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Some Common Fixed Point Results of Tower Mappings in (Pseudo)modular Metric Spaces

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Abstract: In this paper, we prove the existence and uniqueness of common fixed point of tower type contractive mappings in complete metric (pseudo)modular spaces involving the theoretic relation. However, the newly introduced contraction in this paper further characterize and includes in their full strength several existing results in metrical fixed point theory. Some nontrivial supportive examples were given to justify our result. Our results generalize, improve, and unify some existing results.

Keywords: existence; common fixed point; uniqueness of fixed point; partially ordered; complete metric (pseudo)modular space; tower type contraction mappings; binary relation

MSC: 47H09; 47H10; 47H30; 54H25

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1. Introduction

The concept of tower function(s) cut across some areas of science and the problem is sometimes how to identify them. In elementary real analysis, the function $n \mapsto n^n$ for $n \in \mathbb{N}$ is actually a tower function and it is well known that $\lim_{n\to\infty} n^n = \infty$, $\lim_{n\to-\infty} n^n = 0$, $\lim_{n\to 0} n^n = 1$, $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$, etc. However, in the study of matrix algebra, we do encounter the matrices of the type, A^A, $\exp\left(A^A\right)$, where A is a square matrix. Sometimes, the evaluation(s) of these matrices are highly provocative. It is technically believed that whenever a function is raised to power itself or to order function(s), then we can say that such a function possesses a tower function. Some prototypes are of the form $\mathbb{N}^{\mathbb{N}}$, $\mathbb{R}^{\mathbb{N}^{\mathbb{N}}}$, $\mathbb{R}^{\mathbb{R}}$, etc. The ideas above are also applicable in building technology. We can now cast some idea of life science into tower function.

Imagine a pregnant woman went to a radiologist for a scan. Naturally, the woman's chromosome is x; then, the radiologist informs her that she is carrying two female babies. This miracle is indeed a tower function with respect to time when we allow it to be function upon function(s). Suppose that the result of the scan says two baby boys; this is also a tower function with respect to time, i.e., $x \mapsto x^{y^y}$. Finally, if the scan says a boy and a girl, then it is also a tower function with respect to time, i.e., $x \mapsto x^{y^x}$, etc. This concept is naturally a tower function; again, if it is two baby girls, then it is also a tower function with respect to time, i.e., $x \mapsto x^{x^x}$, etc. We allow the positions of the babies to be upwards directed.

In probability theory, the three-tower problem and the concept looks slightly different; a randomly chosen gambler loses a coin and another randomly chosen gambler obtains it. The game continues until one of the three gamblers is ruined.

Somewhere in combinatorial set theory we do encounter tower of infinite ordinals and in elementary calculus, integration of a tower function of the type $x \mapsto x^{x^{x^{\cdot^{\cdot^{\cdot^{x^{\cdot^{\cdot^{\cdot}}}}}}}}}$ is interesting in its own right, and this is also called the infinite Sophomores type dream. When it becomes numeric, it becomes an Ackermann number. In particular, the Sophomores dream tower is of the type $x \mapsto x^{x^x}$. In metrical fixed point theories, especially, contraction mappings involving such towers appeared first in Okeke and Francis [1].

In [2], the celebrated contraction principle due to S. Banach, which appeared in the literature in 1922, is one of the most important and useful results in the metric fixed point theory due to its numerous applications. In Banach's theorem, X is taken to be a complete metric space with a metric d and $f : X \to X$ is required to be a contraction; that is, there must exist $0 \leq L < 1$ such that $d(f(x), f(y)) \leq Ld(x, y)$ for all $x, y \in X$. The conclusion is that f has a fixed point, in fact exactly one of them. This has encountered many extensions/generalizations, as recorded in [3–13] and references therein. Among these generalizations, we prefer the one given by Geraghty [5].

More than a decade ago, Amini-Harandi and Emami [14] characterized the result of Geraghty in the context of a partially ordered complete metric space with some application to ordinary differential equations. Gordji et al. [15] defined the notion of ψ-Geraghty type contraction and supposedly improved and extended the results of Amini-Harandi and Emami [14]. Cho et al. [16] defined the concept of α-Geraghty contraction type maps in the setting of a metric space and proved the existence and uniqueness of a fixed point of such maps in the context of a complete metric space. Popescu [17] generalized the results obtained by Cho et al. [16] and gave other conditions to prove the existence and uniqueness of a fixed point of α-Geraghty contraction type maps in the context of a complete metric space. See also [18,19] for other results for the fixed point theory.

It is interesting to know that some results that involved the Banach contraction mapping principle and its allied results involving partially ordered metric spaces have been optimized to the theoretic relation by replacing the partial order relation with a locally $\mathcal{H}$-transitive relation, which remains an optimal condition of transitivity, as recorded in [20–24] and some of their references therein.

Recently, Okeke and Francis [1] first defined a new class of nonlinear mappings in metric spaces, called metric tower mappings, and proved the existence of a fixed point of Geraghty tower-type mappings in complete metric spaces and gave some nontrivial examples that justified the newly defined contraction mapping. Francis and Okeke [25], defined rational type Geraghty tower contraction mapping and proved the existence of finite and infinite rational Geraghty tower theorem(s) in complete metric spaces.

In 2010, Chistyakov [26] introduced modular metric space as a natural extension and generalization of classical modular in the sense of Nakano [27] and classical metric spaces in the sense of Fre'chet [28]. In fact, metric modular space is a parameterized metric space in extended real line, which may not obey the famous triangular inequality. Chistyakov [29] extended the famous Banach contraction mapping principle in the setting of modular metric space. Furthermore, similar extension have been carried out by Mongkolkeha et al. [30], and while their results contained some bugs, these were eventually solved. In the spirit of modular metric spaces, Chaipunya et al. [31] extended the results in Geraghty [5] by defining more classes satisfying Geraghty functions, while Okeke et al. [32] provided a Geraghty-type class that contained several results in the literature. There are numerous studies on Geraghty contraction in various spaces, such as multiplicative, b metric spaces, partial metric spaces, extended modular metric spaces, G metric spaces, modular G metric spaces, etc. Interested reader should consult [3,13,15,18,19,32] and the references therein for other results for fixed point theory.

In this paper, we give a tower type contraction maps which further characterize and include the results in Amini-Harandi and Emami [14], and some other related contraction types in the literature. We also give a nontrivial supportive example to justify our claims.

The results are new and interesting in their own right. The results we establish in this paper extend, improve, and generalize some existing results in the literature.

2. Preliminaries

Throughout the article, $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$ is the set of nonnegative integers and $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$ is the set of positive real numbers. By a relation (or a binary relation) $\sqsubseteq$ on a set A, we mean a subset of $A \times A$. The following results and definitions will be useful in this paper.

Following [24], A is a set, $\sqsubseteq$ is a relation on A, and f, g are self mappings. We begin with the definitions involving a set theoretic relation.

Definition 1 ([23]). *A pair of elements $u, v \in A$ satisfying either $(u,\ v) \in \sqsubseteq$ or $(v, u) \in \sqsubseteq$ is said to be $\sqsubseteq$-comparative. We shall denote such a pair by $[u,\ v] \in \sqsubseteq$.*

Definition 2 ([33]). *For each pair $u, v \in A$, if $[u,\ v] \in \sqsubseteq$, we say that $\sqsubseteq$ is a complete relation.*

Definition 3 ([33]). *The inverse of $\sqsubseteq$ is a relation $\sqsubseteq^{-1}$ defined by $\sqsubseteq^{-1} := \{(u, v) \in A \times A : (v, u) \in \sqsubseteq\}$.*

Definition 4 ([33]). *The symmetric closure of $\sqsubseteq$ is a relation $\sqsubseteq^s$ defined by $\sqsubseteq^s := \sqsubseteq \cup \sqsubseteq^{-1}$.*

Proposition 1 ([23]). $(u,\ v) \in \sqsubseteq^s \Leftrightarrow [u,\ v] \in \sqsubseteq$.

Definition 5 ([33]). *For any subset $B \subseteq A$, the relation on B defined by $\sqsubseteq |_B := \sqsubseteq \cap B \times B$, which is referred as the restriction of $\sqsubseteq$ on B.*

Definition 6 ([23]). *For each pair of elements $u, v \in A$ with $(u,\ v) \in \sqsubseteq$, if $(fu, fv) \in \sqsubseteq$, then $\sqsubseteq$ is termed as f-closed.*

Remark 1. *We can improve Definition 6 in the following way: For each pair of elements $u, v \in A$ with $(u,\ v) \in \sqsubseteq$, if $(fu, gv) \in \sqsubseteq$, then $\sqsubseteq$ is termed as (f, g)-closed. The idea here coincides with that of Definition 6 if $f = g$.*

Proposition 2 ([21]). *If $\sqsubseteq$ is f-closed, then $\sqsubseteq$ is f^n-closed, $\forall\ n \in\ \mathbb{N}$.*

Definition 7 ([23]). *If a sequence $\{x_n\} \subset A$ verifies $(x_n, x_{n+1}) \in \sqsubseteq\ \forall n \in \mathbb{N}$, then we say that $\{x_n\}$ is $\sqsubseteq$-preserving.*

Definition 8 ([23]). *$\sqsubseteq$ is called ρ-self-closed if each $\sqsubseteq$-preserving convergent sequence in A has a subsequence whose terms are $\sqsubseteq$-comparative with the limit.*

Definition 9 (*f-Transitive relation* [21]). *Given a map f: $X \to X$, we say that a relation $\sqsubseteq$ on X is f-transitive if $(fu,\ fw) \in \sqsubseteq$ for all $u, v, w \in X$ such that $(fu,\ fv), (fv,\ fw) \in \sqsubseteq$.*

From Definition 9, we introduce the following concept.

Definition 10 (*(f, g)-Transitive relation*). *Let f, g: $X \to X$, we say that a relation $\sqsubseteq$ on X is (f, g)-transitive if $(fu,\ gw) \in \sqsubseteq$ for all $u, v, w \in X$ such that $(fu,\ fv), (fv,\ gw), (fu,\ gv), (gv,\ gw) \in \sqsubseteq$.*

Definition 11 ([21,24]). *$\sqsubseteq$ is termed as locally f-transitive if for any $\sqsubseteq$- preserving sequence $\{x_n\} \subset f(\mathrm{A})$, the relation $\sqsubseteq |_{\mathrm{B}}$ (whereas $\mathrm{B} := \{x_n : n \in \mathbb{N}\}$) is transitive.*

Following [21], we have the definitions below.

Definition 12. $\sqsubseteq$ *is called locally* (f, g)*-transitive if for any* $\sqsubseteq$*- preserving sequence* $\{x_n\} \subset f(\mathrm{A}) \subseteq g(A)$ *, the relation* $\sqsubseteq|_{\mathrm{B}}$ *(whereas* $\mathrm{B} := \{x_n : n \in \mathbb{N}\}$*) is transitive.*

Remark 2. *If* $f = g$*, coincide with Definition 11.*

Definition 13 ([22])**.** f *is termed as* $\sqsubseteq$*-continuous at* $x^* \in A$ *if for every* $\sqsubseteq$*-preserving sequence* $\{x_n\} \subset A$ *verifying* $x_n \to x^*$*,* $f(x_n) \to f(x^*)$ *. A* $\sqsubseteq$*-continuous map at each point of A is referred as* $\sqsubseteq$*-continuous.*

Definition 14 ([33])**.** *A binary relation* $\sqsubseteq$ *defined on any nonempty set A is said to be:*

(1) *Reflexive if* $(u,\ u) \in \sqsubseteq$ *for all* $u \in A$*;*
(2) *Symmetric if* $(u, v) \in \sqsubseteq$ *implies* $(v, u) \in \sqsubseteq$*;*
(3) *Transitive if* $(u, v) \in \sqsubseteq, (v, w) \in \sqsubseteq$ *implies* $(u, w) \in \sqsubseteq$*;*
(4) *Dichotomous if* $[u, v] \in \sqsubseteq$ *for all* $u, v \in A$*;*
(5) *Trichotomous if* $[u, v] \in \sqsubseteq$ *or* $u = v$ *for all* $u, v \in A$*;*
(6) *Equivalent if* $\sqsubseteq$ *is reflexive, symmetric, and transitive.*

Definition 15 ([22])**.** *A metric space* $(A,\ \rho)$ *is referred as* $\sqsubseteq$*-complete if each* $\sqsubseteq$*- preserving Cauchy sequence in A converges.*

Definition 16 ([34])**.** *A partially ordered set is a pair* $(A, \sqsubseteq)$*, where A is a set and* $\sqsubseteq$ *is a binary relation on A such that:*

(1) $u \sqsubseteq u$ *for every* $u \in A$*;*
(2) *if u and v belong to A and* $u \sqsubseteq v$ *and* $v \sqsubseteq u$*, then* $u = v$*;*
(3) *if* u, v *and r belong to A and* $u \sqsubseteq v$ *and* $v \sqsubseteq r$*, then* $u \sqsubseteq r$*.*

Definition 17 ([5])**.** *S is the class of functions* $\alpha : \mathbb{R}_+ \to [0, 1)$ *with:*
(i) $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t > 0\}$*;*
(ii) $\alpha(t_n) \to 1 \implies t_n \to 0$ *as* $n \to \infty$*.*

Definition 18 ([31])**.** *For each* $n \in \mathbb{N}$*, let* $\mathcal{S}_n$ *denote the class of n-tuples of functions* $(\beta_1, \beta_2, \beta_3, \cdots, \beta_n)$*, where for each* $i \in \{1, 2, \cdots, n\}$*,* $\beta_i : \mathbb{R}_+ \cup \{\infty\} \to [0, 1)$ *and the following implications holds:* $\beta(t_k) := \beta_1(t_k) + \beta_2(t_k) + \cdots + \beta_n(t_k) \to 1$ *implies* $t_k \to 0$*.*

It follows that, for each $m \in \{1, 2, \cdots, n\}$*, if* $(\beta_1, \beta_2, \beta_3, \cdots, \beta_m) \in \mathcal{S}_m$*, then* $\{\beta_1, \beta_2, \beta_3, \cdots, \beta_m, \underbrace{0, 0, 0 \cdots, 0}_{n-m \text{ entries}}\} \in \mathcal{S}_n$*, where 0 is a zero function.*

Remark 3. *Note that, if* $(\underbrace{\beta, \beta, \cdots, \beta}_{n \text{ entries}}) \in \mathcal{S}_n$*, then we also have the following:* $\beta(t_k) \to \frac{1}{n}$ *implies* $t_k \to 0$.

