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Computational Analysis of the Comprehensive Lifetime Performance Index for Exponentiated Fréchet Lifetime Distribution Products with Multi-Components

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Abstract: The lifetime performance index is commonly used in the manufacturing industry to evaluate the performance of the capabilities of the production process. For products with multiple components, the comprehensive lifetime performance index, which is a monotonically increasing function of the overall process yield, is used to relate to each individual lifetime performance index. For products where the lifetime of the ith component follows an exponentiated Fréchet lifetime distribution, we examine the maximum likelihood estimators for both the comprehensive and individual lifetime performance indices based on the progressive type I interval-censored samples, deriving their asymptotic distributions. By specifying the target level for the comprehensive lifetime performance index, we can set the desired level for individual indices. A testing procedure, using the maximum likelihood estimator as the test statistic, was developed to determine if the comprehensive lifetime performance index meets the target. Given that the lifetime distribution is asymmetric, this study pertains to asymmetrical probability distributions and their applications across diverse fields. We illustrate the power analysis of this testing procedure with figures and summarize key findings. Finally, we demonstrate the application of this testing algorithm with a practical example involving two components to verify if the overall production process achieves the assigned target level.

Keywords: exponentiated Fréchet lifetime distribution; multiple production lines; progressive type I interval censoring; maximum likelihood estimator; lifetime performance index

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1. Introduction

Process capability analysis plays a crucial role in statistical quality control applications. It helps producers determine if their products meet the specified quality standards and requirements. The process capability index (PCI) is a tool used to measure and quantify this capability. Numerous widely recognized PCIs have been proposed in the literature to enhance product quality, including C_p, C_{pk}, C_{pm}, C_{pmk}, and C_L (Juran [1], Kane [2], Hsiang and Taguchi [3], Pearn et al. [4], and Montgomery [5]). Given that product lifetime is a "the-larger-the-better" quality characteristic, the PCI for unilateral tolerance, C_L, is utilized to assess its performance, also known as the lifetime performance index. This investigation focuses on the evaluation of the comprehensive lifetime performance of products with multi-components by using the lifetime performance index C_L. Using this index, Tong et al. [6] employed the uniformly minimum variance unbiased estimator (UMVUE) in the manufacturing of products with a single component. They used this estimator as the test statistic in their hypothesis testing procedure for exponential product lifetimes, relying on the complete sample data set. In real-world situations, researchers often face constraints that prevent them from consistently observing the lifetimes of all tested items. These constraints can include time limitations, budgetary and material restrictions, errors by typists or recorders, and mechanical or experimental difficulties. As a result, incomplete data, such as progressive censoring data, may be collected. For insights on handling such data, refer to Balakrishnan [7], Aggarwala [8], Balakrishnan and Aggarwala [9], Wu et al. [10], Sanjel and Balakrishnan [11], and Lee et al. [12]. To facilitate data collection, the progressive type I interval censoring scheme was adopted in numerous quality control applications. Wu and Lin [13] developed a testing algorithmic procedure to assess the lifetime performance index of products with the Weibull distribution. Wu et al. [14] proposed a computational algorithm for the evaluation of the lifetime performance index of products with Burr XII lifetime distribution under progressive type I interval censoring. Wu and Chang [15] assessed the lifetime performance index of products with lifetime following the exponentiated Fréchet lifetime distribution based on the progressive type I interval-censored sample. Previous studies have focused on the production of products containing a single component. For products with multiple components produced in multiple production lines, Wu and Chiang [16] assessed the overall lifetime performance index of Weibull products in multiple production lines. Wu et al. [17] investigated the comprehensive lifetime performance index of Burr XII products in multiple production lines. This investigation is an extension of the case study by Wu and Chang [15], where they extend beyond products with one component produced in a single production line to products with multiple components produced in multiple production lines. This extended research proposes a testing procedure for the comprehensive lifetime performance index for products with multiple components. Results show that their lifetimes follow the exponentiated Fréchet distribution based on the progressive type I interval-censored sample. Since the lifetime distribution was asymmetric, that study explored the asymmetrical probability distributions and their practical applications across various fields.

In this study, the comprehensive lifetime performance index proposed by Wu et al. [17] for products with multiple components manufactured across multiple production lines is utilized to evaluate whether the comprehensive lifetime performance of multiple components following the exponentiated Fréchet distributions meets a predetermined target level. The maximum likelihood estimator (MLE) for the lifetime performance index is used as the test statistic for the testing procedure, using a progressive type I interval-censored sample. Regarding the maximum likelihood estimation, Rytgaard and Van der Laan [18] investigated the target maximum likelihood estimation for causal inference in survival and competing risk analysis. Luca et al. [19] proposed an evolutionary optimization method for maximum likelihood and approximate maximum likelihood estimation of discrete latent variable models. For recent literature on reliability statistics, Goldengorin et al. [20] provide a comprehensive exploration of reliability in engineering and statistical contexts. Zhuang et al. [21] investigated the life prediction for a two-phase degradation model based on a reparameterized inverse Gaussian process. Xu et al. [22] proposed the Bayesian reliability assessment of a permanent magnet brake for a small sample size.

2. The Lifetime Performance Index and the Maximum Likelihood Estimation

In this section, we utilize the comprehensive lifetime performance index for the lifetime model of exponentiated Fréchet distribution. Section 2.1 clarifies the monotonic relationship among the overall process yield, the comprehensive lifetime performance index, and the individual lifetime performance indices. In Section 2.2, the maximum likelihood estimator and the asymptotic distribution are derived for all lifetime performance indices.

2.1. The Relationship between the Comprehensive Lifetime Performance Index and Individual Lifetime Performance Index

We consider products having d components produced in d production lines. For the ith component, the product lifetime U_i follows an exponentiated Fréchet (EF) lifetime distribution with the scale parameter θ_i and the shape parameter δ_i, as proposed by Pedro et al. [23]. The probability density function (p.d.f.), the cumulative distribution function (c.d.f.), and the hazard function (h.f.) are defined as follows:

$$f_{U_i}(u) = \theta_i \delta_i u^{-\delta_i - 1} e^{-u^{-\delta_i}} \left(1 - e^{-u^{-\delta_i}}\right)^{\theta_i - 1}, u > 0, \delta_i > 0, \theta_i > 0, \quad (1)$$

