academics as it provides a crucial connection between mathematics and physics. In the literature, there are many scientific disciplines that have demonstrated abroad a variety of applications of quantum calculus in the theory of numbers, orthogonal polynomials, combinatorics, relativity theory, and mechanics [2–5], while a number of advancements involving polynomials and q-hypergeometric functions, often employed in number theory and partitioning, began to find practical uses in a range of different scientific areas [6–16]. The generalized q-Apostol–Bernoulli, q-Apostol–Euler, and q-Apostol–Genocchi polynomials in two variables are given in [17], whereas the q-Bernoulli, q-Euler, and q-Genocchi polynomials are examined in [18]. In addition, the theory under concern has also been applied to vector spaces, combinatorial analysis, particle physics, lie theory, nonlinear electric circuit theory, heat conduction theory, mechanical engineering, statistics, and cosmology [19,20]. Anyhow, the significant advancement in the theory of quantum calculus is a creation of the q-analog [3,17,18,21,22]

$$d_q\vartheta(\xi) = \vartheta(\xi) - \vartheta(q\xi),$$

and the q-derivative [1]

$$(D_q\varphi)(\xi) := \frac{d_q\varphi(\xi)}{d_q\xi} := \frac{\varphi(\xi) - \varphi(qx)}{(1-q)\xi}, \xi \neq 0,$$

of a function ϑ for $0 < q < 1$, which opened the door for more developments in this area.

In an effort to expand the applicability of the q-calculus theory, Chakrabarti and Jagannathan [23] recently created the (p,q)-calculus, an enlarged version of the q-calculus. It is important to understand that the actual quantum calculus cannot be created by simply substituting p for q in the q-calculus. However, when p equals 1, it reduces to q-calculus; while several scholars extensively studied and developed the (p,q)-calculus in [17–19,21,23–38], Sadjang explored many concepts of (p,q)-integration, (p,q)-derivative, (p,q)-Taylor formula, and a fundamental theorem of (p,q)-calculus [34,36,37]. Further research on (p,q)-integral transformations has also been conducted in other research projects. Sadjang [39] looked into a number of features of the (p,q)-analogs of the Laplace transforms and how they were used to the solution of specific (p,q)-difference equations. In addition, he examined (p,q)-difference equations and the (p,q)-analogs of the Laplace transform. Later, a few authors used the (p,q)-Aleph function to create (p,q)-analogs of the Laplace and Sumudu transforms. (p,q)-analogs of Laplace-type integral transformations were developed by Jirakulchaiwong et al. in [40], and their findings were expanded to solve multiple (p,q)-differential equations. Hermite–Hadamard inequalities for continuous convex functions via (p,q)-calculus were studied by Prabseang et al. in [33], while Chakrabarti and Jagannathan [23] looked into a (p,q)-oscillator realization of two-parameter quantum algebras. Readers can check more about this subject by using [20,35,36,39–42].

This study discusses several applications and examines some (p,q)-analogs of the gamma integral operator. It develops several convolution theorems and examines some applications of the (p,q)-analogs of the gamma integrals to some special and elementary functions. A few ideas, concepts, and notations from the (p,q)-calculus theory are presented in Sections 1 and 2. The (p,q)-analogs of the gamma integrals of the (p,q)-exponential functions, the (p,q)-trigonometric functions, and a few (p,q)-power functions of various orders are examined in Section 3, whereas results pertaining to differential operators and unit step functions are established in Section 4. Two pairs of convolution products and associated convolution theorems are discussed in Section 5.

2. Preliminaries, Definitions, and Auxiliary Results

In this section, we go over some common concepts and notations in the (p,q)-calculus [33,34]. Assuming $0 < q < p \leq 1$, we consider q to be a fixed real number. Starting with the concept of the (p,q)-analog $d_{p,q}\varphi(x)$ of the differential of a function φ,

$$d_{p,q}\varphi(x) = \varphi(px) - \varphi(qx), \tag{1}$$

the (p,q)-calculus is introduced. Consequently, we obtain the (p,q)-analog of the derivative of $\varphi(x)$ instantaneously, called (p,q)-derivative,

$$(D_{p,q}\varphi)(x) := \frac{d_{p,q}\varphi(x)}{d_{p,q}x} := \frac{\varphi(px) - \varphi(qx)}{(p-q)x}, x \neq 0, \tag{2}$$

$(D_{p,q}\varphi)(0) = \varphi^{'}(0)$ provided $\varphi^{'}(0)$ exists. If φ is differentiable, then $D_{p,q}\varphi$ approaches $\varphi^{'}$ as both p and q tend to the value 1. The (p,q)-numbers $[m]_{p,q}$ and (p,q)-factorials $[m]_{p,q}!$ are defined by [43]

$$[m]_{p,q} = \frac{p^m - q^m}{p-q} \text{ and } [m]_{p,q}! = \prod_{k=1}^{n}[k]_{p,q}, [0]_{p,q} = 1,$$

respectively. The (p,q)-derivative of the product of two functions φ and g satisfies the following (p,q)-analog

$$D_{p,q}(\varphi(x)g(x)) = \varphi(px)D_{p,q}g(x) + g(qx)D_{p,q}\varphi(x). \tag{3}$$

Conversely, the (p,q)-integrals over the intervals $[0,x]$ and $[0,\infty)$ are, respectively, defined in a series form as [36]

$$\int_0^x \varphi(x)d_{p,q}x = (p-q)x\sum_0^{\infty}\frac{q^k}{p^{k+1}}\varphi\left(x\frac{q^k}{p^{k+1}}\right), \left|\frac{p}{q}\right| > 1, \tag{4}$$

$$\int_0^{\infty} \varphi(x)d_{p,q}x = (p-q)\sum_{-\infty}^{\infty}\frac{q^k}{p^{k+1}}\varphi\left(\frac{q^k}{p^{k+1}}\right), \left|\frac{p}{q}\right| > 1, \tag{5}$$

given that, for any real value x, the sums converge absolutely. In a generic interval $[a,b]$, the (p,q)-integral is given by [24]

$$\int_a^b \varphi(x)d_{p,q}x = \int_0^b \varphi(x)d_{p,q}x - \int_0^a \varphi(x)d_{p,q}x. \tag{6}$$

Alike to the q-integration by parts, the (p,q)-integration by parts is defined by ([44], Proposition 2) as follows:

If φ and g are arbitrary functions, then

$$\int_a^b \varphi(px)D_{p,q}g(x)d_{p,q}x = g(b)\varphi(b) - g(a)\varphi(a) - \int_a^b g(qx)D_{p,q}\varphi(x)d_{p,q}x. \tag{7}$$

Note that $b=\infty$ is allowed.

