symmetry

MDPI

Article

Oscillatory Properties of Second-Order Differential Equations with Advanced Arguments in the Noncanonical Case

Zuhur Alqahtani [1], Belgees Qaraad [2,*], Areej Almuneef [1] and Faizah Alharbi [3,*]

[1] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; zumalqahtani@pnu.edu.sa (Z.A.); aaalmuneef@pnu.edu.sa (A.A.)
[2] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[3] Mathematics Department, Faculty of Sciences, Umm Al-Quraa University, Makkah 24227, Saudi Arabia
* Correspondence: belgeesmath2016@gmail.com (B.Q.); fmharbi@uqu.edu.sa (F.A.)

Abstract: This paper focuses on studying certain oscillatory properties of a new class of half-linear second-order differential equations with an advanced argument in a non-canonical case. By employing some new relations between the solution and its higher derivatives and taking into account the symmetry of positive and negative solutions, we have introduced new criteria to test whether all solutions of the studied equation exhibit oscillatory behavior. Our study aims to expand and enhance previous results, helping to understand these properties in the specified context. The results obtained are confirmed and clarified through an example involving Euler-type equations.

Keywords: third-order differential equation; oscillation; nonoscillation; delay

check for updates

Citation: Alqahtani, Z.; Qaraad, B.; Almuneef, A.; Alharbi, F. Oscillatory Properties of Second-Order Differential Equations with Advanced Arguments in the Noncanonical Case. *Symmetry* **2024**, *16*, 1018. https://doi.org/10.3390/sym16081018

Academic Editor: Chuan Li

Received: 15 June 2024
Revised: 28 July 2024
Accepted: 31 July 2024
Published: 9 August 2024

1. Introduction

This study investigates the oscillatory and asymptotic behavior of solutions to second-order advanced differential equations in the following form:

$$\left(r\left(v'\right)^{\alpha}\right)'(\top)+\sum_{i=1}^{k}\Phi_i(\top)v^{\alpha}(\varsigma_i(\top))=0,\quad \top\geq\top_0, \tag{1}$$

where $\alpha>0$ represents the ratio of positive odd integers and $k\geq 1$. We will assume that $r,\varsigma_i\in C^1([\top_0,\infty),(0,\infty))$, $\Phi_i\in C([\top_0,\infty),(0,\infty))$, $\varsigma_i(\top)\geq\top,\varsigma'(\top)\geq 0$, $\varsigma(\top)\geq\varsigma_i(\top)$, Φ_i does not vanish identically, and

$$\int_{\top_0}^{\top} r^{\frac{-1}{\alpha}}(s)\mathrm{d}s<\infty. \tag{2}$$

We restrict our attention to a nontrivial real-valued function $v\in C([t_x,\infty),\mathbb{R})$ of (1) that satisfies (1) on $[t_x,\infty)$ and has the property $r(v')^{\alpha}\in C^1([t_x,\infty),(0,\mathbb{R}))$. We consider only those solutions of (1) that exist on $[t_x,\infty)$ and satisfy

$$\sup\{|v(\top)|:t_v\leq\top<\infty\}>0,\text{ for any }t_v\geq t_x.$$

Symmetry plays a vital role in solving differential equations, as many advanced differential equations have inherent symmetries that can be exploited to find possible solutions. One of the most important of these types is second-order advanced differential equations, which play a significant role in various biological contexts, providing a precise understanding of systems that are inherently dynamic and complex. For example, they are used in most applications related to biology, such as neuroscience, where the Hodgkin–Huxley model, which describes how action potentials in nerve cells start and spread, relies on a system of differential equations. These equations explain the complex dynamics of the voltage across the neuronal cell membrane in response to stimulation [1–5]. Moreover, advanced second-order differential equations provide models in epidemiology, like the SIR model and its derivatives (such as SEIR and SIS), which use differential equations to understand the spread of infectious diseases and simulate how infection and recovery rates affect the disease dynamics within a community. Additionally, many physiological processes, such as cardiac dynamics, calcium signaling in cells, and hormonal regulation, can be designed using second-order differential equations [4,6–13]. These models help understand how changes in one part of the system affect another part and predict system behavior under various scenarios. Second-order differential equations represent the forces, movements, and mechanical behaviors of biological systems, ranging from limb movement to muscle contraction dynamics and blood flow.

There are some examples of biological models that involve second-order differential equations such as the FitzHugh–Nagumo model of neuronal dynamics, which is a simplification of the Hodgkin–Huxley model that describes the electrical properties of the nerve cell. The equations include a second-order differential term when considering the spatial distribution of electrical potentials

$$\frac{\partial^2 V}{\partial x^2} = C\frac{\partial V}{\partial \top^2} + f(V) - W + I,$$

where V is the membrane potential, W represents recovery variables, I is the current, and C is a constant.

Mechanical models of muscle contraction are also among the most important models that include this type of equation

$$m\frac{d^2 x}{d\top^2} + b\frac{dx}{d\top} + kx = F(\top),$$

where x is the muscle length, m is the mass, b is the damping coefficient, k is the spring constant, and $F(\top)$ is the time-dependent external force.

On the other hand, models such as predator–prey equations can be expanded to include second-order systems to include effects such as inertia in population changes or delayed responses

$$\frac{d^2 x}{d\top^2} = x(a - bx - cv) - k\frac{dx}{d\top},$$

where x and v represent the populations of prey and predators, respectively; a , b , and c are constants representing interaction terms; and k is a damping term related to time delay or inertia.

Studies on equations with advanced arguments are limited compared to those dedicated to the study of delayed equations. Despite the existence of a number of studies on equations and special cases or generalizations of Equation (1), most of them highlight cases when

$$\int_{\top_0}^{\top} r^{\frac{-1}{\alpha}}(s)\mathrm{d}s = \infty.$$

Moreover, not taking the effect of the advanced intermediary into account in the special cases of the studied equation has been a common approach in many of the results available to date, such as [14–17].

Recently, some studies have been devoted to finding improved criteria for studying the asymptotic and oscillation behavior of solutions of Equation (1) and its special cases. Jadlovska [18] and Baculíková [19] have used iterative methods to establish criteria that ensure the oscillation of solutions to the equation

$$v''(\top) + \Phi(\top)v(\varsigma(\top)) = 0.$$

On the other hand, Džurina [20] studied the equation

$$\left(r(\top)v'(\top)\right)' + \Phi(\top)v(\varsigma(\top)) = 0,$$

and employed some nonoscillatory monotonic properties to reach oscillation criteria for solutions by using the comparison principle, which depends on a reduced equation of the first order.

In [21], the author was able to provide some relationships that later enabled him to reach the oscillation criteria of the differential equation

$$\left(r(\top)\left|v'(\top)\right|^{\alpha-1}v'(\top)\right)' + \Phi(\top)|v|^{\alpha-1}(\varsigma(\top))v(\varsigma(\top)) = 0,$$

in canonical form and where $\varsigma(\top) \geq \top$.

Most of the available results concerning Equation (1) are under the condition

$$\varsigma(\top) \leq \top,$$

or in canonical form; see, for example, [22–24]. It is well known that there is a clear distinction in classifying nonoscillatory solutions between canonical and noncanonical forms. In canonical form, it is guaranteed that the solution will eventually be increasing, and the first derivative of the solution (e.g., positive) eventually has one sign. In contrast, in the case of canonical equations, we face possibilities for the first derivative of the solution (e.g., positive) to vary. Therefore, dealing with this type of equation becomes more complex. In this paper, we aim to present some recent results that contribute to the development of the oscillation theory for studied equations in noncanonical form.

We provide criteria different from those found in the previous literature, thanks to the new relationships and distinct characteristics we have established. Most available studies rely on two or more conditions to ensure the oscillation of solutions of (1) or its special cases. For example, in reference [25], the authors relied on the condition

$$\int_{\top_0}^{\infty}\left(r^{-1}(\top)\int_{\top_0}^{\top}\pi^{\alpha}(\varsigma(s))\sum_{i=1}^{k}\Phi_i(s)\mathrm{d}s\right)^{1l\alpha}\mathrm{d}\top = \infty, \tag{3}$$

to exclude non-oscillatory decreasing solutions (positive) and added another condition to exclude non-oscillatory increasing solutions (positive). In contrast, we rely on the condition (3) alone to exclude both increasing and decreasing solutions. See Theorem 2. Our diverse conditions, with their varied proof techniques, serve as an effective means to adapt to the different models they are applied to. See Theorems 1–10. Therefore, our results extend the findings presented in some of the literature such as [14–19,26]. Additionally, they improve and simplify some of the results that studied advanced equations in the noncanonical case mentioned in [25,27].