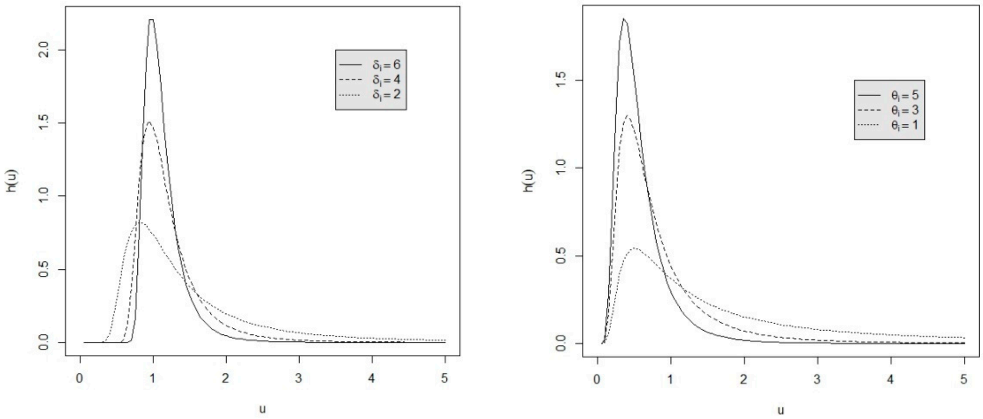
$$F_{U_i}(u) = 1 - (1 - e^{-u^{-\delta_i}})^{\theta_i}, u > 0, \delta_i > 0, \theta_i > 0 \tag{2}$$

and

$$r_{U_i}(u) = \frac{f_{U_i}(u)}{1 - F_{U_i}(u)} = \theta_i \delta_i u^{-\delta_i - 1} e^{-u^{-\delta_i}} (1 - e^{-u^{-\delta_i}})^{-1}, \tag{3}$$

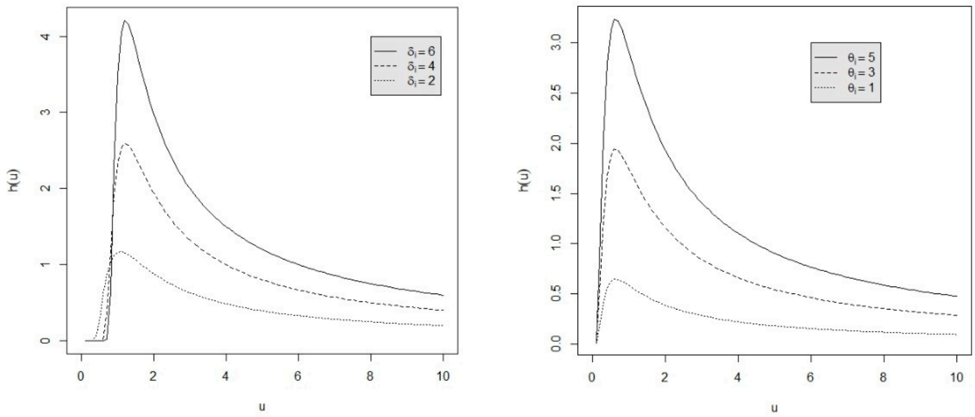
where θ_i is the scale parameter, δ_i is the shape parameter, and $i = 1, \ldots, d$.

The p.d.f. for δ_i = 2, 4, 6 under θ_i = 1 is depicted in Figure 1a and for θ_i = 1, 3, 5 under δ_i = 1 is depicted in Figure 1b. The h.f. for δ_i = 2, 4, 6 under θ_i = 1 is depicted in Figure 2a and for θ_i = 1, 3, 5 under δ_i = 1 is depicted in Figure 2b. From Figure 1a, b, we observe that the shape of the p.d.f. changes when the shape parameter δ_i changes. However, when parameter θ_i changes, it only affects the degree of data dispersion. Additionally, Figure 2a, b show that as the scale parameter θ_i increases, the hazard rate appears to increase, especially for larger values of the shape parameter δ_i, where the increase is more pronounced. For the estimation of different censoring schemes, refer to Pedro et al. [23] and Tabassum et al. [24].



(**a**) (**b**)

Figure 1. (**a**) The p.d.f. for δ_i = 2, 4, 6 under θ_i = 1. (**b**) The p.d.f. for θ_i = 1, 3, 5 under δ_i = 1.



(**a**) (**b**)

Figure 2. (**a**) The h.f. for δ_i = 2, 4, 6 under θ_i = 1. (**b**) The h.f. for θ_i = 1, 3, 5 under δ_i = 1.

Let $Y_i = -ln\left(1 - e^{-U_i^{-\delta_i}}\right)$, then we have $U_i = \left(-ln\left(1 - e^{-Y_i}\right)\right)^{-\frac{1}{\delta_i}}$ with Jacobian of $J = \left(\delta_i u^{-\delta_i - 1} e^{-u^{-\delta_i}}\right)^{-1}$. We can find the probability density function (p.d.f.) of Y_i as

$$f_{Y_i}(y) = f_{U_i}\left(\left(-ln\left(1 - e^{-Y_i}\right)\right)^{-\frac{1}{\delta_i}}\right) J = \theta_i e^{-\theta_i y},\ y > 0, \theta_i > 0. \tag{4}$$

The related cumulative distribution function (c.d.f.) and the hazard function (h.f.) of Y_i are obtained as follows:

$$F_{Y_i}(y) = 1 - e^{-\theta_i y}, y > 0, \theta_i > 0, \tag{5}$$

$$r_{Y_i}(y) = \frac{f_{Y_i}(y)}{1 - F_{Y_i}(y)} = \theta_i. \tag{6}$$

The lifetime of components of products is a "the-larger-the-better" quality characteristic. A longer lifetime indicates better product quality, and products with superior quality have higher market appeal. To assess this characteristic, the lifetime performance index proposed by Montgomery [5] is employed. After assessing consumer demands, manufacturers set a lower specification limit L_{Ui} for the product's lifetime.

According to Montgomery [5], the lifetime performance index for the production of the *i*th component is defined as follows:

$$C_{L_i} = \frac{\mu_i - L_{Ui}}{\sigma_i}, \tag{7}$$

where μ_i denotes the mean lifetime of the *i*th component, σ_i denotes the standard deviation of the lifetime of the *i*th component, and L_{Ui} is the specified lower specification limit for the lifetime of the *i*th component.

The mean and standard deviation of the new lifetime variable is calculated using Y_i are $\mu_i = E(Y_i) = \frac{1}{\theta_i}, \sigma_i = \sqrt{Var(Y_i)} = \frac{1}{\theta_i}$. The lower specification limit becomes $L_i = -ln\left(1 - e^{-L_{Ui}{}^{-\delta_i}}\right)$. The lifetime performance index for the lifetime of the ith component becomes

$$C_{L_i} = \frac{\mu_i - L_i}{\sigma_i} = \frac{\frac{1}{\theta_i} - L_i}{\frac{1}{\theta_i}} = 1 - \theta_i L_i. \tag{8}$$

Equation (8), shows that when $\frac{1}{\theta_i} > L_i$, $C_{L_i} > 0$; conversely, when $\frac{1}{\theta_i} < L_i$, $C_{L_i} < 0$. It can also be noted that as the hazard rate θ_i decreases, the process capability index C_{L_i} increases. This demonstrates that the process capability index C_{L_i} can be used to evaluate lifetime performance.

