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Further Results on Lusin's Theorem for Uncertain Variables

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Abstract: In order to treat the degree of belief rationally, Baoding Liu created uncertainty theory. An uncertain variable, as a measurable function from an uncertainty space to the set of real numbers, is a basic concept in uncertainty theory. It is very meaningful to study its properties. Lusin's theorem is one of the most classical theorems in measure theory that reveals the close relationship between measurable and continuous functions, and has important significance. In this paper, we give three pairs of continuity conditions for uncertain measures, and present that every pair reveals duality, which is a kind of symmetry between objects. Furthermore, it is demonstrated that these continuity conditions are equivalent. And, we also prove that these three pairs of continuity conditions and the condition: if $\{\Lambda_n\}$ is a sequence of open sets and $\Lambda_n \searrow \emptyset$, then $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0$ are equivalent in compact metric spaces. It is shown that Lusin's theorem for uncertain variables holds if and only if the uncertain measure satisfies any of the above continuity conditions in a compact metric space. And, Lusin's theorem can be applied to uncertain variables with symmetric or asymmetric distributions. Finally, we provide several examples to illustrate applications of Lusin's theorem for uncertain variables. As far as we know, our results are new in uncertainty theory.

Keywords: uncertain variable; continuity of uncertain measure; Lusin's theorem; compact metric space

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1. Introduction

The world is always full of indeterminacy, usually one kind is random phenomena and the other is non-random phenomena. For a random phenomenon related to frequency, it can be analyzed by probability theory. To deal with a non-random phenomenon, the concept of a fuzzy set was proposed by Zadeh [1] and possibility theory related to the theory of fuzzy sets was founded by Zadeh [2]. In order to find Pareto optimal solutions to multiple objective optimization problems under an imprecise environment, Garai, Mandal and Roy [3] provided one new set, viz. T-set, to supersede fuzzy set for representing uncertainty. After that, Garai, Mandal and Roy [4] investigated optimization of multi-objective model with fuzzy coefficients in imprecise environment and parametric T-set was provided. For more studies about T-set, the interested readers can refer to [5–7] and so on.

However, when there are no samples to estimate a probability distribution, we must invite some domain experts to assess the belief degree that each event will happen. In order to reasonably deal with the problem of belief degree, uncertainty theory was established in 2007 by Liu [8], and has been extensively investigated by many researchers. At present, uncertainty theory has become a branch of credibility in axiomatic mathematical modeling. In 2009, Liu [9] provided some results in uncertainty theory, including the product measure axiom, an operational law of independent uncertain variables, a concept of entropy of continuous uncertain variables, and so on. Following that, Gao [10] investigated some properties of continuous uncertain measures. Subsequently, Liu [11] introduced some concepts of uncertain sets and so on, proposed an uncertain inference rule, and presented an uncertain system. Some methods for solving linear uncertain differential equations were provided by Chen and Liu [12], and they also proved the existence and uniqueness theorem of solutions to uncertain differential equations. After that, Liu [13] offered a method for solving a specific class of nonlinear uncertain differential equations. A stronger new definition of independence of uncertain sets and its some properties were introduced by Liu [14] in 2013. In 2014, Liu [15] put forward the concept of uncertainty distribution and an independent definition of uncertain processes. In 2018, a concept of totally ordered uncertain set was presented by Liu [16], and he also proved that totally ordered uncertain sets always have membership functions if they are defined on a continuous uncertainty space. Uncertainty theory has also been developed in other fields, such as uncertain risk analysis (see, e.g., Liu [17]), uncertain programming (see, e.g., Zhang and Peng [18]), uncertain finance (see, e.g., Liu [19]), and so on.

An uncertain variable is a measurable function from an uncertainty space to the set of real numbers, and it is typically used to represent quantities with uncertainty. For applying uncertainty theory more effectively, it is highly beneficial to explore its properties. In 2012, the uncertain distribution and expected value of function of an uncertain variable were investigated by Zhu [20]. In order to approximately compute the uncertain distribution, the optimistic value, and expected value of the function of an uncertain variable, Zhu [20] also introduced uncertainty simulations. The sine entropy of uncertain variables and its properties were studied by Yao, Gao, and Dai [21] in 2013. Chen, Li, and Ralescu [22] proved several useful inequalities for uncertain variables and deduced some convergence theorems for continuous uncertain measures in 2014. In 2022, Tian, Zong, and Hu [23] gave the necessary and sufficient condition of the convergence for complex uncertain sequences (Egoroff's theorem), and the sufficient condition of the continuity for complex uncertain variables (Lusin's theorem).

Lusin's theorem is one of the most classical theorems in measure theory. So far, Lusin's theorem has already been generalized to many fields. By applying the regularity and weakly null-additivity of fuzzy measure, Li and Yasuda [24] generalized Lusin's theorem to fuzzy measure spaces in 2004. Kawabe [25] showed that Lusin's theorem remains valid for a weakly null-additive, Riesz space-valued fuzzy Borel measure that has the multiple Egoroff property and is order separable in 2007. Li and Mesiar [26] proved that Lusin's theorem also holds in monotone measure spaces if the monotone measure satisfies condition (E) and has the pseudometric generating property. In 2020, Zong, Hu, and Tian [27] showed that Lusin's theorem for capacities in the framework of g-expectation still holds if g satisfies the Lipschitz condition, $g(t, y, 0) = 0$ and subadditivity. In 2022, Wiesel [28] proved that if Lusin's theorem for sub-additive capacities v in a compact metric space holds, then v is continuous from above. These scholars investigated Lusin's theorem under fuzzy measures and non-additive measures. The conditions of classical Lusin's theorem are sufficient and necessary. However, most scholars only provided sufficient conditions for Lusin's theorem. A lot of surveys showed that human uncertainty does not behave like fuzziness and non-additive measures. The debate focus is that the measure of union of events is not necessarily the maximum of measures of individual events (see, e.g., Liu [29]). Uncertainty theory, as an important branch of mathematics, is an important tool for solving various problems. However, Lusin's theorem for uncertain variables has been rarely studied. Only Tian, Zong, and Hu [23] showed that Lusin's theorem for complex uncertain variables holds if the uncertain measure is strongly order continuous. In our paper, we further derive the sufficient and necessary condition of Lusin's theorem for uncertain variables under uncertainty theory. We have essentially resolved the problem of Lusin's theorem in the framework of uncertainty theory.

