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# Gauge Symmetry of Magnetic and Electric Two-Potentials with Magnetic Monopoles

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**Abstract:** We generalize the U(1) gauge transformations of electrodynamics by means of an analytical extension of their parameter space and observe that this leads naturally to two gauge potentials, one electric, one magnetic, which permit the writing of local Lagrangians describing elementary particles with electric and magnetic charges. Gauge invariance requires a conformal transformation of the metric tensor. We apply this approach, which borrows from Utiyama's methodology, to a model with a massless scalar field and a model with a massless spinor field. We observed that for spinor models non-symmetrized Lagrangians can enable the existence of magnetic monopoles, but this is not possible with symmetrized Lagrangian. Such restrictions do not occur for spinless fields, but the model does not allow spin-one fields interacting with monopoles.

**Keywords:** magnetic monopole; two-potential formalism; gauge symmetry; conformal transformations

## 1. Introduction

The history of magnetism is very long and rich, with some of its earliest scientific discussions attributed to the Greek Thales of Miletus during the 6th century BC. And yet, even today, we do not have a full understanding of why there are two poles in magnets, and it is not even clear whether isolated magnetic poles (or magnetic monopoles) exist, despite much theoretical and experimental work on these questions [1–5]. In 1269, the French scientist Pierre le Pèlerin de Maricourt, or Petrus Peregrinus of Maricourt, wrote his *Epistola de magnete*, in which he described his observations on properties of magnets, such as the presence of two opposite poles in magnets (Maricourt introduced the term *polus*), the repulsion between two like poles, and the fact that cutting a magnet in two halves results in two halves with two further opposite poles [6]. Another breakthrough came in 1864 with James Clerk Maxwell's theory of electrodynamics, based on the discoveries by many precursors such as Ampère and Faraday, and in which Maxwell simply assumed that magnetic charges did not exist, but did not forbid them either [7]. At that time, and for a few years thereafter, the general opinion among physicists was that magnetic monopoles were not elementary particles that existed in reality. In 1894, however, Pierre Curie made the, then bold, suggestion that monopoles could exist in nature, even though none had been observed [8].

It is common to credit Paul Dirac for the first modern investigation of magnetic monopoles, in 1931. For more references pre-dating Dirac's contribution, see the reprint book [9]. Not only did Dirac show that monopoles are compatible with quantum mechanics, but he also demonstrated that the existence of monopoles would explain the observation that electric charge is discrete (or 'quantized', as it is more commonly referred to); that is, a magnetic charge  $g$  would imply that electric charges are given by [10]



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$$e = \frac{2\pi}{g}n. \quad (1)$$

Seventeen years later, Dirac showed that a Lorentz-invariant Lagrangian containing monopoles should be non-local [11]. About twenty years after that, Zwanziger wrote a local Lagrangian with an additional, ‘magnetic photon’ field, by expanding on former ideas by Cabibbo and Ferrari [12]. However, he also needed a constraint in order to reduce the degrees of freedom to that of the on-shell photon, thereby requiring a space-like Lorentz-symmetry-violating four-vector associated with the direction of the Dirac string, itself an unphysical artifact [1,13–15]. (Hereafter, we shall not discuss ‘dyons’, hypothetical objects with both electric and magnetic charges, investigated in Refs. [13,16], and proposed as a phenomenological alternative to quarks in Ref. [17]. Let us just mention that the Dirac quantization condition is generalized, given two dyons with respective electric and magnetic charges  $(e_1, g_1)$  and  $(e_2, g_2)$ , to  $e_1g_2 - e_2g_1 = 4\pi n$ ; limits on their charge and mass were recently established after the first search for dyons at the Large Hadron Colliders [18].) An excellent review on current monopole searches is Ref. [5].

At the time of writing, neither elementary magnetic monopoles nor dyons have been observed yet. An analysis of 13 TeV proton–proton collisions by the MoEDAL Collaboration, during the 2015–2017 Run 2 at the CERN Large Hadron Collider (LHC) led to mass limits in the 1500–3750 GeV range for magnetic charges, and up to  $5g_D$  (where  $g_D = \frac{2\pi}{e}$  is the minimum magnetic charge, called the ‘Dirac magnetic charge’, allowed by Equation (1)), for monopoles with spin 0, 1/2, and 1 [19]. The MoEDAL Collaboration also performed a search for dyons based on a Drell–Yan production and excluded dyons with a magnetic charge up to  $5g_D$  and an electric charge up to  $200e$  for mass limits in the range 870–3120 GeV, and monopoles with magnetic charge up to  $5g_D$  with mass limits in the range 870–2040 GeV [18]. More recently, the ATLAS Collaboration (CERN) also reported on a search for magnetic monopoles and high-electric-charge objects during the LHC’s Run 2 and found no highly ionizing particle candidate [20]. Further limits, if not a discovery, should be obtained after the LHC’s ongoing Run 3, which should be completed in 2026.

We can write the fully symmetric Maxwell equations with magnetic monopoles as follows:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho_e, \\ \nabla \cdot \mathbf{B} &= \rho_m, \\ \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \mathbf{J}_e, \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= -\mathbf{J}_m. \end{aligned} \quad (2)$$

For the sake of simplicity, hereafter we utilize units such that  $c = 1$ . These equations can be written as

$$\nabla \cdot (\mathbf{E} + i\mathbf{B}) = \rho_e + i\rho_m, \quad \nabla \times (\mathbf{E} + i\mathbf{B}) = i\frac{\partial}{\partial t}(\mathbf{E} + i\mathbf{B}) + i(\mathbf{J}_e + i\mathbf{J}_m),$$

which are invariant under the electric–magnetic duality transformation, defined as the following complex phase rotations:

$$\mathbf{E} + i\mathbf{B} \rightarrow e^{i\theta}(\mathbf{E} + i\mathbf{B}), \quad \rho_e + i\rho_m \rightarrow e^{i\theta}(\rho_e + i\rho_m), \quad \mathbf{J}_e + i\mathbf{J}_m \rightarrow e^{i\theta}(\mathbf{J}_e + i\mathbf{J}_m).$$

In this context of classical electromagnetism, this duality transformation was first observed by Heaviside [21]. Afterwards, the concept of duality was extended and extensively investigated in a wide array of the physics literature. The appearance of imaginary numbers in the previous expressions might be seen as a hint for exploiting the analytical extension of U(1) hereafter.

The second expression of Equation (2) appears to be incompatible with  $\mathbf{B} = \nabla \times \mathbf{A}$  if  $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ . However, we can recover Equation (2) by utilizing a *singular* Dirac potential,

$$\mathbf{A}(\mathbf{r}) = \frac{g}{4\pi|\mathbf{r}|} \frac{\mathbf{r} \times \mathbf{n}}{|\mathbf{r} - \mathbf{r} \cdot \mathbf{n}|},$$

as a function of the position  $\mathbf{r}$ , with the unit constant vector  $\mathbf{n}$ , which is parallel to the Dirac string. Other, more suitable, expressions of this potential are discussed in Ref. [1], along with discussions of incidental subtleties.

The continuity equations for electric and magnetic densities and currents are given, respectively, by

$$\nabla \cdot \mathbf{J}_e + \frac{\partial \rho_e}{\partial t} = 0, \quad \nabla \cdot \mathbf{J}_m + \frac{\partial \rho_m}{\partial t} = 0,$$

or, in tensor notation,  $\partial_\mu J_e^\mu = \partial_\mu J_m^\mu = 0$ . In 1948, Dirac attempted to generalize electrodynamics in terms of these four currents,  $J_e^\mu$  and  $J_m^\mu$ , as well as the field strength tensor  $F_{\mu\nu}$  and its dual  $\tilde{F}_{\mu\nu}$ . However, despite these equations being elegant and symmetric, their expressions in terms of potentials require modified expressions of the field strength tensor in terms of the potentials. These, in turn, involve complications such as the ‘Dirac veto’, which states that the trajectory of electric charges in the presence of magnetic charges must not intersect the Dirac string. The duality, that is, the interchange between electric and magnetic objects, such that the possibility of attaching a Dirac string to an electric charge as well as a magnetic charge, led to dual-invariant electrodynamics Lagrangians in terms of two potentials: the four-vector  $A_\mu$  and the pseudovector  $\tilde{A}_\mu$  [12–14]. Without going into detail, let us mention the field strength tensor and its dual from Ref. [12],

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - \varepsilon_{\mu\nu\alpha\beta} \partial^\alpha \tilde{A}^\beta, \\ \tilde{F}_{\mu\nu} &= \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu + \varepsilon_{\mu\nu\alpha\beta} \partial^\alpha A^\beta, \end{aligned}$$

which, whereas they do not require a Dirac string, pose the difficulty of introducing new degrees of freedom, which can be dealt with in various ways. Further discussions on the Cabibbo–Ferrari two potentials are given in Refs. [22–24]. Other constructions of models with two potentials are based on a local Lagrangian formulation by Zwanziger, where the electromagnetic field strength tensor and its dual can be written as [13,14]

$$\begin{aligned} F_{\mu\nu} &= n^\alpha \left[ n_\mu (\partial_\alpha A_\nu - \partial_\nu A_\alpha) - \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} n^\rho (\partial_\alpha \tilde{A}^\sigma - \partial^\sigma \tilde{A}_\alpha) \right], \\ \tilde{F}_{\mu\nu} &= n^\alpha \left[ n_\mu (\partial_\alpha \tilde{A}_\nu - \partial_\nu \tilde{A}_\alpha) - \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} n^\rho (\partial_\alpha A^\sigma - \partial^\sigma A_\alpha) \right], \end{aligned}$$

where the vector  $n_\mu$  is spatially parallel to the Dirac string but with the physics being independent of  $n_\mu$ .

The central objective of this paper is to provide a symmetry argument which leads naturally to the addition of a second potential which will account for the existence of magnetic charges, rather than imposing that second potential by hand. The resulting approach is covariant, valid in any curved spacetime, and should accommodate two-potential Lagrangians of monopoles; note, however, that models that break Lorentz invariance, such as Zwanziger’s Lagrangian with its Dirac string vector, might involve slight modifications to the formalism. Hereafter, we apply an analytical extension of the parameter space of electrodynamics’ U(1) gauge transformations and obtain thereof two potentials which describe elementary particles with electric charge and magnetic charge, the resulting gauge group being effectively the product of the compact Lie group U(1) with the non-compact Lie group of dilations  $\mathbb{R}^+$ . Although the literature on duality is abundant, to the best of our knowledge this is the first attempt to explain the appearance of the dual gauge field by modifying the gauge symmetry group. We thus build gauge theories of magnetic monopoles from first principles: along with the U(1)  $\times$   $\mathbb{R}^+$  symmetry, so that in addition

to the usual (electric) gauge field  $A_\nu$ , there appears a magnetic gauge field  $C_\nu$ , as in Refs. [23,24]; we shall also require conformal invariance. Although this may suggest a similarity with the Weyl approach to gravity, we shall explain how our approach differs from the Weyl theory.

We examine the gauge invariance by means of Utiyama's methodology [25–29]. This approach considers infinitesimal gauge transformations and leads to a set of equations under the assumption of independence of the gauge parameters and their derivatives. The solutions of these equations determine the relevant objects that should be considered in a deductive way, such as the covariant derivative and field strength. We illustrate this approach with a massless scalar field as well as with a spinor field. We comment on the two-potential monopole phenomenology at the end of Section 4.2. We observe that for spinor models a non-symmetrized Lagrangian enables the existence of magnetic monopoles, but this is not the case when we consider a symmetrized Lagrangian. These subtleties do not affect the scalar case, where the magnetic monopole's appearance is clear.