A product in the *i*th production line is classified as a conforming product only if its lifetime exceeds L_{Ui}. The proportion of conforming products in the *i*th production process is defined as the process yield, which is obtained as follows:

$$P_{ri} = P\left(U_i \geq L_{U_i}\right) = P(Y_i \geq L_i) = e^{-\theta_i L_i} = e^{C_{L_i} - 1}, -\infty < C_{L_i} < 1. \tag{9}$$

Equation (9) shows that there is a strict increasing relationship between the process yield P_{ri} and the lifetime performance index C_{L_i}. When the process yield P_{ri} is higher, the lifetime performance index C_{L_i} increases accordingly. This once again demonstrates that the lifetime performance index C_{L_i} can be used to evaluate the product's lifetime performance.

Assuming the lifetimes of d components of products produced in d independent production lines, the overall process yield P_r is defined as

$$P_r = P(Y_i \geq L_i,\ i = 1, 2, \ldots, d) = e^{-\sum_{i=1}^{d} \theta_i L_i} = e^{\sum_{i=1}^{d} C_{L_i} - d}, -\infty < C_{L_i} < 1. \tag{10}$$

We employ the comprehensive lifetime performance index C_T defined by Wu et al. [17] in the following equation:

$$P_r = \exp\{C_T - 1\}, -\infty < C_T < 1. \tag{11}$$

As shown in Equation (11), the comprehensive lifetime performance index C_T still exhibits a strictly increasing relationship with the overall process yield P_r. Furthermore, based on Equations (10) and (11), we can find the relationship between the comprehensive lifetime performance index C_T and the individual lifetime performance index C_{L_i} as follows:

$$C_T = \left(\sum_{i=1}^{d} C_{L_i} - d\right) + 1 = \sum_{i=1}^{d} C_{L_i} - (d-1), -\infty < C_T < 1. \quad (12)$$

A reasonable assumption in this context would be that the capabilities of the production lines are equally important for quality engineers across all components of products. The rational setup of equal individual life performance indices is defined as $C_{L_1} = C_{L_2} = \ldots = C_{L_d} = C_L$. Substituting this back into Equation (12), we obtain the relationship between C_T and C_L as

$$C_L = \frac{C_T + d - 1}{d}, -\infty < C_T < 1. \quad (13)$$

The comprehensive lifetime performance index C_T, the related overall process yield P_r, and the corresponding values of C_L are displayed in Table 1 for $d = 2, 3, 4, 5, 6, 7$. As Table 1 shows, if we require P_r to surpass 0.9753, then C_T must exceed 0.975. Regarding the target value of C_T = 0.975, we can find the corresponding values for the individual lifetime performance index C_L to exceed 0.9875, 0.9917, 0.9938, 0.9950, 0.9958, and 0.9964 for $d = 2, 3, 4, 5, 6, 7$.

Table 1. The corresponding values of C_L for a given value of P_r and C_T.

				Pr				
	0.6873	0.7047	0.7225	0.7408	0.7596	0.7788	0.7985	0.8187
				C_T				
d	0.6250	0.6500	0.6750	0.7000	0.7250	0.7500	0.7750	0.8000
2	0.8125	0.8250	0.8375	0.8500	0.8625	0.8750	0.8875	0.9000
3	0.8750	0.8833	0.8917	0.9000	0.9083	0.9167	0.9250	0.9333
4	0.9063	0.9125	0.9188	0.9250	0.9313	0.9375	0.9438	0.9500
5	0.9250	0.9300	0.9350	0.9400	0.9450	0.9500	0.9550	0.9600
6	0.9375	0.9417	0.9458	0.9500	0.9542	0.9583	0.9625	0.9667
7	0.9464	0.9500	0.9536	0.9571	0.9607	0.9643	0.9679	0.9714
				Pr				
	0.8395	0.8607	0.8825	0.9048	0.9277	0.9512	0.9753	1.0000
				C_T				
d	0.8250	0.8500	0.8750	0.9000	0.9250	0.9500	0.9750	1.0000
2	0.9125	0.9250	0.9375	0.9500	0.9625	0.9750	0.9875	1.0000
3	0.9417	0.9500	0.9583	0.9667	0.9750	0.9833	0.9917	1.0000
4	0.9563	0.9625	0.9688	0.9750	0.9813	0.9875	0.9938	1.0000
5	0.9650	0.9700	0.9750	0.9800	0.9850	0.9900	0.9950	1.0000
6	0.9708	0.9750	0.9792	0.9833	0.9875	0.9917	0.9958	1.0000
7	0.9750	0.9786	0.9821	0.9857	0.9893	0.9929	0.9964	1.0000

2.2. The Maximum Likelihood Estimators for the Life Performance Indices

The progressive type I interval censoring is addressed as follows: Let n products be subjected to a life test with m inspection intervals. This censoring scheme, used in survival analysis and reliability engineering, addresses scenarios where the exact failure times are not observed; instead, the number of failure units X_i is recorded at the ith inspection time point t_i, i = 1, . . ., m. Here, $(t_1, \ldots, t_m)$ are the predetermined inspection time points, and t_m is the ending time of this experiment. When the number of failure units X_i is observed at the ith inspection time point t_i, we remove R_i products from the unexamined units under the removal probability p_i, i = 1, . . ., m. When the experiment is completed, we collect the progressive type I interval-censored sample $(X_1, \ldots, X_m)$ under the censoring scheme of $(R_1, \ldots, R_m)$. We will utilize this sample set to make inferences on the comprehensive lifetime performance index and each individual lifetime performance index. Assuming the products have d components produced in d independent production lines, we observe $(X_{i1}, \ldots, X_{im})$ as the progressive type I interval-censored sample for the lifetimes of the ith component of products at the time points of $(t_1, \cdots, t_m)$ under the progressive censoring scheme of $(R_{i1}, \cdots, R_{im})$. The likelihood function of this censored sample is

$$\begin{aligned} L(\theta_i) &\propto \prod_{j=1}^{m} \left(F_{U_i}(t_j) - F_{U_i}(t_{j-1})\right)^{X_{ij}} \left(1 - F_{U_i}(t_j)\right)^{R_{ij}} \\ &\propto \prod_{j=1}^{m} \left(\left(1 - e^{-t_{j-1}^{-\delta_i}}\right)^{\theta_i} - \left(1 - e^{-t_j^{-\delta_i}}\right)^{\theta_i}\right)^{X_{ij}} \left(\left(1 - e^{-t_j^{-\delta_i}}\right)^{\theta_i}\right)^{R_{ij}} \\ &\propto \prod_{j=1}^{m} \left(1 - \left(\frac{1 - e^{-t_j^{-\delta_i}}}{1 - e^{-t_{j-1}^{-\delta_i}}}\right)^{\theta_i}\right)^{X_{ij}} \left(\left(1 - e^{-t_j^{-\delta_i}}\right)^{\theta_i}\right)^{X_{ij}} \left(\left(1 - e^{-t_j^{-\delta_i}}\right)^{\theta_i}\right)^{R_{ij}}. \end{aligned} \tag{14}$$

