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Neutral Emden–Fowler Differential Equation of Second Order: Oscillation Criteria of Coles Type

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Abstract: In this work, we study the asymptotic and oscillatory behavior of solutions to the second-order general neutral Emden–Fowler differential equation $(a(v)\eta(x(v))z'(v))' + q(v)F(x(g(v))) = 0$, where $v \geq v_0$ and the corresponding function $z = x + p\,(x \circ h)$. Besides the importance of equations of the neutral type, studying the qualitative behavior of solutions to these equations is rich in analytical points and interesting issues. We begin by finding the monotonic features of positive solutions. The new properties contribute to obtaining new and improved relationships between x and z for use in studying oscillatory behavior. We present new conditions that exclude the existence of positive solutions to the examined equation, and then we establish oscillation criteria through the symmetry property between non-oscillatory solutions. We use the generalized Riccati substitution method, which enables us to apply the results to a larger area than the special cases of the considered equation. The new results essentially improve and extend previous results in the literature. We support this claim by applying the results to an example and comparing them with previous findings. Moreover, the reduction of our results to Euler's differential equation introduces the well-known sharp oscillation criterion.

Keywords: differential equations; Emden–Fowler differential equation; oscillation criteria; neutral-type equation

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1. Introduction

Neutral differential equations (NDEs)—in which the largest derivative occurs on the solution with and without delay—are among the most important types of functional differential equations (FDEs). Their importance is due to two main reasons, the first of which is the many practical applications of this type in physics and engineering, and the second is that the study of this type of equation includes many exciting analytical problems. Many applications, such as population dynamics, control, automatic, fluid mixing, and vibrating masses attached to a flexible rod, involve models of NDEs (see Hill [1]). In particular, second-order NDEs are very useful in robotics for building bipedal robots and in biology for explaining the ability of the human body to balance itself [2].

The part of qualitative theory that is concerned with studying the asymptotic and oscillatory features of solutions of FDEs is called oscillation theory. The main objectives of oscillation theory are to characterize the relationship between oscillatory and other fundamental properties of solutions to various classes of differential equations, investigate zero distribution laws, estimate the number of zeros in each interval and the distance between neighboring zeros, and establish conditions for the existence of oscillatory (and non-oscillatory) solutions (see [3,4]). The property of symmetry plays an important role in oscillation theory. In most studies, researchers are interested in obtaining criteria that guarantee the absence of any positive solutions to the examined equations, while the absence of negative solutions is also ensured through symmetry between non-oscillatory solutions. Oscillation theory has been and still is a crucial numerical mathematical technique in many cutting-edge technologies and fields (see [5–7]).

The capacity to forecast whether the solutions of a system will oscillate or remain stable is important in many applications. Understanding oscillation, for example, is useful in mechanical and structural engineering when building systems that can resist periodic forces without experiencing resonance, which could lead to structural failure. For modeling phenomena in biological systems, such as brain activity or heart rhythms, where oscillating patterns are suggestive of normal function or diseased states, oscillation criteria are crucial. In electrical engineering, oscillation criteria are essential for designing oscillators and filters in circuits so that they effectively carry out their intended tasks.

1.1. General NDE of Second Order

In this work, we provide sufficient conditions to verify that all solutions of the NDE

$$\left(a(v)\eta(x(v))z'(v)\right)' + q(v)F(x(g(v))) = 0 \tag{1}$$

are oscillatory, where $v \geq v_0$ and

$$z(v) = x(v) + p(v)x(h(v)).$$

During the study, we assume the following conditions:

A1: $a \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, $p, q \in \mathbf{C}([v_0,\infty),[0,\infty))$, $p(v) \leq p_0 < 1$ and

$$£_{v_0}(\infty) = \infty, \tag{2}$$

where

$$£_{v_0}(v) := \int_{v_0}^{v} a^{-1}(s)\mathrm{d}s;$$

A2: h, $g \in \mathbf{C}([v_0,\infty),\mathbb{R})$, $h(v) \leq v$, $g(v) \leq v$, $g'(v) \geq 0$, $\lim_{v\to\infty} h(v) = \infty$, and $\lim_{v\to\infty} g(v) = \infty$;

A3: $\eta \in \mathbf{C}^1(\mathbb{R},\mathbb{R})$ and $0 < l_0 \leq \eta(x) \leq L^{-1}$ for $x \neq 0$;

A4: $F \in \mathbf{C}(\mathbb{R},\mathbb{R})$, and $xF(x) > 0$ for $x \neq 0$.

For the solution of (1), we mean a function $x \in \mathbf{C}^1([v_x,\infty),\mathbb{R})$, $v_x \in [v_0,\infty)$, which has the properties $a\,\eta\,z' \in \mathbf{C}^1([v_x,\infty),\mathbb{R})$, $\sup\{|x(v)| : v \geq v\} > 0$ for all $v \geq v_x$, and satisfies (1) on $[v_x,\infty)$. A solution x of (1) is called oscillatory if it has arbitrarily large zeros and, otherwise, it is said to be non-oscillatory.

In the remainder of the introduction, we review some relevant previous literature. The main results are then divided into three sections. In Section 2, we present some helpful lemmas. Then, the oscillation results section provides new criteria for testing the oscillation of the considered equation in different cases for the functions F and η. In the next section, we apply our results to some special cases of the equation under study. The Conclusions Section comes at the end of the results.

1.2. Related Literature

Next, we revise some related work that has contributed to studying the oscillatory behavior of solutions to FDEs.

One of the earliest seminal contributions to the study of oscillatory behavior was Coles' paper [8]. He established a method for testing oscillatory behavior based on the definition of a class of positive and locally integrable functions.

In 2000, Manojlovic [9] investigated the oscillatory features of the equation

$$\left(a(v)\eta(v)|x'(v)|^{\alpha-1}x'(v)\right)' + q(v)F(x(v)) = 0,$$

where $\alpha \in \mathbb{R}^+$ and

$$\frac{F'(x)}{\left(\eta(x)|F(x)|^{\alpha-1}\right)^{1/\alpha}} \geq k > 0, \text{ for } x \neq 0.$$

As a generalization and extension of the results in [10,11], Dzurina and Lacková [12] used the Riccati approach to establish criteria for the oscillation of the NDE (1) where $F'(x)$ is nondecreasing in $(-\infty, -v_*)$ and nonincreasing in (v_*, ∞), $v_* \geq 0$. Using Philos-type criteria, Şahiner [13] studied the oscillatory behavior of solutions to (1) and considered the following three cases for F:

(S1) $F'(x)$ exists, and $F'(x) \geq k_1 > 0$.
(S2) $F'(x)$ exists, and $F'(x)/\eta(x) \geq k_2 > 0$ for $x \neq 0$,
(S3) $F(x)/x \geq k_3 > 0$.

Condition (S1) obviously implies (S2), but not the other way around. For instance, (S2) is true for $F(x) = x^3$ and $\eta(x) = x^2$, even though these functions do not meet (S1). It is necessary for F to be differentiable in (S1) and (S2). It is evident that (S3) does not have this requirement.

Using the generalized Riccati transformation, Erbe et al. [14] presented conditions for the oscillation of the NDE

$$\left(a(v)(x(v) + p(v)x(v-\tau))'\right)' + q(v)|x(g(v))|^{\alpha-1}x(g(v)) = 0,$$

where $\alpha \in \mathbb{R}^+$. They obtained oscillation criteria of Kamenev type.

In all the previously mentioned works, researchers always used the traditional relationship $x > (1-p)z$ that links x and z. This relationship is essential in studying the oscillatory behavior of NDEs, as it is the means to transform the NDE into a delay inequality.

There has been a significant surge in the study of FDE oscillation theory lately. Researchers have been interested in addressing some analytical problems in the oscillation theory of NDEs. Alsharidi and Muhib [15] studied the oscillatory features of an NDE with a mixed neutral term. Jadlovská et al. [16,17] provided sharp results for testing the oscillation of equations in the canonical case. For the non-canonical case, Muhib [18] presented more effective conditions for oscillation. Guo et al. [19] studied an NDE with positive and negative coefficients.

