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An Algorithmic Evaluation of a Family of Logarithmic Integrals and Associated Euler Sums

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Abstract: Numerous logarithmic integrals have been extensively documented in the literature. This paper presents an algorithmic evaluation of a specific class of these integrals. Our systematic approach, rooted in logarithmic principles, enables us to extend our findings to other cases within this family of integrals. Furthermore, we explore special cases derived from our main results, thereby enhancing the applicability and significance of our work for a wider audience of researchers.

Keywords: Bessel's equality; logarithmic integrals; Euler sums; Riemann zeta function; binomial sums

MSC: 05A10; 05A19; 33B15; 40A05; 65B10

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1. Introduction and Preliminaries

Beginning with the well-known binomial theorem

$$(1-z)^{-\xi} = \sum_{n=0}^{\infty} \frac{(\xi)_n}{n!}\, z^n \quad (|z|<1), \tag{1}$$

Shen uncovered the following identity (refer to [1], Equation (4)): When $\xi < \frac{1}{2}$,

$$\sum_{n=0}^{\infty} \left(\frac{(\xi)_n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|1-e^{it}\right|^{-2\xi} \mathrm{d}t = \frac{2^{-2\xi}}{\sqrt{\pi}}\, \frac{\Gamma\left(\frac{1}{2}-\xi\right)}{\Gamma(1-\xi)}. \tag{2}$$

Here, $(\xi)_\nu$ denotes the Pochhammer symbol defined (for $\xi,\ \nu \in \mathbb{C}$) in terms of the familiar Gamma function Γ by

$$\begin{aligned}(\xi)_\nu :&= \frac{\Gamma(\xi+\nu)}{\Gamma(\xi)} \\ &= \begin{cases} 1 & (\nu = 0;\ \xi \in \mathbb{C}\setminus\{0\}) \\ \xi(\xi+1)\cdots(\xi+n-1) & (\nu = n \in \mathbb{N};\ \xi \in \mathbb{C}), \end{cases}\end{aligned} \tag{3}$$

it being understood conventionally that $(0)_0 := 1$ and $\mathbb{N}$, $\mathbb{C}$ are the sets of positive integers and complex numbers, respectively. Shen employed a technique wherein he expanded each of the three expressions in (2) as power series centered at $\xi = 0$. By aligning coefficients of equal powers in the resulting Maclaurin series, he unveiled intriguing identities, as illustrated in [1], Equation (19):

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^k\left|1-e^{it}\right| \mathrm{d}t = (-1)^k\, \frac{k!}{2^k} \sum_{n \geqslant \frac{k}{2}}^{\infty} \xi_k^n \quad (k \in \mathbb{N}). \tag{4}$$

Here, the ξ_k^n is defined as

$$\xi_k^n = \sum_{\ell=1}^{k-1} \frac{s_{k-m}^n}{n!}\frac{s_m^n}{n!},$$

where s_k^n represents the Stirling numbers of the first kind (see, e.g., [2], Section 1.6). Several specific instances of (4) were elucidated (refer to [1], Corollary):

$$\frac{1}{2\pi}\int_0^{2\pi} \log|1-e^{it}|\,\mathrm{d}t = 0, \tag{5}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \log^2|1-e^{it}|\,\mathrm{d}t = \frac{1}{2}\zeta(2), \tag{6}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \log^3|1-e^{it}|\,\mathrm{d}t = -\frac{3}{2}\zeta(3), \tag{7}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \log^4|1-e^{it}|\,\mathrm{d}t = \frac{57}{8}\zeta(4). \tag{8}$$

Inspired by Shen's findings [1], researchers in [3] tackled the task of solving a sequence of logarithmic integrals, defined as

$$\frac{1}{2\pi}\int_0^{2\pi} |1+re^{it}|^{2n}\log^m|1+re^{it}|\,\mathrm{d}t. \tag{9}$$

Here, n ranges over non-negative integers, m varies from 0 to 4, and r is constrained to the range between -1 and 1. Here are the formulas being recalled:

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} |1+re^{it}|^{2n}\log|1+re^{it}|\,\mathrm{d}t\\ &\quad= \psi(n+1)\sum_{k=0}^{n}\binom{n}{k}^2 r^{2k} - \sum_{k=0}^{n}\binom{n}{k}^2\psi(k+1)\,r^{2n-2k};\end{aligned} \tag{10}$$

$$\frac{1}{\pi}\int_0^{2\pi} |1+re^{it}|^{2n}\log^2|1+re^{it}|\,\mathrm{d}t = \sum_{j=1}^{3}\mathcal{A}_j(n,r), \tag{11}$$

where

$$\mathcal{A}_1(n,r) = \left\{2\psi^2(n+1)+\psi'(n+1)\right\}\sum_{k=0}^{n}\binom{n}{k}^2 r^{2k},$$

$$\mathcal{A}_2(n,r) = \sum_{k=1}^{\infty}\frac{r^{2n+2k}}{k^2\binom{n+k}{k}^2},$$

and

$$\begin{aligned}\mathcal{A}_3(n,r) &= -4\psi(n+1)\sum_{k=0}^{n}\binom{n}{k}^2\psi(k+1)\,r^{2n-2k}\\ &\quad+\sum_{k=0}^{n}\binom{n}{k}^2\left\{2\psi^2(k+1)-\psi'(k+1)\right\}r^{2n-2k};\end{aligned}$$

$$\frac{2}{\pi}\int_0^{2\pi} |1+re^{it}|^{2n}\log^3|1+re^{it}|\mathrm{d}t = \sum_{j=1}^{3}\mathcal{B}_j(n,r), \tag{12}$$

where

$$\mathcal{B}_1(n,r) = \left\{4\psi^3(n+1)+6\psi(n+1)\psi'(n+1)+\psi^{(2)}(n+1)\right\}\sum_{k=0}^{n}\binom{n}{k}^2 r^{2k},$$

$$\mathcal{B}_2(n,r) = 6\psi(n+1)\sum_{k=1}^{\infty}\frac{r^{2n+2k}}{k^2\binom{n+k}{k}^2} - 6\sum_{k=1}^{\infty}\frac{\psi(k)\,r^{2n+2k}}{k^2\binom{n+k}{k}^2},$$

and

$$\mathcal{B}_3(n,r) = 6\psi(n+1)\sum_{k=0}^{n}\binom{n}{k}^2\left\{2\psi^2(k+1) - \psi'(k+1)\right\}r^{2n-2k}$$
$$-6\left\{2\psi^2(n+1) + \psi'(n+1)\right\}\sum_{k=0}^{n}\binom{n}{k}^2\psi(k+1)r^{2n-2k}$$
$$-\sum_{k=0}^{n}\binom{n}{k}^2\left\{4\,\psi^3(k+1) - 6\,\psi(k+1)\psi'(k+1) + \psi^{(2)}(k+1)\right\}r^{2n-2k};$$

$$\frac{4}{\pi}\int_0^{2\pi}|1+re^{it}|^{2n}\,\log^4|1+re^{it}|\,\mathrm{d}t = \sum_{j=1}^{3}\mathcal{C}_j(n,r), \tag{13}$$

where

$$\mathcal{C}_1(n,r) = \left\{\psi^{(3)}(n+1) + 8\,\psi(n+1)\psi^{(2)}(n+1) + 24\,\psi^2(n+1)\,\psi'(n+1)\right.$$
$$\left. + 6\left\{\psi'(n+1)\right\}^2 + 8\psi^4(n+1)\right\}\sum_{k=0}^{n}\binom{n}{k}^2 r^{2k},$$

$$\mathcal{C}_2(n,r) = 12\left\{2\psi^2(n+1) + \psi'(n+1) - \zeta(2)\right\}\sum_{k=1}^{\infty}\frac{r^{2n+2k}}{k^2\binom{n+k}{k}^2}$$
$$+12\sum_{k=1}^{\infty}\frac{\left\{2\,\psi^2(k) - H_{k-1}^{(2)}\right\}}{k^2\binom{n+k}{k}^2}r^{2k+2n}$$
$$-48\,\psi(n+1)\sum_{k=1}^{\infty}\frac{\psi(k)}{k^2\binom{n+k}{k}^2}r^{2k+2n},$$

$$\mathcal{C}_3(n,r) = -\left\{32\,\psi^3(n+1) + 48\,\psi(n+1)\psi'(n+1) + 8\,\psi^{(2)}(n+1)\right\}$$
$$\times\sum_{k=0}^{n}\binom{n}{k}^2\psi(k+1)\,r^{2n-2k}$$
$$+12\left\{2\,\psi^2(n+1) + \psi'(n+1)\right\}$$
$$\times\sum_{k=0}^{n}\binom{n}{k}^2\left\{2\,\psi^2(k+1) - \psi'(k+1)\right\}r^{2n-2k}$$
$$+8\psi(n+1)\sum_{k=0}^{n}\binom{n}{k}^2\left\{6\,\psi(k+1)\psi'(k+1)\right.$$
$$\left.-4\,\psi^3(k+1) - \psi^{(2)}(k+1)\right\}r^{2n-2k}$$
$$+\sum_{k=0}^{n}\binom{n}{k}^2\left[8\,\psi^4(k+1) + 8\,\psi(k+1)\psi^{(2)}(k+1)\right.$$
$$\left.-24\,\psi^2(k+1)\,\psi'(k+1) + 6\left\{\psi'(k+1)\right\}^2 - \psi^{(3)}(k+1)\right]r^{2n-2k}.$$

In this investigation, we delve deeper into the analysis of these integrals in (9), particularly focusing on the cases where $n \in \mathbb{Z}_{\geqslant 0}$, $m = 5$, 6, and $-1 \leqslant r \leqslant 1$. Our approach is systematic and firmly grounded in algorithmic principles, facilitating a natural extension of our conclusions to situations where $n \in \mathbb{Z}_{\geqslant 0}$, $m \in \mathbb{N}$, with $m \geqslant 7$ and $-1 \leqslant r \leqslant 1$. Also, we elucidate specific instances derived from our primary findings, thus broadening the practicality and significance of our results for a broader spectrum of inquisitive researchers. Furthermore, it is hoped to utilize this algorithmic method to develop a Mathematica symbolic computation package for evaluating the integrals in (9).

To achieve our objective, we bring to mind the following functions and symbols. The psi (or digamma) function, denoted by $\psi(\eta)$, is defined as

$$\psi(\eta) := \frac{d}{d\eta}\{\log\Gamma(\eta)\} = \frac{\Gamma'(\eta)}{\Gamma(\eta)} \quad (\eta \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}). \tag{14}$$

Throughout this discussion, $\mathbb{Z}$ represents the set of integers, and $\mathbb{Z}_{\leqslant \ell}$ ($\mathbb{Z}_{\geqslant \ell}$) denotes the set of integers less than or equal to (greater than or equal to) some ℓ belonging to $\mathbb{Z}$. The polygamma function, denoted as $\psi^{(k)}(\eta)$, is defined by

$$\begin{aligned}\psi^{(\ell)}(\eta) &:= \frac{d^{\ell}}{d\eta^{\ell}}\{\psi(\eta)\} = (-1)^{\ell+1}\ell! \sum_{r=0}^{\infty}\frac{1}{(r+\eta)^{\ell+1}} \\ &= (-1)^{\ell+1}\ell!\,\zeta(\ell+1,\eta) \quad (\ell \in \mathbb{N},\ \eta \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}).\end{aligned} \tag{15}$$

Here, $\psi^{(0)}(\eta) = \psi(\eta)$, and $\zeta(s,\eta)$ represents the generalized (or Hurwitz) zeta function, defined as

$$\zeta(s,\eta) := \sum_{k=0}^{\infty}\frac{1}{(k+\eta)^s} \quad (\Re(s) > 1,\ \eta \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}). \tag{16}$$

Additionally, $\zeta(s,1) =: \zeta(s)$ is the Riemann zeta function. The polygamma functions obey the following functional equation:

$$\psi^{(\nu)}(\eta+m) - \psi^{(\nu)}(\eta) = (-1)^{\nu}\,\nu! \sum_{\ell=1}^{m}\frac{1}{(\eta+\ell-1)^{\nu+1}} \quad (m,\ \nu \in \mathbb{Z}_{\geqslant 0}). \tag{17}$$

The generalized binomial coefficient, denoted as $\binom{u}{v}$, is defined in terms of the gamma function as

$$\binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)} \quad (v \in \mathbb{C},\ u \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant -1}). \tag{18}$$

Given that $\frac{1}{\Gamma(z)} = 0$ ($z \in \mathbb{Z}_{\leqslant 0}$), from Equation (18) we deduce that

$$\binom{n}{k} = \frac{1}{\Gamma(n-k+1)}\cdot\frac{\Gamma(n+1)}{\Gamma(k+1)} = 0\cdot\frac{\Gamma(n+1)}{\Gamma(k+1)} = 0 \quad (n,\ k \in \backslash\mathbb{Z}_{\geqslant 0};\ k > n). \tag{19}$$

The generalized harmonic number $H_n^{(s)}$ of order s is given by

$$H_n^{(s)} = \sum_{k=1}^{n}\frac{1}{k^s} \quad (n \in \mathbb{Z}_{\geqslant 0},\ s \in \mathbb{C}). \tag{20}$$

Here, $H_0^{(s)} = 0$ and $H_n^{(1)} := H_n$ denote the regular harmonic numbers. In all cases, an empty sum is to be understood as zero. Here are the recalled relations rephrased:

$$H_n = \gamma + \psi(n+1) \quad (n \in \mathbb{Z}_{\geqslant 0}), \tag{21}$$

where γ is the Euler–Mascheroni constant (referenced in, for example, [2], Section 1.2);

$$H_n^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\psi^{(m)}(n+1) \quad (m \in \mathbb{N},\ n \in \mathbb{Z}_{\geqslant 0}) \tag{22}$$