This paper is structured as follows: In Section 1, we present the equation under study, along with the general conditions required to achieve the main results of the paper. We also provide an overview of related topics and the motivation behind this study. In Section 2, we introduce some relationships that will be used to derive the oscillation results discussed in the following section. In Section 3, we present various theorems to guarantee the oscillation of the solutions of (1), which can be applied in various forms and with different features. In Section 4, we present examples to corroborate and clarify our results. In the final Section 5, we present a summary of the main findings we have reached. We then pose an open-ended question that could extend the ideas presented in this paper and serve as an inspiration for researchers in the same field.

2. Preliminaries

For the sake of brevity, we will define the following:

$$\Omega_{(a,b)}(\top) = \int_a^b \sum_{i=1}^{k} \Phi_i(s)\mathrm{d}s, \ \ \pi(\top) := \int_\top^\infty r^{-1/\alpha}(s)\mathrm{d}s,$$

and

$$\widetilde{\pi}(\top) := \pi(\top) + \alpha^{-1} \int_\top^\infty \sum_{i=1}^{k} \Phi_i(s)\pi(s)\pi^\alpha(\varsigma(s))\mathrm{d}s.$$

In the following two lemmas, we present some relationships that will play an effective role in proving the main results.

Lemma 1 ([28] Lemma 2.3)**.** *Let*

$$g(h) = S_1 h - S_2(h - S_3)^{(\alpha+1)/\alpha},$$

where $S_1 > 0$, S_2 and S_3 are constants. Then,

$$h^* = S_3 + \left(\frac{\alpha S_1}{(\alpha+1)S_2}\right)^\alpha$$

is the function in which g attains its maximum value on $\mathbb{R}$

$$\max_{h\in\mathbb{R}} g(h) = g(h^*) = S_1 S_3 + \left(\frac{S_1}{\alpha+1}\right)^{\alpha+1}\left(\frac{\alpha}{S_2}\right)^\alpha. \tag{4}$$

Lemma 2. *Suppose that (1) has a solution $v > 0$ on $[\top_1, \infty)$, $\top_1 \in [\top_0, \infty)$, (2) is satisfied and*

$$\Omega_{(\top_0,\infty)}(\top) = \infty. \tag{5}$$

Then,

$$v \text{ is increasing, and } r\left(v'\right)^\alpha \text{ is nonincreasing, } \ \text{on } [\top_1, \infty). \tag{6}$$

Furthermore,

$$v/\pi \text{ is nondecreasing on } [\top_1, \infty). \tag{7}$$

Proof. Let $v > 0$ be a solution of (1) on $[\top_1, \infty)$. From (1), we see that

$$\left(r\left(v'\right)^\alpha\right)'(\top) + \sum_{i=1}^{k} \Phi_i(\top)v^\alpha(\varsigma(\top)) \leq 0.$$

Consequently, either $v' < 0$ or $v' > 0$. In contrast, suppose that (5) holds and there exists $\top_2 \geq \top_1$ such that $v'(\top)$ is positive on $[\top_2, \infty)$. Setting

$$w(\top) := v^{-\alpha}(\varsigma(\top))r(\top)\left(v'(\top)\right)^\alpha > 0. \tag{8}$$

That is,

$$w'(\top) := -\sum_{i=1}^{k} \Phi_i(\top) - w(\top)\varsigma'(\top)\frac{v'(\varsigma(\top))}{v(\varsigma(\top))} \leq -\sum_{i=1}^{k} \Phi_i(\top). \tag{9}$$

Integrating (9), we obtain

$$w(\top) \leq w(\top_2) - \Omega_{(\top_2,\top)}(\top). \tag{10}$$

By (5), it is clear that (10) leads to a contradiction with the sign of function $w(\top)$. Thus, the case $v'(\top) > 0$ cannot be satisfied; hence, v satisfies (6) for $\top \geq \top_1$. Using the fact that $r(v')^\alpha$ is nonincreasing, we obtain

$$\begin{aligned} v(\top) + \int_\top^\infty r^{-1/\alpha}(s) r^{1/\alpha}(s) v'(s) \mathrm{d}s &\geq 0 \\ v(\top) + r^{1/\alpha}(\top) v'(\top) \int_\top^\infty r^{-1/\alpha}(s) \mathrm{d}s &\geq 0. \end{aligned}$$

Then,

$$v(\top) + r^{1/\alpha}(\top) v'(\top) \pi(\top) \geq 0. \tag{11}$$

In view of (11), we have

$$\left(\frac{v}{\pi}\right)'(\top) = \frac{r^{1/\alpha}(\top) v'(\top) + v(\top)}{r^{1/\alpha}(\top) \pi^2(\top)} \geq 0. \tag{12}$$

The proof is complete. □

3. The Main Results

Theorem 1. *Suppose that (1) has a solution $v > 0$ on $[\top_1, \infty)$, $\top_1 \in [\top_0, \infty)$, (2) is satisfied, and*

$$\int_{\top_0}^\infty \left(r^{-1}(\top) \Omega_{(\top_0, \top)}(\top)\right)^{1/\alpha} \mathrm{d}\top = \infty. \tag{13}$$

Then (6) holds for $\top \geq \top_1$ and

$$\lim_{\top \to \infty} v(\top) = 0. \tag{14}$$

Furthermore, there $\exists$ a $\top^* \geq \top_1$ such that

$$v(\top) \begin{cases} \geq M_1 \pi(\top); \\ \leq M_2 e^{\left(-\int_{\top_0}^{\top} \pi(\varsigma(s)) \pi^{-1}(s) r^{-1/\alpha}(s) \left(\Omega_{(\top_0, \top)}(\top)\right)^{1/\alpha} \mathrm{d}s\right)} \end{cases}, M_1, M_2 > 0, \tag{15}$$

$\top \geq \top^*$.

Proof. Let $v > 0$ be a solution of (1) on $[\top_1, \infty)$. According to (13) and (2), it is easy to see that $\Omega_{(\top_0, \top)}(\top)$ is unbounded; that is, (5) satisfied. From Lemma 1, we note that v (6) holds for $\top \geq \top_1$. Since $v' < 0$, there $\exists$ $c \in [0, \infty)$ such that

$$\lim_{\top \to \infty} v(\top) = c.$$

Let us assume that $c > 0$, we obtain

$$\begin{aligned} -\left(r(v')^\alpha\right)'(\top) &= \sum_{i=1}^k \Phi_i(\top) v^\alpha(\varsigma_i(\top)) \\ &\geq c^\alpha \sum_{i=1}^k \Phi_i(\top) \text{ for } \top \geq \top_2,\ \top_2 \in [\top_1, \infty). \end{aligned} \tag{16}$$

Integrating (16) from $\top_2$ to $\top$ leads to

$$-r(\top)\left(v'(\top)\right)^\alpha + r(\top_2)\left(v'(\top_2)\right)^\alpha + c^\alpha \Omega_{(\top_2, \top)}(\top) \geq 0,$$

and

$$v(\top) + c\left(\frac{1}{r(\top)} \Omega_{(\top_2, \top)}(\top)\right)^{1/\alpha} \leq 0. \tag{17}$$

Integrating (17) from $\top_2$ to $\top$, we conclude that

$$v(\top) + v(\top_2) \leq -c \int_{\top_2}^{\top} \left(\frac{1}{r(u)} \Omega_{(\top_2,\top)}(\top) \right)^{1/\alpha} du.$$

Taking (13) into account, we obtain a contradiction, since $v(\top) \to -\infty$ as $\top \to \infty$. Hence, $c = 0$.