Remark 4. *The class of function defined in Definition 17 can equally put to work when we define more functions other than* α *in that class, S.*

Theorem 1 ([5])**.** *Let X be a complete metric space. Let* $f : X \to X$ *with* $d(f(x), f(y)) < d(x, y)$*, for all* $x, y \in X$*. Let* $x_0 \in X$ *and set* $f(x_{n-1}) = x_n$ *for all* $n > 0$*. Then,* $x_n \to x^*$ *in X, with* x^* *a unique fixed point of f, if and only if for any two subsequences* x_{h_n} *and* x_{h_k} *with* $x_{h_n} \neq x_{h_k}$*, we have that* $\Pi_n \to 1$ *only if* $d_n \to 0$*.*

Remark 5. *In Theorem 1, we take for any pair of sequences* x_n *and* y_n *with* $x_n \neq y_n$*, we write* $d_n = d(x_n, y_n)$ *and* $\Pi_n = \frac{d(f(x_n), f(y_n))}{d(x_n, y_n)}$*.*

Theorem 2 ([5]). *Let $f : X \to X$ be a contraction on a complete metric space. Let $x_0 \in X$ and set $f(x_{n-1}) = x_n$ for all $n > 0$. Then $x_n \to x^*$ in X, where x^* a unique fixed point of f in X, if and only if there exists an α in S such that for all n, m:*

$$d(f(x_n), f(x_m)) \leq \alpha(d(x_n, x_m))d(x_n, x_m). \tag{1}$$

Theorem 3 ([5]). *Let (X, ρ) be a complete metric space and $T : X \to X$ such that there is an $\alpha \in S$ satisfying*

$$\rho(Tx, Ty) \leq \alpha(\rho(x, y))\rho(x, y), \tag{2}$$

for all $x, y \in X$. Then, the sequence $\{x_k\}_{k \in \mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X for $k \geq 1$.

The functions ω of the form $\omega : (0, \infty) \times X \times X \to [0, \infty]$, where X is a fixed nonempty set (with at least two elements). Due to the disparity of the arguments, we may (and will) write $\omega_\lambda(x,\ y) = \omega(\lambda,\ x,\ y)$ for all $\lambda > 0$ and $x, y \in X$. In this way, $\omega = \{\omega_\lambda\}_{\lambda > 0}$ is a one-parameter family of functions $\omega_\lambda : X \times X \to [0, \infty]$. On the other hand, given $x, y \in X$, we may set $\omega^{x,y}(\lambda) = \omega(\lambda,\ x,\ y)$ for all $\lambda > 0$, so that $\omega^{x,y} : (0, \infty) \to [0, \infty]$.

Definition 19 ([35]). *A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a metric modular (or simply modular) on X if it satisfies the following three axioms:*
(i) given $x, y \in X, x = y$ if and only if $\omega_\lambda(x,\ y) = 0$ for all $\lambda > 0$;
(ii) $\omega_\lambda(x,\ y) = \omega_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;
(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x,\ z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

Weaker and stronger versions of conditions (i) and (iii) will be of great importance. If, instead of (i), the function ω satisfies (only) a weaker condition:
(i′) $\omega_\lambda(x,\ x) = 0$ for all $\lambda > 0$ and $x \in X$,
then ω is said to be a *pseudomodular* on X. Furthermore, if, instead of (i), the function ω satisfies (i) and a stronger condition:
(i_s) given $x, y \in X$ with $x \neq y, \omega_\lambda(x,\ y) \neq 0$ for all $\lambda > 0$, then ω is called a *strict modular* on X.

A modular (or pseudomodular, or strict modular) ω on X is said to be *convex* if, instead of (iii), it satisfied the (stronger) inequality (iv):
(iv) $\omega_{\lambda+\mu}(x, y) \leq \dfrac{\lambda}{\lambda+\mu}\omega_\lambda(x,\ z) + \dfrac{\mu}{\lambda+\mu}\omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

Remark 6. *(a) The assumption $\omega : (0, \infty) \times X \times X \to (-\infty, +\infty]$ in the definition of a pseudomodular does not lead to a greater generality: in fact, setting $y = x$ and $\mu = \lambda > 0$ in (iii) and taking into account (i′) and (ii), we find:*

$$0 = \omega_{2\lambda}(x, x) \leq \omega_\lambda(x,\ z) + \omega_\lambda(z, x) = 2\omega_\lambda(x,\ z)\ ,$$

thus, $\omega_\lambda(x,\ z) \geq 0$ or $\omega_\lambda(x,\ z) = \infty$ for all $\lambda > 0$ and $x, z \in X$.
(b) If $\omega_\lambda(x,\ y) = \omega_\lambda$ is independent of $x, y \in X$, then, by (i), $\omega \equiv 0$. Note that $\omega \equiv 0$ is only a pseudomodular on X (by virtue of (i)).
If $\omega_\lambda(x,\ y) = \omega(x,\ y)$ does not depend on $\lambda > 0$, then axioms (i)–(iii) mean that ω is an extended metric (extended pseudometric if (i) is replaced by (i′)) on $X; \omega$ is a metric on X if, in addition, it assumes finite values.
(c) Axiom (i) can be written as $(x = y) \iff (\omega_\lambda(x, y) = 0)$ and part $(i_\Leftarrow)$ in it—as $(x \neq y) \Rightarrow (\omega_\lambda(x, y) \neq 0)$. Condition (i_s) says that $(x \neq y) \Rightarrow (\omega_\lambda(x, y) \neq 0)$ for all $\lambda > 0$, and thus, it implies $(i_\Leftarrow)$. In other words, (i_s) means that if $\omega_\lambda(x, y) = 0$ for some $\lambda > 0$ (and not necessarily for all $\lambda > 0$ as in $(i_\Leftarrow)$), then $x = y$. Thus, $(i') + (i_s) \Rightarrow (i) \Rightarrow (i')$.

Following [35], we have the *essential property* of a pseudomodular ω on X is its monotonicity: given $x, y \in X$, the function $\omega^{x,y} : (0,\infty) \to [0,\infty]$ is *nonincreasing* on $(0,\infty)$. In fact, if $0 < \mu < \lambda$, then axioms (iii) (with $z = x$) and (i') imply:

$$\omega_\lambda(x,y) = \omega_{(\lambda-\mu)+\mu}(x,y) \le \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,\ y). \tag{3}$$

As a consequence, given $x, y \in X$, at each point $\lambda > 0$, the limit from the right:

$$(\omega_{+0})_\lambda(x,y) \equiv \omega_{\lambda+0}(x,y) = \lim_{\mu\to\lambda+0} \omega_\mu(x,y) = \sup\{\omega_\mu(x,y) : \mu > \lambda\} \tag{4}$$

and the limit from the left:

$$(\omega_{-0})_\lambda(x,y) \equiv \omega_{\lambda-0}(x,y) = \lim_{\mu\to\lambda-0} \omega_\mu(x,y) = \inf\{\omega_\mu(x,y) : 0 < \mu < \lambda\}, \tag{5}$$

exist in $[0,\infty]$, and the following inequalities hold, for all $0 < \mu < \lambda$:

$$\omega_{\lambda+0}(x,y) \le \omega_\lambda(x,y) \le \omega_{\lambda-0}(x,y) \le \omega_{\mu+0}(x,y) \le \omega_\mu(x,y) \le \omega_{\mu-0}(x,y) \tag{6}$$

To see this, by the monotonicity of ω, for any $0 < \mu < \mu_1 < \lambda_1 < \lambda$, we have:

$$\omega_\lambda(x,y) \le \omega_{\lambda_1}(x,\ y) \le \omega_{\mu_1}(x,y) \le \omega_\mu(x,y) \tag{7}$$

and it remains to pass to the limits as $\lambda_1 \to \lambda - 0$ and $\mu_1 \to \mu + 0$.

Remark 7. *For any $x_i \in X$, the set $X_\omega(x_i) = \{x \in X \text{ such that } \lim_{\lambda\to\infty} \omega_\lambda(x,x_i) = 0\}$ is called a modular metric space generated by x_i and induced by ω. If its generator x_i does not play any role in the situation (that is, X_ω is independent of generators), we shall write X_ω instead of $X_\omega(x_i)$.*

A metric modular on X_ω is said to be nonincreasing with respect to $\lambda > 0$ if for any $x, y \in X_\omega$ and $0 < \mu < \lambda$ such that $\omega_\lambda(x,y) \le \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y)$ for any $x, y \in X$ and $\lambda > 0$. We set $\omega_{\lambda^+}(x,y) := \lim_{\epsilon\uparrow 0} \omega_{\lambda+\epsilon}(x,y)$ and $\omega_{\lambda^-}(x,y) := \lim_{\epsilon\downarrow 0} \omega_{\lambda-\epsilon}(x,y)$

Remark 8. *For any $x, y \in X_\omega$, if a metric modular ω on X_ω has a finite value and $\omega_\lambda(x,y) = \omega_\mu(x,y)$ for all $\lambda, \mu > 0$, then $d(x,y) = \omega_\lambda(x,y)$ is a metric on X_ω.*

Remark 9. *Let ω be a modular on X and let X_ω be any one of the modular sets defined by ω. Then, $d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le \lambda\ \forall\ x,y \in X_\omega\}$ define a metric on X_ω. If the modular ω is convex, then the modular space can be endowed with another metric d^*_ω given by $d^*_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le 1\ \forall\ x,y \in X_\omega\}$. These metrics on the modular set are strongly equivalent: $d_\omega \le d^*_\omega \le 2d_\omega$, as recorded in [26].*

Definition 20 ([35], Sec. 2.1, page 19)**.** *Let (X,ω) be a modular space. Fix $x_0 \in X$. Set $X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) = 0 \text{ as } \lambda \to \infty\}$ and $X^*_\omega = X^*_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) < \infty \text{ for } \lambda > 0\}$, where X_ω and X^*_ω are said to be modular spaces centered at x_0.*

Definition 21 ([26])**.** *A metric modular on X is said to be nonincreasing with respect to $\lambda > 0$ if for any $x, y \in X$ and $0 < \mu < \lambda$ such that $\omega_\lambda(x,y) \le \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y)$ for any $x, y \in X$ and $\lambda > 0$. We set $\omega_{\lambda^+}(x,y) := \lim_{\epsilon\uparrow 0} \omega_{\lambda+\epsilon}(x,y)$ and $\omega_{\lambda^-}(x,y) := \lim_{\epsilon\downarrow 0} \omega_{\lambda-\epsilon}(x,y)$.*

Definition 22 ([31])**.** *Let X_ω be a modular metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_ω. Then*

(a) A point $x \in X_\omega$ is called a limit of $\{x_n\}_{n\in\mathbb{N}}$ if for each $\lambda, \epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\omega_\lambda(x_n, x) < \epsilon$ for all $n \ge N$.

A sequence that has a limit is said to be convergent or converges to x, which we write as $\lim_{n\to\infty} x_n = x$.

(b) *A sequence $\{x_n\}_{n\in\mathbb{N}} \in X_\omega$ is said to be a modular Cauchy sequence if for each $\lambda, \epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\omega_\lambda(x_n, x_m) < \epsilon$ whenever $n, m \geq N$.*

(c) *If every modular Cauchy sequence in X_ω converges in X_ω, then X_ω is said to be complete modular metric space.*

Following [21,31], we give the following definitions via binary relation in metric modular space.

Definition 23. *A modular metric space, X_ω is said to be $\sqsubseteq$-complete if each $\sqsubseteq$- preserving Cauchy sequence in X_ω converges.*

Remark 10. *Here, modular completeness and continuity are replaced by their relational analogs, i.e., $\sqsubseteq$-completeness and $\sqsubseteq$-continuity;*

Definition 24 ([30])**.** *Let ω be a metric modular on X and X_ω a modular metric space induced by ω and $T : X_\omega \to X_\omega$. A mapping T is called a contraction if for each $x, y \in X_\omega$ and for all $\lambda > 0$, there exists $0 \leq k < 1$ such that $\omega_\lambda(Tx, Ty) \leq k\omega_\lambda(x, y)$.*

Definition 25 ([25])**.** *Let (X, ω) be a modular metric space and $T : X_\omega \to X_\omega$ such that there are $\{\alpha, \beta, \gamma, \delta, \epsilon\} \in \mathcal{F}_{Ger}$. Then, T is called a Geraghty metric modular tower contraction map if for all $\lambda > 0$, then:*

$$\omega_\lambda(Tx, Ty) \leq \alpha(\omega_\lambda(x, y))\omega_\lambda(x, y)^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}} ; \tag{8}$$

where $A_\beta := \beta(\omega_\lambda(x, y))\omega_\lambda(x, Tx)$, $B_\gamma := \gamma(\omega_\lambda(x, y))\omega_\lambda(y, Ty)$, $C_\delta := \delta(\omega_\lambda(x, y))\omega_\lambda(x, Ty)$; $D_\epsilon := \epsilon(\omega_\lambda(x, y))\omega_\lambda(y, Tx)$, for all distinct $x, y \in X_\omega$.