(see, for instance, [4], Equation (1.25); and (15)). These Equations (21) and (22) are employed to define extended harmonic numbers $H_\eta^{(m)}$ of order $m \in \mathbb{N}$ with index $\eta \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$ as follows (referenced in [5]):

$$H_\eta^{(m)} := \begin{cases} \gamma + \psi(\eta+1) & (m=1), \\ \zeta(m) + \frac{(-1)^{m-1}}{(m-1)!}\,\psi^{(m-1)}(\eta+1) & (m \in \mathbb{Z}_{\geqslant 2}). \end{cases} \tag{23}$$

Combining Equations (17) and (20), we derive

$$\begin{aligned} &\psi^{(q-p)}(\ell+1) - \psi^{(q-p)}(\ell-j+1) \\ &\quad = (-1)^{q-p}(q-p)!\,\{H_\ell^{(q-p+1)} - H_{\ell-j}^{(q-p+1)}\} \\ &\quad (\ell,\, j,\, q,\, p \in \mathbb{Z}_{\geqslant 0},\, 0 \leqslant p \leqslant q,\, 0 \leqslant j \leqslant \ell). \end{aligned} \tag{24}$$

The polylogarithm function $\mathrm{Li}_n(z)$ is defined by (see, for instance, [2], p. 185; [6])

$$\begin{aligned} \mathrm{Li}_n(z) &:= \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad (|z| \leqq 1,\ n \in \mathbb{Z}_{\geqslant 2}) \\ &= \int_0^z \frac{\mathrm{Li}_{n-1}(t)}{t}\,\mathrm{d}t \quad (n \in \mathbb{Z}_{\geqslant 3}). \end{aligned} \tag{25}$$

Clearly, we have

$$\mathrm{Li}_n(1) = \zeta(n) \quad (n \in \mathbb{Z}_{\geqslant 2}). \tag{26}$$

The dilogarithm function $\mathrm{Li}_2(z)$ is defined by

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\log(1-t)}{t}\,\mathrm{d}t \quad (|z| \leq 1). \tag{27}$$

The polylogarithm function $\mathrm{Li}_z(u)$ of order $z \in \mathbb{C}$ can be extended as follows (refer to, for example, [2], p. 198):

$$\mathrm{Li}_z(u) = \sum_{m=1}^{\infty} \frac{u^m}{m^z} \tag{28}$$

$$(z \in \mathbb{C} \text{ and } |u| < 1;\ \Re(z) > 1 \text{ and } |u| = 1).$$

2. Preliminary Lemmas

This section revisits several findings, presenting them as lemmas, and introduces a new lemma.

Lemma 1. *The following assertions are true:*

(i) $\Gamma(z)$ *and* $\psi(z)$ *are meromorphic functions across the entire complex z-plane, exhibiting simple poles at* $z = -k$ $(k \in \mathbb{Z}_{\geqslant 0})$*. The residues at these poles are as follows:*

$$\operatorname*{Res}_{z=-k} \Gamma(z) = \lim_{z \to -k} (z+k)\Gamma(z) = \frac{(-1)^k}{k!} \quad (k \in \mathbb{Z}_{\geqslant 0}) \tag{29}$$

and

$$\operatorname*{Res}_{z=-k} \psi(z) = \lim_{z \to -k} (z+k)\psi(z) = -1 \quad (k \in \mathbb{Z}_{\geqslant 0}). \tag{30}$$

(ii) *The reciprocal of the gamma function,* $\frac{1}{\Gamma(z)}$*, is an entire function that displays simple zeros at* $z = -k$ $(k \in \mathbb{Z}_{\geqslant 0})$.

(iii) *The Laurent series expansion for* $\psi(z)$ *around* $z = -k$ $(k \in \mathbb{Z}_{\geqslant 0})$ *is expressed as*

$$\psi(z) = -\frac{1}{z+k} + \psi(k+1) + \sum_{n=2}^{\infty} \alpha_n \, (z+k)^{n-1}, \tag{31}$$

where the coefficients α_n *are determined by*

$$\alpha_n = (-1)^n \, \zeta(n) + H_k^{(n)}. \tag{32}$$

(iv) *The Laurent series expansion for the polygamma function* $\psi^{(\ell)}(z)$ *around* $z = -k$ $(k \in \mathbb{Z}_{\geqslant 0})$ *is provided as*

$$\psi^{(\ell)}(z) = \frac{(-1)^{\ell+1} \, \ell!}{(z+k)^{\ell+1}} + \sum_{n=\ell}^{\infty} \{n\}_\ell \, \alpha_{n+1} \, (z+k)^{n-\ell} \quad (\ell \in \mathbb{N}), \tag{33}$$

where $\ell \in \mathbb{N}$, $\{\lambda\}_\ell$ $(\lambda \in \mathbb{C})$ *denotes the falling factorial given by*

$$\{\lambda\}_\ell := \begin{cases} 1 & (\ell = 0) \\ \lambda(\lambda-1)\cdots(\lambda-\ell+1) & (\ell \in \mathbb{N}), \end{cases}$$

and α_n *is determined as in Equation* (32).

Proof. One can consult the proof offered in [7], Lemma 1. Additionally, references [2], pp. 4 and 24, and [8], Section 1.2, provide relevant information. Equation (33) is obtained by iteratively differentiating both sides of (31) ℓ times. Formulas (31) and (33) are documented in reference [9], conveniently placed within a box at the beginning of page 20. □

Lemma 2. *The following statements are valid: For* $k \in \mathbb{Z}_{\geqslant 0}$,

$$\lim_{z \to -k} \frac{\psi(z)}{\Gamma(z)} = (-1)^{k-1} k!, \tag{34}$$

$$\lim_{z \to -k} \frac{\psi'(z)}{\Gamma^2(z)} = (k!)^2, \tag{35}$$

and

$$\lim_{z \to -k} \frac{\gamma + \psi(1-z)}{\{\Gamma(z)\}^2 (z+k)} = 0. \tag{36}$$

Proof. For (34) and (35), one can consult [7], Lemma 2, and [3], Lemma 1.1. For (36), we find from (29) that

$$\begin{aligned} \mathcal{L}_3 : &= \lim_{z \to -k} \frac{\gamma + \psi(1-z)}{\{\Gamma(z)\}^2 (z+k)} = (-1)^k k! \lim_{z \to -k} \frac{\gamma + \psi(1-z)}{\Gamma(z)} \\ &= (-1)^k k! \left\{ \lim_{z \to -k} \frac{\gamma}{\Gamma(z)} + \lim_{z \to -k} \frac{\psi(1-z)}{\Gamma(z)} \right\} \\ &= (-1)^k k! \left\{ 0 + \lim_{z \to -k} \frac{\psi(1-z)}{\Gamma(z)} \right\}. \end{aligned}$$

For $\mathcal{L}_3 = 0$, it suffices to show that

$$\lim_{z \to -k} \frac{\psi(1-z)}{\Gamma(z)} = 0.$$

Recall the familiar identity (see, e.g., [2], p. 3, Equation (12)):

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad (z \in \mathbb{C} \setminus \mathbb{Z}). \tag{37}$$

Taking the logarithmic derivative of both sides of (46), we obtain

$$\frac{\psi(1-z)}{\Gamma(z)} = \frac{\psi(z)}{\Gamma(z)} + \frac{\pi \cos(\pi z)}{\Gamma(z)\,\sin(\pi z)} = \frac{\psi(z)}{\Gamma(z)} + \Gamma(1-z)\,\cos(\pi z),$$

which, upon using (34), gives

$$\lim_{z \to -k} \frac{\psi(1-z)}{\Gamma(z)} = (-1)^{k-1} k! + (-1)^k k! = 0.$$

□

Remark 1. *Considering (23), Equation (36) is an extended version of*

$$\lim_{z \to -k} \frac{H_{-z}}{\{\Gamma(z)\}^2 (z+k)} = 0.$$

Likewise, an analogous limit can be derived:

$$\lim_{z \to -k} \frac{H_{-z}^{(n)}}{\{\Gamma(z)\}^2 (z+k)} = 0 \quad (n \in \mathbb{N}).$$

Lemma 3. *Let $r \in \mathbb{R}$, the set of real numbers, where $|r| \leqslant 1$ and $x > 0$. Also, let $p \in \mathbb{N}$. Then,*

$$\begin{aligned} &\frac{2^{p-1}}{2\pi} \int_0^{2\pi} |1+re^{it}|^{2x} \log^p |1+re^{it}|\, \mathrm{d}t \\ &\quad = \sum_{k=0}^{\infty} \sum_{j=0}^{p-1} \binom{p-1}{j} \left\{ \frac{d^j}{dx^j} \binom{x}{k}^2 \right\} \\ &\qquad \times \left\{ \psi^{(p-1-j)}(x+1) - \psi^{(p-1-j)}(x-k+1) \right\} r^{2k}. \end{aligned} \tag{38}$$

Here,

$$\frac{d}{dx} \binom{x}{k}^2 = 2 \binom{x}{k}^2 \{\psi(x+1) - \psi(x-k+1)\}; \tag{39}$$

$$\begin{aligned} \frac{d^2}{dx^2} \binom{x}{k}^2 = \binom{x}{k}^2 \Big[& 4\{\psi(x+1) - \psi(x-k+1)\}^2 \\ & + 2\{\psi'(x+1) - \psi'(x-k+1)\} \Big]; \end{aligned} \tag{40}$$

$$\begin{aligned} \frac{d^3}{dx^3} \binom{x}{k}^2 = \binom{x}{k}^2 \Big[& 8\{\psi(x+1) - \psi(x-k+1)\}^3 \\ & + 12\{\psi(x+1) - \psi(x-k+1)\}\{\psi'(x+1) - \psi'(x-k+1)\} \\ & + 2\{\psi^{(2)}(x+1) - \psi^{(2)}(x-k+1)\} \Big]; \end{aligned} \tag{41}$$

$$
\begin{aligned}
\frac{d^4}{dx^4}\binom{x}{k}^2 = \binom{x}{k}^2 \Big[& 16\big\{\psi(x+1)-\psi(x-k+1)\big\}^4 \\
& + 48\big\{\psi(x+1)-\psi(x-k+1)\big\}^2\big\{\psi'(x+1)-\psi'(x-k+1)\big\} \\
& + 16\big\{\psi(x+1)-\psi(x-k+1)\big\}\big\{\psi^{(2)}(x+1)-\psi^{(2)}(x-k+1)\big\} \\
& + 12\big\{\psi'(x+1)-\psi'(x-k+1)\big\}^2 + 2\big\{\psi^{(3)}(x+1)-\psi^{(3)}(x-k+1)\big\}\Big];
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
\frac{d^5}{dx^5}\binom{x}{k}^2 = \binom{x}{k}^2 \Big[& 32\big\{\psi(x+1)-\psi(x-k+1)\big\}^5 \\
& + 160\big\{\psi(x+1)-\psi(x-k+1)\big\}^3\big\{\psi'(x+1)-\psi'(x-k+1)\big\} \\
& + 80\big\{\psi(x+1)-\psi(x-k+1)\big\}^2\big\{\psi^{(2)}(x+1)-\psi^{(2)}(x-k+1)\big\} \\
& + 120\big\{\psi(x+1)-\psi(x-k+1)\big\}\big\{\psi'(x+1)-\psi'(x-k+1)\big\}^2 \\
& + 20\big\{\psi(x+1)-\psi(x-k+1)\big\}\big\{\psi^{(3)}(x+1)-\psi^{(3)}(x-k+1)\big\} \\
& + 40\big\{\psi'(x+1)-\psi'(x-k+1)\big\}\big\{\psi^{(2)}(x+1)-\psi^{(2)}(x-k+1)\big\} \\
& + 2\big\{\psi^{(4)}(x+1)-\psi^{(4)}(x-k+1)\big\}\Big].
\end{aligned} \tag{43}
$$

Proof. One can refer to [3], Theorem 2.2 for consultation. Indeed, first, we revisit the content in [3], Theorem 2.1: Let $r \in \mathbb{R}$ with $|r| \leqslant 1$ and $x > 0$. Then,

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} |1+re^{it}|^{2x} \log|1+re^{it}|\,\mathrm{d}t \\
&\quad = \sum_{k=0}^{\infty} \binom{x}{k}^2 \{\psi(x+1)-\psi(x-k+1)\}\, r^{2k}.
\end{aligned} \tag{44}
$$

In particular, when $x = n \in \mathbb{Z}_{\geqslant 0}$,

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} |1+re^{it}|^{2n} \log|1+re^{it}|\,\mathrm{d}t \\
&\quad = \psi(n+1)\sum_{k=0}^{n} \binom{n}{k}^2 r^{2k} - \sum_{k=0}^{n}\binom{n}{k}^2 \psi(k+1)\, r^{2n-2k}.
\end{aligned} \tag{45}
$$

The proof of (44) heavily relies on the following five identities provided in [3], Lemma 1.3:

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad (z \in \mathbb{C}\setminus\mathbb{Z}); \tag{46}$$

$$\psi(z+1) = \psi(z) + \frac{1}{z}; \tag{47}$$

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = O\Big(z^{\alpha-\beta}\Big) \quad (|z|\to\infty,\ |\arg z| < \pi); \tag{48}$$

$$\psi(z) = \log z + O\left(\frac{1}{z}\right) \quad (|z|\to\infty,\ |\arg z| < \pi); \tag{49}$$

$$\psi^{(p)}(z) = O\left(\frac{1}{z^p}\right) \quad (|z|\to\infty,\ |\arg z| < \pi,\ p\in\mathbb{N}). \tag{50}$$

After performing $(p-1)$-fold differentiation on both sides of Equation (44) with respect to the variable x, while applying term-by-term differentiation on the right-hand side, we arrive at Equation (38). This result is established by leveraging the identities (46)–(50), along with Leibnitz's derivative formula for the product of two functions. The proof methodology closely parallels that of (44), although the specific details are omitted for brevity. □