Hence, due to above statement of ([44], Proposition 2), we write

$$\int_a^{\infty} \varphi(px)D_{p,q}g(x)d_{p,q}x = \lim_{b\longrightarrow\infty}(g(b)\varphi(b) - g(a)\varphi(a)) - \int_a^{\infty} g(qx)D_{p,q}\varphi(x)d_{p,q}x. \tag{8}$$

By putting $p=1$ in (6) and (7), the equations, respectively, reduce to the q-integrations

$$\int_a^b \varphi(x)d_qx = \int_0^b \varphi(x)d_qx - \int_0^a \varphi(x)d_qx \tag{9}$$

and

$$\int_0^b \varphi(x)D_qg(x)d_qx = g(b)\varphi(b) - g(a)\varphi(a) - \int_0^b g(qx)D_q\varphi(x)d_qx. \tag{10}$$

The two types of (p,q)-analogs of the exponential function are defined by [24]

$$E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}}\frac{x^n}{[n]_{p,q}!}\ (x\in\mathbb{C}), \tag{11}$$

and

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\frac{n(n-1)}{2}}\frac{x^n}{[n]_{p,q}!}\ (|x|<1). \tag{12}$$

If we replace $p=1$ in (11) and (12), then we attain the q-exponential functions E_p and e_p, respectively. Moreover, the involved (p,q)-derivatives of the (p,q)-exponential functions are given by [24]

$$D_{p,q}e_{p,q}(nt) = ne_{p,q}(npt) \text{ and } D_{p,q}E_{p,q}(nt) = nE_{p,q}(nqt). \tag{13}$$

Consequently, $D_{p,q}e_{p,q}(t) = e_{p,q}(pt)$ and $D_{p,q}E_{p,q}(t) = E_{p,q}(qt)$. On this basis, the (p,q)-gamma functions of the first and second kinds are, respectively, defined by [31]

$$\Gamma_{p,q}(n) = p^{\frac{n(n-1)}{2}}\int_0^{\infty} t^{n-1}E_{p,q}(-qt)d_{p,q}x \text{ and } \tilde{\Gamma}_{p,q}(n) = q^{\frac{n(n-1)}{2}}\int_0^{\infty} t^{n-1}e_{p,q}(-pt)d_{p,q}x. \tag{14}$$

Indeed, (14) and the integration by parts yield

$$\Gamma_{p,q}(n+1) = \tilde{\Gamma}_{p,q}(n+1) = [n]_{p,q}!. \tag{15}$$

In [24], Sadjang has defined the gamma integral operator for functions of certain exponential growth conditions in the form

$$(G_n\varphi)(v) = \frac{n^n}{\delta^n\Gamma(n)}\int_0^\infty \varphi(\tau)\tau^{n-1}e^{\frac{-n\tau}{v}}d\tau, v\in[0,\infty) \text{ and } n\in\mathbb{N}. \tag{16}$$

Herein, we introduce two (p,q)-analogs for the gamma integral operator as follows.

Definition 1. *Let φ be a function of certain exponential growth conditions, then we define the (p,q)-gamma integral operator of the first kind as*

$$G^1_{n,p,q}(\varphi,v) = A\int_0^\infty \varphi(t)t^{n-1}E_{p,q}\left(\frac{-qnt}{v}\right)d_{p,q}t, \tag{17}$$

where $A = \frac{n^n}{v^n\Gamma_{p,q}(n)}$. Alternatively, under the hypothesis of φ, we introduce the (p,q)-gamma integral operator of the second kind as

$$G^2_{n,p,q}(\varphi,v) = A\int_0^\infty \varphi(t)t^{n-1}e_{p,q}\left(\frac{-pnt}{v}\right)d_{p,q}t, \tag{18}$$

where $A = \frac{n^n}{v^n\Gamma_{p,q}(n)}$, provided the two integrals converge.

We now go over a few properties of the previously listed analogs as follows.

Theorem 1. *Let φ, φ_1, and φ_2 be functions of certain exponential growth conditions. Then, we have*
(i) (Linearity) For real numbers α_1, α_1 we have

$$G^1_{n,p,q}(\alpha_1\varphi_1(t)+\alpha_2\varphi_2(t),v) = \alpha_1 G^1_{n,p,q}(\varphi_1,v)+\alpha_2 G^1_{n,p,q}(\varphi_2,v).$$
$$G^2_{n,p,q}(\alpha_1\varphi_1(t)+\alpha_2\varphi_2(t),v) = \alpha_1 G^2_{n,p,q}(\varphi_1,v)+\alpha_2 G^2_{n,p,q}(\varphi_2,v).$$

(ii) (Scaling Property) For a real number β we have

$$G^1_{n,p,q}(\varphi(\beta t),v) = \frac{1}{\beta^n}G^1_{n,p,q}(\varphi(t),\beta v). \quad G^2_{n,p,q}(\varphi(\beta t),v) = \frac{1}{\beta^n}G^2_{n,p,q}(\varphi(t),\beta v).$$

Proof. The proof of the part (i) follows from the definition of the (p,q)-integrals. To prove (ii) let $z=\beta t \Rightarrow d_{p,q}t = \frac{1}{\beta}d_{p,q}z$. Then, considering Equation (17) and inserting the given substitution inside the integral sign yield

$$\begin{aligned}
G^1_{n,p,q}(\varphi(\beta),v) &= A\int_0^\infty \varphi(\beta t)t^{n-1}E_{p,q}\left(\frac{qnt}{v}\right)d_{p,q}t \\
&= A\int_0^\infty \varphi(z)\frac{z^{n-1}}{\beta^{n-1}}E_{p,q}\left(\frac{qnz}{\beta v}\right)\frac{d_{p,q}z}{\beta} \\
&= \frac{1}{\beta^n}A\int_0^\infty \varphi(z)z^{n-1}E_{p,q}\left(\frac{qnz}{\beta v}\right)d_{p,q}z \\
&= \frac{1}{\beta^n}G^1_{n,p,q}(\varphi(t),\beta v).
\end{aligned}$$

The proof of the second equation is alike to that employed for the first equation. This ends the proof of the theorem. □

3. The $G^1_{n,p,q}$ Analog of Differential Operators and Some Convolution Theorems

In the present section, we discuss the value of $G^1_{n,p,q}$ of (p,q)-difference operators of the first degree and extend our results to the second-degree case. It also presents a subsequent pair of definitions, where two operations are thereby defined for their purposes. The proposed products are used to investigate two convolution theorems of the $G^1_{n,p,q}$ analog.