Remark 11. *We can possibly interchange position(s) of $A_\beta, B_\gamma, C_\delta$, and D_ϵ in Definition 25.*

Following the construction by Okeke et al. [32] and Okeke and Francis [1], it will be useful before spelling out the results of Section 3. The following analogy will help. For each $n \in \mathbb{N}$, let $\mathcal{F}_{nGer}$ be the class of n-tuples of functions $\{\mu_1, \mu_2, \cdots \mu_n\}$ and for each $i \in \{1, 2, 3, \cdots, n\}$, the map $\mu_i : \mathbb{R}_+ \cup \{\infty\} \to [0, 1)$, so that we have the following $\mu(t_k) := \mu_1(t_k)\mu_2(t_k)\cdots\mu_n(t_k) \to 1 \implies t_k \to 0$. Now for each $m \in \{1, 2, \cdots, n\}$, suppose that $\{\mu_1, \mu_2, \cdots \mu_m\} \in \mathcal{F}_{mGer}$, then $\{\mu_1, \mu_2, \cdots \mu_m, \underbrace{0, 0, 0 \cdots, 0}_{n-m \text{ times}}\} \in \mathcal{F}_{mGer}$, where 0 is a zero function. Again, if $\underbrace{\{\mu, \mu, \cdots \mu\}}_{n \text{ times}} \in \mathcal{F}_{nGer}$, then $\mu(t_k) \to \frac{1}{n}$ implies $t_k \to 0$ as $k \to \infty$. If $\{\mu_1, \mu_2, \cdots \mu_n\} \in \mathcal{F}_{nGer}$, then $\pi(\mu_1, \mu_2, \cdots \mu_n) \in \mathcal{F}_{nGer}$ is a permutation of $(\mu_1, \mu_2, \cdots \mu_n)$. If $(\mu_1, \mu_2, \cdots \mu_n) \in \mathcal{F}_{nGer}$, then its subsequences i.e; $(\mu_{n_1}, \mu_{n_2}, \cdots \mu_{n_m}) \in \mathcal{F}_{mGer}$ for each $m \in \{1, 2, \cdots, n\}$. $\mu_{n_i} \neq \mu_{n_j}$ for all $i, j \in \{1, 2, \cdots, m\}$, where $\mu_{n_i} \in \{\mu_1, \mu_2, \cdots \mu_n\}$.

We will in this work take the class of functions in Definition 18 as $\mathcal{F}_{Ger}$, the class of all Geraghty functions.

3. Main Results

We start this section with a striking theorem concerning tower mappings in (pseudo) modular metric spaces.

Theorem 4. *Let $(X_\omega, \sqsubseteq)$ be a binary relation on (pseudo)modular set X_ω and suppose that there exists a metric modular d_ω in X_ω such that (X_ω, d_ω) is a $\sqsubseteq$-complete metric (pseudo)modular space. Let f, g be nondecreasing self mappings on X_ω and $\{\alpha, \delta, \epsilon\} \in \mathcal{F}_{Ger}$, $\lambda \in \Gamma := (0, \infty)$ satisfying:*

$$\omega_\lambda(fu, fv) \leq \alpha(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}}} , \tag{9}$$

where $A_\delta := \delta(\omega_\lambda(gu,gv))\omega_\lambda(gu,gv)$; $O_\epsilon := \epsilon(\omega_\lambda(gu,gv))\omega_\lambda(gv,fu)$, for all distinct close $u,v \in X_\omega$ and $\ln^k(\omega_\lambda(gu,gv)) \leq \omega_\lambda(gu,gv)$, $\omega_\lambda^k(gu,gv) \leq \omega_\lambda(gu,gv)$ for all $k \in \mathbb{N}$, $\sqsubseteq$ is (f,g)-closed and locally (f,g)-transitive, and $u,v \in X_\omega$ are $\sqsubseteq$-comparable. Assuming that there exists $x_0 \in X_\omega$ such that $gx_0 \sqsubseteq fx_0$; $f(X) \subseteq g(X)$; f is $(g,\sqsubseteq)$-nondecreasing; g is $\sqsubseteq$-continuous and commutes with f and $\{gu_k\}_{k\in\mathbb{N}}$ $\sqsubseteq$-converges to u^*, so that $gu_k \sqsubseteq gu^*$ for each $k \geq 1$ and f is a $\sqsubseteq$-continuous mapping. Then, f and g have, at least, a coincidence point, that is, there exists $z \in X_\omega$ such that $fz = gz$. Furthermore, for any $u,v \in X_\omega$, there exists $w \in X_\omega$ which is $\sqsubseteq$-comparable to u and v, i.e., $[u,v] \in \sqsubseteq$ so that f and g have a unique common fixed point in X_ω.

Proof. Suppose that $X_\omega = \emptyset$, then there is nothing to prove. Observe that in particular, if g (or f) is injective on the set of all coincidence points of f and g, then f and g have a unique coincidence point, which is also the common fixed point of f and g. Henceforth, we take $X_\omega \neq \emptyset$. Let $x_0 \in X_\omega$ such that $gx_0 \sqsubseteq fx_0$. *If $f(X_\omega) \subseteq g(X_\omega)$, then there exists a Picard sequence of $(f,\ g)$ based on any $x_0 \in X_\omega$.* Indeed, let $x_0 \in X_\omega$. Since $fx_0 \in f(X_\omega) \subseteq g(X_\omega)$, there exists $x_1 \in X_\omega$ such that $gx_1 = fx_0$. Analogously, since $fx_1 \in f(X_\omega) \subseteq g(X_\omega)$, there exists $x_2 \in X_\omega$ such that $gx_2 = fx_1$. Repeating this argument by the inductive hypothesis, we can find a Picard sequence of $(f,\ g)$ based on x_0. Since $f(X_\omega) \subseteq g(X_\omega)$, the above assertion guarantees the existence of a Picard sequence $\{x_n\}$ of $(f,\ g)$, that is, $gx_{n+1} = fx_n$, for all $n \geq 0$. Regarding that f is a $(g,\sqsubseteq)$ nondecreasing mapping, we observe that $gx_0 \sqsubseteq fx_0 = gx_1$ implies $gx_1 = fx_0 \sqsubseteq fx_1 = gx_2$. Inductively, we obtain:

$$gx_0 \sqsubseteq gx_1 \sqsubseteq gx_2 \sqsubseteq gx_3 \sqsubseteq \ldots \sqsubseteq gx_{n-1} \sqsubseteq gx_n \sqsubseteq gx_{n+1} \sqsubseteq \ldots. \tag{10}$$

This implies that $(gx_n, gx_{n+1}) \in \sqsubseteq$. If there exists n_0 such that $gx_{n_0} = gx_{n_{0+1}}$, then $gx_{n_0} = gx_{n_{0+1}} = fx_{n_0}$, that is, f and g have a coincidence point, which completes the existence part of the proof. On the contrary case, assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$, $\lambda \in \Gamma$, that is, $\omega_\lambda(gx_n,\ gx_{n+1}) > 0$ for all $n \geq 0$. Regarding inequality (10), we set $u = x_n$ and $v = x_{n+1}$ in inequality (9). Then we get, for all $n \in \mathbb{N}$:

$$\omega_\lambda(gx_{n+1},gx_{n+2}) = \omega_\lambda(fx_n,fx_{n+1}) = \omega_\lambda(fu,fv) \leq \alpha(\omega_\lambda(gx_n,gx_{n+1}))\omega_\lambda(gx_n,gx_{n+1})^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}}}; \tag{11}$$

where $A_\delta := \delta(\omega_\lambda(gx_n,gx_{n+1}))\omega_\lambda(gx_n,gx_{n+1})$; $O_\epsilon := \epsilon(\omega_\lambda(gx_n,gx_{n+1}))\omega_\lambda(gx_{n+1},fx_n)$.

Thus, $A_\delta := \delta(\omega_\lambda(gx_n,gx_{n+1}))\omega_\lambda(gx_n,gx_{n+1})$; $O_\epsilon := \epsilon(\omega_\lambda(gx_n,gx_{n+1}))\omega_\lambda(gx_{n+1}, gx_{n+1})$.

Therefore, inequality (11) collapses to:

$$\omega_\lambda(gx_{n+1},gx_{n+2}) \leq \alpha(\omega_\lambda(gx_n,gx_{n+1}))\omega_\lambda(gx_n,gx_{n+1}). \tag{12}$$

Then, the sequence $\{\omega_\lambda(gx_n,gx_{n+1})\}_{n\in\mathbb{N}}$ is nonincreasing sequence and bounded below, so $\lim_{n\to\infty} \omega_\lambda(gx_n,gx_{n+1}) = \ell \geq 0$. Assume that $\ell > 0$, then from inequality (12), we obtain:

$$\frac{\omega_\lambda(gx_{n+1},gx_{n+2})}{\omega_\lambda(gx_n,gx_{n+1})} \leq \alpha(\omega_\lambda(gx_n,gx_{n+1})), \tag{13}$$

for $n = 1,2,3,\cdots$. Then, from inequality (13), we get $1 \leq \lim_{n\to\infty} \alpha(\omega_\lambda(gx_n,gx_{n+1}))$ and since $\alpha \in \mathcal{F}_{Ger}$, this implies that $\ell = 0$. Then, $\lim_{n\to\infty} \omega_\lambda(gx_n,gx_{n+1}) = 0$. Now we show that $\{gx_n\}_{n\in\mathbb{N}}$ is a modular preserving Cauchy sequence in X_ω. On the contrary, assume that for each $n,m \in \mathbb{N}$, and $n > m$:

$$\lim_{n,m\to\infty} \omega_\lambda(gx_n,gx_m) > 0. \tag{14}$$

By triangle inequality:

$$\omega_\lambda(gx_n,gx_m) \leq \omega_{\frac{\lambda}{3}}(gx_n,gx_{n+1}) + \omega_{\frac{\lambda}{3}}(gx_{n+1},gx_{m+1}) + \omega_{\frac{\lambda}{3}}(gx_{m+1},gx_m).$$

Hence, from inequality (12), we have $\omega_\lambda(gx_n, gx_m) \leq [1-\alpha(\omega_\lambda(gx_n, gx_m))]^{-1}[\omega_\lambda(gx_n, gx_{n+1}) + \omega_\lambda(gx_{n+1}, gx_m)]$.

Since $\limsup_{n,m\to\infty} \omega_\lambda(gx_n, gx_m) > 0$ and $\lim_{n\to\infty} \omega_\lambda(gx_n, gx_{n+1}) = 0$, then $\lim_{n\to\infty} \limsup_{m\to\infty}(1 - \alpha(\omega_\lambda(gx_n, gx_m)))^{-1} = +\infty$, from which we obtain $\lim_{n,m\to\infty} \sup \alpha(\omega_\lambda(gx_n, gx_m)) = 1$. However, $\alpha \in \mathcal{F}_{Ger}$, we get:

$$\lim_{n,m\to\infty} \sup \omega_\lambda(gx_n, gx_m) = 0. \tag{15}$$

This contradicts inequality (14) and shows that $\{gx_n\}_{n\geq 1}$ is a modular preserving Cauchy sequence in X_ω.

Due to $\sqsubseteq$-completeness of (X_ω, d_ω), then there exists $z \in X_\omega$ such that $\lim_{n\to\infty} gx_n = z$, i.e., for all $\lambda \in \Gamma$, $\lim_{n\to\infty} \omega_\lambda(gx_n, z) \to 0$. Indeed, it suffices to show that *given a sequence* $\{gx_n\} \subset X_\omega$ *and* $z \in X_\omega$, *we have*: $gx_n \to z$ *if and only if* $\omega_\lambda(gx_n, z) \to 0$ *for all* $\lambda > 0$. Now, let $gx_n \to z$. Given $\epsilon > 0$, there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $d_\omega(gx_n, z) < \epsilon$ for all $n \geq n_0(\epsilon)$. By inequality (6) and Theorem 2.2.11(a) of ([35], Sec. 2.2), $\omega_\epsilon(gx_n, z) \leq \omega_{\epsilon-0}(gx_n, z) < \epsilon$ for all $n \geq n_0(\epsilon)$. Hence, if $\lambda > 0$, then, for any $n \geq n_0(\min\{\epsilon, \lambda\})$, we find, by inequality (3), that $\omega_\lambda(gx_n, z) \leq \omega_{\min\{\epsilon,\lambda\}}(gx_n, z) < \min\{\epsilon, \lambda\} \leq \epsilon$, and thus, $\omega_\lambda(gx_n, z) \to 0$. Conversely, if $\epsilon > 0$, then $\omega_\epsilon(gx_n, z) \to 0$, and thus, there is $n_1(\epsilon) \in \mathbb{N}$ such that $\omega_\epsilon(gx_n, z) \leq \epsilon$ for all $n \geq n_1(\epsilon)$. By Remark 9, d_ω implies $d_\omega(gx_n, z) \leq \epsilon$ for $n \geq n_1(\epsilon)$, i.e., $d_\omega(gx_n, z) \to 0$. Therefore, $\lim_{n\to\infty} gx_n = z$ is justified. As g and f are $\sqsubseteq$-continuous, $ggx_n \to gz$ and $fgx_n \to fz$ as $n \to \infty$. On the other hand, recall that g and f commute, so we have that $ggx_{n+1} = gfx_n = fgx_n = fz$ for all $n \geq 0$. Therefore, by the uniqueness of the limit of a preserving modular $\sqsubseteq$-convergent sequence, i.e., we invoke Theorem 4.1.1 of ([35], Sec. 4.1.1) and we conclude that $gz = fz$, that is, z is a coincidence point of f and g.