In this paper, our main result is Lusin's theorem for uncertain variables in a compact metric space. In order to study the necessary and sufficient condition of this main result, we give three pairs of equivalence conditions for the continuity of uncertain measures and show that these conditions and the condition: if $\{\Lambda_n\}$ is a sequence of open sets and $\Lambda_n \searrow \emptyset$, then $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0$, are also equivalent in compact metric spaces. The remainder of this paper is organized as follows: in Section 2, we prove several equivalent conditions of the continuity for uncertain measures. In Section 3, we investigate the necessary and sufficient condition of Lusin's theorem for uncertain variables in a compact metric space including the proof, and give some examples for this Lusin's theorem. In Section 4, we present the conclusions of this paper and our future research plan. In Appendix A, some of Lusin's theorems for fuzzy measure spaces, monotone measure spaces, and capacities are reviewed. In Appendix B, we introduce some definitions and theorems in uncertainty theory that may be used in the paper.

2. Continuity Conditions of Uncertain Measures

In this section, we give some equivalent continuous conditions for uncertain measures.

Lemma 1. *Suppose that $\mathcal{M}$ is an uncertain measure on Γ and $\{\Lambda_n\}$ is a sequence of events. Then, the following continuity conditions:*

(a) *Strongly order continuous: if $\Lambda_n \searrow \Lambda$, and $\mathcal{M}\{\Lambda\} = 0$, then $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0$;*

(b) *Strongly continuous: if $\Lambda_n \nearrow \Lambda$, and $\mathcal{M}\{\Lambda\} = 1$, then $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 1$;*

(c) *Continuous from above: if $\Lambda_n \searrow \Lambda$, then $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = \mathcal{M}\{\Lambda\}$;*

(d) *Continuous from below: if $\Lambda_n \nearrow \Lambda$, then $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = \mathcal{M}\{\Lambda\}$;*

(e) *Continuous from above at $\emptyset$: if $\Lambda_n \searrow \emptyset$, then $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0$;*

(f) *Continuous from below at Γ: if $\Lambda_n \nearrow \Gamma$, then $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 1$*
are equivalent.

Proof. "(a)⇒(e)". Suppose that $\Lambda_n \searrow \Lambda = \emptyset$; by the fact that $\mathcal{M}\{\emptyset\} = 0$, we have

$$\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0, \tag{1}$$

that is, for $\Lambda_n \searrow \emptyset$, we can obtain $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0$. So, from condition (a), we have that it is continuous from above at $\emptyset$.

"(e)⇒(c)". Suppose that $\Lambda'_n \searrow \emptyset$, $\lim_{n\to\infty} \mathcal{M}\{\Lambda'_n\} = 0$. As $\Lambda_n \searrow \Lambda$, we have

$$\Lambda_n - \Lambda \searrow \emptyset. \tag{2}$$

Hence, it follows that

$$\lim_{n\to\infty} \mathcal{M}\{\Lambda_n - \Lambda\} = 0. \tag{3}$$

So, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that

$$\mathcal{M}\{\Lambda_n - \Lambda\} \leq \varepsilon \tag{4}$$

for all $n \geq n_0$. Furthermore, it is clear from the subadditivity of $\mathcal{M}$ that

$$\mathcal{M}\{\Lambda_n\} - \mathcal{M}\{\Lambda\} \leq \mathcal{M}\{\Lambda_n - \Lambda\} \leq \varepsilon \tag{5}$$

for all $n \geq n_0$. That is,

$$\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = \mathcal{M}\{\Lambda\}. \tag{6}$$

So, from condition (e), we can obtain that $\mathcal{M}$ is continuous from above.

"(c)⇒(d)". Suppose that $\Lambda_n \nearrow \Lambda$. Let $\Lambda'_n := \Gamma - \Lambda_n$ for $n = 1, 2, \ldots$, then $\Lambda'_n \searrow \Lambda' := \Gamma - \Lambda$. Using the duality of $\mathcal{M}$ and the condition (c), we have

$$\begin{aligned} 1 - \mathcal{M}\{\Lambda\} = \mathcal{M}\{\Lambda'\} &= \lim_{n\to\infty} \mathcal{M}\{\Lambda'_n\} \\ = \lim_{n\to\infty} \mathcal{M}\{\Gamma - \Lambda_n\} &= 1 - \lim_{n\to\infty} \mathcal{M}\{\Lambda_n\}. \end{aligned} \tag{7}$$

Hence,

$$\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = \mathcal{M}\{\Lambda\}. \tag{8}$$

So, from condition (c), it follows that $\mathcal{M}$ is continuous from below.

"(d)⇒(b)". It is obvious.

"(b)⇒(a)". Suppose that $\Lambda_n \searrow \Lambda$, and $\mathcal{M}\{\Lambda\} = 0$. Let $\Lambda'_n := \Gamma - \Lambda_n$ for $n = 1, 2, \ldots$, then $\Lambda'_n \nearrow \Lambda' := \Gamma - \Lambda$ and $\mathcal{M}\{\Lambda'\} = 1$. Using the duality of $\mathcal{M}$ and the condition (b), it is easy to verify that

$$1 = \lim_{n\to\infty} \mathcal{M}\{\Lambda'_n\} = \lim_{n\to\infty} \mathcal{M}\{\Gamma - \Lambda_n\} = 1 - \lim_{n\to\infty} \mathcal{M}\{\Lambda_n\}. \tag{9}$$

Hence,

$$\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0. \tag{10}$$

So, from condition (b), we can see that it is strongly order continuous.