Moaaz et al. [20] discussed the oscillatory features of the NDE

$$\left(a(v)(z'(v))^{\alpha}\right)' + \sum_{j=1}^{m} q_j(v)x^{\beta}(\sigma_j(v)) = 0,$$

based on the improved relations

$$x(v) > z(v)\sum_{m=1}^{n/2}\frac{1}{p^{2m-1}}\left(1 - \frac{1}{p}\frac{\pi(\tau^{-2m}(v))}{\pi(\tau^{-(2m-1)}(v))}\right), \; n \in \mathbb{Z}^+ \text{ is even},$$

and

$$x(v) > z(v)(1-p)\sum_{m=0}^{(n-1)/2} p^{2m}\frac{\pi\left(\tau^{[2m+1]}(v)\right)}{\pi(v)}, \; n \in \mathbb{Z}^+ \text{ is odd},$$

where
$$f^{[0]}(v) = v \text{ and } f^{[j]}(v) = f\left(f^{[j-1]}(v)\right), \text{ for } j \in \mathbb{N}.$$

Hassan et al. [21] deduced the relation
$$x(v) > z(v) \sum_{\kappa=0}^{(n-1)/2} p^{2\kappa} \left(1 - p\frac{\pi\left(\tau^{[2\kappa+1]}(v)\right)}{\pi(\tau^{2\kappa}(v))}\right), n \in \mathbb{Z}^+ \text{ is odd},$$
and studied the oscillation of the NDE
$$(a(v)(z'(v))^{\alpha})' + q(v)x^{\alpha}(\sigma(v)) = 0. \tag{3}$$

Recently, Bohner et al. [22] established the relationship
$$x(v) > z(v)(1 - p(v))(1 + H_k(v)),$$
and presented improved oscillation criteria for (3), where
$$H_k(v) = \begin{cases} 0 & \text{for } k = 0, \\ \sum_{i=1}^{k} \prod_{j=0}^{2i-1} p\left(\tau^{[j]}(v)\right) & \text{for } \tau(v) \leq v \text{ and } k \in \mathbb{N}, \\ \sum_{i=1}^{k} \frac{\pi\left(\tau^{[2i]}(v)\right)}{\pi(v)} \prod_{j=0}^{2i-1} p\left(\tau^{[j]}(v)\right) & \text{for } \tau(v) \geq v \text{ and } k \in \mathbb{N}, \end{cases}$$

For the third-order NDE
$$(a(v)(z''(v))^{\alpha})' + q(v)x^{\alpha}(\sigma(v)) = 0,$$
Moaaz et al. [23] tested the oscillatory features, and presented the relation
$$x(v) > z(v)(1 - p) \sum_{\kappa=0}^{(n-1)/2} p^{2\kappa} \left(\frac{\tau^{[2\kappa+1]}(v) - v_1}{v - v_1}\right)^2.$$

The study of second-order FDEs is of great importance from a theoretical standpoint, in addition to its various applications. One can observe the reflection of the advancement of the study of the oscillatory behavior of solutions of second-order equations on the study of the oscillation of even-order equations, see [24,25] (for quasi-linear equations), [26,27] (for super-linear equations), and [28] (for equations on Time Scales).

As an extension of the results in [20–23], we obtain new properties for positive solutions of Equation (1) and then employ them to obtain a new relationship between x and z. Using this relationship, we get oscillation conditions for solutions of (1), considering the cases (S1)–(S3). The approach used is an improved extension of the integral averaging technique. The use of improved relationships is directly reflected in the conditions of oscillation, as the new conditions are more efficient in the oscillation test for the studied equations.

2. Preliminary Results

For the sake of simplification, we provide the next notations: $\Psi(v, v_0)$ represents the class of all bounded functions for $v \geq v_0$ that are positive and locally integrable and $\mathbb{S}^+$ represents the class of all eventually positive solutions of (1),
$$\kappa := \frac{1}{l_0 L},$$
$$\widetilde{\psi}(v, \ell) = \int_{\ell}^{v} \psi(s)\mathrm{d}s,$$

$$\Lambda(\varphi; v, \ell) = \frac{1}{\psi(v)} \int_{\ell}^{v} \varphi(s)\psi^2(s)\mathrm{d}s,$$

$$\Theta(\omega; v, \ell) = \frac{1}{\widetilde{\psi}(v,\ell)} \int_{\ell}^{v} \psi(s) \int_{\ell}^{s} \omega(\mu)\mathrm{d}\mu\mathrm{d}s, \text{ for } v \geq \ell \geq v_0,$$

and

$$\Omega_m(v) = \sum_{j=0}^{m} \left(\prod_{k=0}^{2j} p\left(h^{[k]}(v)\right) \right) \left[\frac{1}{p(h^{[2j]}(v))} - 1 \right] \left[\frac{\pounds_{v_1}\left(h^{[2j]}(v)\right)}{\pounds_{v_1}(v)} \right]^{\kappa},$$

where $\ell \in [v_0, \infty)$, $\psi \in \Psi(v, v_0)$, $\varphi \in \mathbf{C}([v_0, \infty), (0, \infty))$, and $\omega \in \mathbf{C}([v_0, \infty), \mathbb{R})$. Moreover, we need the following conditions:

(N1) $-F(-xw) \geq F(xw) \geq F(x)F(w)$ for $xw > 0$.
(N2) $x\eta'(x) > 0$ for $x \neq 0$.

Definition 1 ([29]). *Assume*

$$D_0 = \{(v, s) : v > s > v_0\} \text{ and } D = \{(v, s) : v \geq s \geq v_0\}.$$

A function $H \in \mathbf{C}(D, \mathbb{R})$ belongs to the class $\Im$, if

(i) $H(v, v) = 0$ *for* $v \geq v_0$, $H(v, s) > 0$ *on* D_0;
(ii) $H(v, s)$ *has a continuous and nonpositive partial derivative* $\partial H / \partial s$ *on* D_0 *such that the condition*

$$\frac{\partial H(v, s)}{\partial s} = -h(v, s)\sqrt{H(v, s)}, \tag{4}$$

for all $(v, s) \in D_0$, *is satisfied for some* $h \in \mathbf{C}(D, \mathbb{R})$.

Remark 1. *Given* $\alpha \in \mathbf{C}([v_0, \infty), \mathbb{R})$, *we define an integral operator* Φ. *This operator is defined as*

$$\Phi(\theta; v, \ell) = \int_{\ell}^{v} H(v, s)\theta(s)\alpha(s)\mathrm{d}s \ \text{ for } v \geq v_0,$$

in [30], where $\theta \in \mathbf{C}([v_0, \infty), \mathbb{R})$ *and is specified in terms of* $H(v, s)$ *and* $\alpha(s)$.

Lemma 1 ([31]). *If* $x \in \mathbb{S}^+$, *then*

$$x > \sum_{j=0}^{m} \left(\prod_{k=0}^{2j} p\left(h^{[k]}(v)\right) \right) \left[\frac{z\left(h^{[2j]}(v)\right)}{p\left(h^{[2j]}(v)\right)} - z\left(h^{[2j+1]}(v)\right) \right], \tag{5}$$

for any $m \geq 0$ *is an integer.*

Lemma 2. *If* $x \in \mathbb{S}^+$, *then* z *conforms to*

$$z(v) > 0, \ z'(v) > 0, \text{ and } \left(a(v)\eta(x(v))z'(v)\right)' < 0, \tag{6}$$

eventually.

Proof. Assume that $x \in \mathbb{S}^+$. Based on assumption (A2), there is a $v_1 \in [v_0, \infty)$ whereby $(x \circ h)(v)$ and $(x \circ g)(v)$ are positive for $v \geq v_1$. Consequently, $z(v) > 0$. From (A4), we have $x(g(v)) > 0$. It follows from Equation (1) that

$$\left(a(v)\eta(x(v))z'(v)\right)' = -q(v)F(x(g(v))) < 0.$$

Hence, we conclude that $a(v)\eta(x(v))z'(v)$ has a fixed sign, eventually. This is equivalent to saying that $z'(v) > 0$ or $z'(v) < 0$ for $v \geq v_2$, where v_2 is large enough. However, the case in which $z'(v) < 0$ contradicts condition (2), as shown below:

If $z'(v) < 0$ for $v \geq v_2$, then

$$a(v)\eta(x(v))z'(v) \leq a(v_2)\eta(x(v_2))z'(v_2) := -c < 0.$$

Hence,

$$z'(v) \leq -\frac{c}{a(v)\eta(x(v))}, \tag{7}$$

which with (A3) gives

$$z'(v) \leq -cLa^{-1}(v).$$

Thus,

$$z(v) \leq z(v_2) - cL\int_{v_2}^{v} a^{-1}(s)\mathrm{d}s.$$

But (2) leads to $z(v) \to -\infty$ as $v \to \infty$, a contradiction.
Therefore, the proof is finished. □

Lemma 3. *If $x \in \mathbb{S}^{+}$, then*

$$z(v) \geq La(v)\eta(x(v))z'(v)£_{v_1}(v),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{z(v)}{[£_{v_1}(v)]^{\kappa}}\right) \leq 0$$

is nonincreasing, for $v \geq v_1$.