Suppose that x^* and z^* are coincidence points of f and g respectively, then there exists $w \in X_\omega$ such that $gx^* \sqsubseteq gw$ and $gz^* \sqsubseteq gw$. We claim that $gx^* = gz^*$. In fact, this is immediate from properties of binary relation. Without loss of generality, assume that x^* and z^* are two coincidence points of f and g and let $w \in X_\omega$ be such that $gx^* \sqsubseteq gw$ and $gz^* \sqsubseteq gw$. Let $\{w_n\}$ be a Picard sequence of $(f,\ g)$ based on the point $w_0 = w$. As $x^* \sqsubseteq w$ and $z^* \sqsubseteq w$ and f is a $(g, \sqsubseteq)$ nondecreasing mapping, then $gx^* = fx^* \sqsubseteq fw_0 = gw_1$ and $gz^* = fz^* \sqsubseteq fw_0 = gw_1$. Similarly, by induction, it is easy to prove that $gx^* \sqsubseteq gw_n$ and $gz^* \sqsubseteq gw_n$ for all $n \in \mathbb{N}$. Applying the inequality (9), for all $n \in \mathbb{N}, \lambda \in \Gamma$:

$$\omega_\lambda(gx^*, gw_{n+1}) = \omega_\lambda(fx^*, fw_n) = \omega_\lambda(fu, fv) \leq \alpha(\omega_\lambda(gx^*, gw_n))\omega_\lambda(gx^*, gw_n)^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}}}; \tag{16}$$

where $A_\delta := \delta(\omega_\lambda(gx^*, gw_n))\omega_\lambda(gx^*, gw_n)$; $O_\epsilon := \epsilon(\omega_\lambda(gx^*, gw_n))\omega_\lambda(gw_n, fx^*)$, so that:

$$\omega_\lambda(gx^*, gw_{n+1}) \leq \alpha(\omega_\lambda(gx^*, gw_n))\omega_\lambda(gx^*, gw_n)^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}}}; \tag{17}$$

where $A_\delta := \delta(\omega_\lambda(gx^*, gw_n))\omega_\lambda(gx^*, gw_n)$; $O_\epsilon := \epsilon(\omega_\lambda(gx^*, gw_n))\omega_\lambda(gw_n, gx^*)$. Now we see the possibility of the limiting processes; put $\prod_{d_\omega} := \omega_\lambda(gx^*, gw_{n+1})$, so that:

$$\begin{aligned}\prod_{d_\omega} &\leq \alpha(\omega_\lambda(gx^*, gw_n))\omega_\lambda(gx^*, gw_n)^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}}} \\ &= \alpha(\omega_\lambda(gx^*, gw_n)) \exp\left(\ln\left(\omega_\lambda(gx^*, gw_n)^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}}}\right)\right)\end{aligned}$$

$$= \alpha(\omega_\lambda(gx^*, gw_n)) \exp\left(A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}} \ln\left(\omega_\lambda(gx^*, gw_n) \right)\right)$$

$$= \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \frac{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon^i}}} \ln^i \left(\omega_\lambda(gx^*, gw_n) \right)}{i!}$$

$$\leq \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \frac{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon^i}}} \left(\omega_\lambda(gx^*, gw_n) \right)}{i!}$$

$$= \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \frac{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon^i}}}}{i!} \omega_\lambda(gx^*, gw_n)$$

$$= \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \frac{1}{i!} \exp\left(\ln\left(A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon^i}}} \right)\right) \omega_\lambda(gx^*, gw_n)$$

$$= \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \frac{1}{i!} \exp\left(O_\epsilon^{A_\delta^{O_\epsilon^i}} \ln\left(A_\delta \right)\right) \omega_\lambda(gx^*, gw_n)$$

$$= \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i!j!} O_\epsilon^{A_\delta^{O_\epsilon^{ij}}} \ln^j \left(A_\delta \right) \omega_\lambda(gx^*, gw_n)$$

$$\leq \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i!j!} O_\epsilon^{A_\delta^{O_\epsilon^{ij}}} A_\delta \omega_\lambda(gx^*, gw_n)$$

$$= \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i!j!} O_\epsilon^{A_\delta^{O_\epsilon^{ij}}} \delta(\omega_\lambda(gx^*, gw_n)) \omega_\lambda(gx^*, gw_n) \omega_\lambda(gx^*, gw_n)$$

$$\leq \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i!j!} O_\epsilon^{A_\delta^{O_\epsilon^{ij}}} \delta(\omega_\lambda(gx^*, gw_n)) \omega_\lambda(gx^*, gw_n)$$

$$= \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i!j!} \exp\left(\ln\left(O_\epsilon^{A_\delta^{O_\epsilon^{ij}}} \right)\right) \delta(\omega_\lambda(gx^*, gw_n)) \omega_\lambda(gx^*, gw_n)$$

$$= \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{i!j!k!} A_\delta^{O_\epsilon^{ijk}} \ln^k \left(O_\epsilon \right) \delta(\omega_\lambda(gx^*, gw_n)) \omega_\lambda(gx^*, gw_n)$$

$$\leq \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{i!j!k!} A_\delta^{O_\epsilon^{ijk}} O_\epsilon \delta(\omega_\lambda(gx^*, gw_n)) \omega_\lambda(gx^*, gw_n)$$

$$= \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{i!j!k!} A_\delta^{O_\epsilon^{ijk}} \epsilon(\omega_\lambda(gx^*, gw_n)) \omega_\lambda(gw_n, gx^*) \delta(\omega_\lambda(gx^*, gw_n)) \omega_\lambda(gx^*, gw_n)$$

$$\leq \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{i!j!k!} A_\delta^{O_\epsilon^{ijk}} \epsilon(\omega_\lambda(gx^*, gw_n)) \delta(\omega_\lambda(gx^*, gw_n)) \omega_\lambda(gx^*, gw_n)$$

$$= \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{i!j!k!} \exp\left(\ln\left(A_\delta^{O_\epsilon^{ijk}} \right)\right) \epsilon(\omega_\lambda(gx^*, gw_n)) \delta(\omega_\lambda(gx^*, gw_n)) \omega_\lambda(gx^*, gw_n)$$

$$= \alpha(\omega_\lambda(gx^*, gw_n)) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} O_\epsilon^{ijkl} \ln^l \left(A_\delta \right) \epsilon(\omega_\lambda(gx^*, gw_n)) \delta(\omega_\lambda(gx^*, gw_n)) \omega_\lambda(gx^*, gw_n)$$

$$\leq \alpha(\omega_\lambda(gx^*,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} O_\epsilon^{ijkl} A_\delta \epsilon(\omega_\lambda(gx^*,gw_n))\delta(\omega_\lambda(gx^*,gw_n))\omega_\lambda(gx^*,gw_n)$$
$$= \alpha(\omega_\lambda(gx^*,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} O_\epsilon^{ijkl} \delta(\omega_\lambda(gx^*,gw_n))\omega_\lambda(gx^*,gw_n)\epsilon(\omega_\lambda(gx^*,gw_n))\delta(\omega_\lambda(gx^*,gw_n))\omega_\lambda(gx^*,gw_n)$$
$$= \alpha(\omega_\lambda(gx^*,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} O_\epsilon^{ijkl} \delta(\omega_\lambda(gx^*,gw_n))\epsilon(\omega_\lambda(gx^*,gw_n))\delta(\omega_\lambda(gx^*,gw_n))\omega_\lambda(gx^*,gw_n)$$
$$= \alpha(\omega_\lambda(gx^*,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} O_\epsilon^{ijkl} \delta^2(\omega_\lambda(gx^*,gw_n))\epsilon(\omega_\lambda(gx^*,gw_n))\omega_\lambda(gx^*,gw_n)$$
$$= \alpha(\omega_\lambda(gx^*,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} (\epsilon(\omega_\lambda(gx^*,gw_n))\omega_\lambda(gw_n,gx^*))^{ijkl} \delta^2(\omega_\lambda(gx^*,gw_n))\epsilon(\omega_\lambda(gx^*,gw_n))\omega_\lambda(gx^*,gw_n)$$
$$= \alpha(\omega_\lambda(gx^*,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} \epsilon^{ijkl}(\omega_\lambda(gx^*,gw_n))\omega_\lambda^{ijkl}(gw_n,gx^*) \delta^2(\omega_\lambda(gx^*,gw_n))\epsilon(\omega_\lambda(gx^*,gw_n))\omega_\lambda(gx^*,gw_n)$$
$$\leq \alpha(\omega_\lambda(gx^*,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} \epsilon^{ijkl}(\omega_\lambda(gx^*,gw_n))\omega_\lambda(gw_n,gx^*) \delta^2(\omega_\lambda(gx^*,gw_n))\epsilon(\omega_\lambda(gx^*,gw_n))\omega_\lambda(gx^*,gw_n)$$
$$\leq \alpha(\omega_\lambda(gx^*,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} \epsilon^{ijkl}(\omega_\lambda(gx^*,gw_n)) \delta^2(\omega_\lambda(gx^*,gw_n))\epsilon(\omega_\lambda(gx^*,gw_n))\omega_\lambda(gx^*,gw_n)$$
$$\leq \alpha(\omega_\lambda(gx^*,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\epsilon^{ijkl+1}(\omega_\lambda(gx^*,gw_n))\delta^2(\omega_\lambda(gx^*,gw_n))}{i!j!k!l!}\omega_\lambda(gx^*,gw_n)$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\alpha(\omega_\lambda(gx^*,gw_n))\epsilon^{ijkl+1}(\omega_\lambda(gx^*,gw_n))\delta^2(\omega_\lambda(gx^*,gw_n))}{i!j!k!l!}\omega_\lambda(gx^*,gw_n)$$
$$\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \alpha(\omega_\lambda(gx^*,gw_n))\epsilon^{ijkl}(\omega_\lambda(gx^*,gw_n))\epsilon(\omega_\lambda(gx^*,gw_n))\delta^2(\omega_\lambda(gx^*,gw_n))\omega_\lambda(gx^*,gw_n). \quad (18)$$

Thus, by taking the limit as $n \to \infty$, from inequality (18), we see clearly that:

$$\lim_{n\to\infty} \omega_\lambda(gx^*,gw_n) = 0 \;\forall\; \lambda \in \Gamma. \quad (19)$$

Therefore, from Equation (19), we have that $\{gw_n\} \to gx^*$ as $n \to \infty$. Similarly, replacing gx^* in inequality (18) with gz^*, it can be proved that:

$$\omega_\lambda(gz^*,gw_{n+1}) \leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \alpha(\omega_\lambda(gz^*,gw_n))\epsilon^{ijkl}(\omega_\lambda(gz^*,gw_n))\epsilon(\omega_\lambda(gz^*,gw_n))\delta^2(\omega_\lambda(gz^*,gw_n))\omega_\lambda(gz^*,gw_n). \quad (20)$$

Thus:

$$1 \leq \liminf_{n\to\infty}\Big\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \alpha(\omega_\lambda(gz^*,gw_n))\epsilon^{ijkl}(\omega_\lambda(gz^*,gw_n))\epsilon(\omega_\lambda(gz^*,gw_n))\delta^2(\omega_\lambda(gz^*,gw_n))\Big\}, \quad (21)$$

hence, inequality (21) implies that:

$$\lim_{n\to\infty} \omega_\lambda(gz^*,gw_n) = 0, \quad (22)$$

Therefore, from Equation (22), we get $\{gw_n\} \to gz^*$ as $n \to \infty$. As a consequence, we have that $gx^* = gz^*$, which justified our claim. Next, we show that, for all coincidence point x^* of f and g, the point $\xi = fx^*$ is a common fixed point of f and g. Let $x^* \in X_\omega$ be an arbitrary coincidence point of f and g and let $\xi = fx^* = gx^*$. As f and g commutes, if f and g are commuting mappings and x^* is a coincidence point of f and g, then $\xi = fx^*$ is also a coincidence point of f and g. It follows from $f\xi = fgx^* = gfx^* = g\xi$, so that $\xi = fx^*$ is also a coincidence point of f and g. Then, $f\xi = g\xi$. Moreover, by our previous claim, we have that $gx^* = g\xi$. In particular, $f\xi = g\xi = gx^* = fx^* = \xi$. Therefore, ξ is a common fixed point of f and g.

Finally, we prove that f and g have a unique common fixed point. Let ξ and z be two common fixed points of f and g, that is, $\xi = f\xi = g\xi$ and $z = fz = gz$.