Now, from (7), we see that there is a positive constant c_1 such that

$$v(\top) \geq c_1 \pi(\top) \text{ for } \top \geq \top_3,\ \top_3 \geq \top_2.$$

Integrating (1), we obtain

$$\begin{aligned} -r(\top)\left(v'(\top)\right)^{\alpha} &\geq -r(\top_3)\left(v'(\top_3)\right)^{\alpha} + \int_{\top_3}^{\top} \sum_{i=1}^{k} \Phi_i(s) v^{\alpha}(\varsigma_i(s)) ds \\ &\geq -r(\top_3)\left(v'(\top_3)\right)^{\alpha} + v^{\alpha}(\varsigma(\top))\Omega_{(\top_3,\top)}(\top) \\ &\geq -r(\top_3)\left(v'(\top_3)\right)^{\alpha} + v^{\alpha}(\varsigma(\top))\Omega_{(\top_0,\top)}(\top) \\ &\quad -v^{\alpha}(\varsigma(\top))\Omega_{(\top_0,\top_3)}(\top). \end{aligned}$$

According to $\lim_{\top \to \infty} v(\top) = 0$, there is a $\top_4 \geq \top_3$ such that

$$-r(\top_3)\left(v'(\top_3)\right)^{\alpha} > v^{\alpha}(\varsigma(\top))\Omega_{(\top_0,\top_3)}(\top) \text{ for } \top \geq \top_4.$$

That is

$$-r(\top)\left(v'(\top)\right)^{\alpha} \geq v^{\alpha}(\varsigma(\top))\Omega_{(\top_0,\top)}(\top) \text{ for } \top \geq \top_4.$$

In view of (7), we obtain

$$\begin{aligned} -r(\top)\left(v'(\top)\right)^{\alpha} &\geq \frac{v^{\alpha}(\varsigma(\top))\left(\pi^{\alpha}(\varsigma(\top))\Omega_{(\top_0,\top)}(\top)\right)}{\pi^{\alpha}(\varsigma(\top))} \\ &\geq \frac{v^{\alpha}(\top)\left(\pi^{\alpha}(\varsigma(\top))\Omega_{(\top_0,\top)}(\top)\right)}{\pi^{\alpha}(\top)}. \end{aligned}$$

Thus,

$$\frac{v'(\top)}{v(\top)} \leq -\frac{\pi(\varsigma(\top))\Omega^{1/\alpha}_{(\top_0,\top)}(\top)}{\pi(\top) r^{1/\alpha}(\top)}. \tag{18}$$

Integrating above inequality from $\top_4$ to $\top$, we have

$$\begin{aligned} v(\top) &\leq v(\top_4) e^{\left(-\int_{\top_4}^{\top} \frac{\pi(\varsigma(u))}{\pi(u) r^{1/\alpha}(u)} \Omega^{1/\alpha}_{(\top_0,u)}(\top) du\right)} \\ &= M_2 e^{\left(-\int_{\top_0}^{\top} \frac{\pi(\varsigma(u))}{\pi(u) r^{1/\alpha}(u)} \Omega^{1/\alpha}_{(\top_0,u)}(\top) du\right)}, \end{aligned}$$

where

$$M_2 := v(\top_4) e^{\left(-\int_{\top_0}^{\top_4} \frac{\pi(\varsigma(u))}{\pi(u) r^{1/\alpha}(u)} \Omega^{1/\alpha}_{(\top_0,u)}(\top) du\right)} > 0.$$

The proof is complete. □

Theorem 2. *Suppose that (2) is satisfied. If*

$$\int_{\top_0}^{\infty} \left(r^{-1}(\top) \int_{\top_0}^{\top} \pi^{\alpha}(\varsigma(s)) \sum_{i=1}^{k} \Phi_i(s) ds \right)^{1l\alpha} d\top = \infty, \tag{19}$$

then (1) is oscillatory.

Proof. Let $v(\top) > 0$ be a solution of (1) on $[\top_0, \infty)$. Then, $v(\varsigma(\top)) > 0$ for $\top \in [\top_1, \infty)$. We see that (19) require (5) to be valid. In fact, since $\int_{\top_0}^{\top} \pi^{\alpha}(\varsigma(s)) \sum_{i=1}^{k} \Phi_i(s) \mathrm{d}s$ is an unbounded function due to the facts $\pi' < 0$ and (2), (5) must be satisfied. Therefore, by Lemma 2, (6) holds for $\top \geq \top_1$. From (7), we note that there is a positive constant c such that

$$\frac{v(\top)}{\pi(\top)} \geq c \text{ for } \top \geq \top_2,\ \top_2 \geq \top_1. \tag{20}$$

Using (20) in (1), we obtain

$$-\left(r(v')^{\alpha}\right)'(\top) - c^{\alpha} \sum_{i=1}^{k} \Phi_i(\top)\pi^{\alpha}(\varsigma(\top)) \geq 0. \tag{21}$$

Integrating (21) from $\top_2$ to $\top$, we have

$$-r(\top)\left(v'(\top)\right)^{\alpha} - c^{\alpha} \int_{\top_2}^{\top} \sum_{i=1}^{k} \Phi_i(s)\pi^{\alpha}(\varsigma(s))\mathrm{d}s \geq 0,$$

or

$$-v'(\top) \geq \frac{c}{r^{1l\alpha}(\top)} \left(\int_{\top_2}^{\top} \sum_{i=1}^{k} \Phi_i(s)\pi^{\alpha}(\varsigma(s))\mathrm{d}s \right)^{1l\alpha}. \tag{22}$$

Integrating (22) from $\top_2$ to $\top$ and using (19), we get

$$v(\top_2) \leq v(\top) - \int_{\top_2}^{\top} cr^{-1l\alpha}(u) \left(\int_{\top_2}^{\top} \sum_{i=1}^{k} \Phi_i(s)\pi^{\alpha}(\varsigma(s))\mathrm{d}s \right)^{1l\alpha} \mathrm{d}u \to -\infty.$$

This contradiction completes the proof. □

Remark 1. *In reference [25], the author studied Equation (1) as a special case on the field of time scales. They used the condition (19) to exclude positive, decreasing solutions. Consequently, in their theorems, they had to use more complex additional conditions to exclude potential positive increasing solutions in order to obtain conditions that guarantee the oscillation of the solutions of Equation (1).*

Theorem 3. *Suppose that (2) is satisfied. If*

$$\limsup_{\top \to \infty} \pi^{\alpha}(\varsigma(\top))\Omega_{(\top_1,\top)}(\top) > 1,\ \forall \top_1 \geq \top_0, \tag{23}$$

then (1) is oscillatory.

Proof. Let $v(\top) > 0$ be a solution of (1) on $[\top_0, \infty)$. Then $v(\varsigma(\top)) > 0$ for $\top \in [\top_1, \infty)$. We see that (5) requires (23) and (2). By Lemma 2, we find that (6) holds for $\top \geq \top_1$. Integrating (1) from $\top_1$ to $\top$ and using the property $v' < 0$, we have

$$-r(\top)\left(v'(\top)\right)^{\alpha} + r(\top_1)\left(v'(\top_1)\right)^{\alpha} = \int_{\top_1}^{\top} \sum_{i=1}^{k} \Phi_i(s)v^{\alpha}(\varsigma(s))\mathrm{d}s,$$

that is,

$$-r(\top)\big(v'(\top)\big)^{\alpha} \geq v^{\alpha}(\varsigma(\top))\Omega_{(\top_1,\top)}(\top). \tag{24}$$

In the same way as Lemma 2, we obtain (11), which, along with (24), implies that

$$\begin{aligned} -r(\top)\big(v'(\top)\big)^{\alpha} &\geq -r(\varsigma(\top))\big(v'(\varsigma(\top))\big)^{\alpha}\pi^{\alpha}(\varsigma(\top))\Omega_{(\top_1,\top)}(\top) \\ &\geq -r(\top)\big(v'(\top)\big)^{\alpha}\pi^{\alpha}(\varsigma(\top))\Omega_{(\top_1,\top)}(\top). \end{aligned}$$

This leads to a contradiction:

$$\limsup_{\top\to\infty} \pi^{\alpha}(\varsigma(\top))\Omega_{(\top_1,\top)}(\top) \leq 1.$$

The proof is complete. □

Theorem 4. *Suppose that (2) and (13) are satisfied. If*

$$\limsup_{\top\to\infty} \pi^{\alpha}(\varsigma(\top))\Omega_{(\top_0,\top)}(\top) > 1, \tag{25}$$

then (1) is oscillatory.