Theorem 2. *Let $G^1_{n,p,q}$ be the (p,q)-analog defined by (17). Then, we have*

$$G^1_{n,p,q}\left(D_{p,q}\varphi(t),v\right)=\frac{[n]_{p,q}}{q^{n-1}vp^{n-2}}G^1_n(\varphi,pv)-\frac{q^{n-1}[n-1]_{p,q}}{vp^{n-1}}G^1_{n-1,p,q}(\varphi,pv). \tag{19}$$

Proof. To prove the first part we make use of the definition of the analog $G^1_{n,p,q}$ and insert $q^{n-1}\,q^{1-n}$ inside the integral sign to have

$$\begin{aligned} G^1_{n,p,q}\left(D_{p,q}\varphi(t),v\right) &= A\int_0^\infty t^{n-1}(D_{p,q}\varphi)E_{p,q}\left(-q\frac{nt}{v}\right)d_{p,q}t \\ &= A\int_0^\infty t^{n-1}E_{p,q}\left(-q\frac{nt}{v}\right)(D_{p,q}\varphi)d_{p,q}t \\ &= \frac{A}{q^{n-1}}\int_0^\infty (qt)^{n-1}E_{p,q}\left(-q\frac{nt}{v}\right)(D_{p,q}\varphi)d_{p,q}t. \end{aligned}$$

By putting $a=0$ and rewriting Equation (7) in the form

$$\int_0^\infty g(qx)D_{p,q}\varphi(x)d_{p,q}x=\lim_{b\longrightarrow\infty}(g(b)\varphi(b)-g(0)\varphi(0))-\int_0^\infty \varphi(px)D_{p,q}g(x)d_{p,q}x,$$

we obtain

$$\begin{aligned} G^1_{n,p,q}\left(D_{p,q}\varphi(t),v\right) &= \frac{A}{q^{n-1}}\lim_{b\longmapsto\infty}\varphi(t)t^{n-1}E_{p,q}\left(\frac{-nt}{v}\right)\Big|_0^b \\ &\quad -\frac{A}{q^{n-1}}\int_0^\infty \varphi(pt)D_{p,q}\left(t^{n-1}E_{p,q}\left(-\frac{nt}{v}\right)\right)d_{p,q}t. \end{aligned}$$

Hence, the preceding equation reveals that

$$G^1_{n,p,q}\left(D_{p,q}\varphi(t),v\right)=-\frac{A}{q^{n-1}}\int_0^\infty \varphi(pt)D_{p,q}\left(t^{n-1}E_{p,q}\left(-\frac{nt}{v}\right)\right)d_{p,q}t. \tag{20}$$

However, using the idea of the (p,q)-derivative of the exponential function and the (p,q)-derivative of the product of two functions reveal that

$$D_{p,q}\left(t^{n-1}E_{p,q}\left(\frac{-tn}{v}\right)\right)=(tp)^{n-1}D_{p,q}E_{p,q}\left(\frac{-tn}{v}\right)+E_{p,q}\left(\frac{-tqn}{v}\right)D_{p,q}t^{n-1}.$$

Hence, we rewrite the preceding equation in the form

$$D_{p,q}\left(t^{n-1}E_{p,q}\left(\frac{-tn}{v}\right)\right)=(tp)^{n-1}\left(\frac{-n}{v}\right)E_{p,q}\left(\frac{-tqn}{v}\right)+[n-1]_{p,q}t^{n-2}E_{p,q}\left(\frac{-tqn}{v}\right). \tag{21}$$

Therefore, inserting the preceding value of the derivative yields

$$
\begin{aligned}
G^1_{n,p,q}\left(D_{p,q}\varphi(t),v\right) &= \frac{-A}{q^{n-1}}\int_0^\infty \varphi(pt)(tp)^{n-1}\left(\frac{-n}{v}\right)E_{p,q}\left(\frac{-tq[n]_{p,q}}{v}\right)d_{p,q}t \\
&\quad -\frac{A}{q^{n-1}}[n-1]_{p,q}\int_0^\infty \varphi(pt)t^{n-2}E_{p,q}\left(\frac{-tq[n]_{p,q}}{v}\right)d_{p,q}t \\
&= \frac{[n]_{p,q}}{vq^{n-1}}A\int_0^\infty \varphi(pt)(tp)^{n-1}E_{p,q}\left(\frac{-tqn}{v}\right)d_{p,q}t \\
&\quad -\frac{A[n-1]_{p,q}}{q^{n-1}}\int_0^\infty \varphi(pt)t^{n-2}E_{p,q}\left(\frac{-tqn}{v}\right)d_{p,q}t.
\end{aligned}
$$

Let $pt = z \to t = \dfrac{z}{p} \to d_{p,q}t = \dfrac{d_{p,q}z}{p}$, then we have

$$
\begin{aligned}
G^1_{n,p,q}\left(D_{p,q}\varphi(t),v\right) &= \frac{[n]_{p,q}}{q^{n-1}vp^{n-2}}A\int_0^\infty \varphi(z)z^{n-1}E_{p,q}\left(-\frac{zqn}{pv}\right)d_{p,q}z \\
&\quad -\frac{[n-1]_{p,q}}{q^{n-1}p^{n-1}}A\int_0^\infty \varphi(z)z^{n-2}E_{p,q}\left(-\frac{zqn}{pv}\right)d_{p,q}z.
\end{aligned}
$$

Therefore, we have obtained

$$
G^1_{n,p,q}\left(D_{p,q}\varphi(t),v\right) = \frac{[n]_{p,q}}{q^{n-1}vp^{n-2}}G^1_n(\varphi,pv) - \frac{q^{n-1}[n-1]_{p,q}}{vp^{n-1}}G^1_{n-1,p,q}(\varphi,pv). \tag{22}
$$

This ends the proof of the theorem. □

Theorem 3. *Let $G^1_{n,p,q}$ be the (p,q)-analog defined by (17). Then, we have*

$$
\begin{aligned}
G^1_{n,p,q}\left(D^2_{p,q}\varphi,v\right) &= \left(\frac{[n]_{p,q}}{q^{n-1}vp^{n-2}}\right)^2 G^1_{n,p,q}\left(\varphi,p^2v\right) - \frac{p[n]_{p,q}[n-1]_{p,q}+[n-1]^2_{p,q}}{q^{n-1}vp^{n-2}} \\
&\quad G^1_{n-1,p,q}\left(\varphi,p^2v\right) + \frac{[n-1]_{p,q}[n-2]_{p,q}}{q^{2n-3}vp^{2n-3}}G^1_{n-2,p,q}\left(\varphi,p^2v\right).
\end{aligned}
$$