Proof. Assume that $x \in \mathbb{S}^{+}$. Based on assumption (A2), there is a $v_1 \in [v_0, \infty)$ such that $(x \circ h)(v)$ and $(x \circ g)(v)$ are positive for $v \geq v_1$. Using the results of Lemma 2, we conclude that

$$\begin{aligned} z(v) &= z(v_1) + \int_{v_1}^{v} \frac{1}{a(s)\eta(x(s))} a(s)\eta(x(s))z'(s)\mathrm{d}s \\ &\geq L\int_{v_1}^{v} \frac{1}{a(s)} a(s)\eta(x(s))z'(s)\mathrm{d}s \\ &\geq La(v)\eta(x(v))z'(v)\int_{v_1}^{v} \frac{1}{a(s)}\mathrm{d}s \\ &= La(v)\eta(x(v))z'(v)£_{v_1}(v). \end{aligned}$$

Hence,

$$\begin{aligned} 0 &\geq z'(v) - \frac{1}{La(v)\eta(x(v))£_{v_1}(v)} z(v) \\ &\geq z'(v) - \frac{1}{l_0 L}\frac{[a(v)]^{-1}}{£_{v_1}(v)} z(v). \end{aligned} \tag{8}$$

Since $£_{v_1}(v) \geq 0$, $£_{v_1}(v_1) = 0$ and $£_{v_1}(\infty) = \infty$, there is a $v_2 \geq v_1$ such that $£_{v_1}(v_2) = 1$. From (8), we obtain

$$\begin{aligned} 0 &\geq \frac{\mathrm{d}}{\mathrm{d}v}\left(z(v)\exp\left[-\kappa\int_{v_2}^{v} \frac{[a(s)]^{-1}}{£_{v_1}(s)}\mathrm{d}s\right]\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}v}\left(z(v)\exp[-\kappa \ln £_{v_1}(v)]\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{z(v)}{[£_{v_1}(v)]^{\kappa}}\right). \end{aligned}$$

Therefore, the proof ends. □

Next, we derive a new relationship between x and z that helps improve the oscillation results.

Lemma 4. *If $x \in \mathbb{S}^+$, then Equation (1) can be expressed as the delay form*

$$\left(a(v)\eta(x(v))z'(v)\right)' + q(v)F(\Omega_m(g(v)))F(z(g(v))) \leq 0, \tag{9}$$

for $v \geq v_1$ and any integer $m \geq 0$.

Proof. Assume that $x \in \mathbb{S}^+$. Based on assumption (A2), there is a $v_1 \in [v_0, \infty)$ such that $(x \circ h)(v)$ and $(x \circ g)(v)$ are positive for $v \geq v_1$. From Lemma 1, we find that

$$x(v) > \sum_{j=0}^{m} \left(\prod_{k=0}^{2j} p\left(h^{[k]}(v)\right) \right) \left[\frac{z\left(h^{[2j]}(v)\right)}{p\left(h^{[2j]}(v)\right)} - z\left(h^{[2j+1]}(v)\right) \right] \tag{10}$$

It follows from Lemmas 2 and 3 that

$$z'(v) > 0 \text{ and } \left(\frac{z(v)}{[\pounds_{v_1}(v)]^{\kappa}} \right)' \leq 0.$$

Then,

$$z\left(h^{[2j]}(v)\right) \geq z\left(h^{[2j+1]}(v)\right)$$

and

$$z\left(h^{[2j]}(v)\right) \geq \left[\frac{\pounds_{v_1}\left(h^{[2j]}(v)\right)}{\pounds_{v_1}(v)} \right]^{\kappa} z(v).$$

Using the previous two inequalities in (10), we obtain

$$\begin{aligned} x(v) &> z(v) \sum_{j=0}^{m} \left(\prod_{k=0}^{2j} p\left(h^{[k]}(v)\right) \right) \left[\frac{1}{p\left(h^{[2j]}(v)\right)} - 1 \right] \left[\frac{\pounds_{v_1}\left(h^{[2j]}(v)\right)}{\pounds_{v_1}(v)} \right]^{\kappa} \\ &> \Omega_m(v)z(v). \end{aligned} \tag{11}$$

Combining this relation with Equation (1), we arrive at

$$\begin{aligned} \left(a(v)\eta(x(v))z'(v)\right)' &\leq -q(v)F(\Omega_m(g(v))z(g(v))) \\ &\leq -q(v)F(\Omega_m(g(v)))F(z(g(v))). \end{aligned}$$

Therefore, the proof ends. □

Lemma 5. *Assume that (N1) and (S2) are satisfied. If $x \in \mathbb{S}^+$, then (6) and (9) hold.*

Proof. Assume that $x \in \mathbb{S}^+$. Based on assumption (A2), there is a $v_1 \in [v_0, \infty)$ whereby $(x \circ h)(v)$ and $(x \circ g)(v)$ are positive for $v \geq v_1$. In addition, we observe that (6) and (7) hold for $v \geq v_1$. Now, we may verify that $z'(v) > 0$ for $v \geq v_1$. In fact, multiplying (7) by $F'(z(v)) > 0$ yields

$$F'(z(v))z'(v) \leq -\frac{k_2 L}{a(v)},$$

in light of $x(v) > z(v)$ for $v \geq v_1$. Clearly,

$$F(z(v)) \leq F(z(v_1)) - k_2 L \int_{v_1}^{v} a^{-1}(s)\mathrm{d}s,$$

for $v \geq v_1$. Hence, $F(z(v)) \to -\infty$ as $v \to \infty$ while applying (A1). This goes against (A4); hence, for $v \geq v_1$, $z'(v) > 0$ is required. Next, we find that (9) holds by repeating the procedures from the proofs of Lemmas 3 and 4. The proof is finished. □

Lemma 6. *Assume that (S3) is satisfied. If $x \in \mathbb{S}^+$, then (6) holds for some $v_1 > v_0$. Additionally,*

$$\left(a(v)\eta(x(v))z'(v)\right)' + k_3 q(v)\Omega_m(g(v))z(g(v)) \leq 0. \tag{12}$$

Proof. Assume that $x \in \mathbb{S}^+$. As per the proofs of Lemmas 3 and 4, we then state that (6) and (9) hold for some $v_1 \geq v_0$. Thus, from (1) and (S3), we have

$$\begin{aligned} 0 &= \left(a(v)\eta(x(v))z'(v)\right)' + q(v)F(x(g(v))) \\ &\geq \left(a(v)\eta(x(v))z'(v)\right)' + k_3 q(v)x(g(v)) \\ &= \left(a(v)\eta(x(v))z'(v)\right)' + k_3 q(v)\Omega_m(g(v))z(g(v)). \end{aligned}$$

Note that,

$$\left(a(v)\eta(x(v))z'(v)\right)' + k_3 q(v)\Omega_m(g(v))z(g(v)) \leq 0,\ v \geq v_1. \tag{13}$$

It is evident from (13) that (12) is true. This brings the proof to a close. □

3. Oscillation Results

In this section, we derive new criteria for the oscillation of solutions to Equation (1) in cases (S1)–(S3).

3.1. Oscillation Theorems for Case (S1)

Theorem 1. *Assume that (S1) and (N1) are satisfied. Equation (1) is oscillatory if there are $\psi \in \Psi(v, v_0)$, $\rho \in \mathbf{C}^1([v_0, \infty), \mathbb{R}^+)$, and $\sigma \in \mathbf{C}^1([v_0, \infty), \mathbb{R})$ such that*

$$\frac{\rho'(v)}{\rho(v)} + \frac{2k_1 L g'(v)\sigma(v)}{a(g(v))} := \theta_1(v) \geq 0, \tag{14}$$

$$\int_{v_0}^{\infty} \frac{\widetilde{\psi}^{\varsigma}(s,\ell)}{\Lambda(\varphi_1; s, \ell)} \mathrm{d}s = \infty,\ \varsigma \in [0,1),\ \ell \geq v_0, \tag{15}$$

and

$$\lim_{v \to \infty} \Theta\left(\xi_1 - \frac{1}{4}\varphi_1\theta_1^2\right) = \infty,\ \ell \geq v_0 \tag{16}$$

where

$$\varphi_1(v) = \frac{1}{k_1 L}\rho(v)\frac{a(g(v))}{g'(v)},$$

and

$$\xi_1(v) = \rho(v)\left(q(v)F(\Omega_m(g(v))) + \frac{k_1 L g'(v)}{a(g(v))}\sigma^2(v) - \sigma'(v)\right).$$

