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Symmetries and Invariant Solutions of Higher-Order Evolution Systems

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Abstract: In this paper, we investigate evolution systems in two components, characterized by higher-order spatial derivatives and the presence of two arbitrary functions. Our study begins with an analysis of a fourth-order system. We perform a detailed group classification and identify specific forms of the constitutive functions that allow the system to exhibit additional symmetries in addition to spatial and temporal translations. We extend these results to nth-order systems. Moreover, we derive invariant solutions for these systems. Finally, for each order n, we are able to find non-negative solutions.

Keywords: higher-order evolution systems; Lie symmetries; exact solutions

1. Introduction

In a recent paper [1], the following class of reaction diffusion systems is studied, in the framework of the symmetry groups,

$$\begin{cases} u_t = D_0 u_{xx} + \gamma_1 u(\gamma_2 - u) + g(v), \\ v_t = h(u, v), \end{cases} \tag{1}$$

where D_0, γ_1, and γ_2 are positive constants, while $g(v)$ and $h(u, v)$ are analytical functions of their arguments. As usual, the subscripts t or x denote partial differentiations.

Systems (1) are a special class of the following widely studied second-order diffusion equations in two-components

$$\begin{cases} u_t = D_x[D_1(u, v)u_x] + f(u, v), \\ v_t = D_x[D_2(u, v)v_x] + h(u, v), \end{cases} \tag{2}$$

where the operator D_x denotes the total derivative with respect to x. Later, we will use the operator D_t to represent the total derivative with respect to t.

Diffusive systems model many real phenomena, such as those in the physical, biological, and life sciences fields. For instance, semiconductor devices provide a vast area of research in the field of two-component evolution systems (see e.g., [2,3]). These devices, fundamental to modern electronics, exhibit complex interactions between charge carriers, electric fields, and material properties. To cite just a few more examples, these systems can describe the diffusion in magnetized plasma (see, e.g., [4]) or model the dynamics of predator–prey populations (in the class (2) falls the well-known diffusive Lotka–Voltera system, see, e.g., [5–7]).

However, the majority of studies concern systems in which there is diffusion in both equations (in the framework of the symmetry groups, see, e.g., [8,9]). Subclasses in which a component does not suffer diffusion is quite often considered when describing the evolution of some bacterial colonies (see for instance [10,11] and the references given there).

In the first equation of System (1) D_0 is the constant diffusion coefficient. The reaction term is given by $\gamma_1 u(\gamma_2 - u) + g(v)$. The term $\gamma_1 u(\gamma_2 - u)$ (corresponding to the inhomo-

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geneous term in the Fisher equation [12]) implies that the growth of u is influenced by itself and a threshold value γ_2.

In the second equation of System (1), the function $h(u,v)$ determines how the variable v evolves over time, influenced by u, assuming that v is unaffected by diffusion.

Here, continuing the study carried out in [1], in the framework of the symmetry groups, we examine the following class of evolution systems

$$\begin{cases} u_t = D_0 u_{xxxx} + \gamma_1 u(\gamma_2 - u) + g(v), \\ \\ v_t = h(u,v). \end{cases} \tag{3}$$

In physics, diffusion is typically modeled with the second spatial derivative (u_{xx}), but nonlinear reaction–diffusion equations involving the fourth derivative (u_{xxxx}) appear in many fields (see, e.g., [13–15] and references therein).

Lie symmetry analysis provides a powerful tool for studying partial differential equations (PDEs) and has numerous applications. For instance, it can be used to reduce the number of independent variables, and for PDEs with two independent variables, the reduction process converts any PDE into an ordinary differential equation (ODE). The order of an ODE can be lowered by using Lie symmetries, and for a first-order ODE the reduction can lead to complete integration of the equation. Classical symmetries of a PDE map solutions to solutions, so from a known solution, it is possible to obtain new solutions. Many generalizations of the classical Lie method have been developed to achieve reductions in PDEs. For example, the non-classical symmetry method [16] or the potential symmetry method [17–19].

In this paper, we apply Lie symmetries to search for exact solutions. This technique allows us to derive exact solutions in a methodological way. After determining the Lie symmetries (using the well-known Lie invariance criterion [20]), the first step is to transform the variables $u(t,x)$ and $v(t,x)$ into $u(t,x,U(s))$, $v(t,x,V(s))$, where $s = s(t,x)$, such that the transformed equations are ordinary differential equations in $U(s)$ and $V(s)$. Consequently, this method leads to finding solutions to ODEs.

Motivated by the analogy between the results obtained in [1] for System (1) and those for System (3), here, we also study the following more general system

$$\begin{cases} u_t = D_0 u_{x^n} + \gamma_1 u(\gamma_2 - u) + g(v), \\ \\ v_t = h(u,v), \end{cases} \tag{4}$$

with u_{x^n} the n-th derivative of u with respect to x, where n is an integer such that $n \geq 2$.

While reaction–diffusion systems with second-order diffusion terms are well-studied, systems involving higher-order diffusion terms, such as u_{xxxx} or even more generally u_{x^n} with $n \geq 2$, are less explored. These higher-order terms are crucial in modeling complex physical processes where classical diffusion models are insufficient.

Systems (1), (3) and (4) are characterized not only by a component that does not suffer diffusion, but also, in the evolution equation for u, by a reaction term of the type $f(u)$ $+g(v)$. This suggests that the effects of u and v on the growth of u are separate and cumulative. This situation is realistic when the two effects act independently and additively, for example, in the growth of a population or in enzyme kinetics (see, e.g., [21–23]). In the growth of a population, u represents a population with an intrinsic rate $f(u)$ and the term $g(v)$ could represent a nutrient supply from a source v. In enzyme kinetics, u represents the concentration of a substrate and v the concentration of an enzyme, and $g(v)$ could represent the rate of a reaction catalyzed by the enzyme. A reaction term of the type $f(u)g(v)$ is usually used in predator–prey models. This form can represent situations where the influence of v on u depends on the presence or concentration of u. For example, the growth of u might be amplified or reduced based on the presence of v.