Proof. To prove this theorem, we insert $D^2_{p,q}\varphi$ inside the integral sign of (7) and employ Theorem 1 to write

$$
\begin{aligned}
G^1_{n,p,q}\left(D^2_{p,q}\varphi,v\right) &= A\int_0^\infty t^{n-1}\left(D^2_{p,q}\varphi\right)E_{p,q}\left(\frac{-qnt}{v}\right)d_{p,q}t \\
&= A\int_0^\infty t^{n-1}D_{p,q}\left(D_{p,q}\varphi\right)E_{p,q}\left(\frac{-qnt}{v}\right)d_{p,q}t \\
&= \frac{[n]_{p,q}}{q^{n-1}vp^{n-2}}G^1_{n,p,q}\left(D_{p,q}\varphi,pv\right) - \frac{[n-1]_{p,q}}{q^{n-1}sp^{n-1}}G^1_{n-1,p,q}\left(D_{p,q}\varphi',pv\right) \\
&= \frac{[n]_{p,q}}{q^{n-1}vp^{n-2}}\left(\frac{n}{q^{n-1}vp^{n-2}}G^1_{n,p,q}\left(\varphi,p^2v\right) - \frac{[n-1]_{p,q}}{q^{n-1}vp^{n-1}}G^1_{n-1,p,q}\left(\varphi,p^2v\right)\right) \\
&\quad -\frac{[n-1]_{p,q}}{q^{n-1}vp^{n-1}}\left(\frac{[n-1]_{p,q}}{q^{n-2}vp^{n-3}}G^1_{n-1,p,q}\left(\varphi,p^2v\right) - \frac{n-2}{q^{n-2}vp^{n-2}}G^1_{n-2,p,q}\left(\varphi,p^2v\right)\right).
\end{aligned}
$$

Consequently, performing calculations on the previous equation yields

$$
\begin{aligned}
G^1_{n,p,q}\Big(D^2_{p,q}\varphi, v\Big) &= \left(\frac{[n]_{p,q}}{q^{n-1}vp^{n-2}}\right)^2 G^1_{n,p,q}\Big(\varphi, p^2v\Big) - \frac{[n]_{p,q}[n-1]_{p,q}}{q^{n-1}vp^{n-1}p^{n-2}} G^1_{n-1,p,q}\Big(\varphi, p^2v\Big) \\
&\quad - \frac{\Big([n-1]_{p,q}\Big)^2}{q^{n-1}q^{n-2}vp^{n-1}p^{n-3}} G^1_{n-1,p,q}\Big(\varphi, p^2v\Big) + \\
&\quad \left(\frac{[n-1]_{p,q}[n-2]_{p,q}}{q^{n-1}q^{n-2}vp^{n-1}p^{n-2}} G^1_{n-2,p,q}\Big(\varphi, p^2v\Big)\right) \\
&= \left(\frac{[n]_{p,q}}{q^{n-1}vp^{n-2}}\right)^2 G^1_{n,p,q}\Big(\varphi, p^2v\Big) - \\
&\quad \left(\frac{p^{n-2}[n]_{p,q}[n-1]_{p,q} + [n-1]^2_{p,q}p^{n-3}q^{n-2}}{q^{n-1}q^{n-2}vp^{n-1}p^{n-3}}\right) G^1_{n-1,p,q}\Big(\varphi, p^2v\Big) \\
&\quad + \frac{[n-1]_{p,q}[n-2]_{p,q}}{q^{n-1}q^{n-2}vp^{n-1}p^{n-2}} G^1_{n-2,p,q}\Big(\varphi, p^2v\Big).
\end{aligned}
$$

Additional simplifications result in

$$
\begin{aligned}
G^1_{n,p,q}\Big(D^2_{p,q}\varphi, v\Big) &= \left(\frac{[n]_{p,q}}{q^{n-1}vp^{n-2}}\right)^2 G^1_{n,p,q}\Big(\varphi, p^2v\Big) - \frac{p[n]_{p,q}[n-1]_{p,q} + [n-1]^2_{p,q}}{q^{n-1}vp^{n-2}} \\
&\quad G^1_{n-1,p,q}\Big(\varphi, p^2v\Big) + \frac{[n-1]_{p,q}[n-2]_{p,q}}{q^{2n-3}vp^{2n-3}} G^1_{n-2,p,q}\Big(\varphi, p^2v\Big).
\end{aligned}
$$

The proof is ended. □

Hereafter, we present subsequent pairs of definitions of convolution products.

Definition 2. *Denote by* $\overset{p,q}{*}$ *the* (p,q)*-convolution product defined between two functions* θ_1 *and* θ_2 *as*

$$
\Big(\theta_1 \overset{p,q}{*} \theta_2\Big)(\epsilon) = \int_0^\infty \theta_1\Big(\epsilon t^{-1}\Big)\theta_2(t)t^{-1}d_{p,q}t \tag{23}
$$

provided the integral part exists.

Next, an additional convolution product that aligns with $\overset{p,q}{*}$ is as follows:

Definition 3. *Let* θ_1 *and* θ_2 *be two functions. Then, the* (p,q)*-convolution product* † *between* θ_1 *and* θ_2 *is defined as*

$$
(\theta_1 \dagger \theta_2)(\epsilon) = \int_0^\infty t^{k-1}\theta_1\Big(\frac{\epsilon}{t}\Big)\theta_2(t)d_{p,q}t. \tag{24}
$$

The p,q-convolution theorem of $G^1_{n,p,q}$ is now obtained as follows.

Theorem 4. *Let* $\overset{p,q}{*}$ *and* † *be the* (p,q)*-convolution products defined by* (23) *and* (24)*, respectively. Then, the* (p,q)*-convolution theorem of* $G^1_{n,p,q}$ *is defined for two functions* θ_1 *and* θ_2 *by*

$$
G^1_{n,p,q}\Big(\theta_1 \overset{p,q}{*} \theta_2\Big)(\epsilon) = \Big(G^1_{n,p,q}\theta_1 \dagger \theta_2\Big)(\epsilon).
$$

Proof. Owing to the theorem's hypothesis as above, we write

$$
G^1_{n,p,q}\Big(\theta_1 \overset{p,q}{*} \theta_2\Big)(\epsilon) = A\int_0^\infty \Big(\theta_1 \overset{p,q}{*} \theta_2\Big)(\xi)\xi^{n-1}E_q\Big(\frac{-nq\xi}{\epsilon}\Big)d_{p,q}\xi,
$$

where $A = \frac{n^n}{\epsilon^n \Gamma_{p,q}(n)}$. Hence, inserting the value of the operation in (23) reveals

$$G^1_{n,p,q}\left(\theta_1 \overset{p,q}{*} \theta_2\right)(\epsilon) = A\int_0^\infty \left(\int_0^\infty t^{-1}\theta_1\left(\frac{\xi}{t}\right)\theta_2(t)d_{p,q}t\right)\xi^{n-1}E_q\left(\frac{-nq\xi}{\epsilon}\right)d_{p,q}\xi. \quad (25)$$