Firstly, taking the logarithm of the likelihood function yields the log-likelihood function as follows:

$$\begin{aligned} lnL(\theta_i) &= \sum_{j=1}^{m} X_{ij} ln\left(1 - \left(\frac{1-e^{-t_j^{-\delta_i}}}{1-e^{-t_{j-1}^{-\delta_i}}}\right)^{\theta_i}\right) \\ &-\theta_i \sum_{j=1}^{m} \left(R_{ij} ln\left(1 - e^{-t_j^{-\delta_i}}\right) + X_{ij} ln\left(1 - e^{-t_{j-1}^{-\delta_i}}\right)\right). \end{aligned} \tag{15}$$

Next, differentiating Equation (15) with respect to the parameter θ_i, setting the derivative equal to zero, yields the log-likelihood equation as follows:

$$\begin{aligned} \frac{d}{d\theta_i} lnL(\theta_i) &= \sum_{j=1}^{m} X_{ij} \frac{\left(ln\left(1-e^{-t_{j-1}^{-\delta_i}}\right) - ln\left(1-e^{-t_j^{-\delta_i}}\right)\right)\left(1-\left(\frac{1-e^{-t_j^{-\delta_i}}}{1-e^{-t_{j-1}^{-\delta_i}}}\right)^{\theta_i}\right)}{1-\left(\frac{1-e^{-t_j^{-\delta_i}}}{1-e^{-t_{j-1}^{-\delta_i}}}\right)^{\theta_i}} \\ &- \sum_{j=1}^{m} \left(R_{ij} ln\left(1 - e^{-t_j^{-\delta_i}}\right) + X_{ij} ln\left(1 - e^{-t_{j-1}^{-\delta_i}}\right)\right) \equiv 0. \end{aligned} \tag{16}$$

The maximum likelihood estimator for the parameter θ_i, denoted by $\hat{\theta}_i$, can be obtained by solving Equation (16). However, since Equation (16) does not have a closed-form solution, numerical methods are required to solve it. The maximum likelihood estimator $\hat{k}_i$ has an asymptotic normal distribution. The definition of the Fisher's information number is as follows:

$$I(\theta_i) = -E\left[\frac{d^2 lnL(\theta_i)}{d\theta_i^2}\right].$$

Taking the second derivative of log-likelihood function yields

$$\frac{d^2}{d\theta_i^2} lnL(\theta_i) = -\sum_{j=1}^{m} X_{ij} \frac{\left(ln\left(1 - e^{-t_{j-1}^{-\delta_i}}\right) - ln\left(1 - e^{-t_j^{-\delta_i}}\right)\right)^2 \left(\frac{1-e^{-t_j^{-\delta_i}}}{1-e^{-t_{j-1}^{-\delta_i}}}\right)^{\theta_i}}{\left(1 - \left(\frac{1-e^{-t_j^{-\delta_i}}}{1-e^{-t_{j-1}^{-\delta_i}}}\right)^{\theta_i}\right)^2}. \tag{17}$$

We assume that the number of failures X_{ij} under the progressive type I interval-censored scenario follows a binomial distribution, which is denoted by the following distribution:

$$X_{ij} \middle| X_{i,j-1}, \ldots, X_{i1}, R_{i,j-1}, \cdots, R_{i1} \sim \text{Binomial}\left(n - \sum_{l=1}^{j-1} X_{il} - \sum_{l=1}^{j-1} R_{il}, q_{ij}\right), \tag{18}$$

where $q_{ij} = \frac{F(t_j) - F(t_{j-1})}{1 - F(t_{j-1})} = 1 - \left(\frac{1-e^{-t_j^{-\delta_i}}}{1-e^{-t_{j-1}^{-\delta_i}}}\right)^{\theta_i}, j = 1, \ldots, m, i = 1, \ldots, d.$

Likewise, we assume that the number of progressive censorings at the *i*th observation time point follows a binomial distribution, which is denoted by the following distribution:

$$R_{ij}\big|R_{i,j-1},\ldots,R_{i1},X_{i,j-1},\cdots,X_{i1} \sim \text{Binomial}\Big(n-\sum_{l=1}^{j-1}X_{il}-\sum_{l=1}^{j-1}R_{il},p_j\Big), \quad (19)$$

where p_j represents the progressive removal probabilities.

Through Equations (18) and (19), the expected values are obtained as follows:

$$E\big(X_{ij}\big) = EE\big(X_{ij}\big|X_{i,j-1},\ldots,X_{i1},R_{i,j-1},\cdots,R_{i1}\big)nq_{ij}\prod_{l=1}^{j-1}(1-q_{il})(1-p_{il}), j=1,\ldots,m. \quad (20)$$

$$E\big(R_{ij}\big) = EE\big(R_{ij}\big|R_{i,j-1},\ldots,R_{i1},X_{i,j-1},\cdots,X_{i1}\big) = np_{ij}(1-q_{ij})\prod_{l=1}^{j-1}(1-q_{il})(1-p_{il}),\ j=1,\ldots,m. \quad (21)$$

Based on the above results, the Fisher's information number can be derived as follows:

$$\begin{aligned} I(\theta_i) &= -E\left[\frac{d^2 lnL(\theta_i)}{d\theta_i^2}\right] \\ &= -\sum_{j=1}^{m} nq_{ij}\frac{\left(ln\left(1-e^{-t_{j-1}{}^{-\delta_i}}\right)-ln\left(1-e^{-t_j{}^{-\delta_i}}\right)\right)^2\left(\frac{1-e^{-t_j{}^{-\delta_i}}}{1-e^{-t_{j-1}{}^{-\delta_i}}}\right)^{\theta_i}}{\left(1-\left(\frac{1-e^{-t_j{}^{-\delta_i}}}{1-e^{-t_{j-1}{}^{-\delta_i}}}\right)^{\theta_i}\right)^2}\prod_{l=1}^{j-1}(1-p_{il})(1-q_{il}). \end{aligned} \quad (22)$$

Using the information obtained from Equation (22), we can deduce that the asymptotic normal distribution of $\hat{\theta}_i$ as $\hat{\theta}_i \underset{n\to\infty}{\overset{d}{\to}} \text{N}\big(\theta_i, I^{-1}(\theta_i)\big)$.

For the convenience of data collection, this paper considers the special case where time intervals are all equal. Specifically, the time length in the jth interval is set as $t_j - t_{j-1} = t, j = 1,\ldots,m$, where t is the equal time interval. Simplifying Equation (16) using this assumption, we get

$$\begin{aligned} \frac{d}{d\theta_i}lnL(\theta_i) &= \sum_{j=1}^{m} X_{ij}\frac{\left(ln\left(1-e^{-((j-1)t)^{-\delta_i}}\right)-ln\left(1-e^{-(jt)^{-\delta_i}}\right)\right)\left(1-\left(\frac{1-e^{-(jt)^{-\delta_i}}}{1-e^{-((j-1)t)^{-\delta_i}}}\right)^{\theta_i}\right)}{1-\left(\frac{1-e^{-(jt)^{-\delta_i}}}{1-e^{-((j-1)t)^{-\delta_i}}}\right)^{\theta_i}} \\ &-\sum_{j=1}^{m}\left(R_{ij}ln\left(1-e^{-(jt)^{-\delta_i}}\right)+X_{ij}ln\left(1-e^{-((j-1)t)^{-\delta_i}}\right)\right) \equiv 0. \end{aligned} \quad (23)$$

By solving Equation (23), we obtain the maximum likelihood estimator for the parameter θ_i, denoted as $\hat{\theta}_i$.