## 2. Analytical Extension of the U(1) Gauge Symmetry

Hereafter, we examine and motivate from symmetry arguments the following expressions of  $\mathbf{E}$  and  $\mathbf{B}$  in terms of vector and scalar potentials,  $A_0$ ,  $\mathbf{A}$ ,  $C_0$ , and  $\mathbf{C}$ :

$$\begin{aligned}\mathbf{E} &= -\nabla A_0 - \partial_t \mathbf{A} - \nabla \times \mathbf{C}, \\ \mathbf{B} &= \nabla \times \mathbf{A} - \nabla C_0 - \partial_t \mathbf{C}.\end{aligned}\quad (3)$$

Clearly, their symmetry allows for point-like magnetically charged particles, just as we usually have with the potentials  $A_0$  and  $\mathbf{A}$ , only for electrically charged particles. In Appendix A, we briefly discuss the gauge invariance conditions behind Equation (3), and we write these field equations in terms of the gauge fields.

As we shall explain hereafter, the essence of this section lies in the generalization of the U(1) symmetry by applying an analytical extension,

$$\epsilon \rightarrow \epsilon - i\alpha, \quad (4)$$

so that, for instance, a scalar field would transform *globally* as

$$\phi' = e^{\alpha+i\epsilon} \phi, \quad \phi^{\dagger'} = e^{\alpha-i\epsilon} \phi^\dagger, \quad \alpha, \epsilon \text{ constant.}$$

This amounts to the product of a usual (compact) U(1) phase transformation,  $e^{-i\epsilon}$ , of the field, by a (non-compact) dilation,  $e^\alpha$ , and, as we shall describe, the invariance of the Lagrangian also requires specific conformal transformations related to this dilation factor. As alluded to in the Introduction, this may suggest to some that we are actually constructing Weyl's invariant theories of gravity, but in Appendix B, we explain how our approach is different.

In Sections 2.1 and 2.2, we examine the effects of the analytical extension of the U(1) gauge transformations on the matter field, with their corresponding invariant Lagrangians, and the analogue for the gauge fields, along with their related field strength tensors.

### 2.1. Matter Fields

In order to describe the appearance of the magnetic gauge field  $C_\mu$ , let us consider the Lagrangian of a massless complex scalar field,  $\phi(x)$ , with a quartic potential:

$$L = \sqrt{-g} \left[ g^{\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi + \lambda (\phi^\dagger \phi)^2 \right]. \quad (5)$$

The quartic interaction is a well-studied non-trivial addition to the free theory terms; it follows naturally the quadratic massive term (which vanishes here, in order to maintain global conformal invariance). But first, let us describe its invariance under the global U(1) transformation with analytical extension. We recover the invariance of  $L$  if, in addition to

the transformation on  $\phi, \phi^\dagger$ , we also perform a transformation on the metric tensor; that is, we also perform a rigid conformal transformation on the metric tensor:

$$\phi' = e^{\alpha+i\epsilon}\phi, \quad \phi^{\dagger'} = e^{\alpha-i\epsilon}\phi^\dagger, \quad g'_{\mu\nu} = e^{-n\alpha}g_{\mu\nu},$$

where  $\alpha$  and  $\epsilon$  are constant, and  $n$  depends on the Lagrangian. Note that whereas the general theory should be developed in a non-Minkowskian spacetime, the Minkowski case can be seen as a gauge fixing of the metric tensor. For instance, if we start with a non-flat metric  $g_{\mu\nu}$  that is conformal to  $\eta_{\mu\nu}$ , then the parameter  $\alpha$  can be chosen in such a way that  $g'_{\mu\nu} = \eta_{\mu\nu}$ . (We note that the conformal invariance of Maxwell equations in the presence of magnetic monopoles has been verified in Ref. [30].) In this way, we see that

$$\begin{aligned} L' &= \sqrt{-g'} \left[ g'^{\mu\nu} \partial_\mu \phi^{\dagger'} \partial_\nu \phi' + \lambda (\phi^{\dagger'} \phi')^2 \right] \\ &= \sqrt{-g} \left[ e^{(2-n)\alpha} g^{\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi + e^{2(2-n)\alpha} \lambda (\phi^\dagger \phi)^2 \right]. \end{aligned}$$

For the scalar field described by Equation (5), it is clear that we recover the invariance of the Lagrangian by choosing  $n = 2$ , and then,  $L' = L$ .

In the late 1970s, Kyriakopoulos showed the invariance of the Maxwell equations with monopoles under conformal transformations [30]. That work inspired us to study the role played by conformal transformations in the candidate theories for magnetic monopoles. In particular, one might ponder about its influence at the level of a fundamental symmetry, such as gauge symmetry. One could wonder if the duality transformation,  $\mathbf{E} + i\mathbf{B} \rightarrow e^{i\theta}(\mathbf{E} + i\mathbf{B})$ , would also imply an analogous relation for the potentials involving complex numbers, e.g.,  $A_\mu \rightarrow A_\mu^{(e)} + iA_\mu^{(m)}$ . If that was the case, we could argue that the U(1) symmetry was thereby extended to a symmetry containing a real-phase factor and a complex phase. Moreover, this somewhat curious idea of extending the parameter space with an imaginary factor appears to explain in a natural way the otherwise ad hoc appearance of the additional, so-called ‘magnetic’, potential. In this way, one could see the gauge transformation parameter as being generalized to produce an extended symmetry group such as  $U(1) \times R^+$ . Magnetic monopoles have not been observed yet in the forms proposed in the literature. Hereafter, we suggest a formulation from first principles (gauge theory) aiming to predict monopoles and to offer a possible explanation for the reason why these particles remain elusive to detection.

Now, let us extend the U(1) gauge theory for this type of transformation and see how the magnetic potential results from the analytical extension in Equation (4). We begin with a Lagrangian of  $N$  scalar fields  $\phi^A$ , with  $A = 1, \dots, N$ :

$$L = L(g^{\mu\nu}, \phi^A, \partial_\mu \phi^A).$$

(Were we working with other tensor fields, the standard derivative should be replaced by a spacetime covariant derivative.) We apply the transformation

$$\phi'^A = U^A_B \phi^B = \left( e^{T_a \epsilon^a(x)} \right)^A_B \phi^B, \quad A, B = 1, \dots, N; \quad a = (1), (2), \dots, \dim(\mathbf{G}), \quad (6)$$

where  $\mathbf{G}$  is the gauge group,  $\epsilon^a(x)$  are real spacetime functions, and  $T_a^A_B$  are representation matrices of  $T_a$ , the generators of  $\mathbf{G}$ , with commutation relations  $[T_a, T_b] = f_{ab}^c T_c$ . In order to avoid confusion, the group indices  $a$  are displayed between parentheses, as in Equation (6). The Lagrangian is assumed to be invariant under transformations of the fields:  $\delta L = 0$ .

Henceforth, we will work with an abelian group; that is, such that  $f_{ab}^c = 0$ , with

$$\epsilon^{(1)} = \epsilon, \quad \epsilon^{(2)} = \alpha,$$

(where the  $i$  factor is incorporated within the Lie algebra representation of  $T_2$ ) so that the transformed fields read

$$\phi'^A = \left( e^{T_a \epsilon^a} \right)_B^A \phi^B = \left( e^{T_{(1)} \epsilon + T_{(2)} \alpha} \right)_B^A \phi^B, \quad \epsilon(x), \alpha(x) \in \mathbb{R},$$

in which  $T_a$  now denotes the representation matrices of the corresponding Lie algebra element on the  $\phi$  field. This should not cause confusion in the remainder of this paper.

Here, again, the spacetime metric also transforms as

$$g'^{\mu\nu} = e^{n\alpha} g^{\mu\nu}, \quad g'_{\mu\nu} = e^{-n\alpha} g_{\mu\nu},$$

where  $n$  is determined according to the form of the Lagrangian. (For instance, for the massless complex scalar field Lagrangian

$$L = \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi) \rightarrow L' = \sqrt{-g'} (g'^{\mu\nu} \partial_\mu \phi'^\dagger \partial_\nu \phi') = \sqrt{-g} e^{(-2n+n+2)\alpha} (g^{\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi)$$

and the Lagrangian will be invariant if  $n = 2$ . Similarly, if  $L = \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi)^3$ , then the Lagrangian will be invariant if  $n = -6$ .)

Next, if we take the group parameters,  $\alpha(x)$  and  $\epsilon(x)$ , to be spacetime dependent, we necessitate gauge fields in order to preserve gauge invariance. Since we have two independent parameters, we shall introduce two gauge fields,

$$A_\mu^{(1)} = A_\mu, \quad A_\mu^{(2)} = C_\mu,$$

with the transformation law:  $\delta A_\mu^a = \partial_\mu \epsilon^a$ , for  $a = (1), (2)$ . As usual, these additional fields restore the invariance of the Lagrangian lost due to the derivatives  $\partial_\mu \epsilon^a$ .

We insert these fields within the initial Lagrangian,  $L = L(g^{\mu\nu}, \phi^A, \partial_\mu \phi^A)$ , which results in an invariant Lagrangian:

$$L_I = L_I(g^{\mu\nu}, \phi^A, \partial_\mu \phi^A, A_\mu^a).$$

The variation under infinitesimal transformations is written as [25]

$$\begin{aligned} \delta L_I &= \frac{\partial L_I}{\partial \phi^A} \delta \phi^A + \frac{\partial L_I}{\partial \partial_\mu \phi^A} \delta \partial_\mu \phi^A + \frac{\partial L_I}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial L_I}{\partial A_\mu^a} \delta A_\mu^a \\ &= \left[ \frac{\partial L_I}{\partial \phi^A} T_{(1)}^A{}_B \phi^B + \frac{\partial L_I}{\partial \partial_\mu \phi^A} \partial_\mu (T_{(1)}^A{}_B \phi^B) \right] \epsilon \\ &+ \left[ T_{(2)}^A{}_B \left( \frac{\partial L_I}{\partial \phi^A} \phi^B + \frac{\partial L_I}{\partial \partial_\mu \phi^A} \partial_\mu \phi^B \right) + \frac{\partial L_I}{\partial g^{\mu\nu}} T^{\mu\nu}{}_{\rho\sigma} g^{\rho\sigma} \right] \alpha \\ &+ \frac{\partial L_I}{\partial \partial_\mu \phi^A} T_{(a)}^A{}_B \phi^B \partial_\mu \epsilon^a + \frac{\partial L_I}{\partial A_\mu^a} \partial_\mu \epsilon^a. \end{aligned}$$

If  $L_I$  is invariant under these transformations with  $\epsilon$ ,  $\alpha$ ,  $\partial_\mu \epsilon$ , and  $\partial_\mu \alpha$  considered as independent parameters, then we obtain the symmetry equations:

$$\begin{aligned} T_{(1)}^A{}_B \left( \frac{\partial L}{\partial \phi^A} \phi^B + \frac{\partial L}{\partial \partial_\mu \phi^A} \partial_\mu \phi^B \right) &= 0, \\ T_{(2)}^A{}_B \left( \frac{\partial L}{\partial \phi^A} \phi^B + \frac{\partial L}{\partial \partial_\mu \phi^A} \partial_\mu \phi^B \right) + \frac{\partial L}{\partial g^{\mu\nu}} T^{\mu\nu}{}_{\rho\sigma} g^{\rho\sigma} &= 0, \\ \frac{\partial L}{\partial \partial_\mu \phi^A} T_{(a)}^A{}_B \phi^B + \frac{\partial L_I}{\partial A_\mu^a} &= 0. \end{aligned}$$

The last equation is satisfied as long as the dependence of the Lagrangian with derivatives of the fields  $\phi^A$  and the gauge field is expressed via covariant derivatives,

$$D_\mu \phi^A = \partial_\mu \phi^A - T_a^A{}_B \phi^B A_\mu^a,$$

which implies that the gauge-invariant Lagrangian should be extended as

$$L_I = L_{II}(g^{\mu\nu}, \phi^A, D_\mu \phi^A).$$

The transformation law of this object is described by

$$\delta D_\mu \phi^A = \delta(\partial_\mu \phi^A - T_a^A{}_B \phi^B A_\mu^a) = T_a^A{}_B D_\mu \phi^B \epsilon^a,$$

which ensures that this object is covariant under the symmetry group with generators  $T_a$ , with  $a = (1), (2), \dots, \dim(G)$ .