Lemma 4. *The following assertions are true:*

$$\sum_{k=0}^{\infty} \binom{n}{k}^2 r^{2k} = \sum_{k=0}^{n} \binom{n}{k}^2 r^{2k}; \tag{51}$$

$$\sum_{k=0}^{\infty} \binom{n}{k}^2 \psi(n-k+1)\, r^{2k} = \sum_{k=0}^{n} \binom{n}{k}^2 \psi(k+1)\, r^{2n-2k}; \tag{52}$$

$$\begin{aligned} \sum_{k=0}^{\infty} \binom{n}{k}^2 \psi^2(n-k+1) r^{2k} &= \sum_{k=0}^{n} \binom{n}{k}^2 \psi^2(k+1) r^{2n-2k} \\ &\quad + \sum_{k=1}^{\infty} \frac{r^{2n+2k}}{k^2 \binom{n+k}{k}^2}; \end{aligned} \tag{53}$$

$$\begin{aligned} \sum_{k=0}^{\infty} \binom{n}{k}^2 \psi'(n-k+1) r^{2k} &= \sum_{k=0}^{n} \binom{n}{k}^2 \psi'(k+1) r^{2n-2k} \\ &\quad + \sum_{k=1}^{\infty} \frac{r^{2n+2k}}{k^2 \binom{n+k}{k}^2}; \end{aligned} \tag{54}$$

$$\sum_{k=0}^{\infty} \binom{n}{k}^2 f(n-k+1)\, r^{2k} = \sum_{k=0}^{n} \binom{n}{k}^2 f(k+1)\, r^{2n-2k}. \tag{55}$$

Here, the function $f(z)$ is meromorphic and satisfies

$$f(z) = \frac{c_f}{z+k} + O(z+k) \quad (z \to -k), \tag{56}$$

where c_f is a constant independent of the variable z;

$$\begin{aligned} \sum_{k=0}^{\infty} \binom{n}{k}^2 g(n-k+1)\, r^{2k} &= \sum_{k=0}^{n} \binom{n}{k}^2 g(k+1)\, r^{2n-2k} \\ &\quad + c_g \sum_{k=1}^{\infty} \frac{r^{2n+2k}}{k^2 \binom{n+k}{k}^2}. \end{aligned} \tag{57}$$

Here, the function $g(z)$ is meromorphic and satisfies

$$g(z) = \frac{c_g}{(z+k)^2} + O(z+k) \quad (z \to -k), \tag{58}$$

where c_g is a constant independent of the variable z.

Proof. Upon finding that

$$\sum_{k=0}^{\infty} \binom{n}{k}^2 r^{2k} = \sum_{k=0}^{n} \binom{n}{k}^2 r^{2k} + \sum_{k=n+1}^{\infty} \binom{n}{k}^2 r^{2k},$$

and recognizing that the second sum evaluates to zero according to Equation (19), this confirms (51).

We obtain

$$\sum_{k=0}^{\infty} \binom{n}{k}^2 \psi(n-k+1)\, r^{2k} = \sum_{k=0}^{n} \binom{n}{k}^2 \psi(n-k+1)\, r^{2k} + \sum_{k=n+1}^{\infty} \binom{n}{k}^2 \psi(n-k+1)\, r^{2k}.$$

By letting $k-n-1=k'$ in the last summation and removing the prime from k, with the aid of Equation (34), we derive

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} \binom{n}{k}^2 \psi(n-k+1)\, r^{2k} \\
&= \sum_{k=0}^{\infty} \frac{(n!)^2}{\{(n+k+1)!\}^2} \cdot \frac{1}{\Gamma(-k)} \cdot \lim_{x\to -k} \frac{\psi(x)}{\Gamma(x)} \cdot r^{2n+2+2k} \\
&= \sum_{k=0}^{\infty} \frac{(n!)^2}{\{(n+k+1)!\}^2} \cdot 0 \cdot (-1)^{k-1} k! \cdot r^{2n+2+2k} = 0.
\end{aligned}$$

Thus, we arrive at (52).

Splitting the following summation into two parts, we obtain

$$\sum_{k=0}^{\infty} \binom{n}{k}^2 \psi'(n-k+1) r^{2k} = \sum_{k=0}^{n} \binom{n}{k}^2 \psi'(n-k+1) r^{2k} + \sum_{k=n+1}^{\infty} \binom{n}{k}^2 \psi'(n-k+1) r^{2k}. \tag{59}$$

Letting $k-n-1=k'$, and then, removing the prime on k, we derive

$$\begin{aligned}
\sum_{k=n+1}^{\infty} \binom{n}{k}^2 \psi'(n-k+1) r^{2k} &= \sum_{k=0}^{\infty} \frac{(n!)^2}{\{(n+k+1)!\}^2} \frac{\psi'(-k)}{\Gamma(-k)^2}\, r^{2n+2k+2} \\
&= \sum_{k=0}^{\infty} \frac{(n!)^2}{\{(n+k+1)!\}^2} \lim_{u\to -k} \frac{\psi'(u)}{\Gamma(u)^2}\, r^{2n+2k+2}.
\end{aligned}$$

Using Equation (35), this leads to

$$\sum_{k=n+1}^{\infty} \binom{n}{k}^2 \psi'(n-k+1) r^{2k} = \sum_{k=0}^{\infty} \frac{r^{2n+2k+2}}{(k+1)^2 \binom{n+k+1}{k+1}^2} = \sum_{k=1}^{\infty} \frac{r^{2n+2k}}{k^2 \binom{n+k}{k}^2}.$$

This expression is then substituted into (59) to yield (54).

Dividing the given sum into two distinct parts yields

$$\sum_{k=0}^{\infty} \binom{n}{k}^2 f(n-k+1)\, r^{2k} = \sum_{k=0}^{n} \binom{n}{k}^2 f(n-k+1)\, r^{2k} + \sum_{k=n+1}^{\infty} \binom{n}{k}^2 f(n-k+1)\, r^{2k} \tag{60}$$

By substituting $k-n-1=k'$ in the latter summation and removing the prime notation, using Equations (29) and (56), we derive

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} \binom{n}{k}^2 f(n-k+1)\, r^{2k} \\
&= \sum_{k=0}^{\infty} \frac{(n!)^2}{\{(n+k+1)!\}^2} \cdot \frac{1}{\Gamma(-k)} \cdot \lim_{x\to -k} \frac{1}{(x+k)\,\Gamma(x)} \cdot r^{2n+2+2k} \\
&= \sum_{k=0}^{\infty} \frac{(n!)^2}{\{(n+k+1)!\}^2} \cdot 0 \cdot (-1)^k k! \cdot r^{2n+2+2k} = 0.
\end{aligned}$$

Thus, we deduce (55).

Breaking down the given summation into two segments yields

$$\begin{aligned}
\sum_{k=0}^{\infty} \binom{n}{k}^2 g(n-k+1) r^{2k} = &\sum_{k=0}^{n} \binom{n}{k}^2 g(n-k+1) r^{2k} \\
&+ \sum_{k=n+1}^{\infty} \binom{n}{k}^2 g(n-k+1) r^{2k}.
\end{aligned} \tag{61}$$

By substituting $k-n-1=k'$, and then, removing the prime notation from k, we derive

$$\begin{aligned}
\sum_{k=n+1}^{\infty} \binom{n}{k}^2 g(n-k+1) r^{2k} &= \sum_{k=0}^{\infty} \frac{(n!)^2}{\{(n+k+1)!\}^2} \frac{g(-k)}{\Gamma(-k)^2} r^{2n+2k+2} \\
&= \sum_{k=0}^{\infty} \frac{(n!)^2}{\{(n+k+1)!\}^2} \lim_{u\to -k} \frac{g(u)}{\Gamma(u)^2} r^{2n+2k+2}.
\end{aligned}$$

Using Equations (29) and (58), this simplifies to

$$\sum_{k=n+1}^{\infty} \binom{n}{k}^2 g(n-k+1)\, r^{2k} = c_g \sum_{k=0}^{\infty} \frac{r^{2n+2k+2}}{(k+1)^2 \binom{n+k+1}{k+1}^2} = c_g \sum_{k=1}^{\infty} \frac{r^{2n+2k}}{k^2 \binom{n+k}{k}^2}.$$

This expression is then substituted into Equation (61) to yield Equation (57). □

3. Main Results

We delve deeper into solving these integrals in (9), considering n as a non-negative integer, and allowing m to vary from 5 to 6, while keeping r within the range of -1 to 1. Our methodical approach is firmly grounded in algorithmic principles, facilitating a smooth extension of our results to cases where n is a non-negative integer, m is greater than or equal to 7, and r remains within the boundaries of -1 to 1. Here are the main results.

Theorem 1. *Let $n \in \mathbb{Z}_{\geqslant 0}$ and $r \in \mathbb{R}$ with $|r| \leqslant 1$. Then,*

$$\frac{8}{\pi} \int_0^{2\pi} |1+re^{it}|^{2n} \log^5 |1+re^{it}|\, \mathrm{d}t = \sum_{j=1}^{12} \mathcal{D}_j(n,r), \tag{62}$$

where

$$\begin{aligned}
\mathcal{D}_1(n,r) = \Big[&\psi^{(4)}(n+1) + 10\psi(n+1)\psi^{(3)}(n+1) + 40\psi^2(n+1)\psi^{(2)}(n+1) \\
&+ 20\psi'(n+1)\psi^{(2)}(n+1) + 80\psi^3(n+1)\psi'(n+1) \\
&+ 60\psi(n+1)\{\psi'(n+1)\}^2 + 16\psi^5(n+1)\Big] \sum_{k=0}^{n} \binom{n}{k}^2 r^{2k};
\end{aligned}$$

$$\mathcal{D}_2(n,r) = \Big[-10\psi^{(3)}(n+1) - 80\psi(n+1)\psi^{(2)}(n+1) - 240\psi^2(n+1)\psi'(n+1) - 60\{\psi'(n+1)\}^2 - 80\psi^4(n+1)\Big] \sum_{k=0}^{n} \binom{n}{k}^2 \psi(k+1)\, r^{2n-2k};$$

$$\mathcal{D}_3(n,r) = 40\{\psi^{(2)}(n+1) + 6\psi(n+1)\psi'(n+1) + 4\psi^3(n+1)\} \times \sum_{k=0}^{n} \binom{n}{k}^2 \psi^2(k+1)\, r^{2n-2k};$$

$$\mathcal{D}_4(n,r) = -20\,\{4\psi^3(n+1) + 6\psi(n+1)\psi'(n+1) + \psi^{(2)}(n+1)\} \times \sum_{k=0}^{n} \binom{n}{k}^2 \psi'(k+1)\, r^{2n-2k};$$

$$\mathcal{D}_5(n,r) = -20\,\psi'(n+1) \sum_{k=0}^{n} \binom{n}{k}^2 \times \{\psi^{(2)}(k+1) - 6\,\psi(k+1)\psi'(k+1) + 4\,\psi^3(k+1)\}\, r^{2n-2k};$$

$$\mathcal{D}_6(n,r) = 10\psi(n+1) \sum_{k=0}^{n} \binom{n}{k}^2 \{-\psi^{(3)}(k+1) + 8\psi(k+1)\psi^{(2)}(k+1) - 24\psi^2(k+1)\psi'(k+1) + 6(\psi'(k+1))^2 + 8\psi^4(k+1)\}\, r^{2n-2k};$$

$$\mathcal{D}_7(n,r) = 8\,\psi^2(n+1) \sum_{k=0}^{n} \binom{n}{k}^2 \{-3\psi^{(2)}(k+1) + 30\psi(k+1)\psi'(k+1) - 20\psi^3(k+1) - 2\psi^{(2)}(k+1)\}\, r^{2n-2k};$$

$$\mathcal{D}_8(n,r) = \sum_{k=0}^{n} \binom{n}{k}^2 \{-\psi^{(4)}(k+1) + 10\,\psi(k+1)\psi^{(3)}(k+1) - 40\,\psi^2(k+1)\psi^{(2)}(k+1) + 20\,\psi'(k+1)\psi^{(2)}(k+1) + 80\psi^3(k+1)\psi'(k+1) - 60\,\psi(k+1)(\psi'(k+1))^2 - 16\psi^5(k+1)\}\, r^{2n-2k};$$

$$\mathcal{D}_9(n,r) = 20\,\{4\psi^3(n+1) + 6\psi(n+1)\psi'(n+1) + \psi^{(2)}(n+1)\} \times \sum_{k=1}^{\infty} \frac{r^{2n+2k}}{k^2 \binom{k+n}{k}^2};$$

$$\mathcal{D}_{10}(n,r) = -120\{2\psi^2(n+1) + \psi'(n+1)\} \sum_{k=1}^{\infty} \psi(k)\, \frac{r^{2n+2k}}{k^2 \binom{n+k}{k}^2};$$

$$\mathcal{D}_{11}(n,r) = 120\,\psi(n+1) \sum_{k=1}^{\infty} \{2\,\psi^2(k) - \zeta(2) - H_{k-1}^{(2)}\}\, \frac{r^{2n+2k}}{k^2 \binom{n+k}{k}^2};$$

$$\mathcal{D}_{12}(n,r) = -40 \sum_{k=1}^{\infty} \{2\,\psi^3(k) - 3\,\psi(k)\big(\zeta(2) + H_{k-1}^{(2)}\big) - \zeta(3) + H_{k-1}^{(3)}\} \times \frac{r^{2n+2k}}{k^2 \binom{n+k}{k}^2}.$$

Proof. The proof follows a similar path as that of Theorem 2. Specific details are left out. □