According to the invariance property of the maximum likelihood estimator, we can obtain the maximum likelihood estimator for the lifetime performance index C_{L_i} as follows:

$$\hat{C}_{L_i} = 1-\hat{\theta}_i L_i. \quad (24)$$

By using the delta method, we can obtain

$$\hat{C}_{L_i}\xrightarrow[n\to\infty]{d}N\left(C_{L_i}, L_i{}^2V\big(\hat{\theta}_i\big)\right). \quad (25)$$

According to the invariance property of maximum likelihood estimators, we can obtain the maximum likelihood estimator for the comprehensive lifetime performance index C_T as follows:

$$\hat{C}_T = \sum_{i=1}^{d}\hat{C}_{L_i}-(d-1), -\infty < C_T < 1. \quad (26)$$

It is evident that the asymptotic variance of $\hat{C}_T$ is $\sum_{i=1}^{d} V\left(\hat{C}_{L_i}\right)$, and the estimator of the asymptotic variance is $\sum_{i=1}^{d} \hat{V}\left(\hat{C}_{L_i}\right)$. Apparently, the asymptotic distribution for $\hat{C}_T$ is

$$\hat{C}_T \xrightarrow[n\to\infty]{d} N\left(C_T, \sum_{i=1}^{d} \hat{V}\left(\hat{C}_{L_i}\right)\right). \quad (27)$$

3. The Testing Algorithm Procedure for the Comprehensive Lifetime Performance Index

In Section 3.1, the testing procedure to determine whether the comprehensive lifetime performance index meets the target level and the related power analysis are presented. Section 3.2 includes a numerical example to showcase the testing procedure.

3.1. The Testing Procedure for the Comprehensive Lifetime Performance Index and Power Analysis

Firstly, let us assume that the manufacturer sets a target level c_0 for the comprehensive lifetime performance of products with d components. Based on this target level c_0, the appropriate target level c_0^* for the individual lifetime performance index C_L can be obtained from Table 1. We can then establish a null and alternative hypothesis as follows:

H$_0$: $C_{L_i} \le c_0^*$,*for some i (indicating the process is not capable).*

vs.

H$_1$: $C_{L_i} > c_0^*$, $i = 1, \ldots, d$*(indicating the process is capable).*

This hypothesis test is known as the intersection–union test (IUT) in Casella and Berger [25].

As proposed by Wu et al. [13], the ith test of this hypothesis test is formulated as follows:

$$H_{0i} : C_{L_i} \le c_0^* \text{ and } H_{1i} : C_{L_i} > c_0^*$$

We use the maximum likelihood estimator of the individual lifetime performance index C_{L_i} as $\hat{C}_{L_i} = 1 - \hat{\theta}_i L_i$ to as the test statistic. Let $\alpha' = \alpha^{\frac{1}{d}}$ be the significance level for the ith test, where the critical value is determined as $C_{L_i}^0 = 1 - \left(Z_{1-\alpha'}\sqrt{w(\theta_{i0})} + \theta_{i0}\right)L_i$, and where $w(\theta_{i0}) = I^{-1}(\theta_{i0})$ as per Wu and Chang [11]. The rejection region for the ith test is given by $R_i = \left\{\hat{C}_{L_i} \middle| \hat{C}_{L_i} > C_{L_i}^0\right\}$. For the composite test with $H_0 : C_{L_i} \le c_0^*$, for some i and $H_1 : C_{L_i} > c_0^*$, $i = 1, \ldots, d$, the overall rejection region is determined as $R = \bigcap_{i=1}^{d} R_i = \bigcap_{i=1}^{d}\left\{\hat{C}_{L_i} \;\middle|\; \hat{C}_{L_i} > C_{L_i}^0\right\}$ such that the size of this test is still α.

The proposed testing algorithmic procedure about C_T can be constructed as Algorithm 1:

Algorithm 1. The testing procedure for C_T.

Step 1: For two components with specified lower limits L_1 and L_2, gather the progressive type I interval-censored samples $X_{i1}, \ldots, X_{im}$ at predetermined inspection times $t_1, \ldots, t_m$, using censoring schemes of $R_{i1}, \ldots, R_{im}$ from the exponentiated Fréchet distribution.
Step 2: For a defined process yield P_r and the component count d, locate the appropriate values for C_T (denoted as c_0) and C_L (denoted as c_0^*) from Table 1 to meet each component's required level. Following this, we outline the testing hypothesis $H_0 : C_{L_i} \le c_0^*$ for some i vs.
$H_1 : C_{L_i} > c_0^*, i = 1, \cdots, d$.
Step 3: Quantify the test statistic $\hat{C}_{L_i}$.
Step 4: For the specified significance level α, compute the critical value
$C_{L_i}^0 = 1 - L_i\left(\theta_{i0} + Z_{1-\alpha'}\sqrt{w(\theta_{i0})}\right)$, where $\theta_{i0} = \frac{1-c_0^*}{L_i}$, α'=$\alpha^{1/d}$, $w(\theta_{i0}) = I^{-1}(\theta_{i0})$ and $I_i^{-1}(\theta_{i0})$ is defined in Equation (22).
Step 5: Assess $\hat{C}_{L_i}$ in relation to $C_{L_i}^0$. If $\hat{C}_{L_i}$ exceeds $C_{L_i}^0$ for all i, it can be concluded that the comprehensive lifetime performance index for products with d components produced in d production lines has surpassed the target level c_0.
Furthermore, the test power function $h(c_1)$ of the testing procedure at the point of $C_T = dC_L - (d-1) = c_1 > c_0$ or $C_L = \frac{c_1+d-1}{d} = c_1^*$ in the alternative hypothesisis obtained as follows:

$$
\begin{aligned}
h(c_1) &= P\left(\hat{C}_{L_i} > C^0_{L_i} \middle| c_1^* = 1 - \theta_{i1} L_i\ ,\ i = 1, \ldots, d\right) \\
&= P\left(1 - \hat{\theta}_i L_i > 1 - L_i\left(\theta_{i0} + Z_{1-\alpha'}\sqrt{w(\theta_{i0})}\right) \middle| \theta_{i0} = \frac{1-c_0^*}{L_i}, \theta_{i1} = \frac{1-c_1^*}{L_i}, i = 1, \ldots, d\right) \\
&= \prod_{i=1}^{d} P\left(\hat{\theta}_i < \theta_{i0} + Z_{1-\alpha'}\sqrt{w(\theta_{i0})} \middle| \theta_{i0} = \frac{1-c_0^*}{L_i}, \theta_{i1} = \frac{1-c_1^*}{L_i}\right) \\
&= \prod_{i=1}^{d} P\left(\frac{\hat{\theta}_i - \theta_{i1}}{\sqrt{w(\theta_{i1})}} < \frac{\left(\theta_{i0} + Z_{1-\alpha'}\sqrt{w(\theta_{i0})} - \theta_{i1}\right)}{\sqrt{w(\theta_{i1})}} \middle| \theta_{i0} = \frac{1-c_0^*}{L_i}, \theta_{i1} = \frac{1-c_1^*}{L_i}\right) \\
&= \prod_{i=1}^{d}\left(\Phi\left(\frac{\theta_{i0} - \theta_{i1} + Z_{1-\alpha'}\sqrt{w(\theta_{i0})}}{\sqrt{w(\theta_{i1})}} \middle| \theta_{i0} = \frac{1-c_0^*}{L_i}, \theta_{i1} = \frac{1-c_1^*}{L_i}\right)\right),
\end{aligned} \tag{28}
$$