Naturally, a question that occurs at this point is whether the resulting models reflect the expected magnetic–electric duality. We respond in the affirmative in Appendix C, for any curved spacetime, by making use of the Levi–Civita form defined in terms of the well-known Levi–Civita symbols. In addition, while recovering the duality property we will encounter another motivation for exploiting the analytical extension of the parameter space, Equation (4), which underlies our approach.

## 2.2. Gauge Fields

Here, we analyze the Lagrangian  $L_0$  of the free gauge fields. For the functional dependence of  $L_0$ , as usual, we expect it to depend on the gauge fields and their first derivatives. In addition, since we are dealing with vector fields, the presence of the metric tensor is almost mandatory, otherwise we would not be able to build scalar quantities out of vector fields. Nonetheless, since the metric is not flat a priori, the derivative of the gauge fields should take into account the curvature of the spacetime; in other words, the derivative of the gauge fields should actually appear in  $L_0$  by means of the spacetime covariant derivative. In summary, we propose the functional dependence of  $L_0$  as

$$L_0 = L_0(g^{\mu\nu}, A_\mu^a, \nabla_\nu A_\mu^a).$$

where  $\nabla_\nu A_\mu^a = \partial_\nu A_\mu^a - \Gamma_{\nu\mu}^\rho A_\rho^a$ , with  $\Gamma_{\nu\mu}^\rho$  being the Christoffel symbols. As a matter of consistency with the remainder of the theory, this Lagrangian should be invariant under the gauge transformation of the metric and the gauge fields. Accordingly,

$$\begin{aligned} \delta L_0 &= \frac{\partial L_0}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial L_0}{\partial A_\mu^a} \delta A_\mu^a \\ &+ \frac{\partial L_0}{\partial \nabla_\nu A_\mu^a} \delta \partial_\nu A_\mu^a - \frac{\partial L_0}{\partial \nabla_\nu A_\mu^a} \delta \Gamma_{\nu\mu}^\rho A_\rho^a - \frac{\partial L_0}{\partial \nabla_\nu A_\mu^a} \Gamma_{\nu\mu}^\rho \delta A_\rho^a. \end{aligned}$$

The transformation law for the Christoffel symbols can be straightforwardly evaluated and results in

$$\delta \Gamma_{\mu\nu}^\rho = -n \frac{1}{2} g^{\rho\sigma} (\partial_\mu \alpha g_{\nu\sigma} + \partial_\nu \alpha g_{\mu\sigma} - \partial_\sigma \alpha g_{\mu\nu}).$$

When we impose invariance of the Lagrangian,  $\delta L_0 = 0$ , and assume the independence of the parameters  $\epsilon$  and  $\alpha$  and their derivatives, we find the equations

$$\begin{aligned}
\frac{\partial L_0}{\partial g^{\mu\nu}} n g^{\mu\nu} &= 0, \\
\frac{\partial L_0}{\partial A_\lambda^{(1)}} - \frac{1}{2} \left( \frac{\partial L_0}{\partial \nabla_\nu A_\mu^{(1)}} + \frac{\partial L_0}{\partial \nabla_\mu A_\nu^{(1)}} \right) \Gamma_{\nu\mu}^\lambda &= 0, \\
\frac{\partial L_0}{\partial A_\lambda^{(2)}} - \frac{1}{2} \left( \frac{\partial L_0}{\partial \nabla_\nu A_\mu^{(2)}} + \frac{\partial L_0}{\partial \nabla_\mu A_\nu^{(2)}} \right) \Gamma_{\nu\mu}^\lambda \\
- \frac{1}{2} \left( \frac{\partial L_0}{\partial \nabla_\nu A_\mu^a} + \frac{\partial L_0}{\partial \nabla_\mu A_\nu^a} \right) \left[ -n \frac{1}{2} g^{\rho\sigma} \left( \delta_\mu^\lambda g_{\nu\sigma} + \delta_\nu^\lambda g_{\mu\sigma} - \delta_\sigma^\lambda g_{\mu\nu} \right) \right] A_\rho^a &= 0, \\
\frac{\partial L_0}{\partial \nabla_\nu A_\mu^a} + \frac{\partial L_0}{\partial \nabla_\mu A_\nu^a} &= 0.
\end{aligned}$$

Note that the last three equations consider the symmetry of the lower indices of the Levi-Civita connection and the symmetry of the second derivatives, i.e.,  $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$ . By substituting the last equation into the previous two equations, we can simplify these three equations as follows:

$$\frac{\partial L_0}{\partial g^{\mu\nu}} n g^{\mu\nu} = 0, \quad \frac{\partial L_0}{\partial A_\lambda^a} = 0, \quad \frac{\partial L_0}{\partial \nabla_\nu A_\mu^a} + \frac{\partial L_0}{\partial \nabla_\mu A_\nu^a} = 0.$$

The solutions to the last equations show us that

$$L_0 = L_0(g^{\mu\nu}, F_{\nu\mu}^a),$$

where

$$F_{\nu\mu}^a \equiv \nabla_\nu A_\mu^a - \nabla_\mu A_\nu^a = \partial_\nu A_\mu^a - \partial_\mu A_\nu^a$$

are the field strengths for both  $A_\lambda^{(1)} = A_\lambda$  and  $A_\lambda^{(2)} = C_\lambda$ . In other words,  $L_0$  does not depend explicitly on the gauge fields, but depends on the metric and on the field strengths  $F_{\nu\mu}^a$ . But there is also an extra condition for the Lagrangian of the free gauge field. The first of the equations can be rewritten in terms of a new object  $T_{\mu\nu}^{(0)} \equiv \frac{1}{2} \frac{1}{\sqrt{-g}} \frac{\partial L_0}{\partial g^{\mu\nu}}$ , leading to

$$T^{(0)} \equiv g^{\mu\nu} T_{\mu\nu}^{(0)} = 0.$$

This expression tells us that the energy-momentum tensor  $T_{\mu\nu}^{(0)}$  associated with  $L_0$  has to be traceless.

### 3. Conserved Current

Finally, we can evaluate the existence of conserved currents, as expected by the Noether's theorem. We consider the full Lagrangian of our system, taking into account the contributions of the Lagrangians for the matter fields  $\phi^A$  in interaction with the gauge fields and  $L_0$ :

$$L_T(g^{\mu\nu}, \phi^A, \partial_\mu \phi^A, A_\mu^a, \partial_\mu A_\mu^a) = L_I(g^{\mu\nu}, \phi^A, D_\mu \phi^A) + L_0(g^{\mu\nu}, F_{\nu\mu}^a).$$

The variation in  $L_T$  can be expressed as the sum of a "volume" term and a "surface" term:

$$\delta L_T = V + \partial_\nu S^\nu,$$

where

$$V \equiv \frac{\delta L_T}{\delta \phi^A} \delta \phi^A + \frac{\partial L_T}{\partial g^{\mu\nu}} \delta g^{\mu\nu} - \partial_\mu \frac{\delta L_T}{\delta A_\mu^a} \epsilon^a, \quad S^\nu \equiv \frac{\partial L_I}{\partial D_\nu \phi^A} \delta \phi^A + \frac{\partial L_0}{\partial F_{\nu\mu}^a} \delta A_\mu^a + \frac{\delta L_T}{\delta A_\nu^a} \epsilon^a.$$



According to Utiyama, since the choice of the spacetime volume of evaluation of the action integral is arbitrary, the “volume” and “surface” terms should vanish independently under the symmetry condition, i.e.,  $V = 0$ ,  $\partial_\nu S^\nu = 0$ . We are led to

$$\partial_\nu \left( \frac{\partial L_I}{\partial D_\nu \phi^A} T_a^A{}_B \phi^B + \frac{\delta L_T}{\delta A_\nu^a} \right) = 0, \quad \frac{\partial L_I}{\partial D_\mu \phi^A} T_a^A{}_B \phi^B + \frac{\partial L_T}{\partial A_\mu^a} = 0.$$

We define the currents as

$$J_a^\mu \equiv \frac{\partial L_T}{\partial A_\mu^a},$$

and from the latter equation, we see that this is equal to

$$J_a^\mu = - \frac{\partial L_I}{\partial D_\mu \phi^A} T_a^A{}_B \phi^B,$$

while the former establishes that

$$\partial_\nu J_a^\nu = \partial_\nu \frac{\delta L_T}{\delta A_\nu^a}.$$

We conclude that  $J_a^\mu$  is a conserved quantity under the validity of the field equations,  $\frac{\delta L_T}{\delta A_\nu^a} = 0$ .

#### 4. Applications: Massless Scalar Field and Spinor Field

##### 4.1. Massless Scalar Field with a Quartic Potential

###### Gauge Invariance and Interaction

Consider the Lagrangian in Equation (5) for a massless complex scalar field with a quartic potential:

$$L = \sqrt{-g} \left[ g^{\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi + \lambda (\phi^\dagger \phi)^2 \right].$$

With the notation of Section 2, we have

$$\phi^A = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} = \begin{pmatrix} \phi \\ \phi^\dagger \end{pmatrix},$$

with the analytically extended U(1) transformations, Equation (4),

$$\phi' = e^{\alpha+i\epsilon} \phi, \quad \phi^{\dagger'} = e^{\alpha-i\epsilon} \phi^\dagger,$$

or, in their infinitesimal form,

$$\phi' = (1 + \alpha + i\epsilon)\phi, \quad \phi^{\dagger'} = (1 + \alpha - i\epsilon)\phi^\dagger.$$

Thus, the analytical extension of U(1) implies two parameters,

$$\epsilon^{(1)} = \epsilon, \quad \epsilon^{(2)} = \alpha,$$

and it follows from

$$\delta\phi^A = \phi' - \phi = T_a^A{}_B \phi^B \epsilon^a$$

that

$$\begin{aligned} \delta\phi &= \left( T_{(1)}^1{}_1 \phi + T_{(1)}^1{}_2 \phi^\dagger \right) \epsilon + \left( T_{(2)}^1{}_1 \phi + T_{(2)}^1{}_2 \phi^\dagger \right) \alpha, \\ \delta\phi^\dagger &= \left( T_{(1)}^2{}_1 \phi + T_{(1)}^2{}_2 \phi^\dagger \right) \epsilon + \left( T_{(2)}^2{}_1 \phi + T_{(2)}^2{}_2 \phi^\dagger \right) \alpha. \end{aligned}$$

We can express this in matrix notation by using

$$T_{(1)}^A{}_B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad T_{(2)}^A{}_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which form a 2-dimensional abelian Lie algebra.