Theorem 2. *Let* $n \in \mathbb{Z}_{\geqslant 0}$ *and* $r \in \mathbb{R}$ *with* $|r| \leqslant 1$. *Then,*

$$\begin{aligned}\frac{16}{\pi}\int_0^{2\pi} &|1+re^{it}|^{2n}\,\log^6|1+re^{it}|\,\mathrm{d}t\\ &=\sum_{k=0}^{\infty}\sum_{j=0}^{5}\binom{5}{j}\left\{\frac{d^j}{dx^j}\binom{x}{k}^2\right\}\Bigg|_{x=n}\\ &\quad\times\left\{\psi^{(5-j)}(n+1)-\psi^{(5-j)}(n-k+1)\right\}r^{2k}\\ &=\sum_{j=1}^{11}\Lambda_j(n,r).\end{aligned} \tag{63}$$

Here,

$$\Lambda_1(n,r)=\lambda_1(n)\sum_{k=0}^{n}\binom{n}{k}^2 r^{2k}, \tag{64}$$

where

$$\begin{aligned}\lambda_1(n)=&32\psi(n+1)^6+240\psi(n+1)^4\psi'(n+1)\\ &+360\psi(n+1)^2\psi'(n+1)^2+60\psi'(n+1)^3\\ &+160\psi(n+1)^3\psi^{(2)}(n+1)\\ &+240\psi(n+1)\psi'(n+1)\psi^{(2)}(n+1)+20\psi^{(2)}(n+1)^2\\ &+60\psi(n+1)^2\psi^{(3)}(n+1)+30\psi'(n+1)\psi^{(3)}(n+1)\\ &+12\psi(n+1)\psi^{(4)}(n+1)+\psi^{(5)}(n+1).\end{aligned}$$

$$\begin{aligned}\Lambda_2(n,r)=&\sum_{k=0}^{n}\binom{n}{k}^2\lambda_2(n,k)\,r^{2k}\\ &+\sum_{k=1}^{\infty}\frac{3\pi^4+60\pi^2H_{k-1}^{(2)}+180\{H_{k-1}^{(2)}\}^2}{k^2\binom{n+k}{k}^2}r^{2n+2k}\\ &+\sum_{k=1}^{\infty}\frac{-180\,H_{k-1}^{(4)}+480\,\psi(k)\,H_{k-1}^{(3)}-120\,\pi^2\,\psi(k)^2}{k^2\binom{n+k}{k}^2}r^{2n+2k}\\ &+\sum_{k=1}^{\infty}\frac{-720\,\psi(k)^2\,H_{k-1}^{(2)}+240\,\psi(k)^4-480\,\zeta(3)\,\psi(k)}{k^2\binom{n+k}{k}^2}r^{2n+2k},\end{aligned} \tag{65}$$

where

$$\begin{aligned}\lambda_2(n,k)=&-\psi^{(5)}(n-k+1)+12\psi(n-k+1)\psi^{(4)}(n-k+1)\\ &-60\psi(n-k+1)^2\psi^{(3)}(n-k+1)+30\psi'(n-k+1)\psi^{(3)}(n-k+1)\\ &+160\psi(n-k+1)^3\psi^{(2)}(n-k+1)-240\psi(n-k+1)^4\psi'(n-k+1)\\ &+20\psi^{(2)}(n-k+1)^2-240\psi(n-k+1)\psi'(n-k+1)\psi^{(2)}(n-k+1)\\ &+360\psi(n-k+1)^2\psi'(n-k+1)^2-60\psi'(n-k+1)^3\\ &+32\psi(n-k+1)^6.\end{aligned}$$

$$\Lambda_3(n,r)=\lambda_3(n)\sum_{k=0}^{n}\binom{n}{k}^2\psi(n-k+1)\,r^{2k} \tag{66}$$

where

$$\begin{aligned}\lambda_3(n) =& -12\psi^{(4)}(n+1) - 120\psi(n+1)\psi^{(3)}(n+1) \\ & -480\psi(n+1)^2\psi^{(2)}(n+1) - 240\psi'(n+1)\psi^{(2)}(n+1) \\ & -960\psi(n+1)^3\psi'(n+1) - 720\psi(n+1)\psi'(n+1)^2 \\ & -192\psi(n+1)^5.\end{aligned}$$

$$\begin{aligned}\Lambda_4(n,r) =& \lambda_4(n) \sum_{k=0}^{n} \binom{n}{k}^2 \psi(n-k+1)^2\, r^{2k} \\ & + \lambda_4(n) \sum_{k=1}^{\infty} \frac{r^{2n+2k}}{k^2 \binom{n+k}{k}^2},\end{aligned} \tag{67}$$

where

$$\begin{aligned}\lambda_4(n) =& 60\psi^{(3)}(n+1) + 480\psi(n+1)\psi^{(2)}(n+1) \\ & + 1440\psi(n+1)^2\psi'(n+1) + 360\psi'(n+1)^2 + 480\psi(n+1)^4.\end{aligned}$$

$$\begin{aligned}\Lambda_5(n,r) =& \lambda_5(n) \sum_{k=0}^{n} \binom{n}{k}^2 \psi'(n-k+1)\, r^{2k} \\ & + \lambda_5(n) \sum_{k=1}^{\infty} \frac{r^{2n+2k}}{k^2 \binom{n+k}{k}^2},\end{aligned} \tag{68}$$

where

$$\begin{aligned}\lambda_5(n) =& -30\psi^{(3)}(n+1) - 240\psi(n+1)\psi^{(2)}(n+1) \\ & -240\psi(n+1)^4 - 720\psi(n+1)^2\psi'(n+1) - 180\psi'(n+1)^2.\end{aligned}$$

$$\begin{aligned}\Lambda_6(n,r) =& \psi(n+1) \sum_{k=0}^{n} \binom{n}{k}^2 \lambda_6(n,k)\, r^{2k} \\ & + \psi(n+1) \sum_{k=1}^{\infty} \frac{-480\, H_{k-1}^{(3)} + 240\,\pi^2\,\psi(k)}{k^2 \binom{n+k}{k}^2}\, r^{2n+2k} \\ & + \psi(n+1) \sum_{k=1}^{\infty} \frac{1440\,\psi(k)\, H_{k-1}^{(2)} - 960\,\psi(k)^3}{k^2 \binom{n+k}{k}^2}\, r^{2n+2k} \\ & + 480\,\zeta(3)\,\psi(n+1) \sum_{k=1}^{\infty} \frac{r^{2n+2k}}{k^2 \binom{n+k}{k}^2},\end{aligned} \tag{69}$$

where

$$\begin{aligned}\lambda_6(n,k) =& -12\psi^{(4)}(n-k+1) - 480\psi(n-k+1)^2\psi^{(2)}(n-k+1) \\ & + 120\psi(n-k+1)\psi^{(3)}(n-k+1) + 240\psi'(n-k+1)\psi^{(2)}(n-k+1) \\ & + 960\psi(n-k+1)^3\psi'(n-k+1) - 720\psi(n-k+1)\psi'(n-k+1)^2 \\ & - 192\psi(n-k+1)^5.\end{aligned}$$

$$\begin{aligned}\Lambda_7(n,r) =& \psi(n+1)^2 \sum_{k=0}^{n} \binom{n}{k}^2 \lambda_7(n,k)\, r^{2k} \\ & + \psi(n+1)^2 \sum_{k=1}^{\infty} \frac{1440\,\psi(k)^2 - 720\, H_{k-1}^{(2)}}{k^2 \binom{n+k}{k}^2}\, r^{2n+2k} \\ & - 120\pi^2\,\psi(n+1)^2 \sum_{k=1}^{\infty} \frac{r^{2n+2k}}{k^2 \binom{n+k}{k}^2},\end{aligned} \tag{70}$$

where

$$\lambda_7(n,k) = -60\psi^{(3)}(n-k+1) + 480\psi(n-k+1)\psi^{(2)}(n-k+1) - 1440\psi(n-k+1)^2\psi'(n-k+1) + 360\psi'(n-k+1)^2 + 480\psi(n-k+1)^4.$$

$$\Lambda_8(n,r) = \psi(n+1)^3 \sum_{k=0}^{n} \binom{n}{k}^2 \lambda_8(n,k)\, r^{2k} - 960\psi(n+1)^3 \sum_{k=1}^{\infty} \frac{\psi(k)}{k^2\binom{n+k}{k}^2}\, r^{2n+2k}, \tag{71}$$

where

$$\lambda_8(n,k) = -160\psi^{(2)}(n-k+1) + 960\psi(n-k+1)\psi'(n-k+1) - 640\psi(n-k+1)^3.$$

$$\Lambda_9(n,r) = \psi'(n+1) \sum_{k=0}^{n} \binom{n}{k}^2 \lambda_9(n,k)\, r^{2k} + \psi'(n+1) \sum_{k=1}^{\infty} \frac{720\,\psi(k)^2 - 360\, H_{k-1}^{(2)}}{k^2\binom{n+k}{k}^2} r^{2n+2k} - 60\pi^2\, \psi'(n+1) \sum_{k=1}^{\infty} \frac{r^{2n+2k}}{k^2\binom{n+k}{k}^2}, \tag{72}$$

where

$$\lambda_9(n,k) = -30\psi^{(3)}(n-k+1) + 240\psi(n-k+1)\psi^{(2)}(n-k+1) + 240\psi(n-k+1)^4 - 720\psi(n-k+1)^2\psi'(n-k+1) + 180\psi'(n-k+1)^2.$$

$$\Lambda_{10}(n,r) = \psi^{(2)}(n+1) \sum_{k=0}^{n} \binom{n}{k}^2 \lambda_{10}(n,k)\, r^{2k} - 240\psi^{(2)}(n+1) \sum_{k=1}^{\infty} \frac{\psi(k)}{k^2\binom{n+k}{k}^2} r^{2n+2k}, \tag{73}$$

where

$$\lambda_{10}(n,k) = -160\psi(n-k+1)^3 + 240\psi(n-k+1)\psi'(n-k+1) - 40\psi^{(2)}(n-k+1).$$

$$\Lambda_{11}(n,r) = \psi(n+1)\psi'(n+1) \sum_{k=0}^{n} \binom{n}{k}^2 \lambda_{11}(n,k)\, r^{2k} - 1440\,\psi(n+1)\psi'(n+1) \sum_{k=1}^{\infty} \frac{\psi(k)}{k^2\binom{n+k}{k}^2} r^{2n+2k}, \tag{74}$$

where

$$\lambda_{11}(n,k) = -240\psi^{(2)}(n-k+1) - 960\psi(n-k+1)^3 + 1440\psi(n-k+1)\psi'(n-k+1).$$

Proof. Setting $p = 6$ and $x = n \in \mathbb{Z}_{\geqslant 0}$ in (38), we find

$$\begin{aligned} I_6(n,r) &:= \frac{16}{\pi} \int_0^{2\pi} |1+re^{it}|^{2n} \log^6 |1+re^{it}| \, \mathrm{d}t \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{5} \binom{5}{j} \left\{ \frac{d^j}{dx^j} \binom{x}{k}^2 \right\} \bigg|_{x=n} \\ &\quad \times \left\{ \psi^{(5-j)}(n+1) - \psi^{(5-j)}(n-k+1) \right\} r^{2k} \\ &=: \sum_{j=0}^{5} \Omega_j(n,r). \end{aligned} \tag{75}$$

By using (39)–(43) and expanding the summands, we obtain

$$\Omega_0(n,r) = \sum_{k=0}^{\infty} \binom{n}{k}^2 \left\{ \psi^{(5)}(n+1) - \psi^{(5)}(n-k+1) \right\} r^{2k},$$

$$\begin{aligned} \Omega_1(n,r) = \sum_{k=0}^{\infty} \binom{n}{k}^2 \Big[& 10\psi(n+1)\psi^{(4)}(n+1) \\ & - 10\psi^{(4)}(n+1)\psi(n-k+1) - 10\psi(n+1)\psi^{(4)}(n-k+1) \\ & + 10\psi(n-k+1)\psi^{(4)}(n-k+1) \Big] r^{2k}; \end{aligned}$$

$$\begin{aligned} \Omega_2(n,r) = \sum_{k=0}^{\infty} \binom{n}{k}^2 \Big[& 40\psi^2(n+1)\psi^{(3)}(n+1) \\ & - 80\psi(n+1)\psi^{(3)}(n+1)\psi(n-k+1) + 40\psi^{(3)}(n+1)\psi(n-k+1)^2 \\ & + 20\psi'(n+1)\psi^{(3)}(n+1) - 20\psi^{(3)}(n+1)\psi'(n-k+1) \\ & - 40\psi(n+1)^2\psi^{(3)}(n-k+1) + 80\psi(n+1)\psi(n-k+1)\psi^{(3)}(n-k+1) \\ & - 40\psi(n-k+1)^2\psi^{(3)}(n-k+1) - 20\psi'(n+1)\psi^{(3)}(n-k+1) \\ & + 20\psi'(n-k+1)\psi^{(3)}(n-k+1) \Big] r^{2k}; \end{aligned}$$