Therefore, by using the change in variables $\frac{\xi}{t} = w$ and performing basic calculations on (25) by taking into account (24), we obtain

$$\begin{aligned} G^1_{n,p,q}\left(\theta_1 \overset{p,q}{*} \theta_2\right)(\epsilon) &= A\int_0^\infty \int_0^\infty t^{n-1}\theta(w)\theta_2(t)d_{p,q}tw^{n-1}E_q\left(\frac{-nqwt}{\epsilon}\right)d_{p,q}w \\ i.e. \quad &= A\int_0^\infty t^{n-1}\theta_2(t)\left(\int_0^\infty \theta_1(w)w^{n-1}E_q\left(\frac{-kqwt}{\epsilon}\right)d_{p,q}w\right)d_{p,q}t \\ i.e. \quad &= \int_0^\infty t^{n-1}\theta_2(t)G^1_{n,p,q}\theta_1\left(\frac{\epsilon}{t}\right)d_{p,q}t \\ i.e. \quad &= \left(G^1_{n,p,q}\theta_1 \dagger \theta_2\right)(\epsilon), \end{aligned}$$

where † has the significance of (24). The proof is ended. □

4. (p, q)-Gamma Integral of Elementary Functions

This section presents definitions and discusses characteristics of (p, q)-gamma integrals as well as (p, q)-analogs of exponential functions, trigonometric functions, power functions, and some hyperbolic functions. Further, it applies the (p, q)-analog to some unit step function.

Theorem 5. *Let $G^1_{n,p,q}$ and $G^2_{n,p,q}$ have their usual meaning in* (17) *and* (18)*, respectively. Then, we have*

$$(i)\ \ G^1_{n,p,q}\left(t^{1-n}, v\right) = \frac{[n]^{n-1}_{p,q}}{v^{n-1}\Gamma_{p,q}(n)} \quad (ii)\ \ G^2_{n,p,q}\left(t^{1-n}, v\right) = \frac{[n]^{n-1}_{p,q}}{v^{n-1}\Gamma_{p,q}(n)}. \quad (26)$$

Proof. Let the assumption of the theorem hold. Then, by the (p, q)-gamma integral (17), we find that

$$G^1_{n,p,q}\left(t^{1-n}, v\right) = \frac{[n]^n_{p,q}}{v^n\Gamma_{p,q}(n)}\int_0^\infty E_{p,q}\left(\frac{-qnt}{v}\right)d_{p,q}t.$$

Therefore, by using the scaling property of the (p, q)-integrals

$$\int_0^\infty f(at)d_{p,q}t = \frac{1}{a}\int_0^\infty f(t)d_{p,q}t,\ a \in \mathbb{R},$$

we obtain

$$G^1_{n,p,q}\left(t^{1-n}, v\right) = A\frac{v}{n}\int_0^\infty E_{p,q}(-qt)d_{p,q}t, \quad (27)$$

where $A = \frac{n^n}{v^n\Gamma_{p,q}(n)}$. Hence, by (13) we rewrite (27) in the form

$$G^1_{n,p,q}\left(t^{1-n}, v\right) = \frac{[n]^n_{p,q}}{v^n\Gamma_{p,q}(n)}\frac{-v}{n}\int_0^\infty D_{p,q}E_{p,q}(-t)d_{p,q}t. \quad (28)$$

Thus, (28) can be expressed as

$$\begin{aligned} G^1_{n,p,q}\left(t^{1-n},v\right) &= \frac{[n]^n_{p,q}}{v^n\Gamma_{p,q}(n)}\frac{-v}{n}E_{p,q}(-t)\Big|_0^\infty \\ &= \frac{[n]^n_{p,q}}{v^n\Gamma_{p,q}(n)}\frac{-v}{n}(0-1) \\ &= \frac{[n]^n_{p,q}}{v^n\Gamma_{p,q}(n)}\frac{v}{[n]_{p,q}}. \end{aligned}$$

This proves the first part. To prove the second part, for $A = \frac{n^n}{v^n\Gamma_{p,q}(n)}$, we note that

$$\begin{aligned} G^2_{n,p,q}\left(t^{1-n},v\right) &= A\int_0^\infty e_{p,q}\left(\frac{-pnt}{v}\right)d_{p,q}t \\ &= A\left(\frac{-n}{v}\right)e_{p,q}\left(\frac{-nt}{v}\right)\Big|_0^\infty \\ &= \frac{[n]^n_{p,q}}{v^n\Gamma_{p,q}(n)}\frac{v}{[n]_{p,q}}. \end{aligned}$$

The proof is ended. □

Theorem 6. *Let $G^1_{n,p,q}$ have its usual meaning given by (17), then we have*

$$(i)\ \ G^1_{n,p,q}(1,v) = \frac{v[n-1]_{p,q}}{p^{n-1}[n]_{p,q}}G^1_{n,p,q}\left(t^{-1},v\right). \qquad (ii)\ \ G^1_{n,p,q}(t,v) = \frac{v}{p^n}G^1_{n,p,q}(1,v).$$

Proof. From the definition of $G^1_{n,p,q}$ presented in (17) and inserting p^{n-1} inside the integral part, we have that

$$G^1_{n,p,q}(1,v) = A\int_0^\infty t^{n-1}E_{p,q}\left(\frac{-qnt}{v}\right)d_{p,q}t = \frac{A}{p^{n-1}}\int_0^\infty (pt)^{n-1}E_{p,q}\left(\frac{-qnt}{v}\right)d_{p,q}t.$$

By rearranging the preceding equation in terms of a derivative of an (p,q)-exponential function we obtain

$$G^1_{n,p,q}(1,v) = \frac{A}{p^{n-1}}\int_0^\infty (pt)^{n-1}\left(\frac{-v}{n}\right)D_{p,q}E_{p,q}\left(\frac{-nt}{v}\right)d_{p,q}t. \quad (29)$$