As discussed in Section 2, the above transformation acts on the metric as, on the one hand,

$$\delta g_{\mu\nu} = T_{\mu\nu}{}^{\rho\sigma} g_{\rho\sigma} \alpha,$$

and, on the other hand,

$$g'_{\mu\nu} = (1 - 2\alpha)g_{\mu\nu} \Rightarrow \delta g_{\mu\nu} = -2\alpha g_{\mu\nu},$$

which upon comparison, shows that

$$T_{\mu\nu}{}^{\rho\sigma} = -2\delta_{\mu}^{\rho}\delta_{\nu}^{\sigma}.$$

As discussed in Section 2, if we consider only the local transformations of the fields and the metric tensor

$$\phi' = e^{\alpha+i\epsilon}\phi, \quad \phi^{\dagger'} = e^{\alpha-i\epsilon}\phi^{\dagger}, \quad g'_{\mu\nu} = e^{-2\alpha}g_{\mu\nu},$$

where  $\alpha = \alpha(x)$  and  $\epsilon = \epsilon(x)$ , then the Lagrangian transforms as

$$\delta L = \sqrt{-g}g^{\mu\nu}\phi^{\dagger}\phi\partial_{\mu}(\alpha - i\epsilon)\partial_{\nu}(\alpha + i\epsilon) + \sqrt{-g}g^{\mu\nu}\phi^{\dagger}\partial_{\nu}\phi\partial_{\mu}(\alpha - i\epsilon) + \sqrt{-g}g^{\mu\nu}\phi\partial_{\mu}\phi^{\dagger}\partial_{\nu}(\alpha + i\epsilon),$$

showing that the invariance is lost. We can recover the invariance by adding gauge potentials,  $A_{\mu}^a$ , which interact with the matter fields through the covariant derivative:

$$D_{\mu}\phi^A = \partial_{\mu}\phi^A - T_{(a)}^A{}_B\phi^B A_{\mu}^a.$$

For the current situation,

$$A_{\mu}^{(1)} = A_{\mu}, \quad A_{\mu}^{(2)} = C_{\mu},$$

so that the covariant derivatives are given by

$$D_{\mu}\phi = \partial_{\mu}\phi - iA_{\mu}\phi - C_{\mu}\phi, \quad D_{\mu}\phi^{\dagger} = \partial_{\mu}\phi^{\dagger} + iA_{\mu}\phi^{\dagger} - C_{\mu}\phi^{\dagger}.$$

(In the literature, in standard electrodynamics, it is usual to present the coupling constant such that the covariant derivative reads  $D_{\mu} = \partial_{\mu} - ieA_{\mu}$ . In Utiyama's approach, this is not the case. However, the compatibility between the notation can be obtained as long as we map  $A_{\mu} \rightarrow eA_{\mu}$ . For the case of the field  $C_{\mu}$ , the mapping would read  $C_{\mu} \rightarrow q_m C_{\mu}$ , where  $q_m$  stands for the magnetic-like coupling, or the magnetic charge. Throughout this paper, we will keep Utiyama's notation.)

Then, the Lagrangian in Equation (5) for the interacting fields becomes

$$L_I = \sqrt{-g} \left[ g^{\mu\nu} D_{\mu}\phi^{\dagger} D_{\nu}\phi + \lambda (\phi^{\dagger}\phi)^2 \right].$$

It is straightforward to check that this Lagrangian is gauge-invariant even for non-infinitesimal transformations.

Now, we consider the natural extension of the Maxwell Lagrangian

$$L_0 = L_0(g^{\mu\nu}, F_{\nu\mu}^a) = \sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}^a F_{\rho\sigma} = \sqrt{-g}g^{\mu\rho}g^{\nu\sigma} \left[ F_{\mu\nu}^{(1)} F_{\rho\sigma}^{(1)} + F_{\mu\nu}^{(2)} F_{\rho\sigma}^{(2)} \right],$$

where the gauge fields are decoupled as

$$F_{\mu\nu}^{(1)} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \quad F_{\mu\nu}^{(2)} \equiv \partial_\mu C_\nu - \partial_\nu C_\mu.$$

The corresponding energy–momentum tensor is

$$T_{\alpha\beta} = \frac{-2}{\sqrt{-g}} \frac{\partial L_0}{\partial g^{\alpha\beta}} = g_{\alpha\beta} g^{\mu\rho} g^{\nu\sigma} \left[ F_{\mu\nu}^{(A)} F_{\rho\sigma}^{(A)} + F_{\mu\nu}^{(C)} F_{\rho\sigma}^{(C)} \right] - 4g^{\nu\sigma} \left[ F_{\alpha\nu}^{(A)} F_{\beta\sigma}^{(A)} + F_{\alpha\nu}^{(C)} F_{\beta\sigma}^{(C)} \right],$$

and its trace is null:

$$T = g^{\alpha\beta} T_{\alpha\beta} = 0.$$

We conclude this section by examining the full Lagrangian and its conserved current; that is, we combine the previous Lagrangians:

$$L_T = L_I + L_0 = \sqrt{-g} \left[ g^{\mu\nu} D_\mu \phi^\dagger D_\nu \phi + \lambda (\phi^\dagger \phi)^2 \right] + \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} \left[ F_{\mu\nu}^{(A)} F_{\rho\sigma}^{(A)} + F_{\mu\nu}^{(C)} F_{\rho\sigma}^{(C)} \right].$$

The field equations for the gauge field are obtained from

$$\partial_\nu \left( \frac{\partial L_T}{\partial (\partial_\nu A_\mu^a)} \right) - \frac{\partial L_T}{\partial A_\mu^a} = \partial_\nu (\sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}) - J_a^\mu = 0,$$

which, for  $a = 1$ , leads to a Maxwell-like equation for  $A_\mu$ ,

$$\partial_\nu \left[ \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}^{(A)} \right] - J_A^\mu = 0, \quad \nabla_\nu F^{\mu\nu(A)} = J_A^\mu / \sqrt{-g},$$

and likewise with  $a = 2$  for the gauge field  $C_\mu$ ,

$$\partial_\nu \left[ \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}^{(C)} \right] - J_C^\mu = 0, \quad \nabla_\nu F^{\mu\nu(C)} = J_C^\mu / \sqrt{-g}.$$

The current  $J_A^\mu$  for our situation is

$$J_A^\mu \equiv \frac{\partial L_T}{\partial A_\mu} = \frac{\partial L_I}{\partial A_\mu} = i\sqrt{-g} g^{\rho\mu} \left[ \phi^\dagger D_\rho^e \phi - \phi D_\rho^e \phi^\dagger \right],$$

where

$$D_\rho^e \phi \equiv \partial_\rho \phi - iA_\rho \phi, \quad D_\rho^e \phi^\dagger \equiv \partial_\rho \phi^\dagger + iA_\rho \phi^\dagger.$$

This is the usual “electric” current obtained in scalar electrodynamics. The current  $J_C^\mu$  is similar:

$$J_C^\mu = -\sqrt{-g} g^{\rho\mu} \left[ \phi^\dagger D_\rho^m \phi + \phi D_\rho^m \phi^\dagger \right],$$

with

$$D_\rho^m \phi \equiv \partial_\rho \phi - C_\rho \phi, \quad D_\rho^m \phi^\dagger \equiv \partial_\rho \phi^\dagger - C_\rho \phi^\dagger.$$

Notice the absence of the factor  $i$ . We shall interpret this current as a ‘magnetic current’.

In this example, if we ignore the contribution of the term  $\lambda(\phi^\dagger \phi)^2$ , we have the standard massless scalar field. It is well known that the field equation in curved spacetime, without the gauge fields that we introduced in our approach, is not covariant under conformal transformations. In that context, one solves this problem by adding an extra term to the Lagrangian of the type  $\zeta R \phi^\dagger \phi$ , where  $R$  is the scalar curvature—the conformal symmetry demands  $\zeta$  to be  $1/6$ . In the present approach, however, the addition of this term is not necessary, as discussed in Appendix D. In fact, the presence of the gauge potentials in the covariant derivative is enough to guarantee the invariance of the field equation under internal (gauge) and spacetime (conformal) transformations.

#### 4.2. Example: Free Spinor Field

Consider the Lagrangian of a free massless spinor field  $\phi$ , tentatively described by the non-symmetrized spinor Lagrangian

$$L = i\sqrt{|\eta|}\bar{\psi}\gamma^a\partial_a\psi, \quad \sqrt{|\eta|} = 1.$$

Note that we use the same index 'a' for the flat or tangent spacetime coordinates; this should not cause confusion because we will write the group index in parentheses. Now, if we consider such a fermionic system in a curved spacetime, we substitute

$$\partial_a \rightarrow h_a^\mu \nabla_\mu, \quad \sqrt{|\eta|} \rightarrow h = \det h_\mu^a, \quad \Gamma_\mu = \frac{i}{2}\omega_{\mu ab}\Sigma^{ab}, \quad \Sigma^{ab} = \frac{i}{4}[\gamma^a, \gamma^b],$$

where  $\omega_{\mu ab}$  is the spin connection, so that the Lagrangian becomes

$$L = ih\bar{\psi}\gamma^a h_a^\mu \nabla_\mu \psi.$$

In terms of the notation introduced in Section 2, we now take

$$\phi^A = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix},$$

so that the spinor field transforms as

$$\psi' = e^{\alpha+i\epsilon}\psi, \quad \bar{\psi}' = e^{\alpha-i\epsilon}\bar{\psi},$$

or, in terms of the infinitesimal generators of the transformation,

$$\psi' = (1 + \alpha + i\epsilon)\psi, \quad \bar{\psi}' = (1 + \alpha - i\epsilon)\bar{\psi}.$$

We proceed as in Section 4.1 and utilize the two group parameters

$$\epsilon^{(1)} = \epsilon, \quad \epsilon^{(2)} = \alpha,$$

so that the field transformations

$$\delta\phi^A = \phi' - \phi = T_a^A{}_B\phi^B\epsilon^a$$

result in

$$\begin{aligned} \delta\psi &= \left(T_{(1)}^1{}_1\psi + T_{(1)}^1{}_2\bar{\psi}\right)\epsilon + \left(T_{(2)}^1{}_1\psi + T_{(2)}^1{}_2\bar{\psi}\right)\alpha, \\ \delta\bar{\psi} &= \left(T_{(1)}^2{}_1\psi + T_{(1)}^2{}_2\bar{\psi}\right)\epsilon + \left(T_{(2)}^2{}_1\psi + T_{(2)}^2{}_2\bar{\psi}\right)\alpha. \end{aligned}$$

Again, we may express the generators of transformation as the following matrices:

$$T_{(1)}^A{}_B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad T_{(2)}^A{}_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For the metric, we have on one hand,

$$\delta h_\mu^a = T_\mu^{\nu a} h_\nu^b \alpha,$$

and, on the other hand,

$$h_\mu^a = e^{-n\alpha} h_\mu^a \approx (1 - n\alpha)h_\mu^a \Rightarrow \delta h_\mu^a = -n\alpha h_\mu^a,$$

so that, by comparison, we observe that

$$T_{\mu}^{\nu a} = -n\delta_{\mu}^{\nu}\delta_b^a.$$

Next, following once more the approach described in Section 2, we consider the following local transformations of the fields and the metric tensor:

$$\psi' = e^{\alpha+i\epsilon}\psi, \quad \bar{\psi}' = e^{\alpha-i\epsilon}\bar{\psi}, \quad h_{\mu}^{\prime a} = e^{-n\alpha}h_{\mu}^a,$$

where  $\alpha = \alpha(x)$ ,  $\epsilon = \epsilon(x)$ . So, here again, without covariant derivatives, we find

$$\delta L = -h\bar{\psi}\gamma^a h_a^{\mu}\psi\partial_{\mu}\epsilon + ih\bar{\psi}\gamma^a h_a^{\mu}\psi\partial_{\mu}\alpha$$

so that, as expected, the invariance is still lost.