From the above argument, we show that (a), (b), (c), (d), and (e) are equivalent. In the following, in order to show that (a), (b), (c), (d), (e), and (f) are equivalent, we just have to prove that (e) and (f) are equivalent.

"(e)⇒(f)". Suppose that $\Lambda_n \nearrow \Gamma$. Let $\Lambda'_n := \Gamma - \Lambda_n$ for $n = 1, 2, \ldots$, then $\Lambda'_n \searrow \emptyset$. From the duality of $\mathcal{M}$ and the condition (e), we know that

$$0 = \lim_{n\to\infty} \mathcal{M}\{\Lambda'_n\} = \lim_{n\to\infty} \mathcal{M}\{\Gamma - \Lambda_n\} = 1 - \lim_{n\to\infty} \mathcal{M}\{\Lambda_n\}. \tag{11}$$

Hence,

$$\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 1. \tag{12}$$

So, from condition (e), we have that it is continuous from below at Γ.

"(f)⇒(e)". Suppose that $\Lambda_n \searrow \emptyset$. Let $\Lambda'_n := \Gamma - \Lambda_n$, for $n = 1, 2, \ldots$, then $\Lambda'_n \nearrow \Gamma$. It follows from the duality of $\mathcal{M}$ and the condition (f) that

$$1 = \lim_{n\to\infty} \mathcal{M}\{\Lambda'_n\} = \lim_{n\to\infty} \mathcal{M}\{\Gamma - \Lambda_n\} = 1 - \lim_{n\to\infty} \mathcal{M}\{\Lambda_n\}. \tag{13}$$

Hence,

$$\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0. \tag{14}$$

So, from condition (f), we can see that it is continuous from above at $\emptyset$.

The proof of Lemma 1 is completed. □

Remark 1. *(1) Lemma 1 presents three pairs of continuity conditions for uncertain measures: (a) strong order continuous, (b) strong continuous, (c) continuous from above, (d) continuous from below, (e) continuous from above at $\emptyset$, and (f) continuous from below at Γ, and shows that they are equivalent. From Lemma 1, we know that if, for an uncertain events sequence $\{\Lambda_n\}$ in Γ, $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = \mathcal{M}\{\lim_{n\to\infty} \Lambda_n\}$ holds, then for the sequence $\{\Lambda_n^c := \Gamma - \Lambda_n\}$, $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n^c\} = \mathcal{M}\{\lim_{n\to\infty} \Lambda_n^c\}$ still holds. It is clear that each pair of the above reveals duality. In fact, this duality is a kind of symmetry between objects.*

(2) In probability theory, P not only satisfies continuity from above, but also satisfies continuity from below; thus, P is continuous. Therefore, similar to probability, an uncertain measure $\mathcal{M}$ is said to be continuous if it satisfies both continuity from above and continuity from below. By Lemma 1, if $\mathcal{M}$ satisfies any of the conditions (a), (b), (c), (d), (e), and (f), then it is continuous from above and continuous from below. Thus, $\mathcal{M}$ is continuous if and only if it satisfies any of the above conditions.

In order to study the sufficient and necessary condition of Lusin's theorem for uncertain variables, we assume that (Γ, d) is a compact metric space. The following part of the paper is developed under this assumption.

Lemma 2. *Let (Γ, d) be a compact metric space, and $\mathcal{L}$ be the Borel σ-algebra on the non-empty set Γ. Suppose that $\mathcal{M}$ is an uncertain measure on Γ. Then, we have the following condition:*

(g) If $\{\Lambda_n\}$ is a sequence of open sets and $\Lambda_n \searrow \emptyset$, then $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0$

is equivalent to the condition (e).

Proof. "(g)⇒(e)". For the sequence $\{\Lambda_n\}$ of events satisfying $\Lambda_n \searrow \emptyset$, we can always find the sequence $\{\Lambda'_n\}$ of open sets satisfying $\Lambda_n \subseteq \Lambda'_n$ and $\Lambda'_n \searrow \emptyset$. By the condition (g), we have $\lim_{n\to\infty} \mathcal{M}\{\Lambda'_n\} = 0$. Then, we can easily obtain that $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0$. Hence,

$$\Lambda_n \searrow \emptyset \Rightarrow \lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0. \tag{15}$$

"(e)⇒(g)". It is obvious.

The proof of Lemma 2 is completed. □

Remark 2. *Let* (Γ, d) *be a compact metric space and* $\mathcal{L}$ *be the Borel* σ*-algebra on the non-empty set* Γ*. Suppose that* $\mathcal{M}$ *is an uncertain measure on* Γ*. Then, the conditions (a), (b), (c), (d), (e), (f), and (g) are equivalent.*

3. Lusin's Theorem for Uncertain Variables

In this section, we provide the sufficient and necessary condition of Lusin's theorem for uncertain variables in a compact metric spaces. Before proving Lusin's theorem of uncertain variables, we give the definition and conditions of the regularity of uncertain measures.

Definition 1. *(see [23]) An uncertain measure* $\mathcal{M}$ *is called regular if the following holds: for any* Λ *in* $\mathcal{L}$ *and any* $\varepsilon > 0$*, there exists a closed set* F *and an open set* G*, such that* $F \subseteq \Lambda \subseteq G$*, and* $\mathcal{M}\{G - F\} < \varepsilon$*.*

Lemma 3. *Let* (Γ, d) *be a compact metric space and* $\mathcal{L}$ *be the Borel* σ*-algebra on the non-empty set* Γ*. Suppose that* $\mathcal{M}$ *is an uncertain measure on* Γ *satisfying condition (g). Then,* $\mathcal{M}$ *is regular.*

Proof. We can easily prove Lemma 3 by using a similar method to that of Lemma 3.1 in Tian, Zong, and Hu [23]. Therefore, we omit this proof. □

Theorem 1. *(Lusin's theorem) Let* (Γ, d) *be a compact metric space and* $\mathcal{L}$ *be the Borel* σ*-algebra on the non-empty set* Γ*. Suppose that* $\mathcal{M}$ *is an uncertain measure on* Γ*. Then, for any given uncertain variable* ξ *and for any* $\varepsilon > 0$*, there exists a compact set* $K = K(\xi, \varepsilon) \subseteq \Gamma$*, such that* $\mathcal{M}\{\Gamma - K\} \leq \varepsilon$ *and* ξ *is continuous on* K *if and only if* $\mathcal{M}$ *satisfies the condition (g).*

Proof. "⇐" The proof method refers to the classical Lusin's theorem for measure theory (see, e.g., [30], Proof of Theorem 7.4.3).