Proof. Assume the contrary, that $x \in \mathbb{S}^+$. Consequently, for $v \geq v_1$, (6) and (9) hold, according to Lemmas 3 and 4. Define

$$\omega(v) = \rho(v)\left(\frac{a(v)\eta(x(v))z'(v)}{F(z(g(v)))} + \sigma(v)\right). \tag{17}$$

Then, differentiating (17), it follows that

$$\begin{aligned}\omega'(v) &= \rho(v)\left[\frac{F(z(g(v)))(a(v)\eta(x(v))z'(v))'}{F^2(z(g(v)))}\right.\\&\quad\left.-\frac{a(v)\eta(x(v))z'(v)F'(z(g(v)))z'(g(v))g'(v)}{F^2(z(g(v)))}\right]\\&\quad+\rho(v)\sigma'(v)+\rho'(v)\left(\frac{a(v)\eta(x(v))z'(v)}{F(z(g(v)))}+\sigma(v)\right)\\&= \rho(v)\frac{(a(v)\eta(x(v))z'(v))'}{F(z(g(v)))}\\&\quad-a(v)\rho(v)\eta(x(v))g'(v)\frac{z'(v)F'(z(g(v)))z'(g(v))}{F^2(z(g(v)))}\\&\quad+\rho(v)\sigma'(v)+\frac{\rho'(v)}{\rho(v)}\omega(v).\end{aligned}$$

Using (9), we obtain

$$\begin{aligned}\omega'(v) &= \frac{\rho'(v)}{\rho(v)}\omega(v)-\rho(v)q(v)F(\Omega_m(g(v)))\\&\quad-a(v)\rho(v)\eta(x(v))g'(v)\frac{z'(v)F'(z(g(v)))z'(g(v))}{F^2(z(g(v)))}+\rho(v)\sigma'(v).\end{aligned}$$

Since $g(v)\le v$ and $a(v)\eta(x(v))z'(v)\le 0$, we have

$$a(g(v))\eta(x(g(v)))z'(g(v))\ge a(v)\eta(x(v))z'(v)$$

and

$$z'(g(v))\ge\frac{a(v)\eta(x(v))z'(v)}{a(g(v))\eta(x(g(v)))}.$$

Therefore, we obtain

$$\begin{aligned}\omega'(v) &\le \frac{\rho'(v)}{\rho(v)}\omega(v)-\rho(v)q(v)F(\Omega_m(g(v)))\\&\quad-\frac{k_1\rho(v)g'(v)}{a(g(v))\eta(x(g(v)))}\left(\frac{a(v)\eta(x(v))z'(v)}{F(z(g(v)))}\right)^2+\rho(v)\sigma'(v)\\&\le \frac{\rho'(v)}{\rho(v)}\omega(v)-\rho(v)q(v)F(\Omega_m(g(v)))+\rho(v)\sigma'(v)\\&\quad-\frac{k_1L\rho(v)g'(v)}{a(g(v))}\left(\frac{\omega(v)}{\rho(v)}-\sigma(v)\right)^2\\&= -\xi_1(v)+\theta_1(v)\omega(v)-\frac{1}{\varphi_1(v)}\omega^2(v),\end{aligned}\tag{18}$$

that is,

$$\omega'(v)\le-\xi_1(v)+\frac{1}{4}\varphi_1(v)\theta_1^2(v)-\frac{1}{\varphi_1(v)}\left(\omega(v)-\frac{1}{2}\varphi_1(v)\theta_1(v)\right)^2.$$

For $v\ge\ell\ge\ell_1$, we obtain

$$\begin{aligned}&\omega(v)+\int_\ell^v\frac{1}{\varphi_1(s)}\left(\omega(s)-\frac{1}{2}\varphi_1(s)\theta_1(s)\right)^2\mathrm{d}s\\&\le\ \omega(\ell)-\int_{\ell_1}^v\left(\xi_1(s)-\frac{1}{4}\varphi_1(s)\theta_1^2(s)\right)\mathrm{d}s.\end{aligned}$$

After integrating from ℓ to v and multiplying the above relation by $\psi(v)$, we obtain

$$\begin{aligned} & \int_\ell^v \psi(s)\omega(s)\mathrm{d}s + \int_\ell^v \psi(s)\int_\ell^s \frac{1}{\varphi_1(s)}\left(\omega(\mu) - \frac{1}{2}\varphi_1(\mu)\theta_1(\mu)\right)^2 \mathrm{d}\mu\mathrm{d}s \\ \leq\ & \int_\ell^v \psi(s)\omega(\ell)\mathrm{d}s - \int_\ell^v \psi(s)\int_\ell^s \left(\xi_1(\mu) - \frac{1}{4}\varphi_1(\mu)\theta_1^2(\mu)\right)\mathrm{d}\mu\mathrm{d}s. \end{aligned}$$

Hence,

$$\begin{aligned} & \int_\ell^v \psi(s)\omega(s)\mathrm{d}s + \int_\ell^v \psi(s)\int_\ell^s \frac{1}{\varphi_1(\mu)}\left(\omega(\mu) - \frac{1}{2}\varphi_1(\mu)\theta_1(\mu)\right)^2 \mathrm{d}\mu\mathrm{d}s \\ \leq\ & \widetilde{\psi}(v,\ell)\left[\omega(\ell) - \Theta\left(\xi_1(v) - \frac{1}{4}\varphi_1(v)\theta_1^2(v)\right)\right]. \end{aligned}$$

Based on condition (16), there is a $\ell_1 \geq \ell$ such that

$$\omega(\ell) - \Theta\left(\xi_1(v) - \frac{1}{4}\varphi_1(v)\theta_1^2(v)\right) < 0,$$

for all $v \geq \ell_1$. Then,

$$\begin{aligned} S(v) &= \int_\ell^v \psi(s)\int_\ell^s \frac{1}{\varphi_1(\mu)}\left(\omega(\mu) - \frac{1}{2}\varphi_1(\mu)\theta_1(\mu)\right)^2 \mathrm{d}\mu\mathrm{d}s \\ &\leq -\int_\ell^v \psi(s)\omega(s)\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} S(v) &\leq S(v) + \int_\ell^v \frac{1}{2}\psi(s)\varphi_1(s)\theta_1(s)\mathrm{d}s \\ &< -\int_\ell^v \psi(s)\left(\omega(s) - \frac{1}{2}\varphi_1(s)\theta_1(s)\right)\mathrm{d}s \end{aligned}$$

is obtained by condition (14). As shown, the function $S(v)$ is positive, and we obtain

$$S^2(v) \leq \left(\int_\ell^v \psi(s)\left(\omega(s) - \frac{1}{2}\varphi_1(s)\theta_1(s)\right)\mathrm{d}s\right)^2,\ v \geq \ell_1.$$

We obtain

$$\begin{aligned} S^2(v) &\leq \left(\int_\ell^v \sqrt{\varphi_1(s)}\psi(s)\left(\frac{1}{\sqrt{\varphi_1(s)}}\left[\omega(s) - \frac{1}{2}\varphi_1(s)\theta_1(s)\right]\right)\mathrm{d}s\right)^2 \\ &\leq \left(\int_\ell^v \varphi_1(s)\psi^2(s)\mathrm{d}s\right)\int_\ell^v \left(\frac{1}{\varphi_1(s)}\left[\omega(s) - \frac{1}{2}\varphi_1(s)\theta_1(s)\right]^2\right)\mathrm{d}s \\ &= \Lambda(\varphi_1; v, \ell)S'(v), \end{aligned} \tag{19}$$

by the Schwarz inequality. Note that

$$\begin{aligned} S(v) &= \int_\ell^v \psi(s)\int_\ell^s \frac{1}{\varphi_1(\mu)}\left(\omega(\mu) - \frac{1}{2}\varphi_1(\mu)\theta_1(\mu)\right)^2 \mathrm{d}\mu\mathrm{d}s \\ &\geq \int_\ell^v \psi(s)\int_\ell^{\ell_1} \frac{1}{\varphi_1(\mu)}\left(\omega(\mu) - \frac{1}{2}\varphi_1(\mu)\theta_1(\mu)\right)^2 \mathrm{d}\mu\mathrm{d}s \\ &= P\widetilde{\psi}(v,\ell), \end{aligned} \tag{20}$$

where
$$P=\int_{\ell}^{\ell_1}\frac{1}{\varphi_1(\mu)}\left(\omega(\mu)-\frac{1}{2}\varphi_1(\mu)\theta_1(\mu)\right)^2\mathrm{d}\mu.$$

Using (19) and (20), for every $v\geq\ell_1$ and some ς, $0\leq\varsigma<1$, we obtain
$$P^{\varsigma}\frac{\widetilde{\psi}^{\varsigma}(v,\ell)}{\Lambda(\varphi_1;v,\ell)}\leq S^{\varsigma-2}(v)S'(v).\tag{21}$$

Integrating (21) from ℓ_1 to v, we obtain
$$P^{\varsigma}\int_{\ell_1}^{v}\frac{\widetilde{\psi}^{\varsigma}(s,\ell)}{\Lambda(\varphi_1;s,\ell)}\mathrm{d}s\leq\frac{1}{1-\varsigma}\frac{1}{S^{1-\varsigma}(\ell_1)}<\infty,$$
in contrast to (15). Thus, we have proved the theorem in full. □

Next, utilizing Philos-type integral average conditions, we show some new oscillation criteria for (1).