$$\begin{aligned} \Omega_3(n,r) = \sum_{k=0}^{\infty} \binom{n}{k}^2 \Big[& 80\psi(n+1)^3\psi^{(2)}(n+1) \\ & - 240\psi^{(2)}(n+1)\psi(n+1)^2\psi(n-k+1) + 240\psi(n+1)\psi^{(2)}(n+1)\psi(n-k+1)^2 \\ & - 80\psi^{(2)}(n+1)\psi(n-k+1)^3 + 120\psi(n+1)\psi'(n+1)\psi^{(2)}(n+1) \\ & - 120\psi'(n+1)\psi^{(2)}(n+1)\psi(n-k+1) - 120\psi(n+1)\psi^{(2)}(n+1)\psi'(n-k+1) \\ & + 120\psi^{(2)}(n+1)\psi(n-k+1)\psi'(n-k+1) + 20\psi^{(2)}(n+1)^2 \\ & - 80\psi(n+1)^3\psi^{(2)}(n-k+1) + 240\psi(n+1)^2\psi(n-k+1)\psi^{(2)}(n-k+1) \\ & - 240\psi(n+1)\psi(n-k+1)^2\psi^{(2)}(n-k+1) + 80\psi(n-k+1)^3\psi^{(2)}(n-k+1) \\ & - 120\psi(n+1)\psi'(n+1)\psi^{(2)}(n-k+1) + 120\psi'(n+1)\psi(n-k+1)\psi^{(2)}(n-k+1) \\ & + 120\psi(n+1)\psi'(n-k+1)\psi^{(2)}(n-k+1) \\ & - 120\psi(n-k+1)\psi'(n-k+1)\psi^{(2)}(n-k+1) \\ & - 40\psi^{(2)}(n+1)\psi^{(2)}(n-k+1) + 20\psi^{(2)}(n-k+1)^2 \Big] r^{2k}; \end{aligned}$$

$$\Omega_4(n,r)=\sum_{k=0}^{\infty}\binom{n}{k}^2\Big[80\psi(n+1)^4\psi'(n+1)$$
$$-320\psi(n+1)^3\psi'(n+1)\psi(n-k+1)+480\psi(n+1)^2\psi'(n+1)\psi(n-k+1)^2$$
$$-320\psi(n+1)\psi'(n+1)\psi(n-k+1)^3+80\psi'(n+1)\psi(n-k+1)^4$$
$$+240\psi(n+1)^2\psi'(n+1)^2-480\psi(n+1)\psi'(n+1)^2\psi(n-k+1)$$
$$+240\psi'(n+1)^2\psi(n-k+1)^2+60\psi'(n+1)^3$$
$$-80\psi(n+1)^4\psi'(n-k+1)+320\psi(n+1)^3\psi(n-k+1)\psi'(n-k+1)$$
$$-480\psi(n+1)^2\psi(n-k+1)^2\psi'(n-k+1)+320\psi(n+1)\psi(n-k+1)^3\psi'(n-k+1)$$
$$-80\psi(n-k+1)^4\psi'(n-k+1)-480\psi(n+1)^2\psi'(n+1)\psi'(n-k+1)$$
$$+960\psi(n+1)\psi'(n+1)\psi(n-k+1)\psi'(n-k+1)$$
$$-480\psi'(n+1)\psi(n-k+1)^2\psi'(n-k+1)$$
$$-180\psi'(n+1)^2\psi'(n-k+1)+240\psi(n+1)^2\psi'(n-k+1)^2$$
$$-480\psi(n+1)\psi(n-k+1)\psi'(n-k+1)^2+240\psi(n-k+1)^2\psi'(n-k+1)^2$$
$$+180\psi'(n+1)\psi'(n-k+1)^2-60\psi'(n-k+1)^3+80\psi(n+1)\psi'(n+1)\psi^{(2)}(n+1)$$
$$-80\psi'(n+1)\psi^{(2)}(n+1)\psi(n-k+1)-80\psi(n+1)\psi^{(2)}(n+1)\psi'(n-k+1)$$
$$+80\psi^{(2)}(n+1)\psi(n-k+1)\psi'(n-k+1)-80\psi(n+1)\psi'(n+1)\psi^{(2)}(n-k+1)$$
$$+80\psi'(n+1)\psi(n-k+1)\psi^{(2)}(n-k+1)+80\psi(n+1)\psi'(n-k+1)\psi^{(2)}(n-k+1)$$
$$-80\psi(n-k+1)\psi'(n-k+1)\psi^{(2)}(n-k+1)+10\psi'(n+1)\psi^{(3)}(n+1)$$
$$-10\psi^{(3)}(n+1)\psi'(n-k+1)-10\psi'(n+1)\psi^{(3)}(n-k+1)$$
$$+10\psi'(n-k+1)\psi^{(3)}(n-k+1)\Big]\,r^{2k};$$

$$\Omega_5(n,r)=\sum_{k=0}^{\infty}\binom{n}{k}^2\Big[32\psi(n+1)^6-192\psi(n+1)^5\psi(n-k+1)$$
$$+480\psi(n+1)^4\psi(n-k+1)^2-640\psi(n+1)^3\psi(n-k+1)^3$$
$$+480\psi(n+1)^2\psi(n-k+1)^4-192\psi(n+1)\psi(n-k+1)^5$$
$$+32\psi(n-k+1)^6+160\psi(n+1)^4\psi'(n+1)$$
$$-640\psi(n+1)^3\psi'(n+1)\psi(n-k+1)+960\psi(n+1)^2\psi'(n+1)\psi(n-k+1)^2$$
$$-640\psi(n+1)\psi'(n+1)\psi(n-k+1)^3+160\psi'(n+1)\psi(n-k+1)^4$$
$$+120\psi(n+1)^2\psi'(n+1)^2-240\psi(n+1)\psi'(n+1)^2\psi(n-k+1)$$
$$+120\psi'(n+1)^2\psi(n-k+1)^2-160\psi(n+1)^4\psi'(n-k+1)$$
$$+640\psi(n+1)^3\psi(n-k+1)\psi'(n-k+1)-960\psi(n+1)^2\psi(n-k+1)^2\psi'(n-k+1)$$
$$+640\psi(n+1)\psi(n-k+1)^3\psi'(n-k+1)-160\psi(n-k+1)^4\psi'(n-k+1)$$
$$-240\psi(n+1)^2\psi'(n+1)\psi'(n-k+1)$$
$$+480\psi(n+1)\psi'(n+1)\psi(n-k+1)\psi'(n-k+1)$$
$$-240\psi'(n+1)\psi(n-k+1)^2\psi'(n-k+1)+120\psi(n+1)^2\psi'(n-k+1)^2$$
$$-240\psi(n+1)\psi(n-k+1)\psi'(n-k+1)^2+120\psi(n-k+1)^2\psi'(n-k+1)^2$$
$$+80\psi(n+1)^3\psi^{(2)}(n+1)-240\psi(n+1)^2\psi^{(2)}(n+1)\psi(n-k+1)$$
$$+240\psi(n+1)\psi^{(2)}(n+1)\psi(n-k+1)^2-80\psi^{(2)}(n+1)\psi(n-k+1)^3$$
$$+40\psi(n+1)\psi'(n+1)\psi^{(2)}(n+1)-40\psi'(n+1)\psi^{(2)}(n+1)\psi(n-k+1)$$
$$-40\psi(n+1)\psi^{(2)}(n+1)\psi'(n-k+1)$$

$$
\begin{aligned}
&+40\psi^{(2)}(n+1)\psi(n-k+1)\psi'(n-k+1)-80\psi(n+1)^3\psi^{(2)}(n-k+1)\\
&+240\psi(n+1)^2\psi(n-k+1)\psi^{(2)}(n-k+1)-240\psi(n+1)\psi(n-k+1)^2\psi^{(2)}(n-k+1)\\
&+80\psi(n-k+1)^3\psi^{(2)}(n-k+1)-40\psi(n+1)\psi'(n+1)\psi^{(2)}(n-k+1)\\
&+40\psi'(n+1)\psi(n-k+1)\psi^{(2)}(n-k+1)+40\psi(n+1)\psi'(n-k+1)\psi^{(2)}(n-k+1)\\
&-40\psi(n-k+1)\psi'(n-k+1)\psi^{(2)}(n-k+1)+20\psi(n+1)^2\psi^{(3)}(n+1)\\
&-40\psi(n+1)\psi^{(3)}(n+1)\psi(n-k+1)+20\psi^{(3)}(n+1)\psi(n-k+1)^2\\
&-20\psi(n+1)^2\psi^{(3)}(n-k+1)+40\psi(n+1)\psi(n-k+1)\psi^{(3)}(n-k+1)\\
&-20\psi(n-k+1)^2\psi^{(3)}(n-k+1)+2\psi(n+1)\psi^{(4)}(n+1)\\
&-2\psi^{(4)}(n+1)\psi(n-k+1)-2\psi(n+1)\psi^{(4)}(n-k+1)\\
&+2\psi(n-k+1)\psi^{(4)}(n-k+1)\Big]\,r^{2k}.
\end{aligned}
$$

Then, recollecting the terms in $\Omega_j(n,r)$ ($j = 0, 1, \ldots, 5$) in the eleven sums as follows:

(i) Collecting the constant terms which are not involved in the summation index k, and using (51), we derive (64).

(ii) Collecting the terms which are solely involved in the summation index k, we obtain

$$
\begin{aligned}
\Lambda_2(n,r) &= \sum_{k=0}^{\infty}\binom{n}{k}^2\lambda_2(n,k)\,r^{2k}\\
&= \sum_{k=0}^{n}\binom{n}{k}^2\lambda_2(n,k)\,r^{2k}+\sum_{k=n+1}^{\infty}\binom{n}{k}^2\lambda_2(n,k)\,r^{2k}.
\end{aligned}
$$

We have

$$
\sum_{k=n+1}^{\infty}\binom{n}{k}^2\lambda_2(n,k)\,r^{2k}=\sum_{k=0}^{\infty}\frac{(n!)^2}{\{(k+n+1)!\}^2}\cdot\frac{\lambda_2^{(a)}(-k)}{\Gamma(-k)^2}r^{2n+2+2k},
$$

where

$$
\begin{aligned}
\lambda_2^{(a)}(-k) &= -\psi^{(5)}(-k)+12\psi(-k)\psi^{(4)}(-k)\\
&\quad-60\psi(-k)^2\psi^{(3)}(-k)+30\psi'(-k)\psi^{(3)}(-k)\\
&\quad+160\psi(-k)^3\psi^{(2)}(-k)-240\psi(-k)\psi'(-k)\psi^{(2)}(-k)\\
&\quad+20\psi^{(2)}(-k)^2-240\psi(-k)^4\psi'(-k)\\
&\quad+360\psi(-k)^2\psi'(-k)^2-60\psi'(-k)^3+32\psi(-k)^6.
\end{aligned}
$$

By using (31) and (33), we obtain that, as $z \to -k$,

$$
\begin{aligned}
\lambda_2^{(a)}(z) &= \frac{3\pi^4+60\pi^2H_k^{(2)}+180\{H_k^{(2)}\}^2-180\,H_k^{(4)}+480\,\psi(k+1)\,H_k^{(3)}}{(z+k)^2}\\
&+\frac{-120\,\pi^2\,\psi(k+1)^2-720\,\psi(k+1)^2\,H_k^{(2)}+240\,\psi(k+1)^4}{(z+k)^2}\\
&-\frac{480\,\zeta(3)\,\psi(k+1)}{(z+k)^2}+\frac{280\pi^2H_k^{(3)}+1680H_k^{(2)}H_k^{(3)}+etc}{z+k}+O(z+k).
\end{aligned}
$$

With the aid of (57) and Remark 1, we obtain (65).

(iii) Collecting the terms which are solely involved in $\psi(n-k+1)$, and using (52), we obtain (66).

(iv) Collecting the terms which are solely involved in $\psi(n-k+1)^2$, and using (53), we obtain (67).

(v) Collecting the terms which are solely involved in $\psi'(n-k+1)$, and using (54), we obtain (68).

(vi) Collecting the terms which are solely involved in $\psi(n+1)$, we obtain

$$\Lambda_6(n,r) = \psi(n+1)\sum_{k=0}^{\infty}\binom{n}{k}^2 \lambda_6(n,k)\, r^{2k}$$
$$= \psi(n+1)\left\{\sum_{k=0}^{n}\binom{n}{k}^2 \lambda_6(n,k)\, r^{2k} + \sum_{k=n+1}^{\infty}\binom{n}{k}^2 \lambda_6(n,k)\, r^{2k}\right\}.$$

We find

$$\sum_{k=n+1}^{\infty}\binom{n}{k}^2 \lambda_6(n,k)\, r^{2k} = \sum_{k=0}^{\infty}\frac{(n!)^2}{\{(k+n+1)!\}^2}\frac{\lambda_6^{(a)}(-k)}{\Gamma(-k)^2} r^{2n+2+2k}$$
$$= \sum_{k=0}^{\infty}\frac{(n!)^2}{\{(k+n+1)!\}^2}\lim_{z\to -k}\frac{\lambda_6^{(a)}(z)}{\Gamma(z)^2} r^{2n+2+2k}$$

where

$$\lambda_6^{(a)}(-k) = -12\psi^{(4)}(-k) - 480\psi(-k)^2\psi^{(2)}(-k)$$
$$+ 120\psi(-k)\psi^{(3)}(-k) + 240\psi'(-k)\psi^{(2)}(-k)$$
$$+ 960\psi(-k)^3\psi'(-k) - 720\psi(-k)\psi'(-k)^2$$
$$- 192\psi(-k)^5.$$

By using (31) and (33), we obtain that, as $z \to -k$,

$$\lambda_6^{(a)}(z) = \frac{-480\, H_k^{(3)} + 240\,\pi^2\,\psi(k+1) + 1440\,\psi(k+1)\, H_k^{(2)}}{(z+k)^2}$$
$$+ \frac{480\,\zeta(3) - 960\,\psi(k+1)^3}{(z+k)^2} + \frac{960\,\psi(k+1)^4 - 4800\,\zeta(3)\,\psi(k+1)}{z+k}$$
$$+ \frac{36\pi^4 + 720\pi^2 H_k^{(2)} + 2160\{H_k^{(2)}\}^2 - 2160 H_k^{(4)} + 4800\,\psi(k+1)\, H_k^{(3)}}{z+k}$$
$$- \frac{960\,\pi^2\,\psi(k+1)^2 + 5760\, H_k^{(2)}\,\psi(k+1)^2}{z+k} + O(z+k).$$

Finally, by employing the approach outlined for obtaining $\Lambda_2(n,r)$, we arrive at (69).