(1) Suppose that ξ is an uncertain variable, having the form of an elementary function, i.e., $\xi = \sum_{n=1}^{\infty} a_n I_{A_n}$, where $\{a_n\}$ is a sequence of real numbers and $\{A_n\}$ is a countable partition of Γ. By Lemma 3 and Definition 1, for any $\varepsilon > 0$, there exist open sets G_n and closed sets F_n, such that $F_n \subseteq A_n \subseteq G_n$ and

$$\mathcal{M}\{G_n - F_n\} \leq 2^{-n-2}\varepsilon. \tag{16}$$

Thus, in particular,

$$\Gamma = \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} G_n \subseteq \Gamma, \tag{17}$$

so that $\Gamma = \bigcup_{n=1}^{\infty} G_n$.

Noting the fact that

$$\bigcup_{n=1}^{\infty} F_n - \bigcup_{n=1}^{N} F_n \searrow \emptyset,$$

when $N \to \infty$ and by Lemma 2, we have

$$\lim_{N\to\infty} \mathcal{M}\left\{ \bigcup_{n=1}^{\infty} F_n - \bigcup_{n=1}^{N} F_n \right\} = 0. \tag{18}$$

Hence, there exists a positive integer N_0, such that

$$\mathcal{M}\left\{ \bigcup_{n=1}^{\infty} F_n - \bigcup_{n=1}^{N} F_n \right\} \leq \frac{\varepsilon}{4} \tag{19}$$

for all $N \geq N_0$. For any $\varepsilon > 0$, by (16), (19), and the subadditivity of $\mathcal{M}$, we have

$$\begin{aligned}
&\mathcal{M}\left\{ \bigcup_{n=1}^{\infty} G_n - \bigcup_{n=1}^{N} F_n \right\} \\
&= \mathcal{M}\left\{ \left(\bigcup_{n=1}^{\infty} G_n - \bigcup_{n=1}^{\infty} F_n \right) \bigcup \left(\bigcup_{n=1}^{\infty} F_n - \bigcup_{n=1}^{N} F_n \right) \right\} \\
&\leq \mathcal{M}\left\{ \left(\bigcup_{n=1}^{\infty} (G_n - F_n) \right) \bigcup \left(\bigcup_{n=1}^{\infty} F_n - \bigcup_{n=1}^{N} F_n \right) \right\} \\
&\leq \mathcal{M}\left\{ \bigcup_{n=1}^{\infty} (G_n - F_n) \right\} + \mathcal{M}\left\{ \bigcup_{n=1}^{\infty} F_n - \bigcup_{n=1}^{N} F_n \right\} \\
&\leq \sum_{n=1}^{\infty} \mathcal{M}\{G_n - F_n\} + \mathcal{M}\left\{ \bigcup_{n=1}^{\infty} F_n - \bigcup_{n=1}^{N} F_n \right\} \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}
\end{aligned} \tag{20}$$

for all $N \geq N_0$. It follows from (20) that for all $N \geq N_0$,

$$\begin{aligned}
&\mathcal{M}\left\{ \bigcup_{n=N+1}^{\infty} A_n \right\} = \mathcal{M}\left\{ \Gamma - \bigcup_{n=1}^{N} A_n \right\} = \mathcal{M}\left\{ \bigcup_{n=1}^{\infty} G_n - \bigcup_{n=1}^{N} A_n \right\} \\
&\leq \mathcal{M}\left\{ \bigcup_{n=1}^{\infty} G_n - \bigcup_{n=1}^{N} F_n \right\} \leq \frac{\varepsilon}{2}.
\end{aligned} \tag{21}$$

Hence, by (16), (21), and using the subadditivity of $\mathcal{M}$, we can obtain that for all $N \geq N_0$,

$$\begin{aligned}
&\mathcal{M}\left\{ \Gamma - \bigcup_{n=1}^{N} F_n \right\} = \mathcal{M}\left\{ \bigcup_{n=1}^{\infty} A_n - \bigcup_{n=1}^{N} F_n \right\} \\
&= \mathcal{M}\left\{ \left(\bigcup_{n=1}^{N} A_n - \bigcup_{n=1}^{N} F_n \right) \cup \left(\bigcup_{n=N+1}^{\infty} A_n - \bigcup_{n=1}^{N} F_n \right) \right\} \\
&\leq \mathcal{M}\left\{ \left(\bigcup_{n=1}^{N} A_n - \bigcup_{n=1}^{N} F_n \right) \cup \bigcup_{n=N+1}^{\infty} A_n \right\} \leq \mathcal{M}\left\{ \bigcup_{n=1}^{N} A_n - \bigcup_{n=1}^{N} F_n \right\} + \mathcal{M}\left\{ \bigcup_{n=N+1}^{\infty} A_n \right\} \\
&\leq \mathcal{M}\left\{ \bigcup_{n=1}^{N} G_n - \bigcup_{n=1}^{N} F_n \right\} + \mathcal{M}\left\{ \bigcup_{n=N+1}^{\infty} A_n \right\} \leq \mathcal{M}\left\{ \bigcup_{n=1}^{N} (G_n - F_n) \right\} + \mathcal{M}\left\{ \bigcup_{n=N+1}^{\infty} A_n \right\} \\
&\leq \sum_{n=1}^{N} \mathcal{M}\{G_n - F_n\} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{22}$$

That is, for any $\varepsilon > 0$, we have $\mathcal{M}\left\{ \Gamma - \cup_{n=1}^{N} F_n \right\} \leq \varepsilon$ for all $N \geq N_0$.