where $\Phi(.)$ is the c.d.f. for the standard normal distribution, $\theta_{i0} = \frac{1-c_0^*}{L_i}$, $\theta_{i1} = \frac{1-c_1^*}{L_i}$, $i = 1, \ldots, d$. For the case of $L_1 = \ldots = L_d = L$, this yields the following equation for the power function

$$
h(c_1) = \left(\Phi\left(\frac{\theta_0 - \theta_1 + Z_{1-\alpha'}\sqrt{w(\theta_0)}}{\sqrt{w(\theta_1)}} \middle| \theta_0 = \frac{1-c_0^*}{L_i}, \theta_1 = \frac{1-c_1^*}{L_i}\right)\right)^d
$$

To test $H_0 : C_T \leq 0.8$, the powers $h(c_1)$ are computed using Equation (28) and they are presented in Tables A1–A9 for d = 2, 3, 4 with α = 0.050, 0.075, 0.100, respectively, for c_1 = 0.8, 0.825, 0.85, 0.875, 0.9, 0.925, 0.95, 0.975; m = 2, 4, 6, n = 25, 50, 75,100; and p = 0.050, 0.075, 0.100 with L = 0.05, T = 0.5. The power values depicted in Figures 3–7 serve to exemplify various standard cases. Our observations are as follows: (1) In Figure 3, decreasing p enhances the power when d = 1, n = 25, m = 4, and α = 0.05. This behavior is consistent across different combinations of d, n, m, and α. (2) In Figure 4, as n increases, the power rises with d = 1, m = 4, p = 0.05, and α = 0.05 held constant. Similar patterns emerge across various combinations of d, n, p, and α. (3) In Figure 5, decreasing m enhances the power for fixed values of d = 1, α = 0.05, n = 25 and p = 0.05. (4) In Figure 6, the power increases with α under d = 1, n = 25, m = 4, and p = 0.05. This trend is observed across various combinations of d, n, m, and p. (5) In Figure 7, the power improves as d increases with n = 25, m = 4, p = 0.05, and α = 0.05 held constant. This trend remains consistent across different combinations of n, m, p, and α. The improvement in power is getting not significant when d is greater than 3. (6) Across Figures 3–7, the power increases with the value of c_1 for any combination of d, n, m, p, and α.

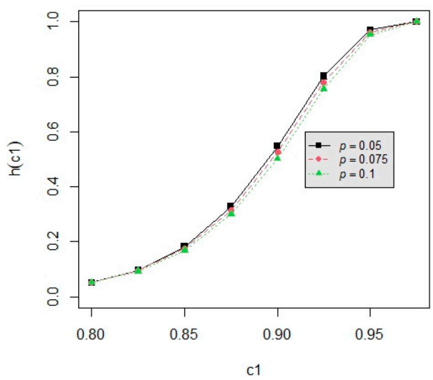


Figure 3. The power curve for d = 1, α = 0.05, n = 25, m = 4, and p = 0.050, 0.075, 0.100.

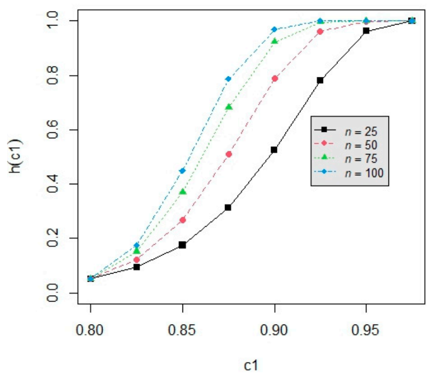


Figure 4. The power curve for $d = 1$, $\alpha = 0.05$, $m = 4$, $p = 0.05$, and $n = 25, 50, 75, 100$.

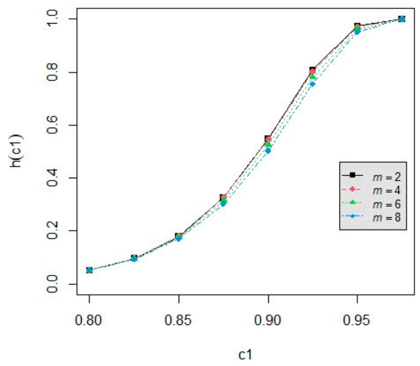


Figure 5. The power curve for $d = 1$, $\alpha = 0.05$, $n = 25$, $p = 0.05$, and $m = 2, 4, 6, 8$.

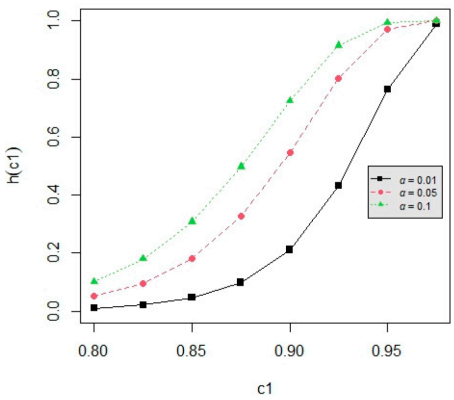


Figure 6. The power curve for $d = 1$, $n = 25$, $m = 4$, $p = 0.05$, and $\alpha = 0.01, 0.05, 0.10$.

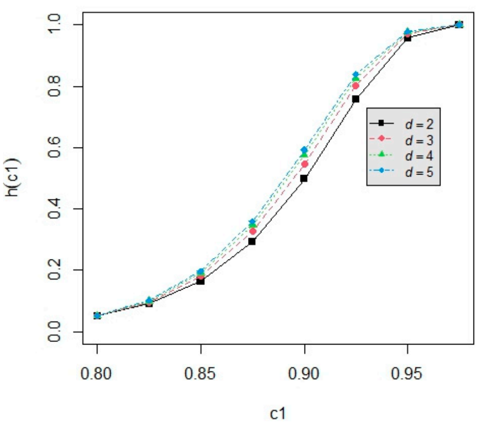


Figure 7. The power curve for $n = 25$, $m = 4$, $p = 0.05$, $\alpha = 0.05$, and $d = 2, 3, 4, 5$.