For any $\xi, z \in X_\omega$, there exists $w \in X_\omega$ which is $\sqsubseteq$-comparable to both ξ and z, i.e., $[\xi, z] \in \sqsubseteq$. Suppose that $\xi < z$ which implies that $\omega_\lambda(\xi, z) > 0$ for all $\lambda \in \Gamma$, so by triangle inequality we have that:

$$\begin{aligned}
\omega_\lambda(\xi, z) =& \omega_\lambda(f(\xi), f(z)) \\
\leq& \omega_{\frac{\lambda}{2}}(\xi, f(w_n)) + \omega_{\frac{\lambda}{2}}(z, f(w_n)) \\
\leq& \omega_\lambda(g(\xi), f(w_n)) + \omega_\lambda(g(z), f(w_n)) \\
=& \omega_\lambda(f(\xi), f(w_n)) + \omega_\lambda(f(z), f(w_n)).
\end{aligned} \tag{23}$$

Monotonicity implies that $f(w_n)$ is $\sqsubseteq$-comparable to $f(\xi) = \xi$ and $f(z) = z$ for $n = 1, 2, 3, \cdots$. Moreover, inequality (23) splits into two fold claims.

Claim a.

$$\begin{aligned}
\omega_\lambda(\xi, f(w_n)) =& \omega_\lambda(f(\xi), f(w_n)) \\
\leq& \alpha(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n)^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}}},
\end{aligned} \tag{24}$$

where $A_\delta := \delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n)$; $O_\epsilon := \epsilon(\omega_\lambda(g\xi, gw_n))\omega_\lambda(gw_n, f\xi)$, and hence, $A_\delta := \delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n)$; $O_\epsilon := \epsilon(\omega_\lambda(g\xi, gw_n))\omega_\lambda(gw_n, g\xi)$.

From inequality (24), we get $\Theta_{d_\omega} := \omega_\lambda(\xi, f(w_n))$. Thus:

$$\begin{aligned}
\Theta_{d_\omega} =& \omega_\lambda(f(\xi), f(w_n)) \\
\leq& \alpha(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n)^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}}} \\
=& \alpha(\omega_\lambda(g\xi, gw_n)) \exp\left(\ln \left(\omega_\lambda(g\xi, gw_n)^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}}} \right) \right) \\
=& \alpha(\omega_\lambda(g\xi, gw_n)) \exp\left(A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}} \ln \left(\omega_\lambda(g\xi, gw_n) \right) \right) \\
=& \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty} \frac{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon^i}}} \ln^i \left(\omega_\lambda(g\xi, gw_n) \right)}{i!} \\
\leq& \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty} \frac{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon^i}}} \left(\omega_\lambda(g\xi, gw_n) \right)}{i!} \\
=& \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty} \frac{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon^i}}}}{i!} \omega_\lambda(g\xi, gw_n) \\
=& \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty} \frac{1}{i!} \exp\left(\ln \left(A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon^i}}} \right) \right) \omega_\lambda(g\xi, gw_n)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty} \frac{1}{i!} \exp\left(O_\epsilon^{A_\delta^{O_\epsilon^i}} \ln\left(A_\delta\right)\right)\omega_\lambda(g\xi, gw_n) \\
&= \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{i!j!} O_\epsilon^{A_\delta^{O_\epsilon^{ij}}} \ln^j\left(A_\delta\right)\omega_\lambda(g\xi, gw_n) \\
&\leq \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{i!j!} O_\epsilon^{A_\delta^{O_\epsilon^{ij}}} A_\delta\omega_\lambda(g\xi, gw_n) \\
&= \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{i!j!} O_\epsilon^{A_\delta^{O_\epsilon^{ij}}} \delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n)\omega_\lambda(g\xi, gw_n) \\
&\leq \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{i!j!} O_\epsilon^{A_\delta^{O_\epsilon^{ij}}} \delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&= \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{i!j!} \exp\left(\ln\left(O_\epsilon^{A_\delta^{O_\epsilon^{ij}}}\right)\right)\delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&= \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{i!j!k!} A_\delta^{O_\epsilon^{ijk}} \ln^k\left(O_\epsilon\right)\delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&\leq \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{i!j!k!} A_\delta^{O_\epsilon^{ijk}} O_\epsilon\delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&= \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{i!j!k!} A_\delta^{O_\epsilon^{ijk}} \epsilon(\omega_\lambda(g\xi, gw_n))\omega_\lambda(gw_n, g\xi)\delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&\leq \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{i!j!k!} A_\delta^{O_\epsilon^{ijk}} \epsilon(\omega_\lambda(g\xi, gw_n))\delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&= \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{i!j!k!} \exp\left(\ln\left(A_\delta^{O_\epsilon^{ijk}}\right)\right)\epsilon(\omega_\lambda(g\xi, gw_n))\delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&= \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} O_\epsilon^{ijkl} \ln^l\left(A_\delta\right)\epsilon(\omega_\lambda(g\xi, gw_n))\delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&\leq \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} O_\epsilon^{ijkl} A_\delta\epsilon(\omega_\lambda(g\xi, gw_n))\delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&= \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} O_\epsilon^{ijkl} \delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n)\epsilon(\omega_\lambda(g\xi, gw_n))\delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&= \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} O_\epsilon^{ijkl} \delta(\omega_\lambda(g\xi, gw_n))\epsilon(\omega_\lambda(g\xi, gw_n))\delta(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&= \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} O_\epsilon^{ijkl} \delta^2(\omega_\lambda(g\xi, gw_n))\epsilon(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&= \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} (\epsilon(\omega_\lambda(g\xi, gw_n))\omega_\lambda(gw_n, g\xi))^{ijkl} \delta^2(\omega_\lambda(g\xi, gw_n))\epsilon(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&= \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} \epsilon^{ijkl}(\omega_\lambda(g\xi, gw_n))\omega_\lambda^{ijkl}(gw_n, g\xi)\delta^2(\omega_\lambda(g\xi, gw_n))\epsilon(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&\leq \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} \epsilon^{ijkl}(\omega_\lambda(g\xi, gw_n))\omega_\lambda(gw_n, g\xi)\delta^2(\omega_\lambda(g\xi, gw_n))\epsilon(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!}\epsilon^{ijkl}(\omega_\lambda(g\xi, gw_n))\delta^2(\omega_\lambda(g\xi, gw_n))\epsilon(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&\leq \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\epsilon^{ijkl+1}(\omega_\lambda(g\xi, gw_n))\delta^2(\omega_\lambda(g\xi, gw_n))}{i!j!k!l!}\omega_\lambda(g\xi, gw_n) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\alpha(\omega_\lambda(g\xi, gw_n))\epsilon^{ijkl+1}(\omega_\lambda(g\xi, gw_n))\delta^2(\omega_\lambda(g\xi, gw_n))}{i!j!k!l!}\omega_\lambda(g\xi, gw_n) \\
&\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \alpha(\omega_\lambda(g\xi, gw_n))\epsilon^{ijkl+1}(\omega_\lambda(g\xi, gw_n))\delta^2(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&= \alpha(\omega_\lambda(g\xi, gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \epsilon^{ijkl}(\omega_\lambda(g\xi, gw_n))\epsilon(\omega_\lambda(g\xi, gw_n))\delta^2(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \alpha(\omega_\lambda(g\xi, gw_n))\epsilon^{ijkl}(\omega_\lambda(g\xi, gw_n))\epsilon(\omega_\lambda(g\xi, gw_n))\delta^2(\omega_\lambda(g\xi, gw_n))\omega_\lambda(g\xi, gw_n). \quad (25)
\end{aligned}
$$

From Inequality (25), we get;

$$
1 \leq \lim_{n\to\infty}\inf\Big\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \alpha(\omega_\lambda(g\xi, gw_n))\epsilon^{ijkl}(\omega_\lambda(g\xi, gw_n))\epsilon(\omega_\lambda(g\xi, gw_n))\delta^2(\omega_\lambda(g\xi, gw_n))\Big\}, \quad (26)
$$

and thus, inequality (26) implies that $\lim_{n\to\infty} \omega_\lambda(g\xi, gw_n) = 0$, for all $\lambda \in \Gamma$. Hence:

$$
\lim_{n\to\infty} \omega_\lambda(\xi, f(w_n)) = 0 = \lim_{n\to\infty} \omega_\lambda(f(\xi), f(w_n)) = \lim_{n\to\infty} \omega_\lambda(g(\xi), f(w_n)). \quad (27)
$$

Again, for any $z \in X_\omega$, there exists $w \in X_\omega$ which is $\sqsubseteq$-comparable to z. Monotonicity implies that $f(w_n)$ is $\sqsubseteq$-comparable to $f(z) = z$ for $n = 1, 2, 3, \cdots$. Moreover, from inequality (23).

Claim b.

$$
\begin{aligned}
\omega_\lambda(z, f(w_n)) =& \omega_\lambda(f(z), f(w_n)) \\
&\leq \alpha(\omega_\lambda(gz, gw_n))\omega_\lambda(gz, gw_n)^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}}}, \quad (28)
\end{aligned}
$$

where $A_\delta := \delta(\omega_\lambda(gz, gw_n))\omega_\lambda(gz, gw_n)$; $O_\epsilon := \epsilon(\omega_\lambda(gz, gw_n))\omega_\lambda(gw_n, fz)$, and hence, $A_\delta := \delta(\omega_\lambda(gz, gw_n))\omega_\lambda(gz, gw_n)$; $O_\epsilon := \epsilon(\omega_\lambda(gz, gw_n))\omega_\lambda(gw_n, gz)$.

From inequality (28), we see that:

$$
\begin{aligned}
\Phi_{d_\omega} =& \omega_\lambda(f(z), f(w_n)) \\
&\leq \alpha(\omega_\lambda(gz, gw_n))\omega_\lambda(gz, gw_n)^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}}} \\
&= \alpha(\omega_\lambda(gz, gw_n)) \exp\left(\ln\left(\omega_\lambda(gz, gw_n)^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}}}\right)\right) \\
&= \alpha(\omega_\lambda(gz, gw_n)) \exp\left(A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}} \ln\left(\omega_\lambda(gz, gw_n)\right)\right) \\
&= \alpha(\omega_\lambda(gz, gw_n)) \sum_{i=0}^{\infty} \frac{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon^i}}} \ln^i\left(\omega_\lambda(gz, gw_n)\right)}{i!}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\frac{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon^{i}}}}\left(\omega_\lambda(gz,gw_n)\right)}{i!}\\
&= \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\frac{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon^{i}}}}}{i!}\omega_\lambda(gz,gw_n)\\
&= \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\frac{1}{i!}\exp\left(\ln\left(A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon^{i}}}}\right)\right)\omega_\lambda(gz,gw_n)\\
&= \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\frac{1}{i!}\exp\left(O_\epsilon^{A_\delta^{O_\epsilon^{i}}}\ln\left(A_\delta\right)\right)\omega_\lambda(gz,gw_n)\\
&= \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{i!j!}O_\epsilon^{A_\delta^{O_\epsilon^{ij}}}\ln^j\left(A_\delta\right)\omega_\lambda(gz,gw_n)\\
&\leq \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{i!j!}O_\epsilon^{A_\delta^{O_\epsilon^{ij}}}A_\delta\omega_\lambda(gz,gw_n)\\
&= \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{i!j!}O_\epsilon^{A_\delta^{O_\epsilon^{ij}}}\delta(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n)\omega_\lambda(gz,gw_n)\\
&\leq \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{i!j!}O_\epsilon^{A_\delta^{O_\epsilon^{ij}}}\delta(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n)\\
&= \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{i!j!}\exp\left(\ln\left(O_\epsilon^{A_\delta^{O_\epsilon^{ij}}}\right)\right)\delta(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n)\\
&= \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{i!j!k!}A_\delta^{O_\epsilon^{ijk}}\ln^k\left(O_\epsilon\right)\delta(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n)\\
&\leq \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{i!j!k!}A_\delta^{O_\epsilon^{ijk}}O_\epsilon\delta(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n)\\
&= \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{i!j!k!}A_\delta^{O_\epsilon^{ijk}}\epsilon(\omega_\lambda(gz,gw_n))\omega_\lambda(gw_n,gz)\delta(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n)\\
&\leq \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{i!j!k!}A_\delta^{O_\epsilon^{ijk}}\epsilon(\omega_\lambda(gz,gw_n))\delta(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n)\\
&= \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{i!j!k!}\exp\left(\ln\left(A_\delta^{O_\epsilon^{ijk}}\right)\right)\epsilon(\omega_\lambda(gz,gw_n))\delta(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n)\\
&= \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{i!j!k!l!}O_\epsilon^{ijkl}\ln^l\left(A_\delta\right)\epsilon(\omega_\lambda(gz,gw_n))\delta(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n)\\
&\leq \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{i!j!k!l!}O_\epsilon^{ijkl}A_\delta\epsilon(\omega_\lambda(gz,gw_n))\delta(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n)\\
&= \alpha(\omega_\lambda(gz,gw_n))\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{i!j!k!l!}O_\epsilon^{ijkl}\delta(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n)\epsilon(\omega_\lambda(gz,gw_n))\delta(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha(\omega_\lambda(gz,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} O_\epsilon^{ijkl} \delta(\omega_\lambda(gz,gw_n))\epsilon(\omega_\lambda(gz,gw_n))\delta(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n) \\
&= \alpha(\omega_\lambda(gz,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} O_\epsilon^{ijkl} \delta^2(\omega_\lambda(gz,gw_n))\epsilon(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n) \\
&= \alpha(\omega_\lambda(gz,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} (\epsilon(\omega_\lambda(gz,gw_n))\omega_\lambda(gw_n,gz))^{ijkl} \delta^2(\omega_\lambda(gz,gw_n))\epsilon(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n) \\
&= \alpha(\omega_\lambda(gz,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} \epsilon^{ijkl}(\omega_\lambda(gz,gw_n))\omega_\lambda^{ijkl}(gw_n,gz) \delta^2(\omega_\lambda(gz,gw_n))\epsilon(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n) \\
&\leq \alpha(\omega_\lambda(gz,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} \epsilon^{ijkl}(\omega_\lambda(gz,gw_n))\omega_\lambda(gw_n,gz) \delta^2(\omega_\lambda(gz,gw_n))\epsilon(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n) \\
&\leq \alpha(\omega_\lambda(gz,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!} \epsilon^{ijkl}(\omega_\lambda(gz,gw_n)) \delta^2(\omega_\lambda(gz,gw_n))\epsilon(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n) \\
&\leq \alpha(\omega_\lambda(gz,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\epsilon^{ijkl+1}(\omega_\lambda(gz,gw_n))\delta^2(\omega_\lambda(gz,gw_n))}{i!j!k!l!}\omega_\lambda(gz,gw_n) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\alpha(\omega_\lambda(gz,gw_n))\epsilon^{ijkl+1}(\omega_\lambda(gz,gw_n))\delta^2(\omega_\lambda(gz,gw_n))}{i!j!k!l!}\omega_\lambda(gz,gw_n) \\
&\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \alpha(\omega_\lambda(gz,gw_n))\epsilon^{ijkl+1}(\omega_\lambda(gz,gw_n))\delta^2(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n) \\
&= \alpha(\omega_\lambda(gz,gw_n)) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \epsilon^{ijkl}(\omega_\lambda(gz,gw_n))\epsilon(\omega_\lambda(gz,gw_n))\delta^2(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \alpha(\omega_\lambda(gz,gw_n))\epsilon^{ijkl}(\omega_\lambda(gz,gw_n))\epsilon(\omega_\lambda(gz,gw_n))\delta^2(\omega_\lambda(gz,gw_n))\omega_\lambda(gz,gw_n), \qquad (29)
\end{aligned}
$$

where $\Phi_{d_\omega} := \omega_\lambda(z, f(w_n))$.