As F_n is a closed set satisfying $F_n \subseteq \Gamma$ for any fixed $n \in \mathbb{N}$, and Γ is a compact set, we can see that F_n is a compact set. $\bigcup_{n=1}^{N_0} F_n$ is the finite union of compact sets F_n, so $\bigcup_{n=1}^{N_0} F_n$ is compact. Take $K := \bigcup_{n=1}^{N_0} F_n$, then K is a compact set and $\mathcal{M}\{\Gamma - K\} \leq \varepsilon$. Since ξ is a constant on each F_n, it is obvious that it is continuous on K.

(2) Let ξ be an arbitrary uncertain variable. Then, ξ can be expressed as the uniform limit of a sequence $\{\xi_n\}$ of uncertain variables, where each ξ_n has only countably many values, i.e., each ξ_n is an elementary function. According to what we have just proved, for each $n \in \mathbb{N}$, we can find a compact subset K_n of Γ, such that $\mathcal{M}\{\Gamma - K_n\} < 2^{-n}\varepsilon$ and ξ_n is continuous on K_n. Take $K := \bigcap_{n=1}^{\infty} K_n$. As K is the infinite intersection of compact subsets K_n, K is a compact subset of Γ. And, we can obtain

$$\begin{aligned}\mathcal{M}\{\Gamma - K\} &= \mathcal{M}\left\{\Gamma - \bigcap_{n=1}^{\infty} K_n\right\} = \mathcal{M}\left\{\bigcup_{n=1}^{\infty} (\Gamma - K_n)\right\} \\ &\leq \sum_{n=1}^{\infty} \mathcal{M}\{\Gamma - K_n\} \leq \sum_{n=1}^{\infty} 2^{-n}\varepsilon = \varepsilon.\end{aligned} \tag{23}$$

That is, for any $\varepsilon > 0$, we have $\mathcal{M}\{\Gamma - K\} \leq \varepsilon$.

At last, we prove that ξ is continuous on K. By the above argument, we know that $\{\xi_n\}$ is continuous, and converges uniformly to ξ on K. From the condition that $\{\xi_n\}$ is continuous, we have that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\xi_n(\gamma) - \xi_n(\gamma_0)| \leq \frac{\varepsilon}{3} \tag{24}$$

whenever $|\gamma - \gamma_0| < \delta$. From the condition that $\{\xi_n\}$ converges uniformly to ξ on K, we have that, for any above $\varepsilon > 0$ and any $\gamma \in K$, there exists a positive integer n_0 such that

$$|\xi_n(\gamma) - \xi(\gamma)| \leq \frac{\varepsilon}{3} \tag{25}$$

for all $n \geq n_0$. That is, for any $\varepsilon > 0$ and any $\gamma, \gamma_0 \in K$, there exists a common positive integer n_0 and $\delta > 0$, such that

$$|\xi_{n_0}(\gamma) - \xi_{n_0}(\gamma_0)| \leq \frac{\varepsilon}{3} \text{ and } |\xi(\gamma) - \xi_{n_0}(\gamma)| \leq \frac{\varepsilon}{3} \tag{26}$$

whenever $|\gamma - \gamma_0| < \delta$. So, it follows that

$$\begin{aligned}&|\xi(\gamma) - \xi(\gamma_0)| = |\, \xi(\gamma) - \xi_{n_0}(\gamma) + \xi_{n_0}(\gamma) - \xi_{n_0}(\gamma_0) + \xi_{n_0}(\gamma_0) - \xi(\gamma_0)| \\ &\leq |\xi(\gamma) - \xi_{n_0}(\gamma)| + |\xi_{n_0}(\gamma) - \xi_{n_0}(\gamma_0)| + |\xi_{n_0}(\gamma_0) - \xi(\gamma_0)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned} \tag{27}$$

That is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\xi(\gamma) - \xi(\gamma_0)| \leq \varepsilon \tag{28}$$

whenever $|\gamma - \gamma_0| < \delta$. Thus, ξ is continuous on K.

"$\Rightarrow$" Fix a sequence $\{\Lambda_n\}$ of open sets such that $\Lambda_n \searrow \emptyset$. Now, we define the sequence of uncertain variables $\{\xi_n\}$ via

$$\xi_n := \begin{cases} 1, & \text{if } \gamma \in \Lambda_n; \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

Then, for each $n \in \mathbb{N}$, there exists a compact $K_n \subseteq \Gamma$, such that

$$\mathcal{M}\{\Gamma - K_n\} \leq 2^{-n-1}\varepsilon \tag{30}$$

and ξ_n is continuous in K_n. Our goal is to prove that $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0$, i.e., for any $\varepsilon > 0$, there exists a positive integer n_0, such that $\mathcal{M}\{\Lambda_n\} \leq \varepsilon$ for all $n \geq n_0$.

For each $n \in \mathbb{N}$, as ξ_n is constant on Λ_n, ξ_n is continuous on Λ_n. But ξ_n is not continuous on $\overline{\Lambda}_n$, because the value of ξ_n on $\partial\Lambda_n$ is 0, where $\overline{\Lambda}_n$ is the closure of Λ_n, and $\partial\Lambda_n$ is the boundary of Λ_n. This means that $\partial\Lambda_n \cap K_n = \emptyset$. Furthermore, $\partial\Lambda_n$ is a compact set, since $\partial\Lambda_n$ is a closed set, $\partial\Lambda_n \subseteq \Gamma$, and Γ is a compact set. Define

$$d_n := \inf\{d(\gamma, \gamma_0) | \gamma \in \partial\Lambda_n, \gamma_0 \in K_n\}, \ \forall n \in \mathbb{N}. \tag{31}$$

Obviously, $d_n > 0$. For each $n \in \mathbb{N}$, choosing a common constant α that satisfies $0 < \alpha < 1$, denote

$$B_n := \{\gamma \in \Lambda_n \mid d(\gamma, \partial\Lambda_n) \geq \alpha d_n\}. \tag{32}$$

Obviously, B_n is a compact set. It follows from (31) and (32) that, for each $n \in \mathbb{N}$, $K_n \cap \Lambda_n = K_n \cap B_n$. Then, $K_n \cap \Lambda_n$ is compact for each $n \in \mathbb{N}$.