That is, upon using the (p,q)-integration by parts (7) and simplifying the the obtained result, we rewrite the preceding equation in the form

$$\begin{aligned} G^1_{n,p,q}(1,v) &= \frac{-Av}{p^{n-1}[n]_{p,q}}t^{n-1}E_{p,q}\left(\frac{-nt}{v}\right)\Big|_{t=0}^\infty - \frac{-Av}{p^{n-1}n}\int_0^\infty E_{p,q}\left(\frac{-nqt}{v}\right)(n-1)t^{n-2}d_{p,q}t \\ &= \frac{-Av}{p^{n-1}[n]_{p,q}}\left((0-0) - \frac{(n-1)}{A}A\int_0^\infty t^{(n-1)-1}E_{p,q}\left(\frac{-nqt}{v}\right)d_{p,q}t\right) \\ &= \frac{v[n-1]_{p,q}}{p^{n-1}[n]_{p,q}}\left(A\int_0^\infty t^{-1}t^{(n-1)}E_{p,q}\left(\frac{-nqt}{v}\right)d_{p,q}t\right) \\ &= \frac{v[n-1]_{p,q}}{p^{n-1}[n]_{p,q}}G^1_{n,p,q}\left(t^{-1},v\right). \end{aligned}$$

This proves the first part. To prove the second part, we employ (7) and insert p^n under the integral sign to have

$$\begin{aligned} G^1_{n,p,q}(t,v) &= A\int_0^\infty t t^{n-1} E_{p,q}\left(\frac{-qnt}{v}\right) d_{p,q}t \\ &= A\int_0^\infty t^n E_{p,q}\left(\frac{-qnt}{v}\right) d_{p,q}t \\ &= \frac{A}{p^n}\int_0^\infty (tp)^n E_{p,q}\left(\frac{-qnt}{v}\right) d_{p,q}t \\ &= \frac{A}{p^n}\int_0^\infty (tp)^n \left(-\frac{v}{n}\right) D_{p,q}E_{p,q}\left(\frac{-nt}{v}\right) d_{p,q}t. \end{aligned}$$

Therefore, computations and the (p,q)-integration by parts (7) yield

$$\begin{aligned} G^1_{n,p,q}(t,v) &= -\frac{vA}{[n]_{p,q}p^n}\int_0^\infty (tp)^n D_{p,q}E_{p,q}\left(\frac{-nt}{v}\right) d_{p,q}t \\ &= \frac{vA}{[n]_{p,q}p^n}\left(t^n E_{p,q}\left(\frac{-nt}{v}\right)\Big|_0^\infty - \int_0^\infty [n]_{p,q} t^{n-1} E_{p,q}\left(\frac{-nqt}{v}\right) d_{p,q}t \right) \\ &= \frac{v[n]_{p,q}}{[n]_{p,q}p^n}\left(A\int_0^\infty t^{n-1} E_{p,q}\left(\frac{-nqt}{v}\right) d_{p,q}t \right). \end{aligned}$$

Hence, we have obtained

$$G^1_{n,p,q}(t,v) = \frac{v}{p^n} G^1_{n,p,q}(1,v). \quad (30)$$

The proof is ended. □

Following corollary is a straightforward consequence of Theorem 4.

Corollary 1. *Let $G^1_{n,p,q}$ have its usual meaning given by* (17)*, then we have*

$$(i)\ \ G^1_{n,p,q}(1,v) = \frac{p^{-n\left(\frac{n-1}{2}\right)} Av[n-1]_{p,q}}{p^{n-1}[n]_{p,q}} \Gamma_{p,\frac{nq}{v}}(n-1). \qquad (ii)\ \ G^1_{n,p,q}(t,v) = p^{-n\frac{(n-1)}{2}} \frac{Av}{p^2} \Gamma_{p,\frac{nq}{v}}(n).$$

Theorem 7. *Let $G^1_{n,p,q}$ and $G^2_{n,p,q}$ have their usual meaning given by* (17) *and* (18)*, respectively. Then, we have*

(i) $G^1_{n,p,q}(t^2,v) = \dfrac{v[n+1]_{p,q}}{[n]_{p,q}p^{(n+1)}} G^1_{n,p,q}(t,v).$

(ii) $G^1_{n,p,q}\left(t^k,v\right) = \dfrac{[n]_{p,q}[n-1+k]_{p,q}}{vp^{(n-1+k)}} G^1_{n,p,q}\left(t^{k-1},v\right), k = 0,1,2,\ldots.$

(iii) $G^2_{n,p,q}(t^2,v) = \dfrac{v[n+1]_{p,q}}{nq^{(n+1)}} G^1_{n,p,q}(t,v).$

(iv) $G^2_{n,p,q}\left(t^k,v\right) = \dfrac{[n]_{p,q}[n-1+k]_{p,q}}{vq^{(n-1+k)}} G^1_{n,p,q}\left(t^{k-1},v\right), k = 0,1,2,\ldots.$

Proof. To prove (i). By considering the definition of $G^1_{n,p,q}$ presented in (7) and the (p,q)-derivative of $E_{p,q}$ given by (13) we write

$$\begin{aligned} G^1_{n,p,q}\left(t^2,v\right) &= A\int_0^\infty t^2 t^{n-1} E_{p,q}\left(\frac{-qnt}{v}\right) d_{p,q}t \\ &= A\int_0^\infty t^{n+1} E_{p,q}\left(\frac{-qnt}{v}\right) d_{p,q}t \\ &= \frac{A}{p^{(n+1)}}\int_0^\infty (pt)^{n+1} E_{p,q}\left(\frac{-qnt}{v}\right) d_{p,q}t. \end{aligned}$$

Hence, we have obtained

$$G^1_{n,p,q}\left(t^2,v\right)=\frac{A}{p^{(n+1)}}\left(\frac{-v}{n}\right)\int_0^\infty (pt)^{n+1}D_{p,q}E_{p,q}\left(\frac{-nt}{v}\right)d_{p,q}t. \tag{31}$$

Thus, the (p,q)-integration by part (7) gives

$$\begin{aligned} G^1_{n,p,q}\left(t^2,v\right) &= \frac{(-v)}{np^{(n+1)}}A\left(t^{n+1}E_{p,q}\left(\frac{-nt}{v}\right)\Big|_0^\infty-\int_0^\infty E_{p,q}\left(\frac{-nqt}{v}\right)(n+1)t^n d_{p,q}t\right) \\ &= \frac{v[n+1]_{p,q}}{np^{(n+1)}}A\int_0^\infty t^nE_{p,q}\left(\frac{-nqt}{v}\right)d_{p,q}t \\ &= \frac{v[n+1]_{p,q}}{[n]_{p,q}p^{(n+1)}}G^1_{n,p,q}(t,v).. \end{aligned}$$

In a similar vein, we expand our work to the $t^k, k=0,1,2,\ldots$ to obtain (ii).

$$\frac{[n]_{p,q}[n-1+k]_{p,q}}{vp^{(n-1+k)}}G^1_{n,p,q}\left(t^{k-1},v\right), k=0,1,2,\ldots. \tag{32}$$

Once again, we proceed to establish the (iii) and (iv) parts. For the (iv) part we may write

$$\frac{[n]_{p,q}[n-1+k]_{p,q}}{vq^{(n-1+k)}}G^1_{n,p,q}\left(t^{k-1},v\right), k=0,1,2,\ldots. \tag{33}$$

This ends the proof of the theorem. □

In terms of the gamma concept, the above theorem can be stated as follows.