In this paper, wishing to extend the analysis initiated in [1], we study a more general class of reaction–diffusion systems characterized by higher-order derivatives in the evolution equation for u. We apply Lie symmetry methods to derive exact solutions for these generalized systems. This approach provides concrete analytical tools to study these types of equations. The obtained exact solutions will serve as valuable benchmarks for future research and practical applications in fields such as population dynamics and enzyme kinetics.

The order of this paper is as follows. In Section 2, we look for infinitesimal symmetry generators of System (3). We obtain a group classification of System (3) with respect to the functions $g(v)$ and $h(u,v)$ (assuming that the equations of System (3) are not decoupled), that is we identify the special forms of the functions $g(v)$ and $h(u,v)$ such that the system admits symmetries beyond spatial and temporal translations. In Sections 3, we extend these results to System (4) for $n \geq 2$. In Sections 4 and 5, we use the results obtained in the previous sections to find invariant solutions. By using the property of invariance with respect to translations in t and x of System (4), in Section 6, we construct non-negative exact solutions. Finally, we present the conclusion in Section 7.

2. Symmetries of Fourth-Order Evolution Systems

In this section, we look for symmetries of the system described by the fourth-order evolution equations (3), and we apply the well-known method (see, e.g., [17,20,24–26]). A symmetry infinitesimal operator for System (3) takes the form

$$X = \xi^1(x,t,u,v)\partial_x + \xi^2(x,t,u,v)\partial_t + \eta^1(x,t,u,v)\partial_u + \eta^2(x,t,u,v)\partial_v. \tag{5}$$

To compute the infinitesimal coordinates ξ^1, ξ^2, η^1, and η^2, we need the fourth extension of operator (5)

$$X^{(4)} = X + \zeta^1_t\frac{\partial}{\partial_{u_t}} + \zeta^1_{xxxx}\frac{\partial}{\partial_{u_{xxxx}}} + \zeta^2_t\frac{\partial}{\partial_{v_t}}. \tag{6}$$

As usual, the expressions for the coordinates ζ^1_t, ζ^1_{xxxx}, ζ^2_t are given by

$$\begin{aligned}
\zeta^1_t &= D_t(\eta^1) - u_t D_t(\xi^1) - u_x D_t(\xi^2),\\
\zeta^1_x &= D_x(\eta^1) - u_t D_x(\xi^1) - u_x D_x(\xi^2),\\
\dots\; &\dots\; \dots\dots\dots\\
\zeta^1_{xxxx} &= D_x(\zeta^1_{xxx}) - u_{xxxt} D_x(\xi^1) - u_{xxxx} D_x(\xi^2),\\
\zeta^2_t &= D_t(\eta^2) - v_t D_t(\xi^1) - v_x D_t(\xi^2).
\end{aligned}$$

Applying operator (6) to System (3), we obtain the following invariance conditions

$$\begin{cases}
-\zeta^1_t + D_0\zeta^1_{xxxx} + (\gamma_1\gamma_2 - 2u\gamma_1)\eta^1 + g_v\eta^2 = 0,\\
-\zeta^2_t + h_u\eta^1 + h_v\eta^2 = 0,
\end{cases} \tag{7}$$

under the constraint that u and v are solutions of System (3). That is, we need to substitute

$$u_{xxxx} = \frac{1}{D_0}(u_t - \gamma_1 u(\gamma_2 - u) - g(v)), \tag{8}$$

$$v_t = h(u,v), \tag{9}$$

and all their differential consequences in (7). Following the well-known procedure, we collect the obtained equations with respect to the derivatives of u and v. By requiring that the corresponding coefficients are zero, we obtain the determining system in the unknowns coordinates ξ^1, ξ^2, η^1, and η^2, involving the constitutive parameters $g(v)$ and $h(u,v)$.

For arbitrary forms of the functions $g(v)$ and $h(u,v)$, we obtain

$$\xi^1 = c_1, \quad \xi^2 = c_2, \quad \eta^1 = 0, \quad \eta^2 = 0. \tag{10}$$

Then, the Principal Lie Algebra (that is, the algebra admitted for every system of class (3)) is spanned by the generators

$$X_1 = \partial_t, \quad X_2 = \partial_x. \tag{11}$$

This means, as expected, that System (3) is invariant under translations in t and x.

We look for special forms of the constitutive functions $g(v)$ and $h(u,v)$ such that System (3) admits additional generators. This corresponds to solving the problem of Lie symmetry classification for System (3). To achieve this, we first solve the determining equations that are independent of the forms of the arbitrary elements g and h, and we obtain the following specializations for the coordinates ξ^1, ξ^2, η^1, and η^2

$$\xi^1 = 4c_1 t + c_2, \quad \xi^2 = c_1 x + c_3, \quad \eta^1 = 2c_1(\gamma_2 - 2u), \quad \eta^2 = \psi(t,x,v), \tag{12}$$

where $c_i,\ i = 1,2,3$ are arbitrary constants and $\psi(t,x,v)$ is an arbitrary function. Substituting these expressions for ξ^1, ξ^2, η^1, and η^2 into the remaining determining equations yields only one equation involving the function $g(v)$

$$2c_1(\gamma_1\gamma_2^2 + 4g) + g_v\psi = 0, \tag{13}$$

and one equation involving the function $h(u,v)$

$$2c_1(2u - \gamma_2)h_u - \psi h_v + (\psi_v - 4c_1)h + \psi_t = 0. \tag{14}$$

From Condition (13), if $g_v \neq 0$, we can find $\psi(t,x,v)$ and substitute it into (14). If $g_v = 0$, we need to distinguish whether the constant function g takes the specific value $-\frac{\gamma_1\gamma_2^2}{4}$ or not. In the latter case, to satisfy Condition (13), it must be $c_1 = 0$. Therefore, we need to distinguish between $g_v \neq 0$, $g(v) = \gamma_0 \neq -\frac{\gamma_1\gamma_2^2}{4}$ and $g(v) = -\frac{\gamma_1\gamma_2^2}{4}$. When the function g is constant, the variable v does not appear in the first equation of System (3). Therefore, if $h_u = 0$, it implies that System (3) consists of two independent equations that can be solved separately. We do not consider this case. In this way, from (13), we consider the following cases.