Next, we choose a new sequence $\{\tilde{K}_n\}$ of compact sets, where $\tilde{K}_n$ can be expressed as $\tilde{K}_n := \bigcap_{m=1}^{n} K_m \subseteq K_n$. Noting the fact that $\tilde{K}_n \cap \Lambda_n$ is a finite intersection of compact sets, we can obtain that $\tilde{K}_n \cap \Lambda_n$ is also a compact set. As $\tilde{K}_n \cap \Lambda_n \subseteq \Lambda_n \searrow \emptyset$, we have $\tilde{K}_n \cap \Lambda_n \searrow \emptyset$ when $n \to \infty$. Because $\tilde{K}_n \cap \Lambda_n$ is a compact set and decreasing, there exists a positive integer n_0, such that $\tilde{K}_n \cap \Lambda_n = \emptyset$ for all $n \geq n_0$. Hence,

$$\mathcal{M}\{\tilde{K}_n \cap \Lambda_n\} = \mathcal{M}\{\emptyset\} = 0 \tag{33}$$

for all $n \geq n_0$, i.e., for any $\varepsilon > 0$, we have

$$\mathcal{M}\{\tilde{K}_n \cap \Lambda_n\} \leq \frac{\varepsilon}{2} \tag{34}$$

for all $n \geq n_0$. Furthermore, by (30) and using the subadditivity of $\mathcal{M}$, we can see that

$$\begin{aligned} &\mathcal{M}\{\Lambda_n - \tilde{K}_n\} \leq \mathcal{M}\{\Gamma - \tilde{K}_n\} = \mathcal{M}\left\{\Gamma - \bigcap_{m=1}^{n} K_m\right\} \\ &= \mathcal{M}\left\{\bigcup_{m=1}^{n} (\Gamma - K_m)\right\} \leq \sum_{m=1}^{n} \mathcal{M}\{\Gamma - K_m\} \leq \sum_{m=1}^{n} 2^{-m-1}\varepsilon \leq \frac{\varepsilon}{2}. \end{aligned} \tag{35}$$

Hence, it follows from (34) and (35) that, for any $\varepsilon > 0$, there exists a positive integer n_0, such that

$$\begin{aligned} \mathcal{M}\{\Lambda_n\} &= \mathcal{M}\left\{\left(\tilde{K}_n \bigcap \Lambda_n\right) \bigcup (\Lambda_n - \tilde{K}_n)\right\} \\ &\leq \mathcal{M}\left\{\tilde{K}_n \bigcap \Lambda_n\right\} + \mathcal{M}\{\Lambda_n - \tilde{K}_n\} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned} \tag{36}$$

for all $n \geq n_0$, by the subadditivity of $\mathcal{M}$.

The proof of Theorem 1 is completed. □

Remark 3. *(1) In Theorem 1, the uncertain variable ξ can have a symmetric distribution, and can also have an asymmetric distribution.*

(2) Let (Γ, d) be a compact metric space and $\mathcal{L}$ be the Borel σ-algebra on the non-empty set Γ. Suppose that $\mathcal{M}$ is an uncertain measure on Γ. From Remark 2, it follows that the Lusin's theorem for uncertain variables holds if and only if $\mathcal{M}$ satisfies any of the continuity conditions (a), (b), (c), (d), (e), (f), and (g).

In the following, several examples are provided to illustrate applications of Theorem 1.

Example 1. *The triplet* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is an uncertain space, where* $\Gamma = [0,1]$ *and* $\mathcal{L}$ *be the Borel* σ*-algebra on* Γ*. Let* $A = \{\gamma_1, \gamma_2, \dots \gamma_n, \dots\} = \left\{1, \frac{1}{2}, \dots \frac{1}{n}, \dots\right\}$ *and the uncertain measure*

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda \cap A} \dfrac{1}{n+1}, & \text{if } \sup\limits_{\gamma_n \in \Lambda \cap A} \dfrac{1}{n+1} < 0.5; \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c \cap A} \dfrac{1}{n+1}, & \text{if } \sup\limits_{\gamma_n \in \Lambda^c \cap A} \dfrac{1}{n+1} < 0.5; \\ 1, & \text{if } \Lambda \cap A = A; \\ 0, & \text{if } \Lambda \cap A = \emptyset. \end{cases} \tag{37}$$

Suppose that ξ *is any uncertain variable. Then, for any given* $\varepsilon > 0$*, there exists a compact set* K*, such that* $\mathcal{M}\{\Gamma - K\} \leq \varepsilon$ *and* ξ *is continuous on* K*.*

Proof. By Theorem 1, we just only need to prove that for any sequence $\{\Lambda_n\}$ of open sets, and $\Lambda_n \searrow \emptyset$, we have $\lim\limits_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0$.

Suppose that $\{\Lambda_n\}$ is a sequence of open sets, and $\Lambda_n \searrow \emptyset$. Now, we prove $\lim\limits_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0$ by the reduction to absurdity.

Assume that

$$\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = a > 0. \tag{38}$$

Then, for a given $\delta > 0$ that satisfies $a - \delta > 0$, there exists a positive integer N_0, such that

$$\mathcal{M}\{\Lambda_n\} \geq a - \delta > 0 \tag{39}$$

for all $n \geq N_0$. Hence, for any $n \geq N_0$, there exists at least one $\gamma_k \in \Lambda_n \cap A$. Noting the fact that $\Lambda_n \searrow \emptyset$ when $n \to \infty$, we can obtain that $\Lambda_n \cap A \searrow \emptyset$. Therefore, $\gamma_k \in \emptyset$. This contradicts to the fact that there are no elements in $\emptyset$. Thus, $\lim\limits_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0$.