Theorem 2. *Assume that (S1) and (N1) are satisfied. Equation (1) is oscillatory if there are $\rho\in\mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, $\sigma\in\mathbf{C}^1([v_0,\infty),\mathbb{R})$, $\alpha\in\mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, and $H\in\Im$ such that*
$$\limsup_{v\to\infty}\frac{1}{H(v,v_0)}\Phi\left(\xi_1-\frac{1}{4}\varphi_1\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)^2;v_0,v\right)=\infty.\tag{22}$$

Proof. Following the steps in the proof of Theorem 1, we observe that (18) is true for all $v\geq\ell\geq\ell_0$. When we apply operator $\Phi(.;v,\ell)$ to (18), we obtain
$$\Phi(\xi_1;\ell,v)+\Phi\big((\varrho-\varphi_1-\alpha\alpha')\omega;\ell,v\big)+\Phi(\varphi_1^{-1}\omega^2;\ell,v)\leq H(v,\ell)\alpha(\ell)\omega(\ell).\tag{23}$$

Completing squares of ω in the above inequality, we obtain
$$\begin{aligned}&\Phi\left(\varphi_1^{-1}\left(\omega+\frac{1}{2}\varphi_1\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)^2\right);\ell,v\right)+\Phi\left(\xi_1-\frac{1}{4}\varphi_1\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)^2;\ell,v\right)\\ \leq\ & H(v,\ell)\alpha(\ell)\omega(\ell).\end{aligned}$$

So,
$$\Phi\left(\xi_1-\frac{1}{4}\varphi_1\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)^2;\ell,v\right)\leq H(v,\ell)\alpha(\ell)\omega(\ell).\tag{24}$$

Thus, we have
$$\begin{aligned}\Phi\left(\xi_1-\frac{1}{4}\varphi_1\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)^2;v_0,v\right)&=\Phi\left(\xi_1-\frac{1}{4}\varphi_1\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)^2;v_0,\ell_0\right)\\&\quad+\Phi\left(\xi_1-\frac{1}{4}\varphi_1\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)^2;\ell_0,v\right)\\&\leq H(v,v_0)\left[\int_{v_0}^{\ell_0}|\xi_1(s)|\alpha(s)\mathrm{d}s+\alpha(\ell_0)|\omega_1(\ell_0)|\right].\end{aligned}$$

We derive a contradiction to condition (22) by dividing both sides of the inequality mentioned above and taking the lim sup in it as $v\to\infty$. The proof of this theorem is therefore finished. □

Theorem 3. *Assume that (S1) and (N1) are satisfied. Equation (1) is oscillatory if there are* $\rho \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, $\sigma \in \mathbf{C}^1([v_0,\infty),\mathbb{R})$, $\alpha \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, $\rho_1, \rho_2 \in \mathbf{C}([v_0,\infty),\mathbb{R}^+)$, *and* $H \in \Im$ *such that*

$$\limsup_{v\to\infty}\frac{1}{H(v,\ell)}\Phi(\xi_1;\ell,v) \geq \rho_1(\ell), \tag{25}$$

and

$$\limsup_{v\to\infty}\frac{1}{H(v,\ell)}\Phi\left(\varphi_1\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)^2;\ell,v\right) \leq \rho_2(\ell), \tag{26}$$

where ρ_1 *and* ρ_2 *satisfy*

$$\liminf_{v\to\infty}\frac{1}{H(v,\ell)}\Phi\left(\varphi_1^{-1}\alpha^{-2}\left(\rho_1-\frac{1}{4}\rho_2\right)^2;\ell,v\right) = \infty. \tag{27}$$

Proof. As in the proof of Theorem 2, we find that (23) and (24) hold. By dividing (24) by $H(v,\ell)$, we may determine that

$$\rho_1(\ell)-\frac{1}{4}\rho_2(\ell) \leq \alpha(\ell)\omega_1(\ell) \text{ for } \ell \geq \ell_0,$$

by the use of (25) and (26). This implies that

$$\frac{1}{\varphi_1(\ell)\alpha^2(\ell)}\left(\rho_1(\ell)-\frac{1}{4}\rho_2(\ell)\right)^2 \leq \frac{1}{\varphi_1(\ell)}\omega^2(\ell). \tag{28}$$

From (23), we obtain

$$\frac{1}{H(v,\ell)}\Phi\left(\varphi_1^{-1}\omega^2+\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)\omega;\ell,v\right) \leq \alpha(\ell)\omega(\ell)-\frac{1}{H(v,\ell)}\Phi(\xi_1;\ell,v).$$

Together with (25), this yields

$$\liminf_{v\to\infty}\frac{1}{H(v,\ell)}\Phi\left(\varphi_1^{-1}\omega^2+\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)\omega_1;\ell,v\right) \leq \alpha(\ell)\omega(\ell)-\rho_1(\ell) \leq c,$$

for $v \geq \ell \geq \ell_0$ and where c is a constant. Currently, we claim that

$$\liminf_{v\to\infty}\frac{1}{H(v,\ell)}\Phi\left(\varphi_1^{-1}\omega^2;\ell,v\right) < \infty. \tag{29}$$

In the case that (29) is not satisfied, there is a sequence $\{v_n\}_{n=1}^{\infty} \subset [v_0,\infty)$ with $\lim_{n\to\infty} v_n = \infty$ such that

$$\lim_{n\to\infty}\frac{1}{H(v_n,\ell)}\Phi\left(\varphi_1^{-1}\omega^2;\ell,v_n\right) = \infty. \tag{30}$$

Observe that (30) and

$$\frac{1}{H(v_n,\ell)}\Phi\left(\varphi_1^{-1}\omega^2;\ell,v_n\right)+\frac{1}{H(v_n,\ell)}\Phi\left(\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)\omega;\ell,v_n\right) \leq c+1,$$

can be used for a large enough n. For a sufficiently large n, this and (30) indicate that

$$1+\frac{\Phi\left(\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)\omega;\ell,v_n\right)}{\Phi\left(\varphi_1^{-1}\omega^2;\ell,v_n\right)} < \frac{1}{2},$$

that is,

$$\frac{\left|\Phi\left(\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)\omega;\ell,v_n\right)\right|}{\Phi\left(\varphi_1^{-1}\omega^2;\ell,v_n\right)}>\frac{1}{2}. \tag{31}$$

Hence,

$$\left(\Phi\left(\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)\omega;\ell,v_n\right)\right)^2\leq\Phi\left(\varphi_1^{-1}\omega^2;\ell,v_n\right)\Phi\left(\varphi_1\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)^2;\ell,v_n\right) \tag{32}$$

following the Schwarz inequality. Our result is

$$\Phi\left(\varphi_1^{-1}\omega^2;\ell,v_n\right)\leq 4\Phi\left(\varphi_1\left(\varrho-\theta_1-\alpha^{-1}\alpha'\right)^2;\ell,v_n\right), \tag{33}$$

based on (31) and (32). Contrary to (30), the right hand side of (33) is bounded by (26). Therefore, using (28), we obtain

$$\begin{aligned}&\liminf_{v\to\infty}\frac{1}{H(v,\ell)}\Phi\left(\varphi_1^{-1}\alpha^{-2}\left(\rho_1-\frac{1}{4}\rho_2\right)^2;\ell,v\right)\\ \leq\ &\liminf_{v\to\infty}\frac{1}{H(v,\ell)}\Phi\left(\varphi_1^{-1}\omega^2;\ell,v\right)\\ <\ &\infty.\end{aligned}$$