As we did in Section 4.1, we easily recover the invariance by adding gauge potentials  $A_{\mu}^a$  which interact with the matter fields through the covariant derivative

$$D_{\mu}\phi^A = \nabla_{\mu}\phi^A - T_a^A{}_B\phi^B A_{\mu}^a.$$

More explicitly, we apply the two gauge fields

$$A_{\mu}^{(1)} = A_{\mu}, \quad A_{\mu}^{(2)} = C_{\mu},$$

so that the covariant derivatives read

$$D_{\mu}\psi = \nabla_{\mu}\psi - iA_{\mu}\psi - C_{\mu}\psi, \quad D_{\mu}\bar{\psi} = \nabla_{\mu}\bar{\psi} + iA_{\mu}\bar{\psi} - C_{\mu}\bar{\psi},$$

or, if we redefine the gauge fields in order to display explicitly the coupling constant for each interaction,

$$D_{\mu}\psi = \nabla_{\mu}\psi - iq_e A_{\mu}\psi - q_m C_{\mu}\psi, \quad D_{\mu}\bar{\psi} = \nabla_{\mu}\bar{\psi} + iq_e A_{\mu}\bar{\psi} - q_m C_{\mu}\bar{\psi}.$$

Then, the new Lagrangian for the interacting fields becomes

$$L_I = ih\bar{\psi}\gamma^a h_a^{\mu} D_{\mu}\psi,$$

and it is straightforward to check that this Lagrangian is gauge-invariant even for non-infinitesimal transformations.

Here, again, as we did in Section 4.1, we combine the previous Lagrangians to obtain the total Lagrangian of our system:

$$L_T = L_I + L_0 = ih\bar{\psi}\gamma^a h_a^{\mu} D_{\mu}\psi + hg^{\mu\rho}g^{\nu\sigma}\left[F_{\mu\nu}^{(A)}F_{\rho\sigma}^{(A)} + F_{\mu\nu}^{(C)}F_{\rho\sigma}^{(C)}\right].$$

We find the equations for the gauge fields from

$$\partial_{\nu}\frac{\partial L_T}{\partial(\partial_{\nu}A_{\mu}^a)} - \frac{\partial L_T}{\partial A_{\mu}^a} = \partial_{\nu}(hg^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}) - J_a^{\mu} = 0,$$

which, with  $a = (1)$ , leads to an equation for  $A_{\mu}$ :

$$\partial_{\nu}\left[hg^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}^{(A)}\right] - J_A^{\mu} = 0, \quad \nabla_{\nu}F^{\mu\nu(A)} = J_A^{\mu}/\sqrt{-g},$$

and the analogue for  $C_{\mu}$  when  $a = (2)$ ,

$$\partial_{\nu}\left[hg^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}^{(C)}\right] - J_C^{\mu} = 0, \quad \nabla_{\nu}F^{\mu\nu(C)} = J_C^{\mu}/\sqrt{-g}.$$

The current  $J_A^\mu$  is the usual electric current,

$$J_A^\mu \equiv \frac{\partial L_T}{\partial A_\mu} = \frac{\partial L_I}{\partial A_\mu} = ih\bar{\psi}\gamma^a h_a^\nu \frac{\partial}{\partial A_\mu} (D_\nu\psi) = q_e h\bar{\psi}\gamma^a h_a^\mu \psi,$$

and  $J_C^\mu$  is a ‘magnetic’ current,

$$J_C^\mu \equiv \frac{\partial L_T}{\partial C_\mu} = \frac{\partial L_I}{\partial C_\mu} = ih\bar{\psi}\gamma^a h_a^\nu \frac{\partial}{\partial C_\mu} (D_\nu\psi) = -iq_m h\bar{\psi}\gamma^a h_a^\mu \psi,$$

which is the electric current multiplied by  $-i\frac{q_m}{q_e}$ .

As regards monopole phenomenology and experiments, the recent study of Ref. [19] analyses the Drell–Yan production of monopoles with the calculations of Ref. [31], which did not exploit Zwanziger’s local two-potential formalism, but an effective U(1) gauge field theory based on conventional models. It is similar for the recent study by Song and Taylor [32], as well as the experiments reviewed in Refs. [2,5]. Laperashvili and Nielsen examined phenomenological aspects of the electric and magnetic fine structure constants with Zwanziger two potentials, by obtaining the renormalization group equations with a non-symmetrized Dirac Lagrangian [33]. This non-symmetrized Lagrangian was also employed by Terning and Verhaaren more recently to cancel spurious poles in observable scattering amplitudes, including for a Dirac dyon [34]. Of interest is that that paper lists the propagators with two potentials. These authors also used the two-potential approach, but restricted their studies to the gauge sector interacting with dark photons, in Refs. [35,36].

We conclude this section by pointing out the absence of magnetic current, with the present formalism, when we apply a symmetrized spinor Lagrangian:

$$L = ih\frac{1}{2}(\bar{\psi}\gamma^a h_a^\mu \nabla_\mu \psi - h_a^\mu \nabla_\mu \bar{\psi}\gamma^a \psi).$$

Indeed, if we consider local transformations of the fields and the metric tensor

$$\psi' = e^{\alpha+i\epsilon}\psi, \quad \bar{\psi}' = e^{\alpha-i\epsilon}\bar{\psi}, \quad h_\mu^a = e^{-n\alpha}h_\mu^a,$$

with the spacetime-dependent parameters  $\alpha = \alpha(x)$ ,  $\epsilon = \epsilon(x)$ , then the transformed Lagrangian reads

$$\begin{aligned} L' &= ih'\frac{1}{2}(\bar{\psi}'\gamma^a e_a^{\prime\mu} \nabla_\mu \psi' - e_a^{\prime\mu} \nabla_\mu \bar{\psi}'\gamma^a \psi') \\ &= e^{(-3n+2)\alpha}L - e^{(-3n+2)\alpha}h\bar{\psi}\gamma^a h_a^\mu \psi\partial_\mu \epsilon. \end{aligned}$$

From the first term on the right-hand side, the closest we come to an invariant Lagrangian is by choosing  $n = \frac{2}{3}$ , for which we still find an additional term,

$$L' = L - h\bar{\psi}\gamma^a h_a^\mu \psi\partial_\mu \epsilon,$$

so that the invariance is lost. Note that there is no dependence on the derivative of  $\alpha$  because the Lagrangian was symmetrized. This has as a consequence that there will be no magnetic charge/current, since only the field  $A_\mu$  needs to be introduced.

We can interpret this result from two different perspectives. On the one hand, if we observe magnetic monopoles produced through processes involving spinor fields, such as leptons and quarks, this may mean that physically, the non-symmetrized Lagrangian is the one that should be considered. On the other hand, if we do not observe magnetic monopoles with the same spinor fields, this could be an indication that the symmetrized Lagrangian is the physical one.

Note that we may not find magnetic monopoles with spinors, but this does not mean they could not be produced with scalar fields. We may argue that we have not observed

monopoles yet because the only elementary scalar particle observed so far is the Higgs boson, which decays very fast. If the new generation of accelerators can provide a high flux of Higgs particles, perhaps we may be able to observe magnetic monopoles through that channel.

## 5. Concluding Remarks

This paper pursues a well-known attempt to circumvent the non-local nature of Dirac-type Lorentz-invariant Lagrangian descriptions of magnetic charges. That idea, originally due to D. Zwanziger, involves not one but two gauge potentials, the second one being ascribed to the magnetic charge (of the monopole or dyon). This leads to *local* Lorentz-invariant Lagrangian descriptions of magnetic charge. Such models were examined quite recently in Ref. [34]: its authors investigated the existence of spurious poles and showed, among other things, that the amplitude for single-photon production of magnetically charged particles by electrically charged particles is equal to zero.

The purpose of the present paper was to propose an elegant motivation, based on symmetry arguments, for the existence of the second ‘magnetic’ gauge potential. As far as we know, that is not found in the literature. We proposed that the magnetic potential appears via an analytical extension of the usual  $U(1)$  group underlying standard electromagnetism with electric charges only, resulting in a non-compact two-dimensional group. We implemented this extension via the Utiyama methodology, which considers infinitesimal gauge transformations and provides conditions due to the independent gauge parameters and their derivatives. These conditions allow us to deduce relevant objects such as the covariant derivative and field strength.

As examples of this approach, we examined a model with a massless scalar field and a model with a massless spinor field. For spinor models, we pointed out that a non-symmetrized Lagrangian enables the existence of magnetic monopoles, but that this is not the case with a symmetrized Lagrangian. These subtleties do not occur with the scalar field. Also, we examine massless spinor fields, and if we add a mass term, in order to describe real leptons and quarks, we find that our model is no longer invariant, since we cannot find a value for  $n$  that makes both the kinetic and massive terms invariant. The masslessness of the fermions is not necessarily a problem since the electroweak interaction associated with the  $SU(2) \times U(1)$  gauge symmetry also demands the same condition before spontaneous symmetry breaking, through which the generation of mass for quarks and leptons is provided by the Higgs mechanism. The same could be applied in the present case.

Another example that could eventually be considered is the spin-one field. However, the standard kinetic term of the Lagrangian,  $\sqrt{-g}F_{\mu\nu}F^{\mu\nu}$  is not invariant under our extended gauge group. This means that the magnetic monopoles will not interact with spin-one fields. Physically, charged spin-one fields could be associated with the vector bosons  $W_{\pm}$ . From this perspective, no experiment involving these particles would be useful to detect magnetic monopoles in the context of our proposal.

To sum up, the detection of magnetic monopoles in the context of the present work is more likely to be achieved with experiments involving scalar fields, which essentially means that they should consider a flux of Higgs particles. As stated before, this may eventually be achieved with a new generation of accelerators. The investigation of how this measurement could be made in experiments, such as MoEDAL and others, is something to be implemented in future works. In this regard, an extension to non-abelian groups accommodating magnetic monopoles could also be attempted. We intend to implement this in a separate paper.

When dealing with particles of the standard model, one usually starts with a  $U(1) \otimes SU(2)$  gauge group in order to deal with the electroweak interactions (or eventually with  $U(1) \otimes SU(2) \otimes SU(3)$  if we also include the strong interaction), where the particles have no masses. This massless character is essential in order to preserve the  $SU(2)$  symmetry. The particles then acquire mass through the Higgs mechanism, which generates mass for both the leptons and the gauge bosons. In the literature, several models predict the

existence of mass for the magnetic monopole. This may eventually happen in our model as well. However, in order to properly address this point, we should extend our analysis to unify our extended group with the  $SU(2)$  group. We expect that the use of the Higgs mechanism may eventually result in a mass for the gauge boson associated with the magnetic monopole. This analysis lies beyond the scope of this paper, but at this point it is not possible to know if the magnetic boson will acquire mass or not and even less to estimate an order of magnitude for it. Some experimental constraints establish a lower limit of a few TeVs for the mass of the monopole. It would be interesting to check if this will be the case for our model, and also if we will face problems with the mass hierarchy or some fine tuning. We intend to consider this elsewhere.