Corollary 2. *Let $G^1_{n,p,q}$ and $G^2_{n,p,q}$ have their usual meaning given by* (17) *and* (18)*, respectively. Then, we have*

(i) $G^1_{n,p,q}(t^2,v)=\dfrac{v[n+1]_{p,q}A}{[n]_{p,q}p^{(n+1)}}p^{-n(n-1)}\Gamma_{p,\frac{nq}{v}}(n+1).$

(ii) $G^1_{n,p,q}\left(t^k,v\right)=\dfrac{v[n-1+k]_{p,q}}{[n]_{p,q}p^{(n-1+k)}}p^{\frac{-n(n-1)}{2}}\Gamma_{p,\frac{[n]_{p,q}q}{v}}(n+k-1).$

(iii) $G^2_{n,p,q}(t^2,v)=\dfrac{v[n+1]_{p,q}A}{[n]_{p,q}q^{(n+1)}}q^{-n(n-1)}\Gamma_{\frac{np}{v},q}(n+1).$

(iv) $G^2_{n,p,q}\left(t^k,v\right)=\dfrac{[n]_{p,q}[n-1+k]_{p,q}}{vq^{[n-1+k]_{p,q}}}q^{\frac{-n(n-1)}{2}}\Gamma_{\frac{np}{v},q}(n+k-1).$

Theorem 8. *Let $G^1_{n,p,q}$ have the significance of* (17)*. Then, its application to $e_{p,q}$ and $E_{p,q}$ is given by*

$$(i)\ \ G^1_{n,p,q}(e_{p,q}(at),v)=\sum_{k=0}^\infty\frac{p^{\binom{k}{2}}a^k}{[k]_{p,q}!}\left(G^1_{n,p,q}t^k,v\right).$$

$$(ii)\ \ G^1_{n,p,q}(E_{p,q}(at),v)=\sum_{k=0}^\infty\frac{q^{\binom{k}{2}}a^k}{[k]_{p,q}!}G^1_{n,p,q}\left(t^k,v\right).$$

Proof. From the definitions of $G^1_{n,p,q}$ and $e_{p,q}$ and simplifying we have

$$\begin{aligned} G^1_{n,p,q}(e_{p,q}(at),v) &= A\int_0^\infty t^{n-1}e_{p,q}(at)E_{p,q}\left(\frac{-qnt}{v}\right)d_{p,q}t \\ &= A\int_0^\infty t^{n-1}E_{p,q}\left(\frac{-qnt}{v}\right)\sum_{k=0}^{\infty}\frac{p^{\binom{k}{2}}(at)^k}{[k]_{p,q}!}d_{p,q}t \\ &= \sum_{k=0}^{\infty}\frac{p^{\binom{k}{2}}a^k}{[k]_{pq}!}\left(A\int_0^\infty t^{n-1}t^kE_{p,q}\left(\frac{-qnt}{v}\right)d_{p,q}t\right). \end{aligned}$$

Hence, we have obtained

$$G^1_{n,p,q}(e_{p,q}(at),v) = \sum_{k=0}^{\infty}\frac{p^{\binom{k}{2}}a^k}{[k]_{p,q}!}\left(G^1_{n,p,q}t^k,v\right). \tag{34}$$

To prove the second part (ii) we have

$$\begin{aligned} G^1_{n,p,q}(E_{pq}(at),v) &= A\int_0^\infty t^{n-1}E(at)E_{p,q}\left(\frac{-qnt}{v}\right)d_{p,q}t \\ &= \sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}a^k}{[k]_{p,q}!}G^1_{n,p,q}\left(t^k,v\right). \end{aligned}$$

Similarly, the following theorem can be established. □

Theorem 9. *Let $G^2_{n,p,q}$ have their usual meaning given by* (18)*. Then, we have*

$$(i)\ \ G^2_{n,p,q}(e_{p,q}(at),v) = \sum_{k=0}^{\infty}\frac{p^{\binom{k}{2}}a^k}{[k]_{p,q}!}G^2_{n,p,q}\left(t^k,v\right).$$

$$(ii)\ \ G^2_{n,p,q}(E_{p,q}(at),v) = \sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}a^k}{[k]_{p,q}!}G^2_{n,p,q}\left(t^k,v\right).$$

Theorem 10. *Let $G^1_{n,p,q}$ and $G^2_{n,p,q}$ have their usual meaning given by* (17) *and* (18)*, respectively. Then, we have*

$(i)\ \ G^1_{n,p,q}(\cos_{p,q}(at,v)) = \sum_{k=0}^{\infty}(-1)^k\frac{p^{2k_2}}{[2k]_{p,q}!}G^1_{n,p,q}\left(t^{2k},v\right).$

$(ii)\ \ G^1_{n,p,q}(Cos_{p,q}(at,v)) = \sum_{k=0}^{\infty}(-1)^k\frac{p^{2k_2}a^{2k}}{[2k]_{p,q}!}G^1_{n,p,q}\left(t^{2k},v\right).$

$(iii)\ \ G^1_{n,p,q}(\sin_{p,q}(at),v) = \sum_{k=0}^{\infty}(-1)^k p\frac{p^{\binom{2k+1}{2}}a^{2k+1}}{[2k+1]_{p,q}!}G^1_{n,p,q}\left(t^{2k+1},v\right).$

$(iv)\ \ G^1_{n,p,q}(Sin_{p,q}(at),v) = \sum_{k=0}^{\infty}(-1)^k\frac{q^{\binom{2k+1}{2}}}{[2k+1]_{p,q}!}a^{2k+1}G^1_{n,p,q}\left(t^{2k+1},v\right).$