1. $g_v \neq 0$;
2. $g(v) = \gamma_0 \neq -\frac{\gamma_1\gamma_2^2}{4}$, $h_u \neq 0$;
3. $g(v) = -\frac{\gamma_1\gamma_2^2}{4}$, $h_u \neq 0$.

We will analyze them in different subsections.

2.1. $g_v \neq 0$

We observe that in the case $g_v \neq 0$, the transformation

$$w = g(v) \tag{15}$$

maps System (3) to the system

$$\begin{cases} u_t = D_0 u_{xxxx} + \gamma_1 u(\gamma_2 - u) + w, \\ w_t = \bar{h}(u,w), \end{cases} \tag{16}$$

where the new function $\bar{h}$ is related to the original function h by the following relation

$$\bar{h}(u,w) = h(u,v) g_v \,|_{v=g^{-1}(w)}, \tag{17}$$

where g^{-1} denotes the inverse function of g. Then, without loss of generality, in this case, we can assume

$$g(v) = v. \tag{18}$$

From Condition (13), we obtain

$$\psi(t,x,v) = -2c_1(\gamma_1\gamma_2^2 + 4v). \tag{19}$$

Consequently, Condition (14) for the function h takes the form

$$c_1\Big((\gamma_1\gamma_2^2 + 4v)h_v + (2u - \gamma_2)h_u - 6h\Big) = 0. \tag{20}$$

By examining Expression (12) with (19), to extend the principal Lie algebra, it is necessary that $c_1 \neq 0$. Then, from (20), we obtain the following special form for function h

$$h(u,v) = (2u-\gamma_2)^3 h_1(\omega), \quad \text{with} \quad \omega = \frac{4v+\gamma_1\gamma_2^2}{(2u-\gamma_2)^2}. \tag{21}$$

Then, we can affirm that in this case, System (3) is also invariant with respect the symmetry generator

$$X_3 = 4t\partial_t + x\partial_x + 2(\gamma_2 - 2u)\partial_u - 2(\gamma_1\gamma_2^2 + 4v)\partial_v. \tag{22}$$

2.2. $g(v) = \gamma_0 \neq -\frac{\gamma_1\gamma_2^2}{4}$, $h_u \neq 0$.

In this case, from Condition (13), we obtain $c_1 = 0$, and (14) becomes

$$-\psi h_v + \psi_v h + \psi_t = 0. \tag{23}$$

In order to extend the principal Lie algebra, it must be

$$h(u,v) = \left(h_1(u) + h_0 \int e^{-h_2(v)} dv\right) e^{h_2(v)}, \tag{24}$$

where h_0 is constant, and h_1, h_2 are functions of u and v, respectively, $h_{1u} \neq 0$. The corresponding system admits the following generator

$$X_3 = e^{h_0 t + h_2(v)} \psi(x) \partial_v. \tag{25}$$

2.3. $g(v) = -\frac{\gamma_1\gamma_2^2}{4}$, $h_u \neq 0$.

In this case, by differentiating Condition (14) with respect to t and u, and with respect to x and u, we have

$$\psi_t h_{uv} - h_u \psi_{tv} = \psi_x h_{uv} - h_u \psi_{xv} = 0. \tag{26}$$

Since $h_u \neq 0$, we can write

$$\psi_t \frac{h_{uv}}{h_u} - \psi_{tv} = \psi_x \frac{h_{uv}}{h_u} - \psi_{xv} = 0. \tag{27}$$

Then, it must be

$$\psi_t \left(\frac{h_{uv}}{h_u}\right)_u = \psi_x \left(\frac{h_{uv}}{h_u}\right)_u = 0. \tag{28}$$

We have the following possibility: $h(u,v) = h_1(v) + h_3(u)e^{h_2(v)}$ or $\psi(t,x,V) = \psi(V)$. We will analyze them separately.

1. If $h(u,v) = h_1(v) + h_3(u)e^{h_2(v)}$, by differentiating Condition (14) with respect to u, we have

$$h_{3u}(h_{2v}\psi - \psi_v) + 2c_1 h_{3uu}(\gamma_2 - 2u) = 0. \tag{29}$$

Since $h_{3u} \neq 0$, we can write

$$h_{2v}\psi - \psi_v + 2c_1 \frac{h_{3uu}}{h_{3u}}(\gamma_2 - 2u) = 0. \tag{30}$$

It must be

$$\left(c_1 \frac{h_{3uu}}{h_{3u}}(\gamma_2 - 2u)\right)_u = 0. \tag{31}$$

We obtain extensions of the principal Lie algebra in the following cases, where we specify the forms of the function h and the corresponding additional generators.