Proof. In the same way as Theorem 3, we obtain (24). According to the fact that $\lim_{\top\to\infty} v(\top) = 0$, there is a $\top_2 > \top_1$ such that

$$-r(\top)\big(v'(\top)\big)^{\alpha} > v^{\alpha}(\varsigma(\top))\Omega_{(\top_0,\top_1)}(\top).$$

Thus,

$$\begin{aligned} -r(\top)\big(v'(\top)\big)^{\alpha} &\geq -r(\top_1)\big(v'(\top_1)\big)^{\alpha} + v^{\alpha}(\varsigma(\top))\Omega_{(\top_0,\top)}(\top) \\ &\quad - v^{\alpha}(\varsigma(\top))\Omega_{(\top_0,\top_1)}(\top) \\ &\geq v^{\alpha}(\varsigma(\top))\Omega_{(\top_0,\top)}(\top), \text{ for } \top \geq \top_2. \end{aligned} \tag{26}$$

In the same way as Lemma 2, we obtain (11), which, along with (24), implies that

$$\begin{aligned} -r(\top)\big(v'(\top)\big)^{\alpha} &\geq -r(\varsigma(\top))\big(v'(\varsigma(\top))\big)^{\alpha}\pi^{\alpha}(\varsigma(\top))\Omega_{(\top_0,\top)}(\top) \\ &\geq -r(\top)\big(v'(\top)\big)^{\alpha}\pi^{\alpha}(\varsigma(\top))\Omega_{(\top_0,\top)}(\top). \end{aligned}$$

This leads to a contradiction:

$$\limsup_{\top\to\infty} \pi^{\alpha}(\varsigma(\top))\Omega_{(\top_0,\top)}(\top) \leq 1.$$

The proof is complete. □

Theorem 5. *Suppose that (2) is satisfied. If*

$$\liminf_{\top\to\infty} \int_{\top}^{\varsigma(\top)} \sum_{i=1}^{k} \Phi_i(s)\pi^{\alpha}(\varsigma(s))\mathrm{d}s > \frac{1}{e}, \tag{27}$$

then (1) is oscillatory.

Proof. Let $v(\top) > 0$ be a solution of (1) on $[\top_0, \infty)$. Then $v(\varsigma(\top)) > 0$ for $\top \in [\top_1, \infty)$. First, we find that (27) and (2) imply (5). Note that (27) implies

$$\int_{\top_0}^{\infty} \sum_{i=1}^{k} \Phi_i(s)\pi^{\alpha}(\varsigma(s))\mathrm{d}s = \infty. \tag{28}$$

Since $\pi' < 0$ and by (28), we see (5) holds. By Lemma 2, (6) is satisfied. From (1) and (11), the first-order advanced differential inequality

$$x'(\top) - g_1(\top)x(\varsigma(\top)) \geq 0, \tag{29}$$

where

$$g_1(\top) = \sum_{i=1}^{k} \Phi_i(\top)\pi^{\alpha}(\varsigma(\top)),$$

has a positive solution x, where $x := -r(v')^{\alpha}$. But, in Theorem 2.4.1 of [29], we note that condition

$$\liminf_{\top\to\infty} \int_{\top}^{\varsigma(\top)} g_1(s)\mathrm{d}s > \frac{1}{e}$$

implies the oscillation of (29). Therefore, it is impossible for (1) to have a positive solution. Thus, the proof is complete. □

Theorem 6. *Suppose that (2) is satisfied. If*

$$\liminf_{\top\to\infty} \int_{\top}^{\varsigma(\top)} \sum_{i=1}^{k} \Phi_i(s)\widetilde{\pi}^{\alpha}(\varsigma(s))\mathrm{d}s > \frac{1}{e}, \tag{30}$$

then (1) is oscillatory.

Proof. Let $v(\top) > 0$ be a solution of (1) on $[\top_0, \infty)$. Then $v(\varsigma(\top)) > 0$ for $\top \in [\top_1, \infty)$. In the same way as Theorem 6, it is clear that (30) and (2) leads to (5). By Lemma 2, we see that (6) holds for $\top \geq \top_1$. Now, we apply the chain rule:

$$\left(r(v')^{\alpha}\right)'(\top) = \alpha\left(r^{1/\alpha}(\top)v'(\top)\right)^{\alpha-1}\left(r^{1/\alpha}v'\right)'(\top).$$

Take into account that

$$\left(v + r^{1/\alpha}v'\pi\right)'(\top) - \pi(\top)\left(r^{1/\alpha}v'\right)'(\top) = 0,$$

and

$$\left(v + r^{1/\alpha}v'\pi\right)'(\top) - \frac{1}{\alpha}\pi(\top)\left(r(\top)(v'(\top))^{\alpha}\right)'\left(r^{1/\alpha}(\top)v'(\top)\right)^{1-\alpha} = 0.$$

From (1), we have

$$\left(v(\top) + r^{1/\alpha}(\top)v'(\top)\pi(\top)\right)' = -\frac{1}{\alpha}\pi(\top)\left(r^{1/\alpha}(\top)v'(\top)\right)^{1-\alpha}\sum_{i=1}^{k}\Phi_i(\top)v^{\alpha}(\varsigma(\top)) < 0. \tag{31}$$

Setting the positive decreasing function

$$\phi(\top) := v(\top) + r^{1/\alpha}(\top)\pi(\top)v'(\top).$$

Integrating (31) and using (11), we obtain

$$\begin{aligned}
\phi(\top) &= \phi(\infty)+\frac{1}{\alpha}\int_{\top}^{\infty}\frac{\pi(s)}{\left(r^{1/\alpha}(s)v'(s)\right)^{\alpha-1}}\sum_{i=1}^{k}\Phi_i(s)v^{\alpha}(\varsigma(s))\mathrm{d}s \\
&\geq \frac{1}{\alpha}\int_{\top}^{\infty}\frac{\pi(s)}{\left(r^{1/\alpha}(s)v'(s)\right)^{\alpha-1}}\sum_{i=1}^{k}\Phi_i(s)v^{\alpha}(\varsigma(s))\mathrm{d}s \\
&\geq \frac{1}{\alpha}\int_{\top}^{\infty}\frac{\pi(s)}{\left(r^{1/\alpha}(s)v'(s)\right)^{\alpha-1}}\pi^{\alpha}(\varsigma(s))\sum_{i=1}^{k}\Phi_i(s)\left(-r^{1/\alpha}\varsigma(s)v'(\varsigma(s))\right)^{\alpha}\mathrm{d}s \\
&\geq \frac{1}{\alpha}\int_{\top}^{\infty}\frac{\pi(s)}{\left(r^{1/\alpha}(s)v'(s)\right)^{\alpha-1}}\pi^{\alpha}(\varsigma(s))\sum_{i=1}^{k}\Phi_i(s)\left(-r^{1/\alpha}(s)v'(s)\right)^{\alpha}\mathrm{d}s \\
&= \frac{1}{\alpha}\int_{\top}^{\infty}\pi(s)\pi^{\alpha}(\varsigma(s))\sum_{i=1}^{k}\Phi_i(s)\left(-r^{1/\alpha}(s)v'(s)\right)\mathrm{d}s.
\end{aligned}$$

This implies

$$\phi(\top)\geq -\alpha^{-1}r^{1/\alpha}(\top)v'(\top)\int_{\top}^{\infty}\pi(s)\pi^{\alpha}(\varsigma(s))\sum_{i=1}^{k}\Phi_i(s)\mathrm{d}s.$$

That is,

$$\begin{aligned}
v(\top) &\geq -\frac{v'(\top)}{r^{1/\alpha}(\top)}\left(\pi(\top)+\alpha^{-1}\int_{\top}^{\infty}\pi(s)\pi^{\alpha}(\varsigma(s))\sum_{i=1}^{k}\Phi_i(s)\mathrm{d}s\right) \\
&= -\frac{v'(\top)\widetilde{\pi}(\top)}{r^{-1/\alpha}(\top)},
\end{aligned} \tag{32}$$

which, together with (1), imply that the first-order advanced differential inequality

$$x'(\top)-g_2(\top)x(\varsigma(\top))\geq 0, \tag{33}$$

where

$$g_2(\top)=\sum_{i=1}^{k}\Phi_i(\top)\widetilde{\pi}^{\alpha}(\varsigma(\top))$$

has a positive solution $x:=-r(v')^{\alpha}$. Referring to the reference Theorem 2.4.1 of [29], we find that condition

$$\liminf_{\top\to\infty}\int_{\top}^{\varsigma(\top)}g_2(s)\mathrm{d}s>\frac{1}{e}$$

leads to the oscillation of (33). So, (1) cannot have a solution $x>0$, which is a contradiction. The proof is complete. □

Theorem 7. *Suppose that (2) and (5) are satisfied. If there is a function ρ that belongs to class $C^1([\top_0,\infty),(0,\infty))$ such that*

$$\limsup_{\top\to\infty}\left\{\pi^{\alpha}(\top)\rho^{-1}(\top)\int_{\top}^{\top}\left(\sum_{i=1}^{k}\Phi_i(s)*\rho(s)\left(\frac{\pi(\varsigma(s))}{\pi(s)}\right)^{\alpha}-\frac{r(s)}{\rho^{\alpha}(s)}\left(\frac{\rho'(s)}{\alpha+1}\right)^{\alpha+1}\right)\mathrm{d}s\right\}>1, \tag{34}$$

$\forall\ \top\in[\top_0,\infty)$, then (1) is oscillatory.