Proof. Proof of Part (i), and by (17) and the fact that [40]

$$\cos_{p,q}(at) = \frac{e_{p,q}(iat)+e_{p,q}(-iat)}{2} = \sum_{k=0}^{\infty}(-1)^k\frac{p^{\binom{2k}{2}}}{[2k]_{p,q}!}a^{2k}t^{2k} \tag{35}$$

we have

$$\begin{aligned} G^1_{n,p,q}(\cos_{p,q}(at,v)) &= A\int_0^\infty \cos(at)t^{n-1}E_{p,q}\left(\frac{-nqt}{v}\right)d_{p,q}t \\ &= \sum_{k=0}^{\infty}(-1)^k\frac{p^{\binom{2k}{2}}}{[2k]!}a^{2k}A\int_0^\infty t^{2k}t^{n-1}E_{p,q}\left(\frac{-nqt}{v}d_{p,q}t\right) \\ &= \sum_{k=0}^{\infty}(-1)^k\frac{p^{\binom{2k}{2}}}{[2k]_{p,q}!}G^1_{n,p,q}\left(t^{2k},v\right). \end{aligned}$$

To prove part (ii), we use (17) and the fact that [40]

$$Cos_{p,q}(at) = \frac{E_{p,q}(iat)+E_{p,q}(-iat)}{2} = \sum_{k=0}^{\infty}(-1)^k\frac{p^{\binom{2k}{2}}}{[2k]_{p,q}!}a^{2k}t^{2k} \tag{36}$$

to obtain

$$G^1_{n,p,q}(Cos_{p,q}(at,v)) = \sum_{k=0}^{\infty}(-1)^k\frac{p^{\binom{2k}{2}}a^{2k}}{[2k]_{p,q}!}G^1_{n,p,q}\left(t^{2k},v\right).$$

Proving (iii) and (iv), we use the facts [40]

$$\sin_{p,q}(at) = \frac{e_{p,q}(iat)-e_{p,q}(-iat)}{2i} = \sum_{k=0}^{\infty}(-1)^k\frac{p^{\binom{2k+1}{2}}a^{2k+1}}{[2k+1]_{p,q}!}t^{2k+1} \tag{37}$$

and

$$Sin_{p,q}(at) = \frac{E_{p,q}(iat)-E_{p,q}(-iat)}{2i} = \sum_{k=0}^{\infty}(-1)^k\frac{q^{\binom{2k+1}{2}}}{[2k+1]_{p,q}!}a^{2k+1}t^{2k+1}. \tag{38}$$

The proof is ended. The above-mentioned findings about $G^2_{n,p,q}$ of the trigonometric functions may be shown using analogous proof. □

Definition 4. *The (p,q)-hyperbolic cosine and sine functions are defined by [39]*

(i) $\cosh_{p,q}(at) = \dfrac{e_{p,q}(at)+e_{p,q}(-at)}{2} = \sum_{k=0}^{\infty}\dfrac{p^{\binom{2k}{2}}}{[2k]_{p,q}!}a^{2k}t^{2k}.$

(ii) $Cosh_{p,q}(at) = \dfrac{E_{p,q}(at)+E_{p,q}(-at)}{2} = \sum_{k=0}^{\infty}\dfrac{p^{\binom{2k}{2}}}{[2k]_{p,q}!}a^{2k}t^{2k}.$

(iii) $\sinh_{p,q}(at) = \dfrac{e_{p,q}(at)-e_{p,q}(-at)}{2} = \sum_{k=0}^{\infty}\dfrac{p^{\binom{2k+1}{2}}}{[2k+1]_{p,q}!}a^{2k+1}t^{2k+1}.$

(iv) $Sinh_{p,q}(at) = \dfrac{E_{p,q}(at)-E_{p,q}(-at)}{2} = \sum_{k=0}^{\infty}\dfrac{q^{\binom{2k+1}{2}}}{[2k+1]_{p,q}!}a^{2k+1}t^{2k+1}.$

By using a similar technique, readers can easily expand the work to (p,q)-hyperbolic cosine and sine functions.

Theorem 11. *Let $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$ be the unit step function. Then, we have*

$$G^1_{n,p,q}(u(t),v) = \frac{v[n-1]_{p,q}}{[n]_{p,q}p^{n-1}}G^1_{n,p,q}\left(t^{-1},v\right). \tag{39}$$

Proof. By considering the definition (17) and that of the unit step function, we obtain

$$
\begin{aligned}
G^{1}_{n,p,q}(u(t),v) &= A\int_0^\infty t^{n-1}u(t)E_{p,q}\left(\frac{-qnt}{v}\right)d_{p,q}t \\
&= \frac{A}{p^{n-1}}\int_0^\infty (pt)^{n-1}u(t)E_{p,q}\left(\frac{-qnt}{v}\right)d_{p,q}t \\
&= -A\frac{v}{np^{n-1}}\int_0^\infty (pt)^{n-1}D_{p,q}E_{p,q}\left(\frac{-nt}{v}\right)d_{p,q}t.
\end{aligned}
$$

Hence, utilizing the concept of the integration by parts, we obtain

$$
\begin{aligned}
G^{1}_{n,p,q}(u(t),v) &= \frac{-Av}{[n]_{p,q}p^{n-1}}\left(t^{n-1}E_{p,q}\left(\frac{-nt}{v}\right)\bigg|_0^\infty - \int_0^\infty E_{p,q}\left(\frac{-nqt}{v}\right)\left(D_{p,q}t^{n-1}\right)d_{p,q}t\right) \\
&= \frac{-Av}{[n]_{p,q}p^{n-1}}\left(-\int_0^\infty [n-1]_{p,q}t^{n-2}E_{p,q}\left(\frac{-nqt}{v}\right)d_{p,q}t\right).
\end{aligned}
$$

Therefore, the definition of $G^{1}_{n,p,q}$ suggests we write

$$
G^{1}_{n,p,q}(u(t),v) = \frac{v[n-1]_{p,q}}{[n-1]_{p,q}p^{n-1}}G^{1}_{n,p,q}\left(t^{-1},v\right). \tag{40}
$$

The proof is ended. □

A simple appropriate change on (40) leads to the following result.

Corollary 3. *Let u be the unit step function. Then, we have*

$$
G^{1}_{n,p,q}(u(t),v) = \frac{v[n-1]_{p,q}}{[n]_{p,q}p^{n-1}}Ap^{-n(n-1)}\Gamma_{p,\frac{qn}{v}}(n-1).
$$

5. Conclusions

In this article, the gamma integral operator's (p,q)-analogs are presented, and their expansion to power functions, (p,q)-exponential functions, and (p,q)-trigonometric functions are covered. It also establishes results about the use of the (p,q)-analogs with unit step functions and first- and second-order (p,q)-differential operators. In addition, two (p,q)-convolution theorems are established and two (p,q)-convolution products are presented for the given (p,q)-analogs.

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