(a)

$$h(u,v) = \left(h_0 + (2u-\gamma_2)^k h_1\right) e^{h_2(v)}, \tag{32}$$

with h_0, $k \neq 0$ and $h_1 \neq 0$ as constants.

$$X_3 = 4t\partial_t + x\partial_x + 2(\gamma_2 - 2u)\partial_u + 4\left(kh_0t + (1-k)\int e^{-h_2(v)}dv\right)e^{h_2(v)}\partial_v, \tag{33}$$

$$X_4 = e^{h_2(v)}\psi(x)\partial_v. \tag{34}$$

(b)

$$h(u,v) = (h_0 + h_1 \ln(2u - \gamma_2))e^{h_2(v)}, \tag{35}$$

with h_0 and $h_1 \neq 0$ as constants.

$$X_3 = 4t\partial_t + x\partial_x + 2(\gamma_2 - 2u)\partial_u + 4\left(\int e^{-h_2(v)}dv - h_1t\right)e^{h_2(v)}\partial_v, \tag{36}$$

$$X_4 = e^{h_2(v)}\psi(x)\partial_v. \tag{37}$$

(c)

$$h(u,v) = \left(h_1(u) + h_0\int e^{-h_2(v)}\,dv\right)e^{h_2(v)}, \tag{38}$$

with h_0 constant, and $h_{1u} \neq 0$.

$$X_3 = e^{h_0t + h_2(v)}\psi(x)\partial_v. \tag{39}$$

2. If $\psi(t,x,V) = \psi(V)$, Condition (14) becomes

$$2c_1(2u - \gamma_2)h_u - \psi h_v + (\psi_v - 4c_1)h = 0. \tag{40}$$

To obtain extensions of the principal Lie algebra, it must me $c_1\psi \neq 0$. We need to distinguish the cases $\psi_v \neq 0$ and $\psi_v = 0$. We obtain extensions of the principal Lie algebra in the following cases, where we write the forms of function h and the corresponding additional generators.

(a)

$$h(u,v) = \frac{e^{-2h_2(v)}}{h_2'(v)}h_1(\omega), \quad \text{with} \quad \omega = (2u-\gamma_2)e^{2h_2(v)}, \quad \text{and} \quad h_2'(v) \neq 0. \tag{41}$$

$$X_3 = 4t\partial_t + x\partial_x + 2(\gamma_2 - 2u)\partial_u + \frac{2}{h_2'(v)}\partial_v. \tag{42}$$

(b)

$$h(u,v) = (2u - \gamma_2)h_1(\omega), \quad \text{with} \quad \omega = h_0\ln(2u - \gamma_2) + v, \tag{43}$$

with h_0 constant.

$$X_3 = 4t\partial_t + x\partial_x + 2(\gamma_2 - 2u)\partial_u + 4h_0\partial_v. \tag{44}$$

We observe that in the case $h(u,v) = \left(h_1(u) + h_0\int e^{-h_2(v)}dv\right)e^{h_2(v)}$ and $g(v) = \gamma_0$, that is constant, we obtain the additional generator $X_3 = e^{h_0t+h_2(v)}\psi(x)\partial_v$ in both cases

$$\gamma_0 = -\frac{\gamma_1\gamma_2}{4}, \quad \gamma_0 \neq -\frac{\gamma_1\gamma_2}{4} \tag{45}$$

Then, in Section 2.2 we can remove $\gamma_0 \neq -\frac{\gamma_1\gamma_2}{4}$.

3. Symmetries of nth-Order Evolution Systems

In this section, we generalize the results obtained in Section 2 for System (3) to System (4). For System (4), we need the following extension of operator (5)

$$X^{(n)} \quad = \quad X + \zeta_t^1\frac{\partial}{\partial_{u_t}} + \zeta_{x^n}^1\frac{\partial}{\partial_{u_{x^n}}} + \zeta_t^2\frac{\partial}{\partial_{v_t}}, \tag{46}$$

where

$$\zeta_{x^n}^1 = D_x(\zeta_{x^{n-1}}^1) - u_{x^{n-1}t}D_x(\xi^1) - u_{x^n}D_x(\xi^2). \tag{47}$$

Applying the generator (46) to (4), we obtain the following invariance conditions

$$-\xi^1_t + D_0\xi^1_{x^n} + (\gamma_1\gamma_2 - 2u\gamma_1)\eta^1 + g_v\eta^2 = 0, \tag{48}$$
$$-\xi^2_t + h_u\eta^1 + h_v\eta^2 = 0, \tag{49}$$

subject to the constraint that u and v are solutions of System (4).

In Condition (48), the highest-order derivatives of u and v only appear in $\xi^1_{x^n}$. From the coefficients of $u_{tx^{n-1}}$ and v_{x^n}, we obtain, respectively,

$$\xi^1_x + u_x\xi^1_u + v_x\xi^1_v = 0, \quad \text{and} \quad \eta^1_v - u_t\xi^1_v - u_x\xi^2_v = 0. \tag{50}$$

So we obtain the following restrictions for ξ^1, ξ^2, and η^1

$$\xi^1 = \alpha(t),\ \xi^2 = \beta(t,x,u),\ \eta^1 = \phi(t,x,u). \tag{51}$$

Substituting in (49), from coefficients of v_x and u_t we obtain, respectively,

$$\beta_u u_t + \beta_t = 0, \quad \beta_u v_x - \eta^2_u = 0. \tag{52}$$

Then

$$\xi^1 = \alpha(t),\ \xi^2 = \beta(x),\ \eta^1 = \phi(t,x,u),\ \eta^2 = \psi(t,x,v), \tag{53}$$

and substituting in (48), from the coefficient of $u_{x^{n-1}}u_x$ we have

$$\phi_{uu} = 0, \tag{54}$$

which implies

$$\phi = u\phi_1(t,x) + \phi_2(t,x). \tag{55}$$

In this way, from the coefficient of u_t in (48), we have

$$\alpha_t - n\beta_x = 0, \tag{56}$$

which implies

$$\alpha = c_2 + nc_1 t, \quad \beta = c_3 + c_1 x. \tag{57}$$

These results imply that from coefficient of $u_{x^{n-1}}$ in (48), we have

$$\phi_{1_x} = 0. \tag{58}$$

To summarize, until now, we have the following results

$$\xi^1 = c_2 + nc_1 t,\ \xi^2 = c_3 + c_1 x,\ \eta^1 = u\phi_1(t) + \phi_2(t,x),\ \eta^2 = \psi(t,x,v). \tag{59}$$