From Inequality (29), we get:

$$1 \leq \lim_{n\to\infty} \inf\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \alpha(\omega_\lambda(gz,gw_n))\epsilon^{ijkl}(\omega_\lambda(gz,gw_n))\epsilon(\omega_\lambda(gz,gw_n))\delta^2(\omega_\lambda(gz,gw_n))\right\}, \qquad (30)$$

and thus, inequality (30) implies that $\lim_{n\to\infty} \omega_\lambda(gz, gw_n) = 0 \;\forall\; \lambda \in \Gamma$. Hence:

$$\lim_{n\to\infty} \omega_\lambda(z, f(w_n)) = 0 = \lim_{n\to\infty} \omega_\lambda(f(z), f(w_n)) = \lim_{n\to\infty} \omega_\lambda(g(z), f(w_n)). \qquad (31)$$

Finally, using inequalities (24) and (28), inequality (23) becomes $\omega_\lambda(\xi, z) \leq \omega_\lambda(\xi, f(w_n)) + \omega_\lambda(z, f(w_n)) = \omega_\lambda(f(\xi), f(w_n)) + \omega_\lambda(f(z), f(w_n)) = \omega_\lambda(g(\xi), f(w_n)) + \omega_\lambda(g(z), f(w_n))$, using Equations (27), (31) and on taking the limit as $n \to \infty$ yields $\omega_\lambda(\xi, z) = 0$ for all $\lambda \in \Gamma$, which is a contradiction. Therefore, $\xi = z$. Hence, $\xi = f(\xi) = g(\xi)$. □

Remark 12. *Theorem 4 is a generalization and further characterization of results in Amini-Harand and Emami [14], Geraghty [5], Banach [2], Alam et al. [20], Alam and Imdad [22,23], and Chisyakov [29]. Again, it is a common practice in analysis that whenever existence proof is made, then the uniqueness follows easily, but the case in this paper is entirely different.*

Example 1. *Let $X_\omega = [0,1] \cup \{\infty\}$. Define the operators $f, g : X_\omega \to X_\omega$ by:*

$$gu = \begin{cases} \dfrac{u}{2}, & \text{if } u \in X_\omega \\ 0, & \text{if } u = 0 \\ \dfrac{1}{2}, & \text{if } u = 1. \end{cases} \tag{32}$$

and:

$$fu = \begin{cases} \dfrac{u}{2}, & \text{if } u \in X_\omega \\ 0, & \text{if } u = 0. \end{cases} \tag{33}$$

Then, the maps satisfy all the conditions in Theorem 4.

Indeed, it suffices to show that inequality (9) and other conditions of Theorem 4 hold, where $A_\delta := \delta(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)$; $O_\epsilon := \epsilon(\omega_\lambda(gu, gv))\omega_\lambda(gv, fu)$, for all distinct close $u, v \in X_\omega$ and for all $\lambda \in \Gamma$.

Now, $\omega_\lambda(fu, fv) = \omega_\lambda(\frac{u}{2}, \frac{v}{2}) = \frac{1}{2}\omega_\lambda(u, v)$:

$$\begin{aligned} \omega_\lambda(gu, gv) &= \omega_\lambda(\frac{u}{2}, \frac{v}{2}) \\ &\leq \frac{1}{2}\omega_\lambda(u, v). \end{aligned}$$

$$\begin{aligned} \omega_\lambda(gv, fu) &= \omega_\lambda(\frac{v}{2}, \frac{u}{2}) \\ &\leq \frac{1}{2}\omega_\lambda(u, v). \end{aligned}$$

Thus, we get:

$$\omega_\lambda(fu, fv) = \frac{1}{2}\omega_\lambda(u, v) \leq \frac{1}{2}\alpha(\frac{1}{2}\omega_\lambda(u, v))\omega_\lambda(u, v)^{\left(\frac{\omega_\lambda(u,v)}{2}\right)^{\left(\frac{\omega_\lambda(u,v)}{2}\right)^{\left(\frac{\omega_\lambda(u,v)}{2}\right)^{\left(\frac{\omega_\lambda(u,v)}{2}\right)}}}}. \tag{34}$$

It is clear that $\delta(\frac{1}{2}\omega_\lambda(u, v)) = \frac{1}{2} < 1$ and $\epsilon(\frac{1}{2}\omega_\lambda(u, v)) = \frac{1}{2} < 1$. Now, we estimate $\alpha(\frac{1}{2}\omega_\lambda(u, v))$. The right hand side of inequality (34) becomes:

$$\begin{aligned} \omega_\lambda(u,v)^{\omega_\lambda(u,v)^{\omega_\lambda(u,v)^{\omega_\lambda(u,v)^{\omega_\lambda(u,v)}}}} &= \exp\left(\omega_\lambda(u,v)^{\omega_\lambda(u,v)^{\omega_\lambda(u,v)^{\omega_\lambda(u,v)}}} \ln\big(\omega_\lambda(u,v)\big)\right) \\ &= \sum_{i=0}^{\infty} \frac{\omega_\lambda(u,v)^{\omega_\lambda(u,v)^{\omega_\lambda(u,v)^{\omega_\lambda^i(u,v)}}}}{i!} \ln^i(\omega_\lambda(u,v)) \\ &\leq \sum_{i=0}^{\infty} \frac{\omega_\lambda(u,v)^{\omega_\lambda(u,v)^{\omega_\lambda(u,v)^{\omega_\lambda^i(u,v)}}}}{i!} \omega_\lambda(u,v) \\ &= \sum_{i=0}^{\infty} \exp\left(\omega_\lambda(u,v)^{\omega_\lambda(u,v)^{\omega_\lambda^i(u,v)}} \ln \omega_\lambda(u,v)\right) \frac{\omega_\lambda(u,v)}{i!} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\omega_\lambda(u,v)^{\omega_\lambda(u,v)^{\omega_\lambda^{ij}(u,v)}} \ln^j \omega_\lambda(u,v)}{i!j!}\omega_\lambda(u,v) \\
&\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\omega_\lambda(u,v)^{\omega_\lambda(u,v)^{\omega_\lambda^{ij}(u,v)}}}{i!j!}\omega_\lambda(u,v)\omega_\lambda(u,v) \\
&\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty} \frac{\omega_\lambda(u,v)^{\omega_\lambda^{ijk}(u,v)}}{i!j!k!}\omega_\lambda(u,v)\omega_\lambda(u,v)\omega_\lambda(u,v) \\
&\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{\omega_\lambda^{ijkl}(u,v)}{i!j!k!l!} \ln^l \omega_\lambda(u,v)\omega_\lambda(u,v)\omega_\lambda(u,v)\omega_\lambda(u,v) \\
&\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{\omega_\lambda^{ijkl+4}(u,v)}{i!j!k!l!} \\
&\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{\omega_\lambda(u,v)}{i!j!k!l!} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!}\omega_\lambda(u,v).
\end{aligned}
$$

Take $\alpha(\frac{1}{2}\omega_\lambda(u,v)) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!}$. *By classical analysis,* $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{i!j!k!l!}$ *is convergent. Then, there exists bounded* $z \in X_\omega$ *such that* $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{|z|}{i!j!k!l!} \leq \frac{1}{2}$. *Therefore, the right hand side of inequality (34) becomes* $\frac{1}{2}\omega_\lambda(u,v)$.

Hence:

$$
\omega_\lambda(fu,fv) = \frac{1}{2}\omega_\lambda(u,v) \leq \alpha(\omega_\lambda(gu,gv))\omega_\lambda(gu,gv)^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}}} = \frac{1}{2}\omega_\lambda(u,v). \tag{35}
$$

Therefore:

$$
\omega_\lambda(fu,fv) \leq \alpha(\omega_\lambda(gu,gv))\omega_\lambda(gu,gv)^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon}}}}; \tag{36}
$$

where $A_\delta := \delta(\omega_\lambda(gu,gv))\omega_\lambda(gu,gv)$; $O_\epsilon := \epsilon(\omega_\lambda(gu,gv))\omega_\lambda(gv,fu)$, *for all distinct close* $u,v \in X_\omega$. *Hence, all the conditions of Theorem 4 are satisfied. The trivial common fixed point is at* $u = 0$.

Corollary 1. *Let* $(X_\omega, \sqsubseteq)$ *be a binary relation on (pseudo)modular set* X_ω *and suppose that there exists a metric modular* d_ω *in* X_ω *such that* (X_ω, d_ω) *is a* $\sqsubseteq$*-complete metric (pseudo)modular space. Let* f,g *be nondecreasing self mappings on* X_ω *and* $\{\alpha,\delta,\epsilon\} \in \mathcal{F}_{Ger}$, $\lambda \in \Gamma := (0,\infty)$ *satisfying:*

$$
\omega_\lambda(fu,fv) \leq \alpha(\omega_\lambda(gu,gv))\omega_\lambda(gu,gv)^{A_\delta^{O_\epsilon}}; \tag{37}
$$

where $A_\delta := \delta(\omega_\lambda(gu,gv))\omega_\lambda(gu,gv)$; $O_\epsilon := \epsilon(\omega_\lambda(gu,gv))\omega_\lambda(gv,fu)$, *for all distinct close* $u,v \in X_\omega$ *and* $\ln^k(\omega_\lambda(gu,gv)) \leq \omega_\lambda(gu,gv)$, $\omega_\lambda^k(gu,gv) \leq \omega_\lambda(gu,gv)$ *for all* $k \in \mathbb{N}$, $\sqsubseteq$ *is* (f,g)*-closed and locally* (f,g)*-transitive and* $u,v \in X_\omega$ *are* $\sqsubseteq$*-comparable. Assuming that there exists* $x_0 \in X_\omega$ *such that* $gx_0 \sqsubseteq fx_0$; $f(X) \subseteq g(X)$; f *is* $(g,\sqsubseteq)$*-nondecreasing;* g *is* $\sqsubseteq$*-continuous and commutes with* f *and* $\{gu_k\}_{k\in\mathbb{N}}$ $\sqsubseteq$*-converges to* u^*, *so that* $gu_k \sqsubseteq gu^*$ *for each* $k \geq 1$ *and* f *is a* $\sqsubseteq$*-continuous mapping. Then,* f *and* g *have, at least, a coincidence point, that is, there exists* $z \in X_\omega$ *such that* $fz = gz$. *Furthermore, for any* $u,v \in X_\omega$, *there exists* $w \in X_\omega$ *which is* $\sqsubseteq$*-comparable to* u *and* v, *so that* f *and* g *have a unique common fixed point in* X_ω.

Corollary 2. *Let $(X_\omega, \sqsubseteq)$ be a binary relation on (pseudo)modular set X_ω and suppose that there exists a metric modular d_ω in X_ω such that (X_ω, d_ω) is a $\sqsubseteq$-complete metric (pseudo)modular space. Let f, g be nondecreasing self mappings on X_ω and $\{\alpha, \delta, \epsilon\} \in \mathcal{F}_{Ger}$, $\gamma \in \mathbb{R}^+$ and $\lambda \in \Gamma$ satisfying:*

$$\omega_\lambda(fu, fv) \le \alpha(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)^{A_\delta^{\gamma O_\epsilon^{A_\delta^{O_\epsilon}}}}; \tag{38}$$

where $A_\delta := \delta(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)$; $O_\epsilon := \epsilon(\omega_\lambda(gu, gv))\omega_\lambda(gv, fu)$, for all distinct close $u, v \in X_\omega$ and $\ln^k(\omega_\lambda(gu, gv)) \le \omega_\lambda(gu, gv)$, $\omega_\lambda^k(gu, gv) \le \omega_\lambda(gu, gv)$. For all $k \in \mathbb{N}$, $\sqsubseteq$ is (f, g)-closed and locally (f, g)-transitive and $u, v \in X_\omega$ are $\sqsubseteq$-comparable. Assuming that there exists $x_0 \in X_\omega$ such that $gx_0 \sqsubseteq fx_0$; $f(X_\omega) \subseteq g(X_\omega)$; f is $(g, \sqsubseteq)$-nondecreasing; g is $\sqsubseteq$-continuous and commutes with f and $\{gu_k\}_{k\in\mathbb{N}}$ $\sqsubseteq$-converges to u^*, so that $gu_k \sqsubseteq gu^*$ for each $k \ge 1$ and f is a $\sqsubseteq$-continuous mapping. Then, f and g have, at least, a coincidence point, that is, there exists $z \in X_\omega$ such that $fz = gz$. Furthermore, for any $u, v \in X_\omega$, there exists $w \in X_\omega$ which is $\sqsubseteq$-comparable to u and v, so that f and g have a unique common fixed point in X_ω.