We obtain a contradiction with (27). The proof is now complete. □

3.2. Oscillation Theorems for Case (S2)

Theorem 4. *Assume that (S2), (N1), and (N2) are satisfied. Equation (1) is oscillatory if there are* $\psi\in\Psi(v_0,v)$, $\rho\in\mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, *and* $\sigma\in\mathbf{C}^1([v_0,\infty),\mathbb{R})$ *such that*

$$\frac{\rho'(v)}{\rho(v)}+\frac{2k_2Lg'(v)\sigma(v)}{a(g(v))}:=\theta_2(v)\geq 0, \tag{34}$$

$$\int_{v_0}^{\infty}\frac{\widetilde{\psi}^{\varsigma}(\ell,s)}{\Lambda(\varphi_2;\ell,s)}\mathrm{d}s=\infty,\ \varsigma\in[0,1),\ \ell\geq v_0, \tag{35}$$

and

$$\lim_{v\to\infty}\Theta\left(\xi_2-\frac{1}{4}\varphi_2\theta_2^2;\ell,v\right)=\infty,\ \ell\geq v_0, \tag{36}$$

where

$$\varphi_2(v)=\frac{1}{k_2L}\rho(v)\frac{a(g(v))}{g'(v)},$$

and

$$\xi_2(v)=\rho(v)\left(q(v)F(\Omega_m(g(v)))+\frac{k_2Lg'(v)}{a(g(v))}\sigma^2(v)-\sigma'(v)\right).$$

Proof. Assume the contrary, that $x\in\mathbb{S}^+$. From Lemma 5, we find that (6) and (9) hold for some $\ell>v_0$. Taking into account the function $\omega(v)$ as stated by (17), we arrive at

$$\begin{aligned}\omega'(v)\ \leq\ &\frac{\rho'(v)}{\rho(v)}\omega(v)-\rho(v)q(v)F(\Omega_m(g(v)))\\ &-\frac{\rho(v)g'(v)F'(z(g(v)))}{a(g(v))\eta(x(g(v)))}\left(\frac{a(v)\eta(x(v))z'(v)}{F(z(g(v)))}\right)^2+\rho(v)\sigma'(v).\end{aligned}$$

Using $x(g(v)) > z(g(v))$ and (N2), we can now determine that

$$\frac{F'(z(g(v)))}{\eta(x(g(v)))} > \frac{F'(z(g(v)))}{\eta(z(g(v)))} \geq k_2.$$

Hence,

$$\begin{aligned} \omega'(v) &\leq \frac{\rho'(v)}{\rho(v)}\omega(v) - \rho(v)q(v)F(\Omega_m(g(v))) \\ &\quad - \frac{k_2\rho(v)g'(v)}{a(g(v))}\left(\frac{a(v)\eta(x(v))z'(v)}{F(z(g(v)))}\right)^2 + \rho(v)\sigma'(v) \\ &= -\xi_2(v) + \theta_2(v)\omega(v) - \frac{1}{\varphi_2(v)}\omega^2(v). \end{aligned} \tag{37}$$

The remaining portions of the proof follow Theorem 1. □

Likewise, as in Case (S1), we can obtain the following two theorems, so their proof has been omitted.

Theorem 5. *Assume that (S2), (N1), and (N2) are satisfied. Equation (1) is oscillatory if there are $\rho \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, $\sigma \in \mathbf{C}^1([v_0,\infty),\mathbb{R})$, $\alpha \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, and $H \in \Im$ such that*

$$\limsup_{v\to\infty} \frac{1}{H(v,v_0)}\Phi\left(\xi_2 - \frac{1}{4}\varphi_2\left(\varrho - \theta_2 - \alpha^{-1}\alpha'\right)^2; v_0, v\right) = \infty. \tag{38}$$

Proof. We begin with inequality (37) and continue as in the Theorem 2 proof. □

For the sake of completeness, we declare an analogous theorem to Theorem 3 below. This may be obtained by following the same technique as in the proof of Theorem 3.

Theorem 6. *Assume that (S2), (N1), and (N2) are satisfied. Equation (1) is oscillatory if there are $\rho \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, $\sigma \in \mathbf{C}^1([v_0,\infty),\mathbb{R})$, $\alpha \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, $\rho_1, \rho_2 \in \mathbf{C}([v_0,\infty),\mathbb{R}^+)$, and $H \in \Im$ such that*

$$\limsup_{v\to\infty} \frac{1}{H(v,\ell)}\Phi(\xi_2; \ell, v) \geq \rho_1(\ell), \tag{39}$$

and

$$\limsup_{v\to\infty} \frac{1}{H(v,\ell)}\Phi\left(\varphi_2\left(\varrho - \theta_2 - \alpha^{-1}\alpha'\right)^2; \ell, v\right) \leq \rho_2(\ell), \tag{40}$$

where ρ_1 and ρ_2 satisfy

$$\liminf_{v\to\infty} \frac{1}{H(v,\ell)}\Phi\left(\varphi_2^{-1}\alpha^{-2}\left(\rho_1 - \frac{1}{4}\rho_2\right)^2; \ell, v\right) = \infty. \tag{41}$$

3.3. Oscillation Theorems for Case (S3)

Theorem 7. *Assume that (S3) is satisfied. Equation (1) oscillates if there are $\psi \in \Psi(v_0, v)$, $\rho \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, and $\sigma \in \mathbf{C}^1([v_0,\infty),\mathbb{R})$ such that*

$$\frac{\rho'(v)}{\rho(v)} + \frac{2k_2Lg'(v)\sigma(v)}{a(g(v))} := \theta_3(v) \geq 0, \tag{42}$$

$$\int_{v_0}^{\infty} \frac{\widetilde{\psi}^{\varsigma}(s,\ell)}{\Lambda(\varphi_3; s,\ell)}\mathrm{d}s = \infty, \varsigma \in [0,1),\ \ell \geq v_0, \tag{43}$$

and

$$\lim_{v\to\infty} \Theta\left(\xi_3 - \frac{1}{4}\varphi_3\theta_3^2; v, \ell\right) = \infty,\ \ell \geq v_0, \tag{44}$$

where

$$\varphi_3(v) = \frac{1}{L}\rho(v)\frac{a(g(v))}{g'(v)},$$

and

$$\xi_3(v) = \rho(v)\left(k_3 q(v)\Omega_m(v) + \frac{Lg'(v)}{a(g(v))}\sigma^2(v) - \sigma'(v)\right).$$

Proof. Assume the contrary, that $x \in \mathbb{S}^+$. From Lemma 6, we find that (6) and (12) hold for some $\ell > v_0$. The function $\omega_1(v)$ is defined by

$$\omega_1 = \rho\left(\frac{a\eta(x)z'}{z(g)} + \sigma\right), \tag{45}$$

for all $v > \ell_0$. After applying (12) and differentiating (45), we obtain

$$\begin{aligned}
\omega_1'(v) \leq\ & \frac{\rho'(v)}{\rho(v)}\omega_1(v) - \rho(v)k_3 q(v)\Omega_m(v) \\
& - \frac{\rho(v)g'(v)}{a(g(v))\eta(x(g(v)))}\left(\frac{a(v)\eta(x(v))z'(v)}{z(g(v))}\right)^2 + \rho(v)\sigma'(v) \\
\leq\ & \frac{\rho'(v)}{\rho(v)}\omega_1(v) - \rho(v)k_3 q(v)\Omega_m(v) \\
& - \frac{L\rho(v)g'(v)}{a(g(v))}\left(\frac{\omega_1(v)}{\rho(v)} - \sigma(v)\right)^2 + \rho(v)\sigma'(v) \\
=\ & -\xi_3(v) + \theta_3(v)\omega_1(v) - \frac{1}{\varphi_3(v)}\omega_1^2(v).
\end{aligned} \tag{46}$$

The type of inequality (46) is the same as that of inequality (18). Therefore, we can finish the Theorem 7 proof using a similar process. □

Likewise, as in Case (S1), we can obtain the following two theorems, so their proof has been omitted.