Considering that no coordinate depends on u in (48), we can also collect terms with respect to u. From the coefficient of u^2, we obtain

$$\phi_1 + nc_1 = 0 \quad \Longrightarrow \phi_1 = -nc_1, \tag{60}$$

and from the coefficient of u we have

$$nc_1\gamma_2 - 2\phi_0 = 0 \quad \Longrightarrow \phi_0 = \frac{n}{2}c_1\gamma_2. \tag{61}$$

Finally, we obtain

$$\xi^1 = c_2 + nc_1 t,\ \xi^2 = c_3 + c_1 x,\ \eta^1 = \frac{n}{2}c_1(\gamma_2 - 2u),\ \eta^2 = \psi(t,x,v), \tag{62}$$

with the conditions

$$\frac{n}{2}c_1(\gamma_1\gamma_2^2 + 4g) + g_v\psi = 0, \tag{63}$$
$$\frac{n}{2}c_1(2u - \gamma_2)h_u - \psi h_v + (\psi_v - nc_1)h + \psi_t = 0. \tag{64}$$

For arbitrary forms of the functions $g(v)$ and $h(u,v)$, System (4) is invariant under translations in t and x. The Principal Lie Algebra is spanned by generator (11).

We look for special forms of the constitutive functions $g(v)$ and $h(u,v)$ such that System (4) admits additional generators. This corresponds to solving the problem of Lie symmetry classification for System (4). The classifying Equations (63) and (64) are very similar to those of System (3). Therefore, by following the same procedure as in Section 2, we can affirm that System (4) admits extensions of the principal Lie algebra in the following cases.

1.
$$g(v) = v, \quad h(u,v) = (2u - \gamma_2)^3 h_1(\omega), \quad \text{with} \quad \omega = \frac{4v + \gamma_1\gamma_2^2}{(2u-\gamma_2)^2}. \tag{65}$$
$$X_3 = nt\partial_t + x\partial_x + \frac{n}{2}(\gamma_2 - 2u)\partial_u - \frac{n}{2}(\gamma_1\gamma_2^2 + 4v)\partial_v. \tag{66}$$

2.
$$g(v) = \gamma_0, \quad h(u,v) = \left(h_1(u) + \int h_0 e^{h_2(v)} dv\right) e^{h_2(v)}, \tag{67}$$
where γ_0, and h_0 are constants and $h_{1u} \neq 0$.
$$X_3 = e^{h_0 t + h_2(v)}\psi(x)\partial_v. \tag{68}$$

3.
$$g(v) = -\frac{\gamma_1\gamma_2^2}{4}, \quad h(u,v) = \left(h_0 + (2u-\gamma_2)^k h_1\right) e^{h_2(v)}, \tag{69}$$
with h_0, $k \neq 0$ and $h_1 \neq 0$ as constants.
$$X_3 = nt\partial_t + x\partial_x + \frac{n}{2}(\gamma_2 - 2u)\partial_u + n\left(kh_0 t + (1-k)\int e^{-h_2(v)} dv\right) e^{h_2(v)}\partial_v, \tag{70}$$
$$X_4 = e^{h_2(v)}\psi(x)\partial_v. \tag{71}$$

4.
$$g(v) = -\frac{\gamma_1\gamma_2^2}{4}, \quad h(u,v) = (h_0 + h_1 \ln(2u - \gamma_2))e^{h_2(v)}, \tag{72}$$
with h_0 and $h_1 \neq 0$ as constants.
$$X_3 = nt\partial_t + x\partial_x + \frac{n}{2}(\gamma_2 - 2u)\partial_u + n\left(\int e^{-h_2(v)} dv - h_1 t\right) e^{h_2(v)}\partial_v, \tag{73}$$
$$X_4 = e^{h_2(v)}\psi(x)\partial_v. \tag{74}$$

5.
$$g(v) = -\frac{\gamma_1\gamma_2^2}{4}, \quad h(u,v) = \frac{e^{-2h_2(v)}}{h_2'(v)} h_1(\omega), \tag{75}$$
with $\omega = (2u - \gamma_2)e^{2h_2(v)}$, and $h_2'(v) \neq 0$.
$$X_3 = nt\partial_t + x\partial_x + \frac{n}{2}(\gamma_2 - 2u)\partial_u + \frac{n}{2h_2'(v)}\partial_v. \tag{76}$$

6.
$$g(v) = -\frac{\gamma_1\gamma_2^2}{4}, \quad h(u,v) = (2u - \gamma_2)h_1(\omega), \tag{77}$$
with $\omega = h_0 \ln(2u - \gamma_2) + v$, and h_0 constant.
$$X_3 = nt\partial_t + x\partial_x + \frac{n}{2}(\gamma_2 - 2u)\partial_u + nh_0\partial_v. \tag{78}$$

4. Exact Solutions of a Fourth-Order Evolution System

In this section, we consider the system of equations from class (3) with

$$g(v) = v \quad \text{and} \quad h(u,v) = (2u-\gamma_2)^3\left(\gamma_4\frac{\gamma_1\gamma_2^2+4v}{(2u-\gamma_2)^2}+\gamma_5\right), \tag{79}$$

where γ_4 and γ_5 are arbitrary constants, that is the system

$$\begin{cases} u_t = D_0u_{xxxx} + \gamma_1 u(\gamma_2 - u) + v, \\ v_t = (2u-\gamma_2)^3\left(\gamma_4\frac{\gamma_1\gamma_2^2+4v}{(2u-\gamma_2)^2}+\gamma_5\right). \end{cases} \tag{80}$$

This system belongs to the class obtained in Section 2.1. Therefore, this system is invariant not only with respect to the generators (11), but also with respect to the generator (22).