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## Appendix A. Conditions on Gauge Fields

Equation (3) is gauge-invariant as long as the new gauge fields  $A'_\mu$  and  $C'_\mu$  satisfy

$$A'_\mu = A_\mu + \partial_\mu f, \quad C'_\mu = C_\mu + \partial_\mu g, \quad (\text{A1})$$

where

$$A^\mu = (A_0, \mathbf{A}), \quad A_\mu = (A_0, -\mathbf{A}),$$

and likewise for  $C_\mu$ , as we apply the flat spacetime metric  $(+, -, -, -)$ .

If we replace the electric and magnetic fields in terms of the potentials in the field equations, we find

$$\begin{aligned} -\nabla^2 A_0 - \partial_t \nabla \cdot \mathbf{A} &= \rho_e \\ \nabla(\partial_t A_0 + \nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} + \partial_t^2 \mathbf{A} &= \mathbf{J}_e \\ -\nabla(\partial_t C_0 + \nabla \cdot \mathbf{C}) + \nabla^2 \mathbf{C} - \partial_t^2 \mathbf{C} &= -\mathbf{J}_m \\ -\nabla^2 C_0 - \partial_t \nabla \cdot \mathbf{C} &= \rho_m \end{aligned} \quad (\text{A2})$$

We have replaced *six* components of  $\mathbf{E}$  and  $\mathbf{B}$  with *eight* components of  $A_\mu$  and  $C_\mu$ . The extra degrees of freedom can be eliminated by gauge fixing terms. For instance, if we use the Lorenz gauge condition for both fields,

$$\partial_t A_0 + \nabla \cdot \mathbf{A} = 0, \quad \partial_t C_0 + \nabla \cdot \mathbf{C} = 0,$$

or, in shorter form,

$$\partial_\mu A^\mu = 0, \quad \partial_\mu C^\mu = 0, \quad (\text{A3})$$



which reduces Equation (A2) into

$$\square A_0 = \rho_e, \quad \square A = \mathbf{J}_e, \quad \square C = \mathbf{J}_m, \quad \square C_0 = \rho_m, \quad (\text{A4})$$

with the d'Alembertian given by  $\square \equiv -\nabla^2 + \partial_t^2$ . Let us mention that there is residual gauge freedom for the gauge transformation in Equation (A1); that is, the gauge fields  $A'$  and  $C'$  also obey Equation (A3) as long as the functions  $f(x)$  and  $g(x)$  in Equation (A1) obey

$$\square f = 0, \quad \square g = 0.$$

## Appendix B. Weyl Theory versus Our Approach

Hereafter, we provide a brief discussion of the Weyl invariant theory of gravity [37,38] adapted for our purpose. Indeed, the fact that we gauge  $\mathbb{R}^+$  might suggest that we apply the Weyl invariant approach to gravity, and we wish to clarify that our approach is different. In Weyl theory, the spacetime connection is not the Levi-Civita connection, given by the Christoffel symbols,  $\left\{ \begin{smallmatrix} \lambda \\ \mu\rho \end{smallmatrix} \right\}$ , but the so-called Weyl connection ( $\overset{W}{\Gamma}$ ), which does not satisfy the metricity condition

$$\nabla_{\mu} g_{\rho\sigma} = \partial_{\mu} g_{\rho\sigma} - \overset{W}{\Gamma}_{\mu\rho}^{\lambda} g_{\lambda\sigma} - \overset{W}{\Gamma}_{\mu\sigma}^{\lambda} g_{\rho\lambda} = n\sigma_{\mu} g_{\rho\sigma} \neq 0.$$

In Weyl theory,  $\sigma_{\mu}$  is a geometrical vector field so that it is part of the spacetime geometry, just like the metric tensor and the connection [37]. The Weyl connection is related to the Levi-Civita connection by

$$\overset{W}{\Gamma}_{\mu\rho}^{\lambda} = \left\{ \begin{smallmatrix} \lambda \\ \mu\rho \end{smallmatrix} \right\} - n \frac{1}{2} g^{\lambda\tau} (\sigma_{\mu} g_{\rho\tau} + \sigma_{\rho} g_{\mu\tau} - \sigma_{\tau} g_{\mu\rho}).$$

We can see that the non-metricity condition chosen above is satisfied with this definition of the connection:

$$\begin{aligned} \overset{W}{\nabla}_{\mu} g_{\rho\sigma} &= \partial_{\mu} g_{\rho\sigma} - \left[ \left\{ \begin{smallmatrix} \lambda \\ \mu\rho \end{smallmatrix} \right\} - n \frac{1}{2} g^{\lambda\tau} (\sigma_{\mu} g_{\rho\tau} + \sigma_{\rho} g_{\mu\tau} - \sigma_{\tau} g_{\mu\rho}) \right] g_{\lambda\sigma} \\ &\quad - \left[ \left\{ \begin{smallmatrix} \lambda \\ \mu\sigma \end{smallmatrix} \right\} - n \frac{1}{2} g^{\lambda\tau} (\sigma_{\mu} g_{\sigma\tau} + \sigma_{\sigma} g_{\mu\tau} - \sigma_{\tau} g_{\mu\sigma}) \right] g_{\rho\lambda} \\ &= \overset{E}{\nabla}_{\mu} g_{\rho\sigma} + n \frac{1}{2} \delta_{\sigma}^{\tau} (\sigma_{\mu} g_{\rho\tau} + \sigma_{\rho} g_{\mu\tau} - \sigma_{\tau} g_{\mu\rho}) \\ &\quad + n \frac{1}{2} \delta_{\rho}^{\tau} (\sigma_{\mu} g_{\sigma\tau} + \sigma_{\sigma} g_{\mu\tau} - \sigma_{\tau} g_{\mu\sigma}) = n\sigma_{\mu} g_{\rho\sigma}, \end{aligned}$$

where we have used the fact that  $\overset{E}{\nabla}_{\mu} g_{\rho\sigma} = \partial_{\mu} g_{\rho\sigma} - \left\{ \begin{smallmatrix} \lambda \\ \mu\rho \end{smallmatrix} \right\} g_{\lambda\sigma} - \left\{ \begin{smallmatrix} \lambda \\ \mu\sigma \end{smallmatrix} \right\} g_{\rho\lambda} = 0$ , which is the standard covariant derivative of Einstein's general relativity, built with the Levi-Civita connection, and vanishes identically.

We can check that the Weyl connection is gauge-invariant under the transformation

$$\bar{g}_{\rho\sigma} = e^{n\alpha} g_{\rho\sigma}, \quad \bar{g}^{\rho\sigma} = e^{-n\alpha} g^{\rho\sigma}, \quad \bar{\sigma}_{\mu} = \sigma_{\mu} + \partial_{\mu}\alpha,$$

as follows:

$$\begin{aligned}
\overline{\Gamma}_{\mu\nu}^{\rho} &= \overline{\left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\}} - n \frac{1}{2} \overline{g}^{\rho\tau} \left( \overline{\sigma}_{\mu} \overline{g}_{\nu\tau} + \overline{\sigma}_{\nu} \overline{g}_{\mu\tau} - \overline{\sigma}_{\tau} \overline{g}_{\mu\nu} \right) \\
&= \frac{1}{2} \overline{g}^{\rho\sigma} \left( \partial_{\mu} \overline{g}_{\nu\sigma} + \partial_{\nu} \overline{g}_{\mu\sigma} - \partial_{\sigma} \overline{g}_{\mu\nu} \right) \\
&\quad - n \frac{1}{2} \overline{g}^{\rho\tau} \left( \overline{\sigma}_{\mu} \overline{g}_{\nu\tau} + \overline{\sigma}_{\nu} \overline{g}_{\mu\tau} - \overline{\sigma}_{\tau} \overline{g}_{\mu\nu} \right) \\
&= \left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\} - n \frac{1}{2} g^{\rho\tau} \left( \sigma_{\mu} g_{\nu\tau} + \sigma_{\nu} g_{\mu\tau} - \sigma_{\tau} g_{\mu\nu} \right) = \overline{\Gamma}_{\mu\nu}^{\rho}.
\end{aligned}$$

Likewise, we see that the non-metricity condition is gauge-invariant, since

$$\begin{aligned}
\overline{\nabla}_{\mu} \overline{g}_{\rho\sigma} &= \partial_{\mu} \overline{g}_{\rho\sigma} - \overline{\Gamma}_{\mu\rho}^{\lambda} \overline{g}_{\lambda\sigma} - \overline{\Gamma}_{\mu\sigma}^{\lambda} \overline{g}_{\lambda\rho} \\
&= \partial_{\mu} (e^{n\alpha} g_{\rho\sigma}) - \overline{\Gamma}_{\mu\rho}^{\lambda} e^{n\alpha} g_{\lambda\sigma} - \overline{\Gamma}_{\mu\sigma}^{\lambda} e^{n\alpha} g_{\lambda\rho} \\
&= n e^{n\alpha} g_{\rho\sigma} \partial_{\mu} \alpha + e^{n\alpha} \left( \partial_{\mu} g_{\rho\sigma} - \overline{\Gamma}_{\mu\rho}^{\lambda} g_{\lambda\sigma} - \overline{\Gamma}_{\mu\sigma}^{\lambda} g_{\lambda\rho} \right) = n \overline{g}_{\rho\sigma} \overline{\sigma}_{\mu}.
\end{aligned}$$

Thus the non-metricity condition preserves its form, so that it is covariant.

We must emphasize that in Weyl theory,  $\sigma_{\mu}$  is a geometric field, so that it has to be included in the dynamics of the gravitational field. If we now compare this to our approach, note that we began our analysis by supposing that we have a Riemannian manifold, which means that we have a manifold equipped with a metric tensor (for distances), a Levi–Civita connection (to define parallel transport), which is free of torsion but presents curvature. This connection allows us to define a covariant derivative, and in our case, the metric and the connection are related by the metricity condition

$$\nabla_{\mu} g_{\rho\sigma} = \partial_{\mu} g_{\rho\sigma} - \left\{ \begin{matrix} \lambda \\ \mu\rho \end{matrix} \right\} g_{\lambda\sigma} - \left\{ \begin{matrix} \lambda \\ \mu\sigma \end{matrix} \right\} g_{\rho\lambda} = 0.$$

If we perform a local conformal transformation of the metric tensor, the Levi–Civita connection and Christoffel symbols transform as follows:

$$\overline{g}_{\rho\sigma} = e^{n\alpha} g_{\rho\sigma}, \quad \overline{g}^{\rho\sigma} = e^{-n\alpha} g^{\rho\sigma}$$

as well as

$$\begin{aligned}
\overline{\left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\}} &= \frac{1}{2} \overline{g}^{\rho\sigma} \left( \partial_{\mu} \overline{g}_{\nu\sigma} + \partial_{\nu} \overline{g}_{\mu\sigma} - \partial_{\sigma} \overline{g}_{\mu\nu} \right) \\
&= \frac{1}{2} e^{-n\alpha} g^{\rho\sigma} \left( \partial_{\mu} (e^{n\alpha} g_{\nu\sigma}) + \partial_{\nu} (e^{n\alpha} g_{\mu\sigma}) - \partial_{\sigma} (e^{n\alpha} g_{\mu\nu}) \right) \\
&= \left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\} + n \frac{1}{2} g^{\rho\sigma} \left( \partial_{\mu} \alpha g_{\nu\sigma} + \partial_{\nu} \alpha g_{\mu\sigma} - \partial_{\sigma} \alpha g_{\mu\nu} \right),
\end{aligned}$$

so that the connection is clearly not gauge-invariant.