Proof. We give the proof in two cases.

(a) Observe that if $\gamma = 0$, then inequality (38) reduced to:

$$\omega_\lambda(fu, fv) \le \alpha(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv) \ \forall\ \lambda \in \Gamma. \tag{39}$$

Observe that in particular, if g (or f) is injective on the set of all coincidence points of f and g, then f and g have a unique coincidence point, which is also the common fixed point of f and g. Let $x_0 \in X_\omega$ such that $gx_0 \sqsubseteq fx_0$. *If $f(X_\omega) \subseteq g(X_\omega)$, then there exists a Picard sequence of $(f,\ g)$ based on any $x_0 \in X_\omega$*. Indeed, let $x_0 \in X_\omega$. Since $fx_0 \in f(X_\omega) \subseteq g(X_\omega)$, there exists $x_1 \in X_\omega$ such that $gx_1 = fx_0$. Analogously, since $fx_1 \in f(X_\omega) \subseteq g(X_\omega)$, there exists $x_2 \in X_\omega$ such that $gx_2 = fx_1$. Repeating this argument by the inductive hypothesis, we can find a Picard sequence of $(f,\ g)$ based on x_0. Since $f(X_\omega) \subseteq g(X_\omega)$, the above assertion guarantees the existence of a Picard sequence $\{x_n\}$ of $(f,\ g)$, that is, $gx_{n+1} = fx_n$, for all $n \ge 0$. Regarding that f is a $(g, \sqsubseteq)$ nondecreasing mapping, we observe that $gx_0 \sqsubseteq fx_0 = gx_1$ implies $gx_1 = fx_0 \sqsubseteq fx_1 = gx_2$. Inductively, we obtain:

$$gx_0 \sqsubseteq gx_1 \sqsubseteq gx_2 \sqsubseteq gx_3 \sqsubseteq \ldots \sqsubseteq gx_{n-1} \sqsubseteq gx_n \sqsubseteq gx_{n+1} \sqsubseteq \ldots. \tag{40}$$

This implies that $(gx_n, gx_{n+1}) \in \sqsubseteq$. If there exists n_0 such that $gx_{n_0} = gx_{n_{0+1}}$, then $gx_{n_0} = gx_{n_{0+1}} = fx_{n_0}$, that is, f and g have a coincidence point, which completes the existence part of the proof. On the contrary case, assume that $gx_n \ne gx_{n+1}$ for all $n \in \mathbb{N}$ and for all $\lambda \in \Gamma$, that is, $\omega_\lambda(gx_n,\ gx_{n+1}) > 0$ for all $n \ge 0$. Regarding inequality (40), we set $u = x_n$ and $v = x_{n+1}$ in inequality (38). Then, we get, for all $n \in \mathbb{N}$:

$$\omega_\lambda(gx_{n+1}, gx_{n+2}) = \omega_\lambda(fx_n, fx_{n+1}) = \omega_\lambda(fu, fv) \le \alpha(\omega_\lambda(gx_n, gx_{n+1}))\omega_\lambda(gx_n, gx_{n+1}). \tag{41}$$

Therefore, inequality (41) collapse to:

$$\omega_\lambda(gx_{n+1}, gx_{n+2}) \le \alpha(\omega_\lambda(gx_n, gx_{n+1}))\omega_\lambda(gx_n, gx_{n+1}). \tag{42}$$

Then, the sequence $\{\omega_\lambda(gx_n, gx_{n+1})\}_{n\in\mathbb{N}}$ is nonincreasing sequence and bounded below, so $\lim_{n\to\infty} \omega_\lambda(gx_n, gx_{n+1}) = \ell \ge 0$. Assume that $\ell > 0$, then from inequality (42), we obtain:

$$\frac{\omega_\lambda(gx_{n+1}, gx_{n+2})}{\omega_\lambda(gx_n, gx_{n+1})} \le \alpha(\omega_\lambda(gx_n, gx_{n+1})), \tag{43}$$

for $n = 1, 2, 3, \cdots$. Then, from inequality (43), we get $i \le \lim_{n\to\infty} \alpha(\omega_\lambda(gx_n, gx_{n+1}))$ and since $\alpha \in \mathcal{F}_{Ger}$ this implies that $\ell = 0$. Then $\lim_{n\to\infty} \omega_\lambda(gx_n, gx_{n+1}) = 0$ for some $\lambda \in \Gamma$. Now, we show that $\{gx_n\}_{n\in\mathbb{N}}$ is a preserving modular Cauchy sequence in X_ω. On the contrary, assume that for each $n, m \in \mathbb{N}$, and $n > m$:

$$\lim_{n,m\to\infty} \omega_\lambda(gx_n, gx_m) > 0. \tag{44}$$

By triangle inequality:

$$\omega_\lambda(gx_n, gx_m) \le \omega_{\frac{\lambda}{3}}(gx_n, gx_{n+1}) + \omega_{\frac{\lambda}{3}}(gx_{n+1}, gx_{m+1}) + \omega_{\frac{\lambda}{3}}(gx_{m+1}, gx_m).$$

Hence, from inequality (42), we have $\omega_\lambda(gx_n, gx_m) \le [1 - \alpha(\omega_\lambda(gx_n, gx_m))]^{-1}[\omega_\lambda(gx_n, gx_{n+1}) + \omega_\lambda(gx_{n+1}, gx_m)]$.

Since $\limsup_{n,m\to\infty} \omega_\lambda(gx_n, gx_m) > 0$ and $\lim_{n\to\infty} \omega_\lambda(gx_n, gx_{n+1}) = 0$, for some $\lambda \in \Gamma$, then $\lim_{n\to\infty} \limsup_{m\to\infty} (1 - \alpha(\omega_\lambda(gx_n, gx_m)))^{-1} = +\infty$, from which we obtain $\lim_{n,m\to\infty} \sup \alpha(\omega_\lambda(gx_n, gx_m)) = 1$. However, with $\alpha \in \mathcal{F}_{Ger}$, we get:

$$\lim_{n,m\to\infty} \sup \omega_\lambda(gx_n, gx_m) = 0. \tag{45}$$

This contradicts inequality (44) and shows that $\{gx_n\}_{n\ge1}$ is a preserving modular Cauchy sequence in X_ω.

Due to the $\sqsubseteq$-completeness of (X_ω, d_ω), there exists $z \in X_\omega$ such that $\lim_{n\to\infty} gx_n = z$, i.e., for all $\lambda \in \Gamma$, $\lim_{n\to\infty} \omega_\lambda(gx_n, z) \to 0$. Indeed, it suffices to show that *given a sequence* $\{gx_n\} \subset X_\omega$ *and* $z \in X_\omega$, *we have*: $gx_n \to z$ *if and only if* $\omega_\lambda(gx_n, z) \to 0$ *for all* $\lambda > 0$. Now, let $gx_n \to z$. Given $\epsilon > 0$, there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $d_\omega(gx_n, z) < \epsilon$ for all $n \ge n_0(\epsilon)$. By inequality (6) and Theorem 2.2.11(a) of [35], $\omega_\epsilon(gx_n, z) \le \omega_{\epsilon-0}(gx_n, z) < \epsilon$ for all $n \ge n_0(\epsilon)$. Hence, if $\lambda > 0$, then, for any $n \ge n_0(\min\{\epsilon, \lambda\})$ we find, by inequality (3), that $\omega_\lambda(gx_n, z) \le \omega_{\min\{\epsilon,\lambda\}}(gx_n, z) < \min\{\epsilon, \lambda\} \le \epsilon$, and thus, $\omega_\lambda(gx_n, z) \to 0$. Conversely, if $\epsilon > 0$, then $\omega_\epsilon(gx_n, z) \to 0$, and thus, there is $n_1(\epsilon) \in \mathbb{N}$ such that $\omega_\epsilon(gx_n, z) \le \epsilon$ for all $n \ge n_1(\epsilon)$. By Remark 9, d_ω implies $d_\omega(gx_n, z) \le \epsilon$ for $n \ge n_1(\epsilon)$, i.e., $d_\omega(gx_n, z) \to 0$. Therefore, $\lim_{n\to\infty} gx_n = z$ is justified. As g and f are $\sqsubseteq$-continuous, $ggx_n \to gz$ and $fgx_n \to fz$ as $n \to \infty$. On the other hand, recall that g and f commute, so we have that $ggx_{n+1} = gfx_n = fgx_n = fz$ for all $n \ge 0$. Therefore, by the uniqueness of the limit of a $\sqsubseteq$-convergent sequence, we conclude that $gz = fz$, that is, z is a coincidence point of f and g.

Suppose that x^* and z^* are coincidence points of f and g respectively, then there exists $w \in X_\omega$ such that $gx^* \sqsubseteq gw$ and $gz^* \sqsubseteq gw$. We claim that $gx^* = gz^*$. In fact, this is immediate from properties of binary relation. Without loss of generality, assume that x^* and z^* are two coincidence points of f and g and let $w \in X_\omega$ be such that $gx^* \sqsubseteq gw$ and $gz^* \sqsubseteq gw$. Let $\{w_n\}$ be a Picard sequence of (f, g) based on the point $w_0 = w$. As $x^* \sqsubseteq w$ and $z^* \sqsubseteq w$ and f is a $(g, \sqsubseteq)$ nondecreasing mapping, then $gx^* = fx^* \sqsubseteq fw_0 = gw_1$ and $gz^* = fz^* \sqsubseteq fw_0 = gw_1$. Similarly, by induction, it is easy to prove that $gx^* \sqsubseteq gw_n$ and $gz^* \sqsubseteq gw_n$ for all $n \in \mathbb{N}$. Applying the inequality (38), for all $k \in \mathbb{N}$:

$$\omega_\lambda(gx^*, gw_{n+1}) = \omega_\lambda(fx^*, fw_n) = \omega_\lambda(fu, fv) \le \alpha(\omega_\lambda(gx^*, gw_n))\omega_\lambda(gx^*, gw_n), \tag{46}$$

so that:

$$\omega_\lambda(gx^*, gw_{n+1}) \le \alpha(\omega_\lambda(gx^*, gw_n))\omega_\lambda(gx^*, gw_n). \tag{47}$$

Thus, by taking the limit as $n \to \infty$, from inequality (47), we can see clearly that:

$$\lim_{n\to\infty} \omega_\lambda(gx^*, gw_n) = 0 \;\forall\; \lambda \in \Gamma. \tag{48}$$

Therefore, from Equation (48) $\{gw_n\} \to gx^*$ as $n \to \infty$. Similarly, replacing gx^* in inequality (47) with gz^*, it can be proved that:

$$\lim_{n\to\infty} \omega_\lambda(gz^*, gw_n) = 0 \;\forall\; \lambda \in \Gamma. \tag{49}$$

Therefore, from Equation (49) $\{gw_n\} \to gz^*$ as $n \to \infty$. As a consequence, we have that $gx^* = gz^*$, which justifies our claim. Next, we show that, for all coincidence point x^* of f and g, the point $\xi = fx$ is a common fixed point of f and g. Let $x^* \in X$ be an arbitrary coincidence point of f and g and let $\xi = fx = gx$. As f and g commute, if f and g are commuting mappings and x^* is a coincidence point of f and g, then $\xi = fx$ is also a coincidence point of f and g. It follows from $f\xi = fgx^* = gfx^* = g\xi$, so that $\xi = fx^*$ is also a coincidence point of f and g. Then, $f\xi = g\xi$. Moreover, by our previous claim, we have that $gx^* = g\xi$. In particular, $f\xi = g\xi = gx^* = fx^* = \xi$. Therefore, ξ is a common fixed point of f and g. Now the conclusion follow

(b) We suppose that $\gamma \neq 0$. Copy the proof of Theorem 4. □

Corollary 3. *Let $(X_\omega, \sqsubseteq)$ be a binary relation on (pseudo)modular set X_ω and suppose that there exists a metric modular d_ω in X_ω such that (X_ω, d_ω) is a $\sqsubseteq$-complete metric (pseudo)modular space. Let f, g be nondecreasing self mappings on X_ω and $\{\alpha, \delta, \epsilon\} \in \mathcal{F}_{Ger}$, $\lambda \in \Gamma$ satisfying:*

$$\omega_\lambda(fu, fv) \leq \alpha(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)^{A_\delta^{A_\epsilon^{A_\delta^{A_\epsilon}}}}; \tag{50}$$

where $A_\delta := \delta(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)$; for all distinct close $u, v \in X_\omega$ and $\ln^k(\omega_\lambda(gu, gv)) \leq \omega_\lambda(gu, gv)$, $\omega_\lambda^k(gu, gv) \leq \omega_\lambda(gu, gv)$ for all $k \in \mathbb{N}, \lambda \in \Gamma$, $\sqsubseteq$ is (f, g)-closed and locally (f, g)-transitive and $u, v \in X_\omega$ are $\sqsubseteq$-comparable. Assuming that there exists $x_0 \in X_\omega$ such that $gx_0 \sqsubseteq fx_0$; $f(X_\omega) \subseteq g(X_\omega)$; f is $(g, \sqsubseteq)$-nondecreasing; g is $\sqsubseteq$-continuous and commutes with f and $\{gu_k\}_{k\in\mathbb{N}}$ $\sqsubseteq$-converges to u^*, so that $gu_k \sqsubseteq gu^*$ for each $k \geq 1$ and f is a $\sqsubseteq$-continuous mapping. Then, f and g have, at least, a coincidence point, that is, there exists $z \in X_\omega$ such that $fz = gz$. Furthermore, for any $u, v \in X_\omega$, there exists $w \in X_\omega$ which is $\sqsubseteq$-comparable to u and v, then f and g have a unique common fixed point in X_ω.