By using infinitesimal symmetries, for PDEs with two independent variables, the reduction process converts any PDE into an ordinary differential equation (ODE).

This reduction procedure can be performed taking into account the characteristic system

$$\frac{dt}{\xi^1} = \frac{dx}{\xi^2} = \frac{du}{\eta^1} = \frac{dv}{\eta^2}. \tag{81}$$

The solutions of this characteristic system transform the variables $u(t,x)$ and $v(t,x)$ into $u(t,x,U(s))$, $v(t,x,V(s))$, with similarity variable $s = s(t,x)$.

By using the generator (22)

$$X_3 = 4t\partial_t + x\partial_x + 2(\gamma_2 - 2u)\partial_u - 2(\gamma_1\gamma_2^2 + 4v)\partial_v, \tag{82}$$

the characteristic system is

$$\frac{dt}{4t} = \frac{dx}{x} = \frac{du}{2(\gamma_2-2u)} = \frac{dv}{-2(\gamma_1\gamma_2^2+4v)}. \tag{83}$$

From this system we obtain the invariant solutions

$$u(t,x) = \frac{1}{2}\gamma_2 + \frac{U(s)}{t}, \quad v(t,x) = -\frac{1}{4}\gamma_1\gamma_2^2 + \frac{V(s)}{x^8}, \quad \text{where} \quad s = \frac{x^4}{t}. \tag{84}$$

By substituting (84) into the System (80), we obtain the reduced system

$$\begin{cases} 256D_0s^5U_{ssss} + 1152D_0s^4U_{sss} + 816D_0U_{ss}s^3 + (24D_0+s)s^2U_s - \gamma_1s^2U^2 + s^2U + V = 0, \\ 8\gamma_4UV + 8\gamma_5s^2U^3 + sV_s = 0. \end{cases} \tag{85}$$

By setting

$$\gamma_4 = \frac{\gamma_1}{4}, \quad \text{and} \quad \gamma_5 = -\frac{\gamma_1^2}{4}, \tag{86}$$

we obtain a specific solution to the reduced System (85)

$$\begin{aligned} U(s) &= -\frac{s}{8D_0\gamma_1} - \frac{3}{2\gamma_1}, \\ V(s) &= \frac{s^4}{64D_0^2\gamma_1} + \frac{5s^3}{8D_0\gamma_1} + \frac{27s^2}{4\gamma_1}. \end{aligned} \tag{87}$$

Returning to (84), we obtain

$$\begin{aligned} u(t,x) &= \frac{\gamma_2}{2} - \frac{x^4}{8D_0\gamma_1t^2} - \frac{3}{2\gamma_1t}, \\ v(t,x) &= -\frac{\gamma_1\gamma_2^2}{4} + \frac{x^8}{64(D_0^2\gamma_1t^4)} + \frac{5x^4}{8(D_0\gamma_1t^3)} + \frac{27}{4\gamma_1t^2}, \end{aligned} \tag{88}$$

as solutions of system

$$\begin{cases} u_t = D_0u_{xxxx} + \gamma_1 u(\gamma_2 - u) + v, \\ v_t = (2u-\gamma_2)^3\left(\frac{\gamma_1}{4}\frac{\gamma_1\gamma_2^2+4v}{(2u-\gamma_2)^2}-\frac{\gamma_1}{4}\right). \end{cases} \tag{89}$$

5. Exact Solutions of a nth-Order Evolution System

Given the analogy between the results obtained in Sections 2 and 3, here we consider the system from class (4) with

$$g(v) = v \quad \text{and} \quad h(u,v) = (2u-\gamma_2)^3\left(\frac{\gamma_1}{4}\frac{\gamma_1\gamma_2^2+4v}{(2u-\gamma_2)^2} - \frac{\gamma_1}{4}\right), \tag{90}$$

which is analogous to the system discussed in Section 4, but of order n with $n \geq 2$

$$\begin{cases} u_t = D_0 u_{x^n} + \gamma_1 u(\gamma_2 - u) + v, \\ v_t = (2u-\gamma_2)^3\left(\frac{\gamma_1}{4}\frac{\gamma_1\gamma_2^2+4v}{(2u-\gamma_2)^2} - \frac{\gamma_1}{4}\right). \end{cases} \tag{91}$$

We recall that this system is invariant with respect to the generators (11) and (66). By using the generator (66)

$$X_3 = nt\partial_t + x\partial_x + \frac{n}{2}(\gamma_2 - 2u)\partial_u - \frac{n}{2}(\gamma_1\gamma_2^2 + 4v)\partial_v, \tag{92}$$

we look for invariant solutions

$$u(t,x) = \frac{1}{2}\gamma_2 + \frac{U(s)}{t}, \quad v(t,x) = -\frac{1}{4}\gamma_1\gamma_2^2 + \frac{V(s)}{x^{2n}}, \quad \text{where} \quad s = \frac{x^n}{t}. \tag{93}$$

If we restrict our attention to functions $U(s)$ that are linear in s, we observe that $U_{ss} = \cdots U_{s^n} = 0$, and in the first equation we can neglect the derivatives of U from the second order. Then, the reduced system for each n becomes

$$\begin{aligned} &(n!D_0 + s)s^2U_s - \gamma_1 s^2U^2 + s^2U + V = 0, \\ &2\gamma_1 UV - 2\gamma_1^2 s^2U^3 + sV_s = 0, \end{aligned} \tag{94}$$

with $U(s) = a_0 + a_1 s$. That is,

$$\begin{aligned} &(n!D_0 + s)s^2a_1 - \gamma_1 s^2(a_0 + a_1 s)^2 + s^2(a_0 + a_1 s) + V = 0, \\ &2\gamma_1(a_0 + a_1 s)V - 2\gamma_1^2 s^2(a_0 + a_1 s)^3 + sV_s = 0, \end{aligned} \tag{95}$$

and we obtain the following solution

$$\begin{aligned} U(s) &= -\frac{3s}{n!D_0\gamma_1} - \frac{3}{2\gamma_1}, \\ V(s) &= \frac{9s^4}{(n!)^2D_0^2\gamma_1} + \frac{15s^3}{n!D_0\gamma_1} + \frac{27s^2}{4\gamma_1}. \end{aligned} \tag{96}$$