Likewise, we examine what happens to the metricity condition:

$$\begin{aligned}
\overline{\nabla}_{\mu} \overline{g}_{\rho\sigma} &= \partial_{\mu} \overline{g}_{\rho\sigma} - \overline{\left\{ \begin{matrix} \lambda \\ \mu\rho \end{matrix} \right\}} \overline{g}_{\lambda\sigma} - \overline{\left\{ \begin{matrix} \lambda \\ \mu\sigma \end{matrix} \right\}} \overline{g}_{\lambda\rho} \\
&= \partial_{\mu} (e^{n\alpha} g_{\rho\sigma}) - \left[ \left\{ \begin{matrix} \lambda \\ \mu\rho \end{matrix} \right\} + n \frac{1}{2} g^{\lambda\tau} \left( \partial_{\mu} \alpha g_{\rho\tau} + \partial_{\rho} \alpha g_{\mu\tau} - \partial_{\tau} \alpha g_{\mu\rho} \right) \right] e^{n\alpha} g_{\lambda\sigma} \\
&\quad - \left[ \left\{ \begin{matrix} \lambda \\ \mu\sigma \end{matrix} \right\} + n \frac{1}{2} g^{\lambda\tau} \left( \partial_{\mu} \alpha g_{\sigma\tau} + \partial_{\sigma} \alpha g_{\mu\tau} - \partial_{\tau} \alpha g_{\mu\sigma} \right) \right] e^{n\alpha} g_{\lambda\rho} = e^{n\alpha} \nabla_{\mu} g_{\rho\sigma}.
\end{aligned}$$

We see, firstly, that the covariant derivative of the metric is covariant under the conformal transformation. Secondly, if the metricity condition is valid prior to the transformation, then the transformed covariant derivative of the metric tensor also satisfies the metricity condition:

$$\nabla_{\mu} g_{\rho\sigma} = 0 \Rightarrow \bar{\nabla}_{\mu} \bar{g}_{\rho\sigma} = 0.$$

From the point of view of a gauge theory for the conformal group, the gauge field  $C_{\mu}$  introduced to recover the gauge invariance of the original theory is not of geometrical nature. Therefore, it can be treated as a matter field whose dynamics takes place in a curved spacetime. If we assume that the matter fields have an insignificant contribution to the spacetime matter content (which can be the case if we are treating a system of two interacting particles), then we can neglect the effect of that field on the spacetime curvature and we can treat our system as a field theory on a curved spacetime whose curvature is determined by other matter content. Because we still consider a conformal transformation of the metric, it would be interesting to consider a gravitational Lagrangian that is also conformally invariant; however, the contribution of the matter content described by our Lagrangians may still be ignored from the point of view of spacetime dynamics.

### Appendix C. Duality of the Model

One of the main features of magnetic monopole models is their duality. Thus, in this appendix, we establish the duality property of our model. As our model involves a curved spacetime, we shall provide a proof that is valid in any curved spacetime. In addition, we will encounter yet another justification for our use of analytical extension.

For this, let us first turn our attention to duality in a flat spacetime. First, define

$$\mathbf{F} \equiv \mathbf{E} + i\mathbf{B}, \quad \mathbf{q} \equiv \rho_e + i\rho_m, \quad \mathbf{J} \equiv \mathbf{J}_e + i\mathbf{J}_m.$$

Then, we find, from Equation (2),

$$\nabla \cdot (\mathbf{E} + i\mathbf{B}) = \rho_e + i\rho_m, \quad \text{so that} \quad \nabla \cdot \mathbf{F} = \mathbf{q},$$

and

$$\nabla \times (\mathbf{E} + i\mathbf{B}) + \frac{1}{i} \frac{\partial}{\partial t} (\mathbf{E} + i\mathbf{B}) = i(\mathbf{J}_e + i\mathbf{J}_m), \quad \text{so that} \quad \nabla \times \mathbf{F} + \frac{1}{i} \frac{\partial \mathbf{F}}{\partial t} = i\mathbf{J}.$$

If we transform the quantities above using the following prescription

$$\mathbf{F} \rightarrow e^{i\theta} \mathbf{F}, \quad \mathbf{q} \rightarrow e^{i\theta} \mathbf{q}, \quad \mathbf{J} \rightarrow e^{i\theta} \mathbf{J}, \quad (\text{A5})$$

where  $\theta$  is a constant, we observe that

$$\nabla \cdot \mathbf{F} = \mathbf{q} \rightarrow \nabla \cdot (e^{i\theta} \mathbf{F}) = e^{i\theta} \mathbf{q} \Rightarrow \nabla \cdot \mathbf{F} = \mathbf{q},$$

$$\nabla \times \mathbf{F} + \frac{1}{i} \frac{\partial \mathbf{F}}{\partial t} = i\mathbf{J} \rightarrow \nabla \times (e^{i\theta} \mathbf{F}) + \frac{1}{i} \frac{\partial (e^{i\theta} \mathbf{F})}{\partial t} = ie^{i\theta} \mathbf{J} \Rightarrow \nabla \times \mathbf{F} + \frac{1}{i} \frac{\partial \mathbf{F}}{\partial t} = i\mathbf{J},$$

which proves the duality property in flat spacetime.

We now turn to the duality in curved spacetimes, such as the ones utilized in this work. We begin with the inhomogeneous Maxwell equations for both gauge fields  $A_{\mu}$  and  $C_{\mu}$  introduced in Equation (3) in curved spacetime:

$$\nabla_{\nu} F^{\mu\nu(A)} = j_{A}^{\mu}, \quad \nabla_{\nu} F^{\mu\nu(C)} = j_{C}^{\mu}.$$

Hereafter, we apply the compact notation

$$F_a^{\mu\nu} = \{F^{\mu\nu(A)}, F^{\mu\nu(C)}\}.$$

We shall define the dual fields in our curved spacetimes as per the prescription in Felsager's book [39]:

$$F_a^{*\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{a\rho\sigma} = \frac{1}{2} \left( -\frac{1}{\sqrt{-g}} \right) \epsilon^{\mu\nu\rho\sigma} F_{a\rho\sigma}, \quad (\text{A6})$$

where  $\varepsilon_{\alpha\beta\gamma\delta}$  is the Levi-Civita form and  $\epsilon_{\mu\nu\rho\sigma}$  is the Levi-Civita symbol, denoted, respectively, by slightly different symbols, and

$$F_{a\mu\nu}^* \equiv \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_a^{\rho\sigma} = \frac{1}{2} \sqrt{-g} \epsilon_{\mu\nu\rho\sigma} F_a^{\rho\sigma}. \quad (\text{A7})$$

The four-dimensional Levi-Civita symbol defined by  $\epsilon^{\alpha\beta\rho\sigma} = +1$  if  $(\alpha, \beta, \rho, \sigma)$  is an even permutation of  $(0, 1, 2, 3)$ ; it is  $-1$  for an odd permutation, and  $0$  otherwise [40].

The Levi-Civita form  $\varepsilon_{\alpha\beta\gamma\delta}$  and Levi-Civita symbol  $\epsilon_{\mu\nu\rho\sigma}$  are related by

$$\varepsilon_{\alpha\beta\gamma\delta} = \sqrt{-g} \epsilon_{\alpha\beta\gamma\delta}, \quad \epsilon^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{-g}} \varepsilon^{\alpha\beta\gamma\delta},$$

and are such that  $\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma}$ .

In particular, the covariant and contravariant representations of the Levi-Civita form are related by  $\varepsilon_{\alpha\beta\gamma\delta} = g_{\mu\alpha} g_{\nu\beta} g_{\rho\gamma} g_{\sigma\delta} \varepsilon^{\mu\nu\rho\sigma}$ . Finally, let us recall an important property of the four-dimensional Levi-Civita form,  $\varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{\rho\sigma\gamma\delta} = 2 \left( \delta_{\rho}^{\alpha} \delta_{\sigma}^{\beta} - \delta_{\sigma}^{\alpha} \delta_{\rho}^{\beta} \right)$ , which allows us to evaluate the dual of the duals:

$$(F_a^*)_{\tau\kappa}^* = \frac{1}{2} \varepsilon_{\tau\kappa\mu\nu} F_a^{*\mu\nu} = \frac{1}{2} \varepsilon_{\tau\kappa\mu\nu} \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{a\rho\sigma} = F_{a\tau\kappa},$$

and

$$(F_a^*)^{*\tau\kappa} = \frac{1}{2} \varepsilon^{\tau\kappa\mu\nu} F_{a\mu\nu}^* = \frac{1}{2} \varepsilon^{\tau\kappa\mu\nu} \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_a^{\rho\sigma} = F_a^{\tau\kappa}.$$

We can check that the Bianchi identity is valid in this curved spacetime version,

$$\nabla_{\mu} F_{a\rho\sigma} + \nabla_{\rho} F_{a\sigma\mu} + \nabla_{\sigma} F_{a\mu\rho} = 0, \quad (\text{A8})$$

where we exploited the relation  $\Gamma_{\mu\nu}^{\rho} = \Gamma_{\nu\mu}^{\rho}$ , valid for a Riemannian manifold. Equivalently, we can see that  $\varepsilon^{\nu\mu\rho\sigma} \nabla_{\mu} F_{a\rho\sigma} = 0$ .