(vii) Collecting the terms which are solely involved in $\psi(n+1)^2$, we obtain

$$\Lambda_7(n,r) = \psi(n+1)^2\sum_{k=0}^{\infty}\binom{n}{k}^2 \lambda_7(n,k)\, r^{2k},$$

where

$$\lambda_7(n,k) = -60\psi^{(3)}(n-k+1) + 480\psi(n-k+1)\psi^{(2)}(n-k+1)$$
$$- 1440\psi(n-k+1)^2\psi'(n-k+1) + 360\psi'(n-k+1)^2$$
$$+ 480\psi(n-k+1)^4.$$

Then,

$$\lambda_7^{(a)}(-k) = -60\psi^{(3)}(-k) + 480\psi(-k)\psi^{(2)}(-k)$$
$$- 1440\psi(-k)^2\psi'(-k) + 360\psi'(-k)^2 + 480\psi(-k)^4.$$

By using (31) and (33), we obtain that, as $z \to -k$,

$$\begin{aligned}\lambda_7^{(a)}(z) =& \frac{-120\pi^2 - 720H_k^{(2)} + 1440\,\psi(k+1)^2}{(z+k)^2} \\ &+ \frac{-2400H_k^{(3)} + 960\pi^2\psi(k+1) + 5760\psi(k+1)H_k^{(2)}}{z+k} \\ &+ \frac{-1920\psi(k+1)^3 + 2400\zeta(3)}{z+k} + O(z+k).\end{aligned}$$

By following the method detailed for deriving $\Lambda_2(n,r)$, we reach Equation (70).

(viii) Collecting the terms which are solely involved in $\psi(n+1)^3$, we obtain

$$\Lambda_8(n,r) = \psi(n+1)^3 \sum_{k=0}^{\infty} \binom{n}{k}^2 \lambda_8(n,k)\, r^{2k},$$

where

$$\begin{aligned}\lambda_8(n,k) =& -160\psi^{(2)}(n-k+1) + 960\psi(n-k+1)\psi'(n-k+1) \\ &- 640\psi(n-k+1)^3.\end{aligned}$$

Then,

$$\begin{aligned}\lambda_8^{(a)}(-k) =& -160\psi^{(2)}(-k) + 960\psi(-k)\psi'(n-k+1) \\ &- 640\psi(-k)^3.\end{aligned}$$

By using (31) and (33), we obtain that, as $z \to -k$,

$$\begin{aligned}\lambda_8^{(a)}(z) =& \frac{-320\pi^2 - 1920H_k^{(2)} + 1920\psi(k+1)^2}{z+k} \\ &- \frac{960\psi(k+1)}{(z+k)^2} + O(z+k).\end{aligned}$$

By adhering to the prescribed procedure for deriving $\Lambda_2(n,r)$, we arrive at Equation (71).

(ix) Collecting the terms which are solely involved in $\psi'(n+1)$, we obtain

$$\Lambda_9(n,r) = \psi'(n+1) \sum_{k=0}^{\infty} \binom{n}{k}^2 \lambda_9(n,k)\, r^{2k},$$

where

$$\begin{aligned}\lambda_9(n,k) =& -30\psi^{(3)}(n-k+1) + 240\psi(n-k+1)\psi^{(2)}(n-k+1) \\ &+ 240\psi(n-k+1)^4 - 720\psi(n-k+1)^2\psi'(n-k+1) \\ &+ 180\psi'(n-k+1)^2.\end{aligned}$$

Then,

$$\begin{aligned}\lambda_9^{(a)}(-k) =& -30\psi^{(3)}(-k) + 240\psi(-k)\psi^{(2)}(-k) \\ &+ 240\psi(-k)^4 - 720\psi(-k)^2\psi'(-k) \\ &+ 180\psi'(-k)^2.\end{aligned}$$

By using (31) and (33), we have that, as $z \to -k$,

$$
\begin{aligned}
\lambda_9(z) = & \frac{-60\pi^2 - 360H_k^{(2)} + 720\,\psi(k+1)^2}{(z+k)^2} \\
& + \frac{-1200H_k^{(3)} + 480\pi^2\psi(k+1) + 2880\psi(k+1)H_k^{(2)}}{z+k} \\
& + \frac{-960\psi(k+1)^3 + 1200\zeta(3)}{z+k} + O(z+k).
\end{aligned}
$$

By following the designated procedure for deriving $\Lambda_2(n,r)$, we reach Equation (72).

(x) Collecting the terms which are solely involved in $\psi^{(2)}(n+1)$, we obtain

$$
\Lambda_{10}(n,r) = \psi^{(2)}(n+1) \sum_{k=0}^{\infty} \binom{n}{k}^2 \lambda_{10}(n,k)\, r^{2k},
$$

where

$$
\begin{aligned}
\lambda_{10}(n,k) = & -160\psi(n-k+1)^3 + 240\psi(n-k+1)\psi'(n-k+1) \\
& -40\psi^{(2)}(n-k+1).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\lambda_{10}^{(a)}(-k) = & -160\psi(-k)^3 + 240\psi(-k)\psi'(-k) \\
& -40\psi^{(2)}(-k).
\end{aligned}
$$

By using (31) and (33), we find that, as $z \to -k$,

$$
\begin{aligned}
\lambda_{10}^{(a)}(z) = & \frac{-80\pi^2 - 480H_k^{(2)} + 480\,\psi(k+1)^2}{z+k} \\
& - \frac{240\,\psi(k+1)}{(z+k)^2} + O(z+k).
\end{aligned}
$$

By adhering to the specified procedure for deriving $\Lambda_2(n,r)$, we arrive at Equation (73).

(xi) Collecting the terms which are solely involved in

$$
\psi(n+1)\psi'(n+1),
$$

we obtain

$$
\Lambda_{11}(n,r) = \psi(n+1)\psi'(n+1) \sum_{k=0}^{\infty} \binom{n}{k}^2 \lambda_{11}(n,k)\, r^{2k},
$$

where

$$
\begin{aligned}
\lambda_{11}(n,k) = & -240\psi^{(2)}(n-k+1) - 960\psi(n-k+1)^3 \\
& + 1440\psi(n-k+1)\psi'(n-k+1).
\end{aligned}
$$

$$
\lambda_{11}^{(a)}(-k) = -240\psi^{(2)}(-k) - 960\psi(-k)^3 + 1440\psi(-k)\psi'(-k).
$$

By using (31) and (33), we obtain that, as $z \to -k$,

$$
\begin{aligned}
\lambda_{11}(z) = & \frac{-480\pi^2 - 2880\,H_k^{(2)} + 2880\,\psi(k+1)^2}{z+k} \\
& - \frac{1440\,\psi(k+1)}{(z+k)^2} + O(z+k).
\end{aligned}
$$

Following the prescribed method to determine $\Lambda_2(n,r)$ leads us to Equation (74).

□

It is worth noting that the integral formulas presented here are undoubtedly novel and capable of generating various specific cases. Additionally, the algorithmic method can be used to evaluate the integrals

$$\int_0^{2\pi} |1+re^{it}|^{2n} \log^m |1+re^{it}| \,\mathrm{d}t$$

for $n \in \mathbb{Z}_{\geqslant 0}$ and $m = 7, 8, 9, \ldots$.

4. Certain Variants of Euler Sums

The following series involving harmonic numbers,

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n}{n^2} = \zeta(3), \tag{76}$$

was discovered by Euler in 1775 and has a long history (see, e.g., [10], [p. 252 et seq.]; see also [11]). By applying Parseval's identity to a Fourier series and the contour integral to a generating function, D. Borwein and J. M. Borwein [12] established the following interesting identity (see also [13], [Equation (2.16)]; [14], [p. 280]; [15], [Equation (9)]):

$$\sum_{n=1}^{\infty} \frac{H_n^2}{(n+1)^2} = \frac{11}{17} \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \frac{11}{4}\,\zeta(4). \tag{77}$$

Euler initiated this line of investigation in the course of his correspondence with Goldbach from 1742 and he was the first to consider the linear harmonic sums (see, e.g., [9,16])

$$\mathbf{S}_{p,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}. \tag{78}$$

Euler, whose investigations were completed by Nielsen in 1906 (see [17]), showed that the linear harmonic sums in (78) can be evaluated in the following cases: $p = 1$; $p = q$; $p + q$ odd; $p + q$ even, but with the pair (p, q) being the set $\{(2, 4),\ (4.2)\}$. Of these special cases, in the ones with $p \neq q$, if $\mathbf{S}_{p,q}$ is known, then $\mathbf{S}_{q,p}$ can be found by means of the symmetry relation

$$\mathbf{S}_{p,q} + \mathbf{S}_{q,p} = \zeta(p)\,\zeta(q) + \zeta(p+q) \tag{79}$$

and vice versa (see, e.g., [18]). A rather extensive numerical search for linear relations between linear Euler sums and polynomials in zeta values (see [9,19]) strongly suggests that Euler found all the possible evaluations of linear harmonic sums; for example,

$$2\,\mathbf{S}_{1,q} = (q+2)\,\zeta(q+1) - \sum_{j=1}^{q-2} \zeta(q-j)\,\zeta(j+1) \quad (q \in \mathbb{Z}_{\geqslant 2}). \tag{80}$$

The nonlinear harmonic sums involve products of at least two (generalized) harmonic numbers. Let $P = (p_1, \ldots, p_k)$ be a partition of an integer p into k summands, so that $p = p_1 + \cdots + p_k$ and $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k$. The Euler sum of index P, q is defined by

$$\mathbb{S}_{P;q} := \sum_{n=1}^{\infty} \frac{H_n^{(p_1)}\,H_n^{(p_2)} \cdots H_n^{(p_k)}}{n^q}, \tag{81}$$

where the quantity $q + p_1 + \cdots + p_k$ is called the weight, and the quantity k is the degree. For simplicity, repeated summands in partitions are denoted by powers, for example,

$$\mathbb{S}_{1^2,2^3,5;q} = \mathbb{S}_{1,1,2,2,2,5;q} = \sum_{n=1}^{\infty} \frac{H_n^2\,\{H_n^{(2)}\}^3\,H_n^{(5)}}{n^q}.$$

The alternating version of (81) is given by

$$\bar{\mathbb{S}}_{P;q} := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(p_1)} H_n^{(p_2)} \cdots H_n^{(p_k)}}{n^q}. \tag{82}$$

To explore a wide range of the literature concerning multiple zeta values and Euler sums of varying degrees, interested readers are encouraged to consult the comprehensive survey and expository paper [20]. Since its publication, research in this field has remained vibrant and dynamic, as evidenced by numerous subsequent works: For instance, Eie and Wei [21] evaluated several exceptional quadruple Euler sums of even weight by using identities among multiple zeta values with variables and the relation obtained from the shuffle formula of two multiple zeta values; Espinosa and Moll [22] presented an explicit formula for the Tornheim double series, expressed through integrals involving the Hurwitz zeta function; Freitas [23] demonstrated that two types of integrals involving polylogarithmic functions satisfy specific recurrence relations, enabling their expression in terms of Euler sums; Furdui [24] proved two series involving products of harmonic numbers. One of these series is given by

$$\sum_{n=1}^{\infty} \frac{H_n}{n} \cdot \frac{H_{n+1}}{n+1} = \frac{\pi^2}{6} + 2\zeta(3);$$

Li and Chu [25] explored two summation theorems for the ${}_2F_1\left(\frac{1}{2}\right)$-series by Gauss and Bailey using the coefficient extraction method. They evaluated forty infinite series involving harmonic numbers and binomial/multinomial coefficients in closed form, including eight conjectures; by selecting various kernel functions and base functions, Li and Qin [26] derived certain Euler sums with parameters; Mezö [27] investigated formulas for nonlinear Euler sums that involve multiple zeta values. He used these formulas to derive new closed-form expressions for several nonlinear Euler series; Pilehrood et al. [28] introduced new binomial identities for multiple harmonic sums under specific conditions. They demonstrated several congruences for these sums modulo a prime p; Qin et al. [29] introduced a kernel function incorporating a complex parameter. They applied this function to derive identities involving linear extended Euler sums with parameters, as well as novel Euler sums. These findings extend the known identities of standard Euler sums; Qin et al. [30] presented certain intriguing identities on the Hurwitz zeta function and some extended Euler sums: For example,

$$\sum_{n=1}^{\infty} \frac{H_{2n}}{n^{2m}} = \frac{(2m+1)\zeta(2m+1)}{4} - \sum_{j=1}^{m} 2^{2j} \zeta(2j+1)\zeta(2m-2j) \quad (m \in \mathbb{N});$$

Si et al. [31] used integrals of polylogarithm functions to investigate the analytic representations of specific types of quadratic- and cubic Euler-related sums involving harmonic numbers and reciprocal binomial coefficients. An example of their findings is expressed as

$$\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{n(n+\ell)} = \frac{1}{\ell} \left\{ 2\zeta(3) + \frac{H_\ell^3 + 3H_\ell H_\ell^{(2)} + 2H_\ell^{(3)}}{3} - \frac{H_\ell^2 + H_\ell^{(2)}}{\ell} \right\} \quad (\ell \in \mathbb{N});$$

Sofo [32] introduced novel closed-form expressions for the sums involving quadratic alternating harmonic numbers and reciprocal binomial coefficients; Sofo [33] presented an explicit analytical representation for Euler-type sums of harmonic numbers with multiple arguments; Wang and Lyu [34] employed Bell polynomials along with generating functions and integration techniques to construct diverse mixed Euler sums and Stirling sums. They also introduced a unified method for evaluating unknown Euler sums; Xu [35] derived various series expressions that incorporate harmonic numbers and Stirling numbers of the first kind, expressed in terms of multiple zeta values. Additionally, Xu uncovered novel connections between multiple zeta values and multiple zeta star values. An instance of his findings is given as

$$\mathbb{S}_{1^2,2;4} = \tfrac{193}{96}\zeta(8) + 2\zeta(3)\zeta(5) - 2\zeta(2)\zeta^2(3) + \tfrac{3}{2}\,\mathbf{S}_{2;6} \approx 1.29069;$$