Returning to (93), we obtain a solution for System (91)

$$\begin{aligned} u(t,x) &= \frac{1}{2}\gamma_2 - \frac{1}{t}\left(\frac{3x^n}{n!D_0\gamma_1 t} + \frac{3}{2\gamma_1}\right), \\ v(t,x) &= \frac{9x^{2n}}{(n!)^2D_0^2\gamma_1 t^4} + \frac{15x^n}{n!D_0\gamma_1 t^3} + \frac{27}{4\gamma_1 t^2} - \frac{1}{4}\gamma_1\gamma_2^2. \end{aligned} \tag{97}$$

Of course, for $n = 4$ we obtain the solution given by (88).

6. Positivity of Solutions

Since all systems (4) are invariant with respect to translations in t and x, from (97) we can also affirm that System (91) admits the following solutions

$$\begin{aligned} u(t,x) &= \frac{1}{2}\gamma_2 - \frac{1}{t-t_0}\left(\frac{3(x-x_0)^n}{n!D_0\gamma_1(t-t_0)} + \frac{3}{2\gamma_1}\right), \\ v(t,x) &= \frac{9(x-x_0)^{2n}}{(n!)^2D_0^2\gamma_1(t-t_0)^4} + \frac{15(x-x_0)^n}{n!D_0\gamma_1(t-t_0)^3} + \frac{27}{4\gamma_1(t-t_0)^2} - \frac{1}{4}\gamma_1\gamma_2^2, \end{aligned} \tag{98}$$

with t_0, x_0 as constants, obtained from solutions (97) by applying the transformation

$$x \to x - x_0, \quad t \to t - t_0. \tag{99}$$

This can be useful when, as often happens, it is required that the solution be positive or non-negative. For example, in the biological context, $u(t,x)$ and $v(t,x)$ can represent densities and should be non-negative within a suitable domain $[t_1;t_2] \times [x_1;x_2]$, where $t_1 < t_2$ and $x_1 < x_2$.

Writing the solution of System (91) in the form (98), it is possible to select t_0 and x_0 such that $u(t,x) \geq 0$ and $v(t,x) \geq 0$ in $[t_1;t_2] \times [x_1;x_2]$, for each $D_0 > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$ and for each integer $n \geq 2$. In order to demonstrate this, we consider two distinct cases.

1. Case n odd
   In this case, in order to have $u(t,x) \geq 0$ and $v(t,x) \geq 0$ in $[t_1;t_2] \times [x_1;x_2]$, it is sufficient to choose t_0 and x_0 such that

$$t_0 \geq t_2, \quad \text{and} \quad x_0 \geq x_2 + \left(\frac{D_0\gamma_1^2\gamma_2^2 n!}{60}(t_0 - t_1)^3\right)^{\frac{1}{n}}. \tag{100}$$

   If (100) are satisfied, for all $(t,x) \in [t_1;t_2] \times [x_1;x_2]$, it holds that

$$t_1 - t_0 \leq t - t_0 \leq t_2 - t_0 \leq 0; \quad x_1 - x_0 \leq x - x_0 \leq x_2 - x_0 \leq 0, \tag{101}$$

   and we immediately obtain that $u(t,x)$, given by (98), is non-negative on $[t_1;t_2] \times [x_1;x_2]$. Regarding $v(t,x)$ we observe that the quantity

$$\frac{9(x-x_0)^{2n}}{(n!)^2 D_0^2 \gamma_1 (t-t_0)^4} + \frac{27}{4\gamma_1 (t-t_0)^2} \tag{102}$$

   is non-negative, while we can write

$$\begin{aligned} \frac{15(x-x_0)^n}{n! D_0 \gamma_1 (t-t_0)^3} - \frac{1}{4}\gamma_1\gamma_2^2 &= \frac{15}{n! D_0 \gamma_1 (t_0-t)^3}\left[(x_0-x)^n - \frac{D_0\gamma_1^2\gamma_2^2 n!}{60}(t_0-t)^3\right] \geq \\ &\geq \frac{15}{n! D_0 \gamma_1 (t_0-t)^3}\left[(x_0-x_2)^n - \frac{D_0\gamma_1^2\gamma_2^2 n!}{60}(t_0-t)^3\right] \geq \\ &\geq \frac{15}{n! D_0 \gamma_1 (t_0-t)^3}\left[(x_0-x_2)^n - \frac{D_0\gamma_1^2\gamma_2^2 n!}{60}(t_0-t_1)^3\right]. \end{aligned} \tag{103}$$