With this in mind, let us consider the covariant derivative of the dual electromagnetic tensor, defined in Equation (A6):

$$\nabla_{\mu} F_a^{*\mu\nu} = \frac{1}{2} \nabla_{\mu} \varepsilon^{\mu\nu\rho\sigma} F_{a\rho\sigma} - \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \nabla_{\mu} F_{a\rho\sigma} = \frac{1}{2} \nabla_{\mu} \varepsilon^{\mu\nu\rho\sigma} F_{a\rho\sigma}. \quad (\text{A9})$$

Since  $\nabla_{\mu} g_{\rho\sigma} = 0$ , we find that  $\nabla_{\mu} \varepsilon^{\mu\nu\rho\sigma} = 0$  in all coordinate systems, and we conclude from Equation (A9) that  $\nabla_{\mu} F_a^{*\mu\nu} = 0$ , which is valid for both  $a = (A)$  and  $a = (C)$ ; that is,

$$\nabla_{\mu} F_{(A)}^{*\mu\nu} = 0, \quad \nabla_{\mu} F_{(C)}^{*\mu\nu} = 0. \quad (\text{A10})$$

(We understand  $F_{(A)}^{*\mu\nu} \equiv F^{*\mu\nu(A)}$ , and so on.) Consequently, we can express the field equations

$$\nabla_{\nu} F_{(A)}^{\mu\nu} = j_A^{\mu}, \quad \nabla_{\nu} F_{(C)}^{\mu\nu} = j_C^{\mu},$$

without any loss of generality, as follows:

$$\nabla_{\nu} F_{(A)}^{\mu\nu} + \nabla_{\nu} F_{(C)}^{*\mu\nu} = j_A^{\mu}, \quad \nabla_{\nu} F_{(C)}^{\mu\nu} + \nabla_{\nu} F_{(A)}^{*\mu\nu} = j_C^{\mu}. \quad (\text{A11})$$

With the following definitions,

$$\mathcal{F}^{\mu\nu} \equiv \left( F_{(A)}^{\mu\nu} + F_{(C)}^{*\mu\nu} \right) + i \left( F_{(C)}^{\mu\nu} + F_{(A)}^{*\mu\nu} \right), \quad \mathcal{J}^\mu \equiv j_A^\mu + ij_C^\mu,$$

we find, by using Equation (A11),

$$\nabla_\nu \mathcal{F}^{\mu\nu} = \mathcal{J}^\mu.$$

In order to observe the duality property of our model, we perform the duality transformation analogous to Equation (A5),

$$\mathcal{F}^{\mu\nu} \rightarrow \mathcal{F}'^{\mu\nu} = e^{i\theta} \mathcal{F}^{\mu\nu}, \quad \mathcal{J}^\mu \rightarrow \mathcal{J}'^\mu = e^{i\theta} \mathcal{J}^\mu.$$

Then, the field equation for the duality-transformed objects coincides exactly with the original field equations; that is,  $\nabla_\nu \mathcal{F}'^{\mu\nu} = \mathcal{J}'^\mu$ , which becomes  $\nabla_\nu (e^{i\theta} \mathcal{F}^{\mu\nu}) = e^{i\theta} \mathcal{J}^\mu$ ,  $e^{i\theta} \nabla_\nu \mathcal{F}^{\mu\nu} = e^{i\theta} \mathcal{J}^\mu$ , which leads to  $\nabla_\nu \mathcal{F}^{\mu\nu} = \mathcal{J}^\mu$ , as proposed.

As already mentioned, this structure motivates our foundational application of the analytical extension, so that the duality may be seen as a guide for the existence of two potentials. Note that

$$\begin{aligned} \mathcal{F}^{\mu\nu} &\equiv \left( F_{(A)}^{\mu\nu} + F_{(C)}^{*\mu\nu} \right) + i \left( F_{(C)}^{\mu\nu} + F_{(A)}^{*\mu\nu} \right) \\ &= g^{\mu\rho} g^{\nu\sigma} \left( F_{\rho\sigma}^{(A)} + i F_{\rho\sigma}^{(C)} \right) + i \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \left( F_{\rho\sigma}^{(A)} - i F_{\rho\sigma}^{(C)} \right). \end{aligned}$$

From the expressions  $F_{\rho\sigma}^{(A)} = \partial_\rho A_\sigma - \partial_\sigma A_\rho$  and  $F_{\rho\sigma}^{(C)} = \partial_\rho C_\sigma - \partial_\sigma C_\rho$ , with the definition  $Z_\rho \equiv A_\rho + iC_\rho$ , we find

$$\begin{aligned} \mathcal{F}^{\mu\nu} &= g^{\mu\rho} g^{\nu\sigma} (\partial_\rho Z_\sigma - \partial_\sigma Z_\rho) + i \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} (\partial_\rho Z_\sigma^\dagger - \partial_\sigma Z_\rho^\dagger) \\ &= F_{(Z)}^{\mu\nu} + i F_{(Z)}^{*\dagger\mu\nu}, \end{aligned}$$

which shows consistency.

As an example, let us define the electric and magnetic fields' charge densities and vector currents in terms of the following components of the electromagnetic field strength tensor:

$$\begin{aligned} \mathbf{E}^i &\equiv F_{(A)}^{i0} + F_{(C)}^{*i0}, & \rho_e &\equiv -j_A^0, \\ \mathbf{B}^i &\equiv F_{(C)}^{i0} + F_{(A)}^{*i0}, & \rho_m &\equiv -j_C^0, \\ j_e^i &\equiv j_A^i, & j_m^i &\equiv j_C^i, \end{aligned}$$

so that the  $0i$  component of the analytically extended electromagnetic tensor is

$$\mathcal{F}^{0i} \equiv \left( F_{(A)}^{0i} + F_{(C)}^{*0i} \right) + i \left( F_{(C)}^{0i} + F_{(A)}^{*0i} \right) = -\mathbf{E}^i - i\mathbf{B}^i,$$

which, with the analytically extended four-current

$$\mathcal{J}^\mu = j_A^\mu + ij_C^\mu,$$

leads to

$$\nabla_\nu \mathcal{F}^{0\nu} = \nabla_i \mathcal{F}^{0i} = \mathcal{J}^0, \quad \nabla_i (\mathbf{E}^i + i\mathbf{B}^i) = \rho_e + i\rho_m.$$

Clearly, this is compatible with Equation (2).

#### Appendix D. Is the Term $\zeta R\phi^\dagger\phi$ Necessary?

In the literature, the study of the massless scalar field in the context of conformal invariance demands the inclusion of a term of the type  $\zeta R\phi^\dagger\phi$  in the Lagrangian, where  $R$  is the scalar curvature and  $\zeta$  is a constant parameter. This discussion can be found, for instance, in Wald's book [41]. In summary, the field equation for the massless scalar field in a  $d$ -dimensional curved spacetime is given by

$$\square\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0.$$

Note that the covariant derivative above is the spacetime covariant derivative,  $\nabla_\mu = \partial_\mu - \Gamma_\mu$ , where  $\Gamma_\mu$  is the Levi-Civita connection.

Now, we analyze if the transformed equation satisfies the same equation:

$$(\square\phi)' = (g^{\mu\nu}\nabla_\mu\nabla_\nu\phi)' = g'^{\mu\nu}\nabla'_\mu\nabla'_\nu\phi' = e^{2\alpha}g^{\mu\nu}(\partial_\mu\nabla_\nu\phi' - \Gamma'^{\sigma}_{\mu\nu}\nabla_\sigma\phi').$$

Since the connection transforms as

$$\Gamma'^{\nu}_{\rho\mu} = \Gamma^{\nu}_{\rho\mu} - (\delta^{\nu}_{\mu}\partial_\rho\alpha + \delta^{\nu}_{\rho}\partial_\mu\alpha - g^{\nu\sigma}g_{\rho\mu}\partial_\sigma\alpha),$$

we have

$$(\square\phi)' = e^{3\alpha}[\square\phi + (4-d)g^{\mu\nu}\nabla_\mu\alpha\nabla_\nu\phi + \phi(3-d)g^{\mu\nu}\nabla_\mu\alpha\nabla_\nu\alpha + \phi\square\alpha].$$

This shows that this equation is not covariant; that is,  $\square\phi = 0$  does not lead to  $(\square\phi)' = 0$  necessarily.

In order to solve this problem, it is usual to introduce the term  $\zeta R\phi$  in the field equation (which corresponds to the term  $\zeta R\phi^\dagger\phi$  in the Lagrangian). This term is motivated by several arguments stemming from quantum corrections to renormalization conditions (see, for instance, [42,43]). We consider the equation

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \zeta R\phi = 0,$$

and reproduce the same steps of the calculation above, i.e.,

$$(\square\phi - \zeta R\phi)' = g'^{\mu\nu}\nabla'_\mu\nabla'_\nu\phi' - \zeta R'\phi'.$$

With the transformation law for the scalar curvature,

$$R' = e^{2\alpha}[R + 2(d-1)g^{\mu\nu}\nabla_\mu\nabla_\nu\alpha - (d-2)(d-1)g^{\mu\nu}\nabla_\mu\alpha\nabla_\nu\alpha],$$

we end up with

$$\begin{aligned} (\square\phi - \zeta R\phi)' = e^{3\alpha}\{ & \square\phi - \zeta R\phi - (d-4)g^{\mu\nu}\nabla_\mu\alpha\nabla_\nu\phi \\ & - \phi[(d-3) - \zeta(d-2)(d-1)]g^{\mu\nu}\nabla_\mu\alpha\nabla_\nu\alpha \\ & + \phi[1 - 2\zeta(d-1)]\square\alpha\}. \end{aligned}$$

In  $d = 4$  dimensions, we obtain

$$(\square\phi - \zeta R\phi)' = e^{3\alpha}[\square\phi - \zeta R\phi - \phi(1 - 6\zeta)g^{\mu\nu}\nabla_\mu\alpha\nabla_\nu\alpha + \phi(1 - 6\zeta)\square\alpha].$$

This equation is not covariant for arbitrary values of  $\zeta$ . However, by choosing  $\zeta = \frac{1}{6}$  the equation is made covariant, since

$$\left(\square\phi - \frac{1}{6}R\phi\right)' = e^{3\alpha}\left(\square\phi - \frac{1}{6}R\phi\right)$$

so that if  $\square\phi - \frac{1}{6}R\phi = 0$ , then the equation is satisfied in its same form for any gauge fixing. We conclude that with the addition of the term  $\zeta R\phi$ , the field equation becomes covariant whenever we fix the parameter  $\zeta$  to be  $1/6$ .

At this point, it is important to highlight that these considerations are made under the assumptions that the field equation contains only the spacetime covariant derivative and not the gauge covariant derivative involved in our model.

Now, we analyze the field equation for the massless scalar field in our model after introducing the gauge fields. The Lagrangian

$$L = \sqrt{-g}g^{\mu\nu}D_\mu\phi^\dagger D_\nu\phi$$

leads to the following field equation:

$$g^{\mu\nu}\mathcal{D}_\nu D_\mu\phi^\dagger = 0,$$

where

$$\mathcal{D}_\nu \equiv \nabla_\nu - I_{(a)}^A{}_B\phi^B A_\nu^a = \nabla_\nu + iA_\nu + C_\nu.$$

We emphasize the presence of the spacetime covariant derivative and the presence of the gauge fields  $A_\nu, C_\nu$  in this equation.

Now, we analyze the transformed equation:

$$\begin{aligned} (g^{\mu\nu}\mathcal{D}_\nu D_\mu\phi^\dagger)' &= g'^{\mu\nu}\mathcal{D}'_\nu (D_\mu\phi^\dagger)' \\ &= g'^{\mu\nu}(\nabla'_\nu + iA'_\nu + C'_\nu)(e^{\alpha-i\epsilon}D_\mu\phi^\dagger) \\ &= e^{3\alpha-i\epsilon}\left[2g^{\mu\nu}\partial_\nu\alpha D_\mu\phi^\dagger + 2g^{\mu\nu}\partial_\nu\alpha D_\mu\phi^\dagger - g^{\mu\nu}4\partial_\nu\alpha D_\mu\phi^\dagger + g^{\mu\nu}\mathcal{D}_\nu D_\mu\phi^\dagger\right] \\ &= e^{3\alpha-i\epsilon}g^{\mu\nu}\mathcal{D}_\nu D_\mu\phi^\dagger \end{aligned}$$

This shows that the transformed equation is covariant, and hence, it is satisfied in any gauge:

$$(g^{\mu\nu}\mathcal{D}_\nu D_\mu\phi^\dagger)' = e^{3\alpha-i\epsilon}(g^{\mu\nu}\mathcal{D}_\nu D_\mu\phi^\dagger) = 0.$$

We conclude that there is no need to modify this equation by introducing an extra term like  $\zeta R\phi^\dagger$  due to lack of symmetry with respect to gauge transformations. Perhaps, by quantum corrections or renormalizability conditions, this equation may be modified and a term like  $\zeta R\phi^\dagger$  may eventually emerge. However, this is beyond the scope of the present work.

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