Proof. Let $v(\top)>0$ be a solution of (1) on $[\top_0,\infty)$. Then $v(\varsigma(\top))>0$ for $\top\in[\top_1,\infty)$. From Lemma 2, we note that (6) holds. Moreover, we define

$$w:=\rho\left(r(v')^{\alpha}v^{-\alpha}+\pi^{-\alpha}\right)\geq 0\ \text{ on } [\top_1,\infty). \tag{35}$$

Differentiating (35), we obtain

$$\begin{aligned} w' &= \frac{w}{\rho}\rho' + \rho v^{-\alpha}\left(r(v')^{\alpha}\right)' - \alpha\rho r\left(\frac{v'}{v}\right)^{\alpha+1} + \alpha\rho r^{-1/\alpha}\pi^{-(\alpha+1)} \\ &\leq \frac{w}{\rho}\rho' + \rho v^{-\alpha}\left(r(v')^{\alpha}\right)' - \left(w - \frac{\rho}{\pi^{\alpha}}\right)^{(\alpha+1)/\alpha}\alpha(\rho r)^{-1/\alpha} \\ &\quad + \pi^{-(\alpha+1)}\alpha\rho r^{-1/\alpha}. \end{aligned} \tag{36}$$

By virtue of (1) and (7), we conclude that

$$\left(r(\top)(v'(\top))^{\alpha}\right)' + \left(\frac{\pi(\varsigma(\top))}{\pi(\top)}\right)^{\alpha}\sum_{i=1}^{k}\Phi_i(\top)v^{\alpha}(\top) \leq 0 \text{ for } \top \geq \top_2, \tag{37}$$

where $\top_2 \in [\top_1, \infty)$. Which, together with (36), gives

$$\begin{aligned} w'(\top) \leq & -\rho(\top)\sum_{i=1}^{k}\Phi_i(\top)\pi^{\alpha}(\varsigma(\top))\pi^{-\alpha}(\top) + \frac{1}{\rho(\top)}\rho'(\top)w(\top) \\ & -\alpha(\rho r(\top))^{-1/\alpha}\left(w(\top) - \frac{\rho(\top)}{\pi^{\alpha}(\top)}\right)^{(\alpha+1)/\alpha} + \alpha\rho(\top)r^{-1/\alpha}(\top)\pi^{-(\alpha+1)}(\top). \end{aligned}$$

From (4) and

$$S_1 := \rho'(\top)\frac{1}{\rho(\top)}, \quad S_2 := \alpha(r(\top)\rho(\top))^{-1/\alpha}, \quad S_3 := \pi^{-\alpha}(\top)\rho(\top),$$

we have

$$\begin{aligned} w'(\top) &\leq -\rho(\top)\sum_{i=1}^{k}\Phi_i(\top)\pi^{-\alpha}(\top)\pi^{\alpha}(\varsigma(\top)) + \pi^{-\alpha}(\top)\rho'(\top) \\ &\quad + (\alpha+1)^{-(\alpha+1)}r(\top)(\rho')^{\alpha+1}(\top)\rho^{-\alpha}(\top) + \alpha\rho(\top)r^{-1/\alpha}(\top)\pi^{-(\alpha+1)}(\top) \\ &= -\rho(\top)\sum_{i=1}^{k}\Phi_i(\top)\pi^{-\alpha}(\top)\pi^{\alpha}(\varsigma(\top)) + \left(\frac{\rho}{\pi^{\alpha}}\right)'(\top) \\ &\quad + (\alpha+1)^{-(\alpha+1)}r(\top)(\rho'(\top))^{\alpha+1}\rho^{-\alpha}(\top). \end{aligned} \tag{38}$$

Integrating (38), we obtain that

$$\begin{aligned} & \int_{\top_2}^{\top}\left(\rho(s)\sum_{i=1}^{k}\Phi_i(s)\left(\frac{\pi(\varsigma(s))}{\pi(s)}\right)^{\alpha} - (\alpha+1)^{-(\alpha+1)}r(s)(\rho'(s))^{\alpha+1}\rho^{-\alpha}(s)\right)\mathrm{d}s \\ & -\frac{\rho(s)}{\pi^{\alpha}(s)} + \frac{\rho(\top_2)}{\pi^{\alpha}(\top_2)} \\ \leq\; & w(\top_2) - w(\top). \end{aligned}$$

From (35), we obtain

$$\begin{aligned} & \int_{\top_2}^{\top}\left(\rho(s)\sum_{i=1}^{k}\Phi_i(s)\left(\frac{\pi(\varsigma(s))}{\pi(s)}\right)^{\alpha} - r(s)(\rho'(s))^{\alpha+1}(\alpha+1)^{-(\alpha+1)}\rho^{-\alpha}(s)\right)\mathrm{d}s \\ \leq\; & \rho(\top_2)r(\top_2)(v'(\top_2))^{\alpha}v^{-\alpha}(\top_2) - \rho(\top)r(\top)(v'(\top))^{\alpha}v^{-\alpha}(\top). \end{aligned} \tag{39}$$

In view of (11), we obtain

$$-\rho(\top)\pi^{-\alpha}(\top) \leq \rho(\top)r(\top)(v'(\top))^{\alpha}v^{-\alpha}(\top) \leq 0.$$

Into (39), we are led to

$$\int_{\top_2}^{\top}\left(\rho(s)\sum_{i=1}^{k}\Phi_i(s)\left(\frac{\pi(\varsigma(s))}{\pi(s)}\right)^{\alpha}-(\alpha+1)^{-(\alpha+1)}r(s)\left(\rho'(s)\right)^{\alpha+1}\rho^{-\alpha}(s)\right)\mathrm{d}s$$
$$\leq \pi^{-\alpha}(\top)\rho(\top). \tag{40}$$

That is,

$$\limsup_{\top\to\infty}\left\{\pi^{\alpha}(\top)\rho^{-1}(\top)\int_{\top}^{\top}\left(\rho(s)\sum_{i=1}^{k}\Phi_i(s)\left(\frac{\pi(\varsigma(s))}{\pi(s)}\right)^{\alpha}-\frac{r(s)}{\rho^{\alpha}(s)}\left(\frac{\rho'(s)}{\alpha+1}\right)^{\alpha+1}\right)\mathrm{d}s\right\}\leq 1.$$

This is a contradiction with (34). The proof is complete. □

We observe that the presence of the function ρ provides us with multiple opportunities to test the oscillation of (1). In the following results, we have chosen $\rho(\top)=1$, $\rho(\top)=\pi(\top),\rho(\top)=\pi^{\alpha}(\top)$.

Corollary 1. *Suppose that (2) and (5) are satisfied. If*

$$\limsup_{\top\to\infty}\pi^{\alpha}(\top)\int_{\top}^{\top}\sum_{i=1}^{k}\Phi_i(s)\left(\frac{\pi(\varsigma(s))}{\pi(s)}\right)^{\alpha}\mathrm{d}s>1, \tag{41}$$

$\forall\,\top\in[\top_0,\infty)$, then (1) is oscillatory.

Corollary 2. *Suppose that (2) and (5) are satisfied. If*

$$\limsup_{\top\to\infty}\int_{\top}^{\top}\sum_{i=1}^{k}\Phi_i(s)\pi^{\alpha}(\varsigma(s))-r^{-1/\alpha}(s)\pi^{-1}(s)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\mathrm{d}s>1, \tag{42}$$

$\forall\,\top\in[\top_0,\infty)$, then (1) is oscillatory.

Corollary 3. *Suppose that (2) and (5) are satisfied. If*

$$\limsup_{\top\to\infty}\pi^{\alpha-1}(\top)\int_{\top}^{\top}\left(\sum_{i=1}^{k}\Phi_i(s)\pi^{\alpha}(\varsigma(s))\pi^{1-\alpha}(\top)-\frac{r^{-1/\alpha}(s)\pi^{-\alpha}(s)}{(\alpha+1)^{\alpha+1}}\right)\mathrm{d}s>1, \tag{43}$$

$\forall\,\top\in[\top_0,\infty)$, then (1) is oscillatory.