3.2. Practical Example

We consider the case of products with two components (d = 2) in this section. The lifetime data for the first component consists of the failure times (number of cycles) of n = 36 electrical appliances in Lawless [26]. The data of 36 failure times U_{1j}, j = 1, ..., 36, is listed as follows: 0.001, 0.0035, 0.0049, 0.017, 0.0329, 0.0381, 0.0707, 0.0958, 0.1062, 0.1167, 0.1594, 0.1925, 0.199, 0.2223, 0.3270, 0.2400, 0.2451, 0.2471, 0.2551, 0.2565, 0.2568, 0.2702, 0.2761, 0.2831, 0.3034, 0.3034, 0.3059, 0.3113, 0.3214, 0.3478, 0.3504, 0.4329, 0.6367, 0.6976, 0.7846, 1.3403. The p-value of the G test based on the Gini statistic (see Gail and Gastwirth [27]) is a function of the values of δ_1. The Gini statistic is obtained by $G_n = \frac{\sum_{j=1}^{n-1} j(n-j)\left(Y_{1(j+1)} - Y_{1(j)}\right)}{(n-1)\sum_{j=1}^{n}(n-j+1)\left(Y_{1(j)} - Y_{1(j-1)}\right)}$, where $Y_{1(j)} = -ln\left(1 - e^{-U_{1j}^{-\delta_1}}\right)$, j = 1, ..., 36. Figure 8 depicts the p-values vs. the values of δ_1. From this figure, it reveals that the p-value peaks at 0.961 when δ_1 = 0.6, so δ_1 is concluded to be 0.6.

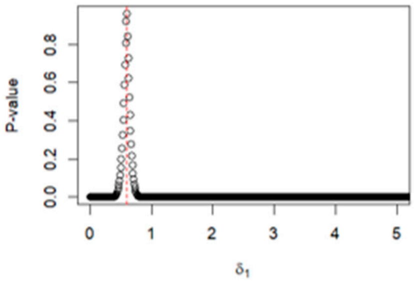


Figure 8. The p-values vs. the values of δ_1.

The data for the lifetimes of the second component consists of n = 50 failure times in Aarset [28]. The data of U_{2j}, j = 1, ..., 50, is listed as follows: 0.001, 0.002, 0.010, 0.010, 0.010, 0.010, 0.020, 0.030, 0.060, 0.070, 0.110, 0.120, 0.180, 0.180, 0.180, 0.180, 0.180, 0.210, 0.320, 0.360, 0.400, 0.450, 0.450, 0.470, 0.500, 0.550, 0.600, 0.630, 0.630, 0.670, 0.670, 0.670, 0.670, 0.670, 0.720, 0.750, 0.790, 0.820, 0.820, 0.830, 0.840, 0.840, 0.840, 0.850, 0.850, 0.850, 0.850, 0.850, 0.860, 0.860. Similarly, Figure 9 shows the p-values vs. the values of δ_2. Figure 9 reveals that the p-value peaks at 0.9881 when δ_2 = 0.87, so δ_2 is concluded to be 0.87.

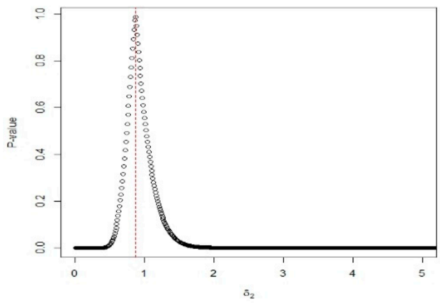


Figure 9. The p-values vs. the values of δ_2.

High p-values indicate that two datasets fit the exponentiated Fréchet distribution very well. We use these two datasets to illustrate how to implement the testing procedure proposed in Section 3.1 with an end time of T = 2 and the number of inspections m = 8 under the pre-specified removal percentages of (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (0.05, 0.05, 0.05, 0.05, 0.05, 0.05, 0.05, 0.05, 1.00, respectively).

The testing procedure is implemented in the following order according to the Algorithm 1 given in Section 3.1:

Step 1: For two components with specified lower limits $L_1 = L_2 = L$ = 0.05, observe the progressive type I interval-censored sample ($X_{11}, X_{12}, X_{13}, X_{14}, X_{15}, X_{16}, X_{17}, X_{18}$) = (18, 13, 2, 0, 0, 1, 0, 0, respectively) and ($X_{21}, X_{22}, X_{23}, X_{24}, X_{25}, X_{26}, X_{27}, X_{28}$) = (19, 6, 9, 12, 0, 0, 0, 0, respectively) for each component at the preset times (t_1, ..., t_8) = (0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0, respectively) with censoring schemes of $(R_{11}, R_{12}, R_{13}, R_{14}, R_{15}, R_{16}, R_{17}, R_{18})$ = (1, 0, 1, 0, 0, 0, 0, 0, respectively and $(R_{21}, R_{22}, R_{23}, R_{24}, R_{25}, R_{26}, R_{27}, R_{28})$ = (1, 1, 1, 0, 0, 0, 0, 1, respectively) from the exponentiated Fréchet distribution.

Step 2: For a defined conforming rate P_r = 0.9048 and the component count d = 2, the appropriate values for c_0 = 0.9 and c_0^* = 0.95 are found from Table 1. The value of c_0^* = 0.95 is the required level for each individual lifetime performance index. We then outline the testing hypotheses $H_0 : C_{L_i} \leq 0.95$ for some i vs. $H_1 : C_{L_i} > 0.95, i = 1, 2.$

Step 3: Obtain the maximum likelihood estimators $\hat{\theta}_1 = 7.376792$ and $\hat{\theta}_2 = 5.062244$ for two components. Compute the values of the test statistics $\hat{C}_{L_1} = 0.9823$ and $\hat{C}_{L_2} = 0.9969$, respectively.

Step 4: For the level of significance α = 0.05, we have $\alpha' = (0.05)^{\frac{1}{2}} = 0.2236$. Compute $\theta_{10} = \frac{1-c_0^*}{L_1} = \frac{1-0.95}{0.0024} = 20.8333$ and $\theta_{20} = \frac{1-c_0^*}{L_2} = \frac{1-0.95}{0.0006} = 83.3333$. Then we can calculate the critical values of $C_{L_1}^0 = 0.9578$ and $C_{L_2}^0 = 0.9578$.

Step 5: Since $\hat{C}_{L_1} = 0.9823 > C_{L_1}^0 = 0.9578$ and $\hat{C}_{L_2} = 0.9969 > C_{L_2}^0 = 0.9578$, we can infer that both individual lifetime performance indices reach the desired target values, so that the comprehensive lifetime performance index for products with two components surpasses the target level c_0 = 0.9.