Proof. Observe that in particular, if g (or f) is injective on the set of all coincidence points of f and g, then f and g have a unique coincidence point, which is also the common fixed point of f and g. Let $x_0 \in X_\omega$ such that $gx_0 \sqsubseteq fx_0$. *If $f(X_\omega) \subseteq g(X_\omega)$, then there exists a Picard sequence of $(f,\ g)$ based on any $x_0 \in X_\omega$*. Indeed, let $x_0 \in X_\omega$. Since $fx_0 \in f(X_\omega) \subseteq g(X_\omega)$, there exists $x_1 \in X_\omega$ such that $gx_1 = fx_0$. Analogously, since $fx_1 \in f(X_\omega) \subseteq g(X_\omega)$, there exists $x_2 \in X_\omega$ such that $gx_2 = fx_1$. Repeating this argument by the inductive hypothesis, we can find a Picard sequence of $(f,\ g)$ based on x_0. Since $f(X_\omega) \subseteq g(X_\omega)$, the above assertion guarantees the existence of a Picard sequence $\{x_n\}$ of $(f,\ g)$, that is, $gx_{n+1} = fx_n$, for all $n \geq 0$. Regarding that f is a $(g, \sqsubseteq)$-nondecreasing mapping, we observe that $gx_0 \sqsubseteq fx_0 = gx_1$ implies $gx_1 = fx_0 \sqsubseteq fx_1 = gx_2$. Inductively, we obtain:

$$gx_0 \sqsubseteq gx_1 \sqsubseteq gx_2 \sqsubseteq gx_3 \sqsubseteq \ldots \sqsubseteq gx_{n-1} \sqsubseteq gx_n \sqsubseteq gx_{n+1} \sqsubseteq \ldots. \tag{51}$$

This implies that $(gx_n, gx_{n+1}) \in \sqsubseteq$. If there exists n_0 such that $gx_{n_0} = gx_{n_{0+1}}$, then $gx_{n_0} = gx_{n_{0+1}} = fx_{n_0}$, that is, f and g have a coincidence point, which completes the existence part of the proof. On the contrary case, assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$, that is, $\omega_\lambda(gx_n,\ gx_{n+1}) > 0$ for all $n \geq 0, \lambda \in \Gamma$. Regarding inequality (51), we set $u = x_n$ and $v = x_{n+1}$ in inequality (50). Then, we get, for all $n \in \mathbb{N}$:

$$\omega_\lambda(gx_{n+1}, gx_{n+2}) = \omega_\lambda(fx_n, fx_{n+1}) = \omega_\lambda(fu, fv) \leq \alpha(\omega_\lambda(gx_n, gx_{n+1}))\omega_\lambda(gx_n, gx_{n+1})^{A_\delta^{A_\epsilon^{A_\delta^{A_\epsilon}}}}; \tag{52}$$

where $A_\delta := \delta(\omega_\lambda(gx_n, gx_{n+1}))\omega_\lambda(gx_n, gx_{n+1})$.

Thus, $A_\delta := \delta(\omega_\lambda(gx_n, gx_{n+1}))\omega_\lambda(gx_n, gx_{n+1})$. Carefully following the proof of Theorem 4, the conclusion follows. □

Corollary 4. *Let $(X_\omega, \sqsubseteq)$ be a binary relation on (pseudo)modular set X_ω and suppose that there exists a metric modular d_ω in X_ω such that (X_ω, d_ω) is a $\sqsubseteq$-complete metric (pseudo)modular* *space. Let f, g be nondecreasing self mappings on X_ω for all $\lambda \in \Gamma$ and $\{\alpha, \delta, \epsilon\} \in \mathcal{F}_{Ger}$, $\gamma \in \mathbb{R}^+$ satisfying:*

$$\omega_\lambda(fu, fv) \leq \alpha(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)^{A_\delta^{\gamma A_\epsilon^{A_\delta^{A_\epsilon}}}}; \tag{53}$$

where $A_\delta := \delta(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)$; for all distinct close $u, v \in X_\omega$ and $\ln^k(\omega_\lambda(gu, gv)) \leq \omega_\lambda(gu, gv)$, $\omega_\lambda^k(gu, gv) \leq \omega_\lambda(gu, gv) < \infty$ for all $k \in \mathbb{N}$, $\sqsubseteq$ is (f, g)-closed and locally (f, g)-transitive and $u, v \in X_\lambda$ are $\sqsubseteq$-comparable. Assuming that there exists $x_0 \in X_\omega$ such that $gx_0 \sqsubseteq fx_0$; $f(X_\omega) \subseteq g(X_\omega)$; f is $(g, \sqsubseteq)$ nondecreasing; g is $\sqsubseteq$-continuous and commutes with f and $\{gu_k\}_{k\in\mathbb{N}}$ $\sqsubseteq$-converges to u^*, so that $gu_k \sqsubseteq gu^*$ for each $k \geq 1$ and f is a $\sqsubseteq$-continuous mapping. Then, f and g have, at least, a coincidence point, that is, there exists $z \in X_\omega$ such that $fz = gz$. Furthermore, for any $u, v \in X_\omega$, there exists $w \in X_\omega$, which is $\sqsubseteq$-comparable to u and v, then f and g have a unique common fixed point in X_ω.

Proof. We give the proof in three cases.

(a) Observe that if $\gamma = 0$, then inequality (53) is reduced to:

$$\omega_\lambda(fu, fv) \leq \alpha(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)\ \forall\ \lambda \in \Gamma. \tag{54}$$

Observe that in particular, if g (or f) is injective on the set of all coincidence points of f and g, then f and g have a unique coincidence point, which is also the common fixed point of f and g. Let $x_0 \in X_\omega$ such that $gx_0 \sqsubseteq fx_0$. *If $f(X_\omega) \subseteq g(X_\omega)$, then there exists a Picard sequence of $(f,\ g)$ based on any $x_0 \in X_\omega$*. Indeed, let $x_0 \in X_\omega$. Since $fx_0 \in f(X_\omega) \subseteq g(X_\omega)$, there exists $x_1 \in X_\omega$ such that $gx_1 = fx_0$. Analogously, since $fx_1 \in f(X_\omega) \subseteq g(X_\omega)$, there exists $x_2 \in X_\omega$ such that $gx_2 = fx_1$. Repeating this argument by the inductive hypothesis, we can find a Picard sequence of $(f,\ g)$ based on x_0. Since $f(X_\omega) \subseteq g(X_\omega)$, the above assertion guarantees the existence of a Picard sequence $\{x_n\}$ of $(f,\ g)$, that is, $gx_{n+1} = fx_n$, for all $n \geq 0$. Regarding that f is a $(g, \sqsubseteq)$ nondecreasing mapping, we observe that $gx_0 \sqsubseteq fx_0 = gx_1$ implies $gx_1 = fx_0 \sqsubseteq fx_1 = gx_2$. Part (a) of Corollary 2 finishes the proof of (a).

(b) Observe also that if $\gamma = 1$, then inequality (53) is reduced to:

$$\omega_\lambda(fu, fv) \leq \alpha(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)^{A_\delta^{A_\epsilon^{A_\delta^{A_\epsilon}}}}; \tag{55}$$

and the Corollary 3 finishes the proof of (b).

(c) Now we suppose that $\gamma \in \mathbb{R}$. With little effort, Corollary 3 concludes the proof of (c). □

Corollary 5. *Let $(X_\omega, \sqsubseteq)$ be a binary relation on (pseudo)modular set X_ω and suppose that there exists a metric modular d_ω in X_ω such that (X_ω, d_ω) is a $\sqsubseteq$-complete metric (pseudo)modular space. Let f, g be nondecreasing self mappings on X_ω and $\{\alpha, \delta, \epsilon\} \in \mathcal{F}_{Ger}$, $\lambda \in \Gamma$ satisfying:*

$$\omega_\lambda(fu, fv) \leq \alpha(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)^{A_\delta^{\min\left\{\frac{O_\epsilon}{1+O_\epsilon}, \frac{A_\delta}{1+O_\epsilon}\right\}}}, \tag{56}$$

where $A_\delta := \delta(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)$; $O_\epsilon := \epsilon(\omega_\lambda(gu, gv))\omega_\lambda(gv, fu)$, for all distinct close $u, v \in X_\omega$ and $\ln^k(\omega_\lambda(gu, gv)) \leq \omega_\lambda(gu, gv)$, $\omega_\lambda^k(gu, gv) \leq \omega_\lambda(gu, gv)$ for all $k \in \mathbb{N}$, $\sqsubseteq$ is (f, g)-closed and locally (f, g)-transitive and $u, v \in X_\omega$ are $\sqsubseteq$-comparable. Assuming that there exists $x_0 \in X_\omega$ such that $gx_0 \sqsubseteq fx_0$; $f(X_\omega) \subseteq g(X\omega)$; f is $(g, \sqsubseteq)$-nondecreasing; g is $\sqsubseteq$-continuous and commutes with f and $\{gu_k\}_{k\in\mathbb{N}}$ $\sqsubseteq$-converges to u^*, so that $gu_k \sqsubseteq gu^*$ for each $k \geq 1$ and f is a $\sqsubseteq$-continuous mapping. Then, f and g have, at least, a coincidence point, that is, there exists $z \in X_\omega$ such that $fz = gz$. Furthermore, for any $u, v \in X_\omega$, there exists $w \in X_\omega$ which is $\sqsubseteq$-comparable to u and v, so that f and g have a unique common fixed point in X_ω.

Proof. It follows from proof of part (a) of Corollary 2. □

Corollary 6. *Let $(X_\omega, \sqsubseteq)$ be a binary relation on (pseudo)modular set X_ω and suppose that there exists a metric modular d_ω in X_ω such that (X_ω, d_ω) is a $\sqsubseteq$-complete metric (pseudo)modular space. Let f, g be nondecreasing self mappings on X_ω for all $\lambda \in \Gamma$ and $\{\alpha, \delta, \epsilon\} \in \mathcal{F}_{Ger}$ satisfying:*

$$\omega_\lambda(fu, fv) \leq \alpha(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)^{A_\delta^{\max\left\{\frac{O_\epsilon}{1+O_\epsilon}, \frac{A_\delta}{1+O_\epsilon}\right\}}}; \tag{57}$$

where $A_\delta := \delta(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)$; $O_\epsilon := \epsilon(\omega_\lambda(gu, gv))\omega_\lambda(gv, fu)$, for all distinct close $u, v \in X_\omega$ and $\ln^k(\omega_\lambda(gu, gv)) \leq \omega_\lambda(gu, gv)$, $\omega_\lambda^k(gu, gv) \leq \omega_\lambda(gu, gv)$ for all $k \in \mathbb{N}$, $\sqsubseteq$ is (f, g)-closed and locally (f, g)-transitive and $u, v \in X_\omega$ are $\sqsubseteq$-comparable. Assuming that there exists $x_0 \in X_\omega$ such that $gx_0 \sqsubseteq fx_0$; $f(X_\omega) \subseteq g(X\omega)$; f is $(g, \sqsubseteq)$-nondecreasing; g is $\sqsubseteq$-continuous and commutes with f and $\{gu_k\}_{k\in\mathbb{N}}$ $\sqsubseteq$-converges to u^*, so that $gu_k \sqsubseteq gu^*$ for each $k \geq 1$ and f is a $\sqsubseteq$-continuous mapping. Then, f and g have, at least, a coincidence point, that is, there exists $z \in X_\omega$ such that $fz = gz$. Furthermore, for any $u, v \in X_\omega$, there exists $w \in X_\omega$, which is $\sqsubseteq$-comparable to u and v, so that f and g have a unique common fixed point in X_ω.

Question: Is it possible to have a common unique fixed point in Theorem 4 if we replace inequality (9) with an infinite type of the form? That is:

$$\omega_\lambda(fu, fv) \leq \alpha(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)^{A_\delta^{O_\epsilon^{A_\delta^{O_\epsilon^{\cdot^{\cdot^{\cdot}}}}}}}, \tag{58}$$

where $A_\delta := \delta(\omega_\lambda(gu, gv))\omega_\lambda(gu, gv)$; $O_\epsilon := \epsilon(\omega_\lambda(gu, gv))\omega_\lambda(gv, fu)$, for $u, v \in X_\omega$.

4. Conclusions

The results established in this paper are new, novel, and interesting. Our findings here further characterize and include in their full strength some results in the literature on the playground of binary relational metric pseudo(modular) spaces. Particularly, results such as Banach contraction mapping and Geraghty contraction, i.e., Amini-Harand and Emami [14], Geraghty [5], Banach [2], Alam et al. [20], Alam and Imdad [22,23], Chisyakov [29], etc. become special cases.

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