Xu [36] utilized contour integral representations, residue calculus, and integral representations of series to investigate the analytic representations of parametric Euler sums involving harmonic numbers. This exploration connected these sums with zeta values and rational function series, both in linear and nonlinear forms; Xu [37] demonstrated that sums of multiple harmonic numbers with specified indices can be expressed using multiple zeta values, multiple harmonic numbers, and Stirling numbers of the first kind. He also provided an explicit formula for these sums; Xu [38] established some relations involving cubic, quadratic, and linear Euler sums. An example of his findings is offered as

$$\begin{aligned}\mathbb{S}_{1^2,4;4} =& \tfrac{7749}{160}\zeta(10) - 14\zeta(2)\zeta(3)\zeta(5) + \tfrac{3}{2}\zeta^2(3)\zeta(4) - \tfrac{125}{8}\zeta(3)\zeta(7)\\ &- 16\zeta^2(5) + 11\mathbf{S}_{2,8} + \tfrac{5}{2}\zeta(2)\mathbf{S}_{2,6} \approx 1.23696;\end{aligned}$$

Xu [39] introduced a new set of identities for Euler sums and polylogarithm integrals using generating function techniques and series integral representations. He then utilized these identities to derive the closed forms of all quadratic Euler sums with a weight of ten. An instance of his findings is given as

$$\begin{aligned}\mathbb{S}_{4^2;2} =& -\tfrac{203}{120}\zeta(10) - 80\zeta(2)\zeta(3)\zeta(5) - 2\zeta^2(3)\zeta(4) + 98\zeta(3)\zeta(7)\\ &+ 30\zeta^2(5) - 14\mathbf{S}_{2,8} + 20\zeta(2)\mathbf{S}_{2,6} \approx 1.74226;\end{aligned}$$

Xu and Cai [40] explored the analytic representations of Euler sums in terms of the values of the polylogarithm function and the Riemann zeta function; Xu and Cheng [41] developed a method for evaluating Euler sums that incorporate both harmonic numbers and alternating harmonic numbers; Xu et al. [42] developed an approach to evaluate Euler sums and integrals of polylogarithm functions based on computations using the simple Cauchy product formula. An example of their findings is given as

$$\begin{aligned}\mathbb{S}_{1,2;6} =& -\tfrac{799}{72}\zeta(9) + 3\zeta(2)\zeta(7) + \tfrac{23}{6}\zeta(3)\zeta(6)\\ &+ \tfrac{13}{4}\zeta(4)\zeta(5) - \tfrac{2}{3}\zeta^3(3) \approx 1.03381;\end{aligned}$$

Xu et al. [43] explored some explicit formulae for double nonlinear Euler sums involving harmonic numbers and alternating harmonic numbers. As applications of these formulae, they gave new closed-form representations of several quadratic Euler sums through Riemann zeta function and linear sums: An instance of their findings is offered as

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^2} = \frac{41}{16}\zeta(4) + \frac{1}{2}\zeta(2)\,\ln^2 2 - \frac{1}{12}\ln^4 2 - \frac{7}{4}\zeta(3)\,\ln 2 - 2\,\mathrm{Li}_4\left(\frac{1}{2}\right).$$

This section delves into specific variations of Euler sums, which will be referenced in the subsequent section. These sums are presented in the following theorems and corollary.

Theorem 3. *The following formula holds:*

$$\sum_{k=1}^{\infty}\frac{H_k^{(n)}}{k(k+1)} = \zeta(n+1) \quad (n \in \mathbb{N}). \tag{83}$$

Proof. Let $\mathcal{L}_A$ be the left-hand side of (83). Then, we have

$$
\begin{aligned}
\mathcal{L}_A &= \sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j^n} - \frac{1}{k+1}\left\{\sum_{j=1}^{k+1}\frac{1}{j^n} - \frac{1}{(k+1)^n}\right\}\right] \\
&= \lim_{N\to\infty}\sum_{k=1}^{N}\left[\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j^n} - \frac{1}{k+1}\sum_{j=1}^{k+1}\frac{1}{j^n}\right] + \sum_{k=2}^{\infty}\frac{1}{k^{n+1}} \\
&= \lim_{N\to\infty}\left(1 - \frac{1}{N+1}\sum_{j=1}^{N+1}\frac{1}{j^n}\right) + \sum_{k=2}^{\infty}\frac{1}{k^{n+1}}.
\end{aligned}
$$

Note that

$$
\sum_{j=1}^{N+1}\frac{1}{j^n} \leqslant \sum_{j=1}^{N+1}\frac{1}{j} = H_{N+1} \quad (n \in \mathbb{N}),
$$

and

$$
H_{N+1} \sim \log(N+1) \quad \text{as } N \to \infty.
$$

We, finally, obtain

$$
\mathcal{L}_A = 1 + \sum_{k=2}^{\infty}\frac{1}{k^{n+1}} = \zeta(n+1).
$$

□

Remark 2. *The particular case of (83) when $n = 1$ can be demonstrated through the identity*

$$
2\zeta(2) = \int_0^1 \frac{\log^2(1-x)}{x^2}\,dx = 2\sum_{k=1}^{\infty}\frac{H_k}{k(k+1)}.
$$

Particular cases of (83) can be derived through Mathematica.

Theorem 4. *The following formulas hold:*

$$
\sum_{k=1}^{\infty}\frac{H_k^2}{k(k+1)} = 3\,\zeta(3); \tag{84}
$$

$$
\sum_{k=1}^{\infty}\frac{H_k^3}{k(k+1)} = 10\,\zeta(4); \tag{85}
$$

$$
\sum_{k=1}^{\infty}\frac{H_k^4}{k(k+1)} = 30\,\zeta(5) + 6\,\zeta(2)\zeta(3). \tag{86}
$$

Proof. We prove only (86). The other two can be verified similarly. Let $\mathcal{L}_B$ be the left-hand side of (86). Then, as in the proof of Theorem 3, we find

$$
\begin{aligned}
\mathcal{L}_B = \lim_{N\to\infty}\sum_{k=1}^{N}\left(\frac{1}{k}H_k^4 - \frac{1}{k+1}H_{k+1}^4\right) + 4\sum_{k=1}^{\infty}\frac{H_k^3}{(k+1)^2} \\
+ 6\sum_{k=1}^{\infty}\frac{H_k^2}{(k+1)^3} + 4\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^4} + \sum_{k=2}^{\infty}\frac{1}{k^5}.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\mathcal{L}_B =& 1 - \lim_{N\to\infty}\frac{1}{N+1}H_{N+1}^4 \\
&+ 4\{-\zeta(5) + 3\,\mathbf{S}_{1,4} - 3\,\mathbb{S}_{1^2;3} + \mathbb{S}_{1^3;2}\} \\
&+ 6\{\mathbb{S}_{1^2;3} - 2\,\mathbf{S}_{1,4} + \zeta(5)\} + 4\{\mathbf{S}_{1,4} - \zeta(5)\} + \sum_{k=2}^{\infty}\frac{1}{k^5},
\end{aligned}
$$

which, upon employing the results in Remark 6, leads to the right-hand side of (86). □

Remark 3. *Formula (85) represents a specific instance derived from the overarching identity found in [44], p. 378. Formula (84) can be obtained through Mathematica 13.0.*

Likewise, we can evaluate the following general sum,

$$\sum_{k=1}^{\infty} \frac{H_k^p}{k(k+1)} \quad (p \in \mathbb{Z}_{\geqslant 5}), \tag{87}$$

in terms of a finite combination of Riemann zeta functions, provided that the involved linear and nonlinear Euler sums of weight $p+1$ are already known.

Theorem 5. *The following formulas hold:*

$$\sum_{k=1}^{\infty} \frac{H_k\, H_k^{(2)}}{k(k+1)} = 2\,\zeta(4); \tag{88}$$

$$\sum_{k=1}^{\infty} \frac{H_k^2\, H_k^{(2)}}{k(k+1)} = 5\,\zeta(5); \tag{89}$$

$$\sum_{k=1}^{\infty} \frac{\{H_k^{(2)}\}^2}{k(k+1)} = 6\,\zeta(2)\zeta(3) - 10\,\zeta(5); \tag{90}$$

$$\sum_{k=1}^{\infty} \frac{H_k\, H_k^{(3)}}{k(k+1)} = \tfrac{15}{2}\zeta(5) - 3\zeta(2)\zeta(3). \tag{91}$$

Proof. By following the same procedures as outlined in Theorems 3 and 4, we can derive the results here. The detailed steps are omitted. □

Remark 4. *Formula (88) can be obtained as a particular case of Problem 4.19 in [44], [p. 290].*

Similarly, we can evaluate the following general sum,

$$\sum_{k=1}^{\infty} \frac{H_k^p\, H_k^{(q)}}{k(k+1)} \quad (p,\, q \in \mathbb{Z}_{\geqslant 0}), \tag{92}$$

in terms of a finite combination of Riemann zeta functions, provided that the involved linear and nonlinear Euler sums of weight $p+q+1$ are already known.

The following corollary presents some identities that will be used in the subsequent section.

Corollary 1. *The following formulas hold:*

$$\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)^2} = 2\,\zeta(2) - 3. \tag{93}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^3(k+1)^2} = \zeta(3) - 3\zeta(2) + 4. \tag{94}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^4(k+1)^2} = \zeta(4) - 2\zeta(3) + 4\zeta(2) - 5. \tag{95}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^5(k+1)} = \zeta(5) - \zeta(4) + \zeta(3) - \zeta(2) + 1. \tag{96}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^5(k+1)^2} = \zeta(5) - 2\zeta(4) + 3\zeta(3) - 5\zeta(2) + 6. \quad (97)$$

$$\sum_{k=1}^{\infty} \frac{1}{k^6(k+1)^2} = \zeta(6) - 2\zeta(5) + 3\zeta(4) - 4\zeta(3) + 6\zeta(2) - 7. \quad (98)$$

$$\sum_{k=1}^{\infty} \frac{1}{k^4(k+1)} = \zeta(4) - \zeta(3) + \zeta(2) - 1. \quad (99)$$

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2(k+1)^2} = 3\zeta(3) - 2\zeta(2). \quad (100)$$

$$\sum_{k=1}^{\infty} \frac{H_k}{k^3(k+1)^2} = \tfrac{5}{4}\zeta(4) - 5\zeta(3) + 3\zeta(2). \quad (101)$$

$$\sum_{k=1}^{\infty} \frac{H_k}{k^4(k+1)^2} = 3\zeta(5) - \tfrac{5}{2}\zeta(4) + 7\zeta(3) - 4\zeta(2) - \zeta(2)\zeta(3). \quad (102)$$

$$\sum_{k=1}^{\infty} \frac{H_k}{k^5(k+1)^2} = \tfrac{7}{4}\zeta(6) - \tfrac{1}{2}\zeta(3)^2 + 2\zeta(2)\zeta(3) - 6\zeta(5) + \tfrac{15}{4}\zeta(4) - 9\zeta(3) + 5\zeta(2). \quad (103)$$

$$\sum_{k=1}^{\infty} \frac{H_k^2}{k^2(k+1)^2} = 7\zeta(4) - 6\zeta(3). \quad (104)$$

$$\sum_{k=1}^{\infty} \frac{H_k^2}{k^3(k+1)^2} = \tfrac{7}{2}\zeta(5) - \tfrac{45}{4}\zeta(4) + 9\zeta(3) - \zeta(2)\zeta(3). \quad (105)$$

$$\sum_{k=1}^{\infty} \frac{H_k^2}{k^4(k+1)^2} = \tfrac{97}{24}\zeta(6) - 2\zeta(3)^2 - 7\zeta(5) + 2\zeta(2)\zeta(3) + \tfrac{31}{2}\zeta(4) - 12\zeta(3). \quad (106)$$

$$\sum_{k=1}^{\infty} \frac{H_k^3}{k^2(k+1)^2} = \tfrac{35}{2}\zeta(5) - 20\zeta(4) + 2\zeta(2)\zeta(3). \quad (107)$$

$$\sum_{k=1}^{\infty} \frac{H_k^3}{k^3(k+1)^2} = \tfrac{93}{16}\zeta(6) - \tfrac{5}{2}\zeta(3)^2 - 3\zeta(2)\zeta(3) - \tfrac{55}{2}\zeta(5) + 30\zeta(4). \quad (108)$$

$$\sum_{k=1}^{\infty} \frac{H_k^4}{k^2(k+1)^2} = \tfrac{919}{12}\zeta(6) + 6\,\zeta(3)^2 - 12\,\zeta(2)\zeta(3) - 60\,\zeta(5). \quad (109)$$

$$\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^2(k+1)^2} = \tfrac{5}{2}\zeta(4) - 2\zeta(3). \quad (110)$$

$$\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^3(k+1)^2} = 3\zeta(2)\zeta(3) - \tfrac{9}{2}\zeta(5) - \tfrac{17}{4}\zeta(4) + 3\zeta(3). \quad (111)$$

$$\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^4(k+1)^2} = 6\,\zeta(4) - \tfrac{1}{3}\,\zeta(6) - 4\,\zeta(3) - 6\,\zeta(2)\zeta(3) + \zeta(3)^2 + 9\,\zeta(5). \quad (112)$$

$$\sum_{k=1}^{\infty} \frac{H_k^{(3)}}{k^2(k+1)^2} = 10\zeta(5) - 4\zeta(2)\zeta(3) - 2\zeta(4). \quad (113)$$

$$\sum_{k=1}^{\infty}\frac{H_k^{(3)}}{k^3(k+1)^2}=\tfrac{1}{2}\zeta(6)+3\zeta(4)+6\zeta(2)\zeta(3)+\tfrac{1}{2}\zeta(3)^2-\tfrac{31}{2}\zeta(5). \tag{114}$$

$$\sum_{k=1}^{\infty}\frac{H_k^{(4)}}{k^2(k+1)^2}=\tfrac{31}{6}\zeta(6)-2\,\zeta(3)^2-2\,\zeta(5). \tag{115}$$

$$\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^2(k+1)^2}=\tfrac{9}{2}\zeta(5)-4\zeta(4). \tag{116}$$

$$\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^3(k+1)^2}=6\,\zeta(4)-\tfrac{101}{48}\zeta(6)-\zeta(2)\zeta(3)+\tfrac{5}{2}\zeta(3)^2-\tfrac{11}{2}\zeta(5). \tag{117}$$

$$\sum_{k=1}^{\infty}\frac{H_kH_k^{(3)}}{k^2(k+1)^2}=\tfrac{197}{24}\zeta(6)+6\,\zeta(2)\zeta(3)-3\,\zeta(3)^2-15\,\zeta(5). \tag{118}$$

$$\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(2)}}{k^2(k+1)^2}=\tfrac{55}{6}\,\zeta(6)+\zeta(3)^2-10\,\zeta(5). \tag{119}$$

$$\sum_{k=1}^{\infty}\frac{\{H_k^{(2)}\}^2}{k^2(k+1)^2}=\tfrac{13}{4}\zeta(6)-12\,\zeta(2)\zeta(3)+20\,\zeta(5). \tag{120}$$