   If the conditions (100) are satisfied, we can affirm that this quantity is non-negative.Therefore, we can conclude that $v(t,x)$, defined by (98), is non-negative on $[t_1;t_2] \times [x_1;x_2]$.
2. Case n even
   In this case, if we choose

$$t_0 \leq t_1 - \frac{3}{D_0\gamma_1\gamma_2 n!}\left(D_0 n! + \sqrt{D_0^2 (n!)^2 + \frac{2}{3} D_0 \gamma_1 \gamma_2 n! (x_2 - x_0)^n}\right), \tag{104}$$

   it holds that

$$0 \leq t_1 - t_0 \leq t - t_0 \leq t_2 - t_0. \tag{105}$$

   And if we choose

$$x_0 \leq x_1 - \left(\frac{D_0\gamma_1^2\gamma_2^2 n!}{60}(t_2 - t_0)^3\right)^{\frac{1}{n}}, \tag{106}$$

   it will be that

$$0 \leq x_1 - x_0 \leq x - x_0 \leq x_2 - x_0. \tag{107}$$

   Regarding $u(t,x)$, given by (98), we obtain

$$u(t,x) = \frac{1}{2}\gamma_2 - \frac{1}{t-t_0}\left(\frac{3(x-x_0)^n}{n!D_0\gamma_1(t-t_0)} + \frac{3}{2\gamma_1}\right) = \\ \frac{3}{D_0\gamma_1 n!(t-t_0)^2}\left[\frac{D_0\gamma_1\gamma_2 n!}{6}(t-t_0)^2 - \frac{D_0 n!(t-t_0)}{2} - (x-x_0)^n\right] \geq \\ \frac{3}{D_0\gamma_1 n!(t-t_0)^2}\left[\frac{D_0\gamma_1\gamma_2 n!}{6}(t-t_0)^2 - \frac{D_0 n!(t-t_0)}{2} - (x_2-x_0)^n\right]. \tag{108}$$

Clearly,

$$\frac{3}{D_0\gamma_1 n!(t-t_0)^2} \geq 0. \tag{109}$$

Taking into account that $t - t_0 \geq 0$, the second factor

$$\left[\frac{D_0\gamma_1\gamma_2 n!}{6}(t-t_0)^2 - D_0 n!(t-t_0) - (x_2-x_0)^n\right] \tag{110}$$

will be non-negative if

$$t - t_0 \geq \frac{3}{D_0\gamma_1\gamma_2 n!}\left(D_0 n! + \sqrt{D_0^2(n!)^2 + \frac{2}{3}D_0\gamma_1\gamma_2 n!(x_2-x_0)^n}\right). \tag{111}$$

But, from (104) and (105),

$$t - t_0 \geq t_1 - t_0 \geq \frac{3}{D_0\gamma_1\gamma_2 n!}\left(D_0 n! + \sqrt{D_0^2(n!)^2 + \frac{2}{3}D_0\gamma_1\gamma_2 n!(x_2-x_0)^n}\right), \tag{112}$$

then (111) is satisfied and $u(t,x)$, given by (98), is non-negative in $[t_1;t_2] \times [x_1;x_2]$ when (104) and (106) are satisfied.
Regarding $v(t,x)$, we observe that

$$\frac{9(x-x_0)^{2n}}{(n!)^2 D_0^2\gamma_1(t-t_0)^4} + \frac{27}{4\gamma_1(t-t_0)^2} \tag{113}$$

is non-negative, while we can write

$$\frac{15(x-x_0)^n}{n!D_0\gamma_1(t-t_0)^3} - \frac{1}{4}\gamma_1\gamma_2^2 = \frac{15}{n!D_0\gamma_1(t-t_0)^3}\left[(x-x_0)^n - \frac{D_0\gamma_1^2\gamma_2^2 n!}{60}(t-t_0)^3\right] \geq \\ \geq \frac{15}{n!D_0\gamma_1(t-t_0)^3}\left[(x_1-x_0)^n - \frac{D_0\gamma_1^2\gamma_2^2 n!}{60}(t-t_0)^3\right] \geq \\ \geq \frac{15}{n!D_0\gamma_1(t-t_0)^3}\left[(x_1-x_0)^n - \frac{D_0\gamma_1^2\gamma_2^2 n!}{60}(t_2-t_0)^3\right]. \tag{114}$$

From hypothesis (106), we obtain

$$x_1 - x_0 \geq \left(\frac{D_0\gamma_1^2\gamma_2^2 n!}{60}(t_2-t_0)^3\right)^{\frac{1}{n}}, \tag{115}$$

then

$$\frac{15}{n!D_0\gamma_1(t-t_0)^3}\left[(x_1-x_0)^n - \frac{D_0\gamma_1^2\gamma_2^2 n!}{60}(t_2-t_0)^3\right] \geq 0. \tag{116}$$

We can conclude that $v(t,x)$, given by (98), is non-negative in $[t_1;t_2] \times [x_1;x_2]$ when (104) and (106) are satisfied.

It is a simple matter that if we use strict inequalities in the conditions (100) for the odd cases and in (104) and (106) for the even cases, we can conclude that $u(t,x)$ and $v(t,x)$ are positive in $[t_1;t_2] \times [x_1;x_2]$.

For the sake of clarity, in the odd case, conditions (100) have been obtained to ensure (101) and

$$\left[(x_0 - x_2)^n - \frac{D_0 \gamma_1^2 \gamma_2^2 n!}{60}(t_0 - t_1)^3\right] \geq 0. \quad (117)$$

A similar approach was taken in the even case.

7. Conclusions

In this paper, we considered System (3) and its generalization (4). The difference between these two systems lies in the highest order of the spatial derivative, which can have various biological implications, such as increased dispersion or smoothing of spatial profiles of the variable u. We obtained a group classification of Systems (3) and (4) with respect to the functions $g(v)$ and $h(u,v)$ (assuming that the equations of systems are not decoupled), that is, we identified the special forms of the constitutive functions $g(v)$ and $h(u,v)$ such that the systems admit symmetries other than spatial and temporal translations. After that, we computed exact solutions. For special forms of g and h, we were able to obtain exact solutions for Systems (3) and (4) for any $n \geq 2$. Finally, considering the system in a biological context and using the property of invariance with respect to translations in t and x of System (4), we obtained non-negative exact solutions that are relevant to real-world scenarios.

The results of this work encourage further research by extending the symmetry analysis to other types of nonlinearities in the reaction term $f(u) + g(v)$, as well as to more complex multi-component systems that better model biological processes, such as interacting species or enzyme–substrate reactions.

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