We complete the proof of Example 1. □

Example 2. *The triplet* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is an uncertain space, where* $\Gamma = [a, b]$*, and* a*,* b *are two real numbers satisfying* $a < b$*, and* $\mathcal{L}$ *be the Borel* σ*-algebra on* Γ*. The uncertain measure* $\mathcal{M}$ *is defined by the following uniform distribution with parameters* a *and* b*, i.e.,*

$$\mathcal{M}\{\Lambda\} = \int_\Lambda \frac{1}{b-a} d\gamma. \tag{40}$$

Suppose that ξ *is any uncertain variable. Then, for any given* $\varepsilon > 0$*, there exists a compact set* K*, such that* $\mathcal{M}\{\Gamma - K\} \leq \varepsilon$ *and* ξ *is continuous on* K*.*

Proof. We can easily prove that the uncertain measure $\mathcal{M}$ is also a probability measure. Therefore, the uncertain measure $\mathcal{M}$ satisfies the condition (c). Thus, by Lemma 1, Lemma 2 and Theorem 1, we can complete the proof of Example 2. □

At last, we give an example to show that Lusin's theorem for uncertain variables does not necessarily hold if $\mathcal{M}$ is not continuous.

Example 3. *The triplet* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is an uncertain space, where* $\Gamma = (0,1)$*, and* $\mathcal{L}$ *is the Borel* σ*-algebra on* Γ*. The uncertain measure* $\mathcal{M}$ *is defined by*

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1, & \text{if } \Lambda = \Gamma \\ 0.5, & \text{otherwise.} \end{cases} \tag{41}$$

The uncertain variable is defined by

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases} \tag{42}$$

Then, for the sequence $\{\Lambda_n := (0, 1 - \frac{1}{n})\}$, *it is clear that* $\Lambda_n \nearrow \Gamma$. *But, we have* $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0.5$. *By Lemma 1, we know that* $\mathcal{M}$ *is not continuous. It is easy to see that* ξ *is not continuous.*

4. Conclusions

Lusin's theorem, a classical theorem in measure theory, reveals the close connection between measurable functions and continuity, which makes many problems concerning measurable functions reduce to the discussion of continuous functions. Uncertainty theory, as an important branch of mathematics, has developed rapidly in the fields of uncertain finance, uncertain statistics, uncertainty risk analysis, and so on. However, so far, little attention has been paid to Lusin's theorem for uncertain variables. In this paper, we investigate the sufficient and necessary condition for Lusin's theorem to hold for uncertain variables. Before proving Lusin's theorem, we first give three pairs of continuity conditions for uncertain measures: strong order continuous, strong continuous, continuous from above, continuous from below, continuous from above at $\emptyset$, and continuous from below at Γ, and prove that they are equivalent. It follows from Remark 1 that every pair mentioned above discloses duality. As we all know, this duality is a sort of symmetry between objects, so every pair can be symmetrically transformed into each other, for example, in Lemma 1, condition (c) can transform to (d), and (d) can also transform to (c). Then, we prove that above conditions and the condition: if $\{\Lambda_n\}$ is a sequence of open sets and $\Lambda_n \searrow \emptyset$, then $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0$, are equivalent in compact metric spaces. Finally, a sufficient and necessary condition of Lusin's theorem for uncertain variables is shown. Furthermore, we obtain seven sufficient and necessary conditions for uncertain Lusin's theorem. And some examples are provided to illustrate applications of Lusin's theorem for uncertain variables.

In this paper, in order to prove that our Lusin's theorem holds, we need the continuity condition $\Lambda_n \searrow \emptyset$, then $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0$. In classical probability theory, for Lusin's theorem, this continuity condition is essential. Based on the condition that (Γ, d) is a compact metric space, we prove that Lusin's theorem for uncertain variables holds is equivalent to the continuity condition $\Lambda_n \searrow \emptyset$ holding, thus $\lim_{n\to\infty} \mathcal{M}\{\Lambda_n\} = 0$.

In many cases, randomness and human uncertainty coexist in a complex system. In order to better deal with this case, Liu [31] presented a new concept of uncertain random variable, and combined probability measure and uncertain measure into a chance measure in 2013. Meanwhile, an operational law of uncertain random variables, an expected value formula and uncertain random programming were firstly provided based on chance theory by Liu [32]. Lusin's theorem for uncertain random variables in chance measure spaces has not yet been thoroughly examined. In the forthcoming study, we will explore the conditions that make Lusin's theorem of uncertain random variables hold.

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Appendix A. Some Results of Lusin's Theorem

In the following, we review the results of Lusin's theorem on fuzzy measure spaces and monotone measure spaces, as well as capacities.

Definition A1. *(See [24,33]) A set function $\mu : \mathcal{B} \to [0, \infty]$ is called a monotone measure, if it satisfies that*

(1) $\mu(\emptyset) = 0$ (vanishing at $\emptyset$);
(2) $\mu(A) \leq \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{B}$ (monotonicity).
μ is called a fuzzy measure if it fulfills (1), (2) and the following two properties:
(3) If $A_n \nearrow A$, then $\lim\limits_{n\to\infty} \mu(A_n) = \mu(A)$ (continuity from below);
(4) If $A_n \searrow A$, then $\lim\limits_{n\to\infty} \mu(A_n) = \mu(A)$ (continuity from above).