Lemma 3. *Suppose that (2) and (5) are satisfied. Moreover,*

$$\gamma<1-\delta, \tag{44}$$

where $\gamma,\delta\geq 0$ are constants,

$$\frac{\gamma}{\sum_{i=1}^{k}\Phi_i(\top)\pi^{\alpha}(\varsigma(\top))\pi(\top)r^{1/\alpha}(\top)}\leq 1, \tag{45}$$

and

$$\frac{\delta}{\pi(\varsigma(\top))\Omega^{\alpha}_{(\top_1,\top)}(\top)}\leq 1. \tag{46}$$

Then, there exists a $\top_*\in[\top_1,\infty)$ such that

$$v/\pi^{1-\gamma}\text{ is nondecreasing,} \tag{47}$$

and

$$v/\pi^{\delta} \text{ is nonincreasing} \tag{48}$$

on $[\top_*, \infty)$.

Proof. From Lemma 2, we see that (6) holds. According to (1), (11), and (45), we have

$$\begin{aligned}\left(-r(v')^{\alpha}\pi^{\gamma}\right)'(\top) &= -\left(r(v')^{\alpha}\right)'(\top)\pi^{\gamma}(\top) + \gamma r(\top)\left(v'(\top)\right)^{\alpha}\frac{\pi^{\gamma-1}(\top)}{r^{1/\alpha}(\top)} \\ &= \sum_{i=1}^{k}\Phi_i(\top)v^{\alpha}(\varsigma(\top))\pi^{\gamma}(\top) + \gamma r(\top)\left(v'(\top)\right)^{\alpha}\pi^{\gamma-1}(\top)r^{-1/\alpha}(\top) \\ &\geq -\sum_{i=1}^{k}\Phi_i(\top)\pi^{\gamma}(\top)\pi^{\alpha}(\varsigma(\top))r(\top)\left(v'(\top)\right) \\ &\quad +\gamma r(\top)\left(v'(\top)\right)^{\alpha}\pi^{\gamma-1}(\top)r^{-1/\alpha}(\top).\end{aligned}$$

Hence

$$\left(-r(v')^{\alpha}\pi^{\gamma}\right)'(\top) = -r(\top)\left(v'(\top)\right)^{\alpha}\pi^{\gamma}(\top)\left(\sum_{i=1}^{k}\Phi_i(\top)\pi^{\alpha}(\varsigma(\top)) - \gamma\pi^{-1}(\top)r^{-1/\alpha}(\top)\right).$$

It follows that

$$\left(-r(v')^{\alpha}\pi^{\gamma}\right)'(\top) \geq 0 \text{ for } \top \geq \top_2,\ \top_2 \in [\top_1, \infty). \tag{49}$$

Using (49), we have

$$\begin{aligned} v(\top) &\geq -\int_{\top}^{\infty}\frac{r^{1/\alpha}(s)\pi^{\gamma}(s)}{r^{1/\alpha}(s)\pi^{\gamma}(s)}v'(s)\mathrm{d}s \\ v(\top) &\geq -\int_{\top}^{\infty}r^{1/\alpha}(s)\pi^{\gamma}(s)r^{-1/\alpha}(s)\pi^{-\gamma}(s)v'(s)\mathrm{d}s \\ &\geq -r^{1/\alpha}(s)v'(\top)\pi^{\gamma}(\top)\int_{\top}^{\infty}r^{-1/\alpha}(s)\pi^{-\gamma}(s)\mathrm{d}s. \end{aligned} \tag{50}$$

Take into consideration that

$$\int_{\top}^{\infty}r^{-1/\alpha}(s)\pi^{-\gamma}(s)\mathrm{d}s = \frac{1}{1-\gamma}\pi^{1-\gamma}(\top), \tag{51}$$

hence,

$$v(\top) \geq -\frac{1}{1-\gamma}r^{1/\alpha}(\top)v'(\top)\pi(\top). \tag{52}$$

That is,

$$\left(\frac{v}{\pi^{1-\gamma}}\right)'(\top) = \frac{1}{(1-\gamma)}r^{-1/\alpha}(\top)\pi^{2-\gamma}(s)\left(r^{1/\alpha}(\top)v'(\top)\pi(\top) + v\right).$$

Therefore,

$$\left(\frac{v}{\pi^{1-\gamma}}\right)'(\top) \geq 0.$$

As in the proof of Theorem 1,we obtain to (18); that is,

$$v(\top) + r^{1/\alpha}(\top)v'(\top)\frac{\pi(\top)}{\pi(\varsigma(\top))}\Omega^{-1/\alpha}_{(\top_1,\top)}(\top) \leq 0. \tag{53}$$

Substituting (53) in the equality

$$\left(\frac{v}{\pi^{\delta}}\right)'(\top) = \frac{v'(\top)}{\pi^{\delta}(\top)} + \frac{\delta v(\top)}{\pi^{\delta+1}(\top) r^{1/\alpha}(\top)}.$$

In view of (46), we obtain

$$\begin{aligned}\left(\frac{v}{\pi^{\delta}}\right)'(\top) &\leq v'(\top)\pi^{-\delta}(\top) - \delta v'(\top)\pi^{-\delta}(\top)\pi^{-1}(\varsigma(\top))\Omega^{-1/\alpha}_{(\top_1,\top)}(\top) \\ &= v'(\top)\pi^{-\delta}(\top)\Big(1 - \delta\pi^{-1}(\varsigma(\top))\Omega^{-1/\alpha}_{(\top_1,\top)}(\top)\Big) \leq 0.\end{aligned}$$

Thus,

$$\left(\frac{v}{\pi^{\delta}}\right)'(\top) \leq 0.$$

The proof is complete. □

Theorem 8. *Suppose that (2) is satisfied, and γ and δ are constants satisfying (44)–(46). If*

$$\frac{1}{(1-v)^{\alpha}} \limsup_{\top\to\infty} \frac{\pi^{\gamma\alpha}(\top)}{\pi^{\alpha(\gamma+\delta-1)}(\varsigma(\top))} \int_{\top_1}^{\top} \pi^{\delta\alpha}(\varsigma(s)) \sum_{i=1}^{k} \Phi_i(s) ds > 1, \tag{54}$$

$\forall\, \top_1 \geq \top_0$, *then (1) is oscillatory.*

Proof. Let $v(\top) > 0$ be a solution of (1) on $[\top_0, \infty)$. Then, $v(\varsigma(\top)) > 0$ for $\top \in [\top_1, \infty)$. Note that in order for (19) to hold, (5) must hold; we see that (2) gives

$$\frac{\pi^{\gamma}(\top)}{\pi^{\gamma+\delta+1}(\varsigma(s))} \leq \frac{1}{\pi^{\delta-1}(\top)} \to 0 \text{ as } \top \to \infty. \tag{55}$$

From (55), we conclude that

$$\int_{\top_0}^{\top} \pi^{\delta\alpha}(\varsigma(s)) \sum_{i=1}^{k} \Phi_i(s) ds,$$

and $\Omega_{(\top_0,\top)}(\top)$ must be unbounded. Therefore, from Lemma 2, (6) holds. Now, integrating (1) from $\top_1$ to $\top$ and using Lemma 3, we find that $\left(v/\pi^{\delta}\right)' \leq 0$ and $\left(v/\pi^{1-\gamma}\right)' \geq 0$; consequently

$$\begin{aligned}-r(\top)\big(v'(\top)\big)^{\alpha} &= -r(\top_1)\big(v'(\top_1)\big)^{\alpha} + \int_{\top_1}^{\top} \sum_{i=1}^{k} \Phi_i(s) v^{\varsigma}(\varsigma(s)) ds \\ &\geq \Big(v(\varsigma(\top))\pi^{-\delta}(\varsigma(\top))\Big)^{\alpha} \int_{\top_1}^{\top} \pi^{\delta\alpha}(\varsigma(s)) \sum_{i=1}^{k} \Phi_i(s) ds \\ &= \Big(v(\varsigma(\top))\pi^{1-\gamma}(\varsigma(\top))\pi^{-\delta}(\varsigma(\top))\pi^{\gamma-1}(\varsigma(\top))\Big)^{\alpha} \int_{\top_1}^{\top} \pi^{\delta\alpha}(\varsigma(s)) \sum_{i=1}^{k} \Phi_i(s) ds,\end{aligned}$$

that is,

$$-r(\top)\big(v'(\top)\big)^{\alpha} \geq \Big(v(\top)\pi^{1-\gamma-\delta}(\varsigma(\top))\pi^{\gamma-1}(\top)\Big)^{\alpha} \int_{\top_1}^{\top} \pi^{\delta\alpha}(\varsigma(s)) \sum_{i=1}^{k} \Phi_i(s) ds. \tag{56}$$