Theorem 8. *Assume that (S3) is satisfied. Equation (1) oscillates if there are* $\rho \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, $\sigma \in \mathbf{C}^1([v_0,\infty),\mathbb{R})$, $\alpha \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, *and* $H \in \Im$ *such that*

$$\limsup_{v\to\infty} \frac{1}{H(v,v_0)}\Phi\left(\xi_3 - \frac{1}{4}\varphi_3\left(\varrho - \theta_3 - \alpha^{-1}\alpha'\right)^2; v_0, v\right) = \infty. \tag{47}$$

Theorem 9. *Assume that (S3) is satisfied. Equation (1) oscillates if there are* $\rho \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, $\sigma \in \mathbf{C}^1([v_0,\infty),\mathbb{R})$, $\alpha \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, $\rho_1, \rho_2 \in \mathbf{C}([v_0,\infty),\mathbb{R}^+)$, *and* $H \in \Im$ *such that*

$$\limsup_{v\to\infty} \frac{1}{H(v,\ell)}\Phi(\xi_3;\ell,v) \geq \rho_1(\ell), \tag{48}$$

and

$$\limsup_{v\to\infty} \frac{1}{H(v,\ell)}\Phi\left(\varphi_3\left(\varrho - \theta_3 - \alpha^{-1}\alpha'\right)^2; \ell, v\right) \leq \rho_2(\ell), \tag{49}$$

where ρ_1 *and* ρ_2 *satisfy*

$$\liminf_{v\to\infty} \frac{1}{H(v,\ell)}\Phi\left(\varphi_3^{-1}\alpha^{-2}\left(\rho_1 - \frac{1}{4}\rho_2\right)^2; \ell, v\right) = \infty. \tag{50}$$

4. Special Cases

In this section, to facilitate the application of the results, we present some results by identifying special cases of functions Ψ and H.

Corollary 1. *Assume that (S1) and (N1) are satisfied. Equation (1) is oscillatory if there are $\Psi \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, and $\sigma \in \mathbf{C}^1([v_0,\infty),\mathbb{R})$ such that (14) holds,*

$$\int_{v_0}^{\infty} \frac{1}{\varphi_1(s)} \mathrm{d}s = \infty,$$

and

$$\int_{v_0}^{\infty} \left(\xi_1(s) - \frac{1}{4}\varphi_1(s)\theta_1^2(s) \right) \mathrm{d}s = \infty. \tag{51}$$

Proof. Taking the function $\Psi(v) = 1/\varphi_1(v)$, then

$$\begin{aligned} \lim_{v\to\infty} \int_{\ell}^{v} \frac{\widetilde{\psi}^{\varsigma}(s,\ell)}{\Lambda(\varphi_1;s,\ell)} \mathrm{d}s &= \lim_{v\to\infty} \int_{\ell}^{v} \frac{1}{\varphi_1(s)} \left(\int_{\ell}^{s} \left(\frac{1}{\varphi_1(\mu)} \right)^{\varsigma-1} \mathrm{d}\mu \right) \mathrm{d}s \\ &= \frac{1}{\varsigma} \lim_{v\to\infty} \left(\int_{\ell}^{v} \frac{1}{\varphi_1(s)} \mathrm{d}s \right)^{\varsigma} \\ &= \infty, \end{aligned}$$

and

$$\begin{aligned} &\lim_{v\to\infty} \Theta\left(\xi_1 - \frac{1}{4}\varphi_1(v)\theta_1^2(v);\ell,v \right) \\ = &\lim_{v\to\infty} \left(\int_{\ell}^{v} \frac{1}{\varphi_1(s)} \mathrm{d}s \right)^{-1} \int_{\ell}^{v} \frac{1}{\varphi_1(s)} \int_{\ell}^{s} \left(\xi_1(\mu) - \frac{1}{4}\varphi_1(\mu)\theta_1^2(\mu) \right) \mathrm{d}\mu \mathrm{d}s \\ = &\lim_{v\to\infty} \int_{v_0}^{v} \left(\xi_1(s) - \frac{1}{4}\varphi_1(s)\theta_1^2(s) \right) \mathrm{d}s \\ = &\infty. \end{aligned}$$

Then, Equation (1) oscillates according to Theorem 1. □

Corollary 2. *Assume that (S1) and (N1) are satisfied. Equation (1) oscillates if there are $\Psi \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, and $\sigma \in \mathbf{C}^1([v_0,\infty),\mathbb{R})$ such that (14) holds and*

$$\lim_{v\to\infty} \frac{1}{v^2} \int_{\ell_0}^{v} \varphi_1(s) \mathrm{d}s = 0,$$

and

$$\lim_{v\to\infty} \frac{1}{v} \int_{v_0}^{v} \int_{v_0}^{s} \left(\xi_1(\mu) - \frac{1}{4}\varphi_1(\mu)\theta_1^2(\mu) \right) \mathrm{d}\mu \mathrm{d}s = \infty.$$

Proof. Assume that $\Psi(s) = 1$. The oscillation of Equation (1) can be inferred from Theorem 1. □

Corollary 3. *Assume that (N1) and (S1) are satisfied. Let $\lim_{v\to\infty} \rho(v) = \infty$ and*

$$\liminf_{v\to\infty} \rho(v) \int_{v}^{\infty} [q(s)F(\Omega_m(g(s)))] \mathrm{d}s > \frac{1}{4}, \tag{52}$$

where

$$\rho(v) = \int_{v_0}^{v} \frac{k_1 L g'(s)}{a(g(s))} \mathrm{d}s.$$

Then, Equation (1) oscillates.

Proof. According to (52), two numbers $\ell > v_0$ and $\epsilon > 1/(4c)$ exist such that

$$\rho(v) \int_{v}^{\infty} [q(s)F(\Omega_m(g(s)))] \mathrm{d}s \geq \epsilon, \quad v \geq \ell.$$

Let $H(v,s) = (\rho(v) - \rho(s))^2$, $\alpha(v) = 1$, and $\sigma(v) = -1/2\rho(v)$. Hence,

$$h(v,s) = \frac{2\rho'(v)}{\rho(v) - \rho(s)},\ \theta_1(v) = \frac{\rho(v)}{\rho'(v)} \text{ and } \theta_1(v) = 0.$$

Then,

$$\begin{aligned}&\Phi(\xi_1 - \frac{1}{4}\varphi_1\Big(\varrho - \theta_1 - \alpha^{-1}\alpha'\Big)^2; \ell, v)\\ &= \int_\ell^v (\rho(v) - \rho(s))^2 \rho(s)\left[q(s)F(\Omega_m(g(s))) - \frac{\rho'(s)}{4\rho^2(s)}\right]\mathrm{d}s - \frac{1}{2}\Big(\rho^2(v) - \rho^2(\ell)\Big).\end{aligned}$$

Define $\mathcal{K}(v) = \int_v^\infty q(s)F(\Omega_m(g(s)))\mathrm{d}s$. Then,

$$\begin{aligned}&\Phi(\xi_1 - \frac{1}{4}\varphi_1\Big(\varrho - \theta_1 - \alpha^{-1}\alpha'\Big)^2; \ell, v)\\ &= \int_\ell^v (\rho(v) - \rho(s))^2 \rho(s)\mathrm{d}s\left[-\mathcal{K}(s) + \frac{1}{4\rho(s)}\right] - \frac{1}{2}\Big(\rho^2(v) - \rho^2(\ell)\Big)\\ &= (\rho(v) - \rho(\ell))^2\rho(\ell)\left[\mathcal{K}(\ell) + \frac{1}{4\rho(\ell)}\right] - \frac{1}{2}\Big(\rho^2(v) - \rho^2(\ell)\Big)\\ &+ \int_\ell^v \left(\rho(s)\mathcal{K}(s) - \frac{1}{4}\right)\left[-4\rho(v) + 3\rho(s) + \frac{\rho^2(v)}{\rho(s)}\right]\rho'(s)\mathrm{d}s\\ &\geq \left(\epsilon - \frac{1}{4}\right)\int_\ell^v \left[-4\rho(v) + 3\rho(s) + \frac{\rho^2(v)}{\rho(s)}\right]\rho'(s)\mathrm{d}s - \frac{1}{2}\Big(\rho^2(v) - \rho^2(\ell)\Big)\\ &\geq \left(\epsilon - \frac{1}{4}\right)\left[\ln\left(\frac{\rho(v)}{\rho(\ell)}\right) - \frac{5}{2}\right]\rho^2(v) - \frac{1}{2}\Big(\rho^2(v) - \rho^2(\ell)\Big).\end{aligned}$$

It follows that

$$\lim_{v\to\infty} \frac{1}{H(v,\ell)}\Phi\left(\xi_1 - \frac{1}{4}\varphi_1\Big(h - \theta_1 - \alpha^{-1}\alpha'\Big)^2; \ell, v\right) = \infty,$$

which is the same as (22). Equation (1) oscillates, as Theorem 2 indicates. □

We define

$$H(v,s) = (v - s)^n,$$

where n is an integer and $n > 1$. Furthermore,

$$h(v,s) = n(v - s)^{n-1}.$$

As such, we derive the following oscillation criteria as a result of Theorems 2 and 3.

Corollary 4. *Assume that (N1) and (S1) are satisfied. If there exist a $\rho \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$ and an integer $n > 1$ such that*

$$\limsup_{v\to\infty} \frac{1}{(v-\ell)^n}\int_{v_0}^v \left((v-s)^n\xi_1(s) - \frac{n^2}{4}(v-s)^{n-2}\varphi_1(s)\right)\mathrm{d}s = \infty,$$

then Equation (1) oscillates.