Proof. Formulas (93)–(95), (97) and (99) can be proven through the application of partial fractions to their respective summands:

$$\frac{1}{k^2(k+1)^2}=\frac{1}{k^2}+\frac{1}{(k+1)^2}+\frac{2}{k+1}-\frac{2}{k}, \tag{121}$$

$$\frac{1}{k^3(k+1)^2}=\frac{1}{k^3}-\frac{2}{k^2}-\frac{1}{(k+1)^2}+\frac{3}{k}-\frac{3}{k+1}, \tag{122}$$

$$\frac{1}{k^4(k+1)^2}=\frac{1}{k^4}-\frac{2}{k^3}+\frac{3}{k^2}+\frac{1}{(k+1)^2}-\frac{4}{k}+\frac{4}{k+1}, \tag{123}$$

$$\frac{1}{k^5(k+1)^2}=\frac{1}{k^5}-\frac{2}{k^4}+\frac{3}{k^3}-\frac{4}{k^2}-\frac{1}{(k+1)^2}+\frac{5}{k}-\frac{5}{k+1}, \tag{124}$$

and

$$\frac{1}{k^4(k+1)}=\frac{1}{k^4}-\frac{1}{k^3}+\frac{1}{k^2}-\frac{1}{k}+\frac{1}{k+1}. \tag{125}$$

Using Equation (122) to rewrite Formula (105), we derive

$$\sum_{k=1}^{\infty}\frac{H_k^2}{k^3(k+1)^2}=\mathbb{S}_{1^2;3}-3\mathbb{S}_{1^2;2}+2\mathbf{S}_{1,3}-\zeta(4)+3\sum_{k=1}^{\infty}\frac{H_k^2}{k(k+1)},$$

where all the sums on the right-hand side, along with (84), are previously established.

Similarly, the other formulas can be verified, though the specific details are not included here. □

Remark 5. *Formulas (102), (104) and (110) can be obtained through Mathematica.*

Similarly, we can evaluate the following general sum,

$$\sum_{k=1}^{\infty}\frac{H_k^p\,H_k^{(q)}}{k^r(k+1)^s}\quad (p,\,q\in\mathbb{Z}_{\geqslant 0};\,r,\,s\in\mathbb{N}), \tag{126}$$

in terms of a finite combination of Riemann zeta functions, provided that the involved linear and nonlinear Euler sums of weight $p+q+j$, where $j\in\mathbb{N}$ and $j\leqslant\max\{r,s\}$, are already known, along with the known evaluation of the general sum in (92).

In this context, the evaluations of the general sums given in (83), (87) and (92) are essential for evaluating the general sum presented in (126), such as those identities in Corollary 1.

5. Particular Cases and Remarks

This section delves into specific instances of the main findings outlined in Section 3, supplemented by pertinent observations, as needed.

We begin with the following corollary.

Corollary 2. *The following identities hold: For* $-1 \leqslant r \leqslant 1$,

$$\begin{aligned}\frac{1}{5\pi}\int_0^{2\pi} \log^5|1+re^{it}|\,\mathrm{d}t &= 3\,\mathrm{Li}_5\left(r^2\right)\\ &\quad -9\sum_{k=1}^{\infty}\frac{H_k}{k^4}r^{2k}+6\sum_{k=1}^{\infty}\frac{H_k^2}{k^3}r^{2k}-2\sum_{k=1}^{\infty}\frac{H_k^3}{k^2}r^{2k}\\ &\quad -3\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3}r^{2k}-\sum_{k=1}^{\infty}\frac{H_k^{(3)}}{k^2}r^{2k}+3\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^2}r^{2k},\end{aligned} \tag{127}$$

and

$$\begin{aligned}\frac{1}{5\pi}\int_0^{2\pi} \log^5|1\pm e^{it}|\,\mathrm{d}t &= \frac{1}{160\pi}\int_0^{2\pi} \log^5\left\{2(1\pm\cos t)\right\}\mathrm{d}t\\ &= -3\,\zeta(2)\zeta(3)+\frac{39}{2}\,\zeta(5).\end{aligned} \tag{128}$$

Proof. Setting $n=0$ in the result in Theorem 1 yields (127). Further putting $r=\pm 1$ in (127) gives (128). Here, for the six linear and nonlinear Euler sums of weight 5, refer to the subsequent remark. □

Remark 6. *It is intriguing to note that when* $r=\pm 1$ *in (127), the resulting right-hand side encompasses all possible six linear and nonlinear Euler sums of weight 5. These sums have already been assessed as follows:*

$$\begin{aligned}&\mathbf{S}_{1,4}=3\zeta(5)-\zeta(2)\zeta(3); &&\mathbf{S}_{2,3}=3\zeta(2)\zeta(3)-\tfrac{9}{2}\zeta(5);\\ &\mathbf{S}_{3,2}=\tfrac{11}{2}\zeta(5)-2\zeta(2)\zeta(3); &&\mathbb{S}_{1^2;3}=\tfrac{7}{2}\zeta(5)-\zeta(2)\zeta(3);\\ &\mathbb{S}_{1^3;2}=\zeta(2)\zeta(3)+10\,\zeta(5); &&\mathbb{S}_{1,2;2}=\zeta(2)\zeta(3)+\zeta(5).\end{aligned}$$

Corollary 3. *Let* $r\in\mathbb{R}$ *with* $|r|\leqslant 1$. *Then,*

$$\begin{aligned}&\frac{16}{\pi}\int_0^{2\pi} \log^6|1+re^{it}|\,\mathrm{d}t\\ &= 1800\,\mathrm{Li}_6\left(r^2\right)-2880\sum_{k=1}^{\infty}\frac{H_k}{k^5}\,r^{2k}+2160\sum_{k=1}^{\infty}\frac{H_k^2}{k^4}\,r^{2k}\\ &\quad -960\sum_{k=1}^{\infty}\frac{H_k^3}{k^3}\,r^{2k}+240\sum_{k=1}^{\infty}\frac{H_k^4}{k^2}\,r^{2k}-1080\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4}\,r^{2k}\\ &\quad +1440\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^3}\,r^{2k}-720\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(2)}}{k^2}\,r^{2k}\\ &\quad +180\sum_{k=1}^{\infty}\frac{\left\{H_k^{(2)}\right\}^2}{k^2}\,r^{2k}-480\sum_{k=1}^{\infty}\frac{H_k^{(3)}}{k^3}\,r^{2k}\\ &\quad +480\sum_{k=1}^{\infty}\frac{H_kH_k^{(3)}}{k^2}\,r^{2k}-180\sum_{k=1}^{\infty}\frac{H_k^{(4)}}{k^2}\,r^{2k}\end{aligned} \tag{129}$$

and

$$\begin{aligned} \frac{16}{\pi}\int_0^{2\pi}\log^6|1\pm e^{it}|\,\mathrm{d}t &= \frac{1}{4\pi}\int_0^{2\pi}\log^6\{2(1\pm\cos t)\}\,\mathrm{d}t \\ &= \frac{275\,\pi^6}{42}+720\,\zeta(3)^2. \end{aligned} \tag{130}$$

Proof. When we substitute $n=0$ into the outcome stated in Theorem 2, we obtain Equation (129). Moreover, if we substitute $r=\pm1$ into Equation (129), we arrive at Equation (130). □

Corollary 4. *The following integral formula holds:*

$$\begin{aligned} &\int_0^{2\pi}|1\pm e^{it}|^2\log^5|1\pm e^{it}|\,\mathrm{d}t \\ &= \frac{1}{16}\int_0^{2\pi}(1\pm\cos t)\,\log^5\{2(1\pm\cos t)\}\,\mathrm{d}t \\ &= 225\pi-90\gamma\pi+90\gamma^2\pi-30\gamma^3\pi-\frac{125\pi^3}{6} \\ &\quad+10\gamma\pi^3-10\gamma^2\pi^3+\frac{10\gamma^3\pi^3}{3}-\frac{29\pi^5}{24}+\frac{2\pi^7}{9} \\ &\quad+30\pi\zeta(3)-5\pi^3\zeta(3)-90\pi\zeta(5). \end{aligned} \tag{131}$$

Proof. By substituting $n=1$ and $r=\pm1$ into Equation (62), and utilizing the identities provided in Section 4, we can readily obtain the result presented here. □

Corollary 5. *The following integral formula holds:*

$$\begin{aligned} &\int_0^{2\pi}|1\pm e^{it}|^2\log^6|1\pm e^{it}|\,\mathrm{d}t \\ &= \frac{1}{32}\int_0^{2\pi}(1\pm\cos t)\,\log^6\{2(1\pm\cos t)\}\,\mathrm{d}t \\ &= -\frac{2595\pi}{2}+1620\gamma\pi-1035\gamma^2\pi+280\gamma^3\pi-30\gamma^4\pi \\ &\quad+\frac{195\,\pi^3}{4}-80\gamma\pi^3+75\gamma^2\pi^3-40\gamma^3\pi^3+10\gamma^4\pi^3-\frac{3\pi^5}{2} \\ &\quad-\frac{11\gamma\pi^5}{2}+\gamma^2\pi^5+\frac{103\,\pi^7}{112}+20\pi\zeta(3)-420\gamma\pi\zeta(3) \\ &\quad+360\gamma^2\pi\zeta(3)-25\,\pi^3\zeta(3)+20\gamma\pi^3\zeta(3)+50\pi\zeta(3)^2 \\ &\quad+270\pi\zeta(5). \end{aligned} \tag{132}$$

Proof. By substituting $n=1$ and $r=\pm1$ in Equation (63), and using the identities provided in Section 4, we can readily derive the result presented here. □

Remark 7. *It is interesting to observe that when $r=\pm1$ in Equation (129), the resulting right-hand side includes all possible eleven linear and nonlinear Euler sums of weight 6. These sums have already been evaluated as follows:*

$$\begin{aligned} &\mathbf{S}_{1,5}=\tfrac{7}{4}\zeta(6)-\tfrac{1}{2}\zeta^2(3); && \mathbf{S}_{2,4}=-\tfrac{1}{3}\zeta(6)+\zeta^2(3); \\ &\mathbf{S}_{3,3}=\tfrac{1}{2}\zeta(6)+\tfrac{1}{2}\zeta^2(3); && \mathbf{S}_{4,2}=\tfrac{37}{12}\zeta(6)-\zeta^2(3); \\ &\mathbb{S}_{1^2;4}=\tfrac{97}{24}\zeta(6)-2\zeta^2(3); && \mathbb{S}_{1^3;3}=\tfrac{93}{16}\zeta(6)-\tfrac{5}{2}\zeta^2(3); \\ &\mathbb{S}_{1^4;2}=\tfrac{979}{24}\zeta(6)+3\zeta^2(3); && \mathbb{S}_{1,2;3}=-\tfrac{101}{48}\zeta(6)+\tfrac{5}{2}\zeta^2(3); \\ &\mathbb{S}_{1,3;2}=\tfrac{227}{48}\zeta(6)-\tfrac{3}{2}\zeta^2(3); && \mathbb{S}_{1^2,2;2}=\tfrac{41}{12}\zeta(6)+2\zeta^2(3); \\ &\mathbb{S}_{2^2;2}=\tfrac{19}{24}\zeta(6)+\zeta^2(3). \end{aligned}$$

6. Concluding Remarks

In this investigation, we conducted an in-depth analysis of specific integrals discussed in the Abstract. Our systematic approach, firmly rooted in algorithmic principles, allows for a natural extension of our conclusions to cases where $n \in \mathbb{Z}_{\geqslant 0}$, $m \in \mathbb{N}$ with $m \geqslant 7$, and $-1 \leqslant r \leqslant 1$. Additionally, we elucidated specific instances derived from our primary findings, thereby broadening the applicability and significance of our results for a wider range of researchers.

Remarks 6 and 7 highlight the close relationship between the integrals considered here and Euler sums. Notably, it is serendipitous that the specific integrals in Corollaries 2 and 3 encompass all possible linear and nonlinear Euler sums of weights 5 and 6, respectively.

It is noted that some of the involved computations, particularly those in Equations (65), (69) and (70), are nearly impossible without the assistance of Mathematica.

It is noted that interested researchers, including the authors, are hoped to develop a Mathematica symbolic computation package for evaluating the integrals discussed in this paper in a future publication, utilizing the algorithmic method presented here.

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