Combining (56) with (52), we have

$$-r(\top)\big(v'(\top)\big)^{\alpha} \geq -r(\top)\big(v'(\top)\big)^{\alpha} \left(\frac{1}{1-\gamma}\pi^{\gamma}(\top)\pi^{1-\gamma-\delta}(\varsigma(\top))\right)^{\alpha} \int_{\top_1}^{\top} \pi^{\delta\alpha}(\varsigma(s)) \sum_{i=1}^{k} \Phi_i(s) ds.$$

Thus,

$$\left(\frac{1}{1-\gamma}\pi^{\gamma}(\top)\pi^{1-\gamma-\delta}(\varsigma(\top))\right)^{\alpha}\int_{\top_1}^{\top}\pi^{\delta\alpha}(\varsigma(s))\sum_{i=1}^{k}\Phi_i(s)\mathrm{d}s\leq 1$$

or

$$\limsup_{\top\to\infty}\pi^{\gamma\alpha}(\top)\pi^{\alpha(1-\gamma-\delta)}(\varsigma(\top))\int_{\top_1}^{\top}\pi^{\delta\alpha}(\varsigma(s))\sum_{i=1}^{k}\Phi_i(s)\mathrm{d}s\leq(1-\upsilon)^{\alpha}.$$

This is a contradiction with (54). This ends the proof. □

Theorem 9. *Suppose that (2) holds,*

$$0\leq\gamma<1, \tag{57}$$

and (45) is satisfied. If

$$e(1-\gamma)^{-\alpha}\liminf_{\top\to\infty}\int_{\top}^{\varsigma(\top)}\sum_{i=1}^{k}\Phi_i(s)\pi^{\varsigma}(\varsigma(s))\mathrm{d}s>1,$$

then (1) is oscillatory.

Proof. According to the proof of Theorem 5, use (11) instead of (52). This ends the proof. □

Theorem 10. *Suppose that (2), (5) and (57) are satisfied. Also, suppose that (45) holds. If there is a function $\rho\in C^1([\top_0,\infty),(0,\infty))$ such that*

$$\limsup_{\top\to\infty}\left\{\frac{\pi^{\varsigma}(\top)}{\rho(\top)}\int_{\top}^{\top}\left(\rho(s)\sum_{i=1}^{k}\Phi_i(s)\left(\pi(\varsigma(s))\pi^{-1}(s)\right)^{\alpha(1-\upsilon)}-\frac{r(s)\rho^{-\alpha}(\top)(\rho'(\top))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right)\mathrm{d}s\right\}>1, \tag{58}$$

$\top\in[\top_0,\infty)$, *then (1) is oscillatory.*

Proof. In (37) in the proof of Theorem 7, it suffices to replace (7) by (47). The proof is complete. □

Corollary 4. *Suppose that (2) is satisfied, (57) and (45) hold. If*

$$\limsup_{\top\to\infty}\int_{\top}^{\top}\left(\sum_{i=1}^{k}\Phi_i(s)\pi^{\alpha(1-\upsilon)}(\varsigma(s))\pi^{\alpha\gamma}(s)-\frac{(\alpha/\alpha+1)^{\alpha+1}r^{-1/\alpha}(s)}{\pi(s)}\right)\mathrm{d}s>1,\forall\top\in[\top_0,\infty), \tag{59}$$

then (1) is oscillatory.

4. Examples and Discussion

Example 1. *Setting in (1)*

$$r(\top)=\top^{\alpha+1},\ \Phi(\top)=\Phi_0>0,\ \varsigma_i(\top)=\lambda_i\top,\ \lambda_i\in(0,1],\ i=1,2,\ldots,n,\ \top\geq 1. \tag{60}$$

We see that

$$\pi(\top)=\alpha\frac{1}{\top^{1/\alpha}}.$$

(1) *By Theorem 1, it is easy to verify that (13) holds. Therefore, any nonoscillatory of (60) tends to zero as $\top\to\infty$, and its lower and upper bounds are $M_1\top^{-1/\alpha}$ and $M_2\top^{-(\Phi_0/\lambda)^{1/\alpha}}$, respectively, for some positive constants M_1, M_2.*

(2) *By Theorem 2, we note that (19) is not satisfied; so Theorem 2 and the results in [25] fail.*

(3) *By Theorem 3, if*

$$\Phi_0>\frac{\lambda}{\alpha^{\alpha}}, \tag{61}$$

then (1) with (60) is oscillatory.

(4) *By Theorem 5, if*

$$\Phi_0 > \frac{\lambda}{e\alpha^\alpha \ln \lambda}, \tag{62}$$

then (1) with (60) is oscillatory.

(5) *By Theorem 6, if*

$$\Phi_0 > \frac{\lambda}{e\alpha^\alpha (1 + \alpha^{\alpha-1}\lambda^{-1}\Phi_0)^\alpha \ln \lambda}, \tag{63}$$

then (1) with (60) is oscillatory. Note that (63) improves (62).

(6) *By Corollary 2, (1) with (60) is oscillatory if*

$$\Phi_0 > \lambda^\alpha (\alpha + 1)^{-(\alpha+1)}. \tag{64}$$

Furthermore, employ the results presented in the lemma 3, set

$$\gamma := \alpha^{\alpha+1}\frac{\Phi_0}{\lambda} \text{ and } \delta := \alpha\left(\frac{\Phi_0}{\lambda}\right)^{1/\alpha},$$

and assume that (44) is satisfied. We obtain the result that

Theorem 8 improves Theorem 3, where (61) is replaced by

$$\Phi_0 > \frac{\alpha^{-\alpha}(1-\gamma)^\alpha (1-\delta)}{\lambda \gamma^{-1}}.$$

Theorem 9 improves Theorem 5, where (62) is replaced by

$$\Phi_0 > \frac{\alpha^{-\alpha}\lambda(1-\gamma)}{e \ln \lambda}.$$

Theorem 10 improves Theorem 7, where (64) is replaced by

$$\Phi_0 > (\alpha+1)^{-(\alpha+1)}\lambda^{\alpha(1-\gamma)}.$$

Example 2. *Consider the following equation:*

$$\left(\top^2 v'(\top)\right)' + \Phi_0 \sum_{i=1}^{k} v(\lambda_i \top) = 0. \tag{65}$$

That is,

$$r(\top) = \top^2,\ \Phi(\top) = \Phi_0 > 0,\ \alpha = 1,\ \varsigma_i(\top) = \lambda_i \top,\ \lambda_i \in (0,1],\ i = 1,2,...,n,\ \top \geq 1.$$

By taking specific values for λ, we also provide critical values for Φ_0, and we obtain their oscillation criteria. From Figure 1, we observe that

Theorems 5 and 6 are more efficient when λ has larger values.

Theorems 5 and 6 are not applicable in ordinary cases.

In Theorem 5, Condition (64) takes a sharp oscillation condition of the form

$$\Phi_0 > 1/4.$$

for smaller values of λ. Theorem 7 provides the most efficient condition.

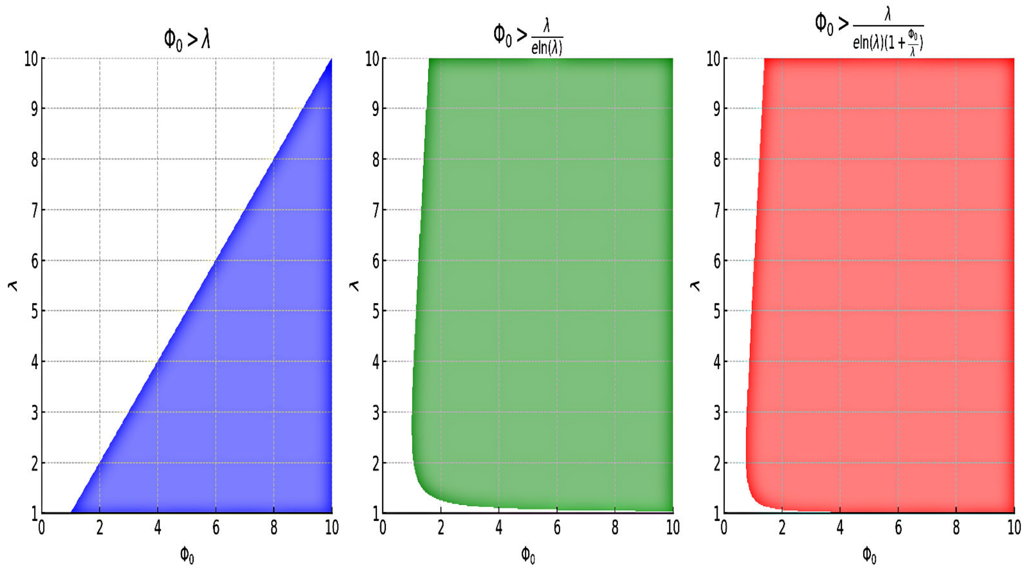

Figure 1. Test of the advantage of criteria for Equation (65).