Corollary 5. *Assume that (S1) and (N1) are satisfied. Equation (1) is oscillatory if there exists a $\rho \in \mathbf{C}^1([v_0,\infty),\mathbb{R}^+)$, $\rho_1, \rho_2 \in \mathbf{C}([v_0,\infty),\mathbb{R}^+)$, and an integer $n > 1$ such that*

$$\limsup_{v\to\infty} \frac{1}{(v-\ell)^n}\int_\ell^v (v-s)^n\xi_1(s)\mathrm{d}s \geq \rho_1(\ell),$$

and

$$\limsup_{v\to\infty}\frac{1}{(v-\ell)^n}\int_\ell^v (v-s)^{n-2}\varphi_1(s)\mathrm{d}s \le \rho_2(\ell),$$

where ρ_1 *and* ρ_2 *satisfy*

$$\liminf_{v\to\infty}\frac{1}{(v-\ell)^n}\int_\ell^v \frac{(v-s)^n}{\varphi_1(s)}\left(\rho_1(s)-\frac{1}{4}\rho_2(s)\right)^2\mathrm{d}s = \infty.$$

In the following example, we apply our results to a form of (1) and also compare the results with previous results in the literature.

Example 1. *Consider*

$$\left(\frac{1}{1+\delta\sin^2(x(v))}\left[(x(v)+p_0x(\lambda v))'\right]\right)' + \frac{q_0}{v^2}\left[\beta x^3(\mu v)+x(\mu v)\right] = 0, \tag{53}$$

where $\lambda,\mu\in(0,1]$, $\delta,\beta\ge 0$, *and* $q_0>0$. *It is easy to see that* $k_1=1$, $L=1$, $l_0=\frac{1}{1+\delta}$, $\kappa=1+\delta$, $h^{[2j]}(v)=\lambda^{2j}v$, *and*

$$\Omega_m(v) = (1-p_0)\left[1+\sum_{j=1}^{m}p_0^{2j}\lambda^{2(1+\delta)j}\right] = \widehat{p}_0.$$

For Corollary 1, if we choose $\rho(v)=v$ *and* $\sigma(v)=-1/(2\mu v)$, *then* $\theta_1(v)=0$, $\varphi_1(v)=\frac{1}{\mu}v$, *and*

$$\xi_1(v) = \frac{1}{v}\left(q_0\widetilde{p}_0-\frac{3}{4\mu}\right),$$

where $\widetilde{p}_0 := \beta\widehat{p}_0^3+\widehat{p}_0$. *So,*

$$\int_{v_0}^{\infty}\frac{1}{\varphi_1(s)}\mathrm{d}s = \int_{v_0}^{\infty}\frac{\mu}{s}\mathrm{d}s = \infty,$$

and

$$\begin{aligned}\int_{v_0}^{\infty}\left(\xi_1(s)-\frac{1}{4}\varphi_1(s)\theta_1^2(s)\right)\mathrm{d}s &= \int_{v_0}^{\infty}\left(q_0\widetilde{p}_0-\frac{3}{4\mu}\right)\frac{1}{s}\mathrm{d}s\\ &= \infty,\end{aligned}$$

if

$$q_0 > \frac{3}{4\mu\widetilde{p}_0}. \tag{54}$$

Therefore, Equation (53) oscillates if (54) holds.
On the other hand, using Corollary 3, we find $\rho(v)=\mu v$, *and*

$$\begin{aligned}\liminf_{v\to\infty}\rho(v)\int_v^{\infty}[q(s)F(\Omega_m(g(s)))]\mathrm{d}s &= \liminf_{v\to\infty}\mu v\int_v^{\infty}\left[\frac{q_0}{s^2}\widetilde{p}_0\right]\mathrm{d}s\\ &> \frac{1}{4},\end{aligned}$$

if

$$q_0 > \frac{1}{4\mu\widetilde{p}_0}. \tag{55}$$

Therefore, Equation (53) oscillates if (55) holds. We note that Corollary 3 provides a more efficient criterion than Corollary 1. Figure 1 shows one of the numerical solutions to (53).

Remark 2. *Consider the special case of Equation (53) when $\delta = \beta = 0$. In this case, condition (55) reduces to*

$$q_0 > \frac{1}{4\mu\widehat{p}_0}. \tag{56}$$

In 2018, Grace et al. [32] investigated the oscillatory behavior of solutions of the neutral differential equation (NDE)

$$\left(a(v)\left(z'(v)\right)^{\alpha}\right)' + q(v)x^{\alpha}(g(v)) = 0,$$

where α is a ratio of odd natural numbers. They used the Riccati method and the comparison method with a first-order equation. Their results are essentially an improvement on results in the literature that preceded their work. For testing the oscillation of

$$(x(v) + p_0x(\lambda v))'' + \frac{q_0}{v^2}x(\mu v) = 0,$$

the best results they obtained (in Example 3 [32]) were

$$(1 - p_0)q_0\mu^{\epsilon} > \frac{1}{4}, \tag{57}$$

where

$$\epsilon = \frac{1}{1 + (1 - p_0)q_0\mu}.$$

In the case where $\mu = 0.5$, $p_0 = 0.5$, and $\lambda = 0.9$, criteria (56) and (57) reduce to $q_0 \gtrapprox 0.79750$ and $q_0 \gtrapprox 0.88227$, respectively. Therefore, our results improve the results in [32].

Remark 3. *It is worth noting that if Equation (53) is reduced by setting $\delta = \beta = p_0 = 0$, we obtain Euler's equation*

$$x''(v) + \frac{q_0}{v^2}x(\mu v) = 0.$$

Thus, condition (56) becomes $q_0 > \frac{1}{4\mu}$. Thus, in the ordinary case $(\mu = 1)$, we obtain the sharp criterion for the oscillation of Euler's equation $q_0 > 1/4$.

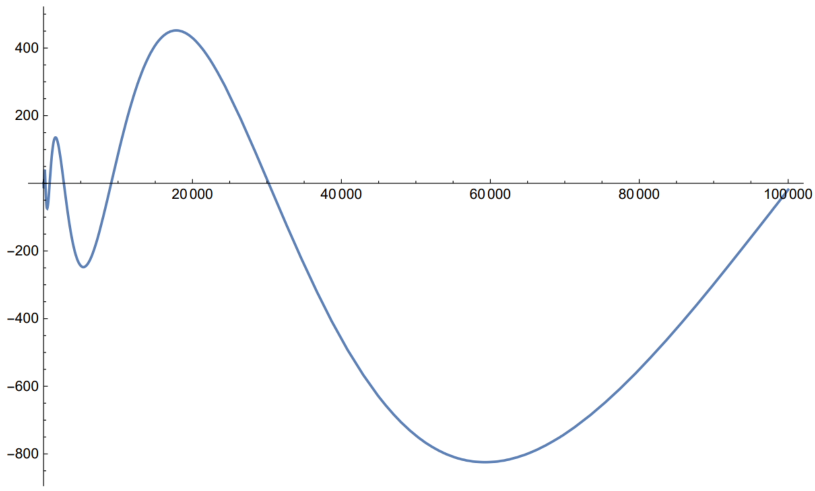

Figure 1. A numerical oscillatory solution to Equation (53).

5. Conclusions

The investigation of the oscillatory features of delay equations is rich in analytical issues. Therefore, there are always attempts to improve the relationships, inequalities, and techniques used in oscillation theory. In this study, we investigated the oscillatory properties of solutions to a class of neutral-type FDEs. We deduced a new relationship between x and z, which is one of the vital relations in the oscillation theory of NDEs. Next, we utilized the Riccati method to derive Coles-type oscillation criteria. Moreover, we introduced many conditions for the oscillation of the examined equation by considering cases (S1)–(S3). It is worth noting that our results are a direct improvement on previous relevant results. This is clear by setting $M = 0$, so we find that

$$\begin{aligned} \Omega_m(v) &\geq \Omega_0(v) = \sum_{j=0}^{0} p(v)\left[\frac{1}{p(v)} - 1\right]\left[\frac{£_{v_1}(v)}{£_{v_1}(v)}\right]^{\kappa} \\ &= 1 - p(v), \end{aligned}$$

and the relation (11) reduces to the traditional relationship $x > (1 - p)z$. In future work, we look forward to using the same approach to study oscillatory behavior in the non-canonical case as well as the oscillation of higher-order equations. Moreover, we will work in the future to obtain oscillation standards with fewer restrictions than those imposed here.

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