Definition A2. *(See [34]) μ is called weakly null-additive, if for any $E, F \in \mathcal{B}$,*

$$\mu(E) = \mu(F) = 0 \Rightarrow \mu(E \cup F) = 0.$$

Theorem A1. *(See [24]) (Lusin's theorem on fuzzy measure spaces) Let μ be weakly null-additive fuzzy measure on $\mathcal{B}$. If f is a real-valued measurable function on X then, for every $\varepsilon > 0$, there exists a closed subset $F_\varepsilon \in \mathcal{B}$ such that f is continuous on F_ε and $\mu(X - F_\varepsilon) < \varepsilon$.*

Definition A3. *(See [35]) A set function $\mu : \mathcal{B} \to [0, +\infty]$ is said to fulfill condition (E), if for every double sequence $\{E_n^{(m)} | n, m \in \mathbb{N}\} \subset \mathcal{B}$ satisfying the conditions: for any fixed $m = 1, 2, \ldots$,*

$$E_n^{(m)} \searrow E^{(m)} \text{ and } \mu(\bigcup_{m=1}^{\infty} E^{(m)}) = 0,$$

there exist increasing sequences $\{n_i\}_{i\in\mathbb{N}}$ and $\{m_i\}_{i\in\mathbb{N}}$ of natural numbers, such that

$$\lim_{k\to\infty} \mu(\bigcup_{i=k}^{\infty} E_{n_i}^{(m_i)}) = 0.$$

Definition A4. *(See [36]) μ is said to have pseudometric generating property (for short (p.g.p.)), if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $E, F \in \mathcal{B}, \mu(E) \vee \mu(F) < \delta$ implies $\mu(E \cup F) < \varepsilon$.*

Theorem A2. *(See [26]) (Lusin's theorem on monotone measure spaces) Let a monotone measure μ fulfill condition (E) and have p.g.p, and f be a real-valued measurable function on X. Then, for each $\varepsilon > 0$, there exists a closed subset $F_\varepsilon \in \mathcal{B}$ such that $\mu(X - F_\varepsilon) < \varepsilon$ and f is continuous on F_ε.*

Definition A5. *(see [28]) Let (X, d) be a metric space and $\mathcal{B}$ denote its Borel σ-algebra.*
A capacity v: $\mathcal{B} \to \mathbb{R}$ is a set function satisfying:

(1) $v(\emptyset) = 0$, $v(X) = 1$;
(2) If $A \subset B$, then $v(A) \leq v(B)$;
(3) If $A_n \nearrow A$, then $\lim_{n\to\infty} v(A_n) = v(A)$.
A capacity v is said to be sub-additive, if $v(A \cup B) \leq v(A) + v(B)$ for all $A, B \in \mathcal{B}$.

Theorem A3. *(See [28]) (Lusin's theorem for capacities) Let (X, d) be a compact metric space and v be a sub-additive capacity satisfying the following property: for all $\varepsilon > 0$ and for all Borel measurable* *function $f : X \to \mathbb{R}$, there exists a compact set $K = K(f, \varepsilon) \in \mathcal{B}$, such that $v(X - K) \leq \varepsilon$ and f is continuous on K.*

Then, for all $\{O_n\}_{n\in\mathbb{N}}$, O_n open, and $O_n \searrow O$, we have $\lim_{n\to\infty} v(O_n) = v(O)$.

Appendix B. Uncertain Variable

In this section, some definitions and theorems of uncertainty theory that may be used in this paper are introduced.

Definition A6. *(See [8]) Let $\mathcal{L}$ be a σ-algebra on a non-empty set Γ. A set function is $\mathcal{M}$ called an uncertain measure if it satisfies the following axioms:*

Axiom 1 (normality axiom). $\mathcal{M}(\Gamma) = 1$ for the universal set;
Axiom 2 (duality axiom). $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$, where $\Lambda^c := \Gamma - \Lambda$;
Axiom 3 (subadditivity axiom). For every countable sequence of $\{\Lambda_i\} \subseteq \mathcal{L}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertain space, and each element Λ in $\mathcal{L}$ is called an event.
Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ Write

$$\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots$$

which is the set of all ordered tuples of the form $(\gamma_1, \gamma_2, \ldots)$, where $\gamma_k \in \Gamma_k$ for $k = 1, 2, \ldots$ A measurable rectangle in Γ is a set

$$\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots$$

where $\Lambda_k \in \mathcal{L}_k$ for $k = 1, 2, \ldots$. The smallest σ-algebra containing all measurable rectangles of Γ is called the product σ-algebra, denoted by

$$\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots .$$

Then, the product uncertain measure $\mathcal{M}$ on the product σ-algebra $\mathcal{L}$ is defined by the following product axiom in [9].

Axiom 4 (product axiom). Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Theorem A4. *(See [8]) (Monotonicity theorem) Uncertain measure $\mathcal{M}$ is a monotone increasing set function. That is, for any events $\Lambda_1 \subseteq \Lambda_2$, we have*

$$\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}.$$

Theorem A5. *(See [8]) Suppose that $\mathcal{M}$ is an uncertain measure. Then, the empty set $\emptyset$ has an uncertain measure zero, i.e.,*

$$\mathcal{M}\{\emptyset\} = 0.$$

Theorem A6. *(See [8]) Suppose that $\mathcal{M}$ is an uncertain measure. Then, for any event Λ, we have*

$$0 \leq \mathcal{M}\{\Lambda\} \leq 1.$$

Definition A7. *(See [8]) An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set of B of real numbers, the set*

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

Definition A8. *(See [8]) The uncertainty distribution Φ of an uncertain variable ξ is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

Definition A9. *(See [9]) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Definition A10. *(See [8]) The uncertain sequence $\{\xi_n\}$ is said to be convergent almost surely (a.s.) to ξ if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{n\to\infty} |\xi_n(\gamma) - \xi(\gamma)| = 0$$

for every $\gamma \in \Lambda$. In that case, we write $\xi_n \to \xi$, a.s.

Definition A11. *(See [37]) The sequence $\{\xi_n\}$ is said to be convergent uniformly almost surely to ξ if there exists $\{E_k\}$, $\mathcal{M}\{E_k\} \to 0$ such that $\{\xi_n\}$ converges uniformly to ξ on $E_k^c := \Gamma - E_k$, for any fixed $k \in \mathbb{N}$.*

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