5. Conclusions

In Theorem 1, we provide a condition that ensures that any non-oscillatory solution converges to zero, as well as the lower and upper bounds for these solutions. In Theorems 2 and 3, we provide different conditions such that if we cannot apply the first condition, we apply the second condition. In Theorem 4, we present a result to remove the dependency on the constant $\top_1$ mentioned in Theorem 3. In Theorem 5 and Theorem 6, we present results that can be applied to ordinary differential equations using the comparison principle with advanced first-order differential inequalities. In Theorem 7, we present criteria that guarantee the oscillation of the solutions of (1). These criteria are notable for their applicability to a wide range of applications based on the specification of the function ρ. In Lemma 3, we provide stronger estimates for development and subsequently employ them to improve the criteria mentioned in the previous literature through Theorems 8–10.

This paper presents a detailed analysis of the asymptotic and oscillatory behavior of a specific class of half-linear second-order differential equations with advanced arguments in a noncanonical case. Using a comparison method with a first-order equation (previously extensively studied in the available literature) and the Riccati technique, we have established new and precise criteria for determining whether the solutions of these equations exhibit oscillatory behavior. Our results not only deepen the current understanding of this differential equation but also contribute to expanding the literature on advanced differential equations and enriching oscillation theory.

It is worth noting that the techniques used in this paper can be utilized to test the asymptotic and oscillatory behavior of solutions for a broader class of differential equations

$$\left(r(\top)\left((v(\top)+P(\top)v(\nu(\top)))'\right)^{\alpha}\right)'+\sum_{i=1}^{k}\Phi_i(\top)v^{\alpha}(\varsigma_i(\top))=0.$$

Therefore, this extension may support increasing the effectiveness of the methods used in this paper, thereby ensuring the continued advancement in this field of study.

Author Contributions: Conceptualization, Z.A., F.A. and B.Q.; methodology, Z.A., B.Q. and A.A.; validation, Z.A., B.Q. and A.A.; investigation, Z.A., B.Q. and A.A.; resources, B.Q. and Z.A.; data curation, Z.A., B.Q. and A.A.; writing—original draft preparation, Z.A., F.A. and B.Q.; writing—review and editing, B.Q., Z.A., F.A. and A.A.; visualization, Z.A., F.A. and B.Q.; supervision, A.A., F.A. and B.Q.; project administration, B.Q. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Researchers Supporting Project number (PNURSP2024R518), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Data Availability Statement: Data are contained within the article.

Acknowledgments: Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2024R518), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Hale, J.K. Functional differential equations. In *Oxford Applied Mathematical Sciences*; Springer: New York, NY, USA, 1971.
2. Rihan, F.A. *Delay Differential Equations and Applications to Biology*; Springer Nature Singapore Pte Ltd.: Singapore, 2021.
3. Li, T.; Rogovchenko, Y.V. Oscillation of second-order neutral differential equations. *Math. Nachr.* **2015**, *288*, 1150–1162. [CrossRef]
4. Li, T.; Rogovchenko, Y.V. Oscillation criteria for second-order superlinear Emden-Fowler neutral differential equations. *Monatsh. Math*. **2017**, *184*, 489–500. [CrossRef]
5. Trench, W.F. Canonical forms and principal systems for general disconjugate equations. *Trans. Amer. Math. Soc*. **1973**, *189*, 319–327. [CrossRef]
6. Philos, C.G. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays. *Arch. Math.* **1981**, *36*, 168–178. [CrossRef]
7. Erbe, L.; Kong, Q.; Zhang, B. *Oscillation Theory for Functional Differential Equations*; Marcel Dekker: New York, NY, USA, 1995.
8. Tang, X. Oscillation for first order superlinear delay differential equations. *J. London Math. Soc.* **2002**, *65*, 115–122. [CrossRef]
9. Baculíková, B. Properties of third order nonlinear functional differential equations with mixed arguments. *Abstr. Appl. Anal*. **2011**, *2011*, 1–15. [CrossRef]
10. Salim, A.; Mesri, F.; Benchohra, M.; Tunç, C. Controllability of Second Order Semilinear Random Differential Equations in Fréchet Spaces. *Mediterr. J. Math.* **2023**, *20*, 84. [CrossRef]
11. Tunç, C. Stability to vector Liénard equation with constant deviating argument. *Nonlinear Dyn.* **2013**, *73*, 1245–1251. [CrossRef]
12. Li, T.; Rogovchenko, Y.V.; Zhang, C. Oscillation results for second-order nonlinear neutral differential equations. *Adv. Differ. Equ.* **2013**, *2013*, 1–13. [CrossRef]
13. Li, T.; Thandapani, E.; Graef, J.R.; Tunc, E. Oscillation of second-order Emden–Fowler neutral differential equations. *Nonlinear Stud.* **2013**, *20*, 1–8. [CrossRef]
14. Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. Even-order half-linear advanced differential equations: Improved criteria in oscillatory and asymptotic prop- erties. *Appl. Math. Comput*. **2015**, *266*, 481–490. [CrossRef]
15. Džurina, J. Oscillation of second order differential equations with advanced argument. *Math. Slovaca* **1995**, *45*, 263–268.
16. Koplatadze, R.; Kvinikadze, G.; Stavroulakis, I.P. Properties A and B of n th order linear differential equations with deviating argument. *Georgian Math. J*. **1999**, *6*, 553–566.
17. Kusano, T.; Naito, M. Comparison theorems for functional differential equations with deviating arguments. *J. Math. Soc. Jpn*. **1981**, *33*, 509–532. [CrossRef]
18. Jadlovská, I. Iterative oscillation results for second-order differential equations with advanced argument. *Electron. J. Diff. Equ*. **2017**, *2017*, 1–11.
19. Baculíková, B. Oscillatory behavior of the second order functional differential equations. *Appl. Math. Lett*. **2017**, *72*, 35–41. [CrossRef]
20. Džurina, J. A comparison theorem for linear delay differential equations. *Arch. Math. Brno* **1995**, *31*, 113–120.
21. Chatzarakis, G.E.; Grace, S.R.; Jadlovská, I.R.E.N.A. A sharp oscillation criterion for second-order half-linear advanced differential equations. *Acta Math. Hung*. **2021**, *163*, 2. [CrossRef]
22. Agarwal, R.P.; Bohner, M.; Saker, S.H. Oscillation of second order delay dynamic equation. *Canadian Appl. Math. Quart*. **2005**, *13*, 1–18.
23. Zhang, B.G.; Shanliang, Z. Oscillation of second-order nonlinear delay dynamic equations on time scales. *Comput. Math. Appl* . **2005**, *49*, 599–609. [CrossRef]
24. Şahı, Y. Oscillation of second-order delay dynamic equations on time scales. *Nonlinear Anal. Theor. Meth. Appl.* **2005**, *63*, e1073–e1080. [CrossRef]
25. Hassan, T.S. Kamenev-type oscillation criteria for second order nonlinear dynamic equations on time scales. *Appl. Math. Comput.* **2011**, *217*, 5285–5297. [CrossRef]

26. Agarwal, R.P.; Grace, S.R.; O'Regan, D. *Oscillation Theory for Second Order Dynamic Equations*; CRC Press: London, UK, 2003; Volume 416. [CrossRef]
27. Agarwal, R.P.; Zhang, C.; Li, T. New Kamenev-type oscillation criteria for second-order nonlinear advanced dynamic equations. *Appl. Math. Comput*. **2013**, *225*, 822–828. [CrossRef]
28. Wu, H.; Erbe, L.; Peterson, A. Oscillation of solution to second-order half-linear delay dynamic equations on time scales. *Electron. J. Diff. Equ*. **2016**, *2016*, 1–15.
29. Ladde, G.S.; Lakshmikantham, V.; Zhang, B.G. *Oscillation Theory of Differential Equations with Deviating Arguments*; Marcel Dekker: New York, NY, USA, 1987; Volume 110.