4. Conclusions

Assessing lifetime performance indices for products, particularly those with exponentiated Fréchet lifetime distribution, is essential across various manufacturing sectors. For products composed of multiple components produced on separate production lines, each with lifetimes following an exponentiated Fréchet distribution, we employed a comprehensive lifetime performance index, which is a monotonically increasing function of both the overall process yield and the individual lifetime performance indices. Using the progressive type I interval-censored samples, we developed a hypothesis testing procedure to evaluate whether the comprehensive index meets the target level by testing multiple hypotheses for each individual lifetime performance index. We analyzed the effects of the number of inspection intervals, sample size, removal probability, significance level, and the number of components on test power and demonstrated the proposed testing process with a numerical example involving two-component products for a specified target value and a given level of significance. In the last few years, developments related to the detailed study of families of classical probability distributions and their modifications have been proposed in relation to the saturation characteristic regarding the Hausdorff distance. We will study this topic for exponentiated Fréchet distributions in the future. On the other hand, we will also work on the optimal experimental design related to the proposed testing procedure for the comprehensive lifetime performance index to obtain the given test power or to find the minimum total experimental cost under a specific cost structure. The current lifetime performance index has some limitations in addressing asymmetrical distributions. Future research should consider the new lifetime performance index by replacing the mean with the median.

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Data Availability Statement: Data is available in a publicly accessible repository. The data presented in this study is openly available in Lawless [26] and Aarset [28].

Conflicts of Interest: The authors declare no conflicts of interest.

Nomenclature

θ_i	the scale parameter for the exponentiated Fréchet lifetime distribution;
δ_i	the shape parameter for the exponentiated Fréchet lifetime distribution;
C_{L_i}	the lifetime performance index for the ith component;
P_{ri}	the process yield for the ith component;
C_T	the comprehensive lifetime performance index;
P_r	the overall process yield.

Appendix A

Table A1. The power $h(c_1)$ at d = 2, α = 0.01.

			c_1							
m	n	p	0.8	0.825	0.85	0.875	0.9	0.925	0.95	0.975
2	25	0.05	0.0100	0.0184	0.0355	0.0719	0.1513	0.3217	0.6400	0.9670
		0.075	0.0100	0.0182	0.0348	0.0698	0.1459	0.3096	0.6213	0.9609
		0.1	0.0100	0.0180	0.0341	0.0678	0.1406	0.2976	0.6021	0.9537
	50	0.05	0.0100	0.0271	0.0736	0.1917	0.4405	0.7865	0.9840	1.0000
		0.075	0.0100	0.0267	0.0718	0.1859	0.4279	0.7729	0.9812	1.0000
		0.1	0.0100	0.0263	0.0700	0.1801	0.4151	0.7587	0.9780	1.0000
	75	0.05	0.0100	0.0358	0.1193	0.3347	0.6928	0.9559	0.9997	1.0000
		0.075	0.0100	0.0353	0.1161	0.3253	0.6791	0.9507	0.9996	1.0000
		0.1	0.0100	0.0347	0.1130	0.3158	0.6651	0.9449	0.9995	1.0000
	100	0.05	0.0100	0.0449	0.1705	0.4767	0.8502	0.9927	1.0000	1.0000
		0.075	0.0100	0.0441	0.1659	0.4649	0.8400	0.9914	1.0000	1.0000
		0.1	0.0100	0.0433	0.1614	0.4530	0.8291	0.9898	1.0000	1.0000
4	25	0.05	0.0100	0.0191	0.0379	0.0775	0.1621	0.3376	0.6521	0.9672
		0.075	0.0100	0.0186	0.0358	0.0716	0.1471	0.3049	0.6022	0.9490
		0.1	0.0100	0.0181	0.0339	0.0661	0.1331	0.2739	0.5508	0.9236
	50	0.05	0.0100	0.0282	0.0780	0.2020	0.4541	0.7922	0.9835	1.0000
		0.075	0.0100	0.0272	0.0729	0.1858	0.4198	0.7554	0.9750	1.0000
		0.1	0.0100	0.0262	0.0681	0.1706	0.3864	0.7156	0.9632	1.0000
	75	0.05	0.0100	0.0374	0.1256	0.3474	0.7020	0.9562	0.9996	1.0000
		0.075	0.0100	0.0357	0.1168	0.3215	0.6651	0.9412	0.9993	1.0000
		0.1	0.0100	0.0342	0.1086	0.2967	0.6268	0.9225	0.9986	1.0000
	100	0.05	0.0100	0.0468	0.1787	0.4897	0.8548	0.9925	1.0000	1.0000
		0.075	0.0100	0.0446	0.1659	0.4575	0.8267	0.9884	1.0000	1.0000
		0.1	0.0100	0.0425	0.1538	0.4258	0.7954	0.9827	1.0000	1.0000
6	25	0.05	0.0100	0.0190	0.0371	0.0747	0.1539	0.3172	0.6176	0.9538
		0.075	0.0100	0.0181	0.0340	0.0658	0.1316	0.2683	0.5371	0.9132
		0.1	0.0100	0.0174	0.0312	0.0580	0.1122	0.2246	0.4569	0.8517
	50	0.05	0.0100	0.0278	0.0755	0.1927	0.4319	0.7660	0.9769	1.0000
		0.075	0.0100	0.0262	0.0678	0.1684	0.3791	0.7034	0.9579	1.0000
		0.1	0.0100	0.0247	0.0609	0.1468	0.3298	0.6356	0.9283	0.9997
	75	0.05	0.0100	0.0366	0.1208	0.3314	0.6767	0.9450	0.9993	1.0000
		0.075	0.0100	0.0341	0.1076	0.2920	0.6164	0.9152	0.9981	1.0000
		0.1	0.0100	0.0319	0.0958	0.2559	0.5547	0.8757	0.9952	1.0000
	100	0.05	0.0100	0.0457	0.1714	0.4690	0.8349	0.9894	1.0000	1.0000
		0.075	0.0100	0.0423	0.1521	0.4189	0.7855	0.9800	0.9999	1.0000
		0.1	0.0100	0.0392	0.1349	0.3713	0.7299	0.9647	0.9998	1.0000

Table A2. The power $h(c_1)$ at $d = 2$, $\alpha = 0.05$.

			c_1							
m	n	p	0.8	0.825	0.85	0.875	0.9	0.925	0.95	0.975
2	25	0.05	0.0500	0.0887	0.1598	0.2859	0.4903	0.7555	0.9587	0.9999
		0.075	0.0500	0.0881	0.1576	0.2808	0.4812	0.7452	0.9544	0.9998
		0.1	0.0500	0.0874	0.1554	0.2757	0.4721	0.7344	0.9497	0.9998
	50	0.05	0.0500	0.1172	0.2584	0.5021	0.7931	0.9697	0.9997	1.0000
		0.075	0.0500	0.1161	0.2543	0.4937	0.7842	0.9666	0.9996	1.0000
		0.1	0.0500	0.1149	0.2502	0.4852	0.7749	0.9632	0.9995	1.0000
	75	0.05	0.0500	0.1429	0.3501	0.6674	0.9248	0.9969	1.0000	1.0000
		0.075	0.0500	0.1413	0.3443	0.6582	0.9195	0.9964	1.0000	1.0000
		0.1	0.0500	0.1397	0.3386	0.6488	0.9137	0.9958	1.0000	1.0000
	100	0.05	0.0500	0.1673	0.4343	0.7841	0.9742	0.9997	1.0000	1.0000
		0.075	0.0500	0.1653	0.4273	0.7756	0.9716	0.9997	1.0000	1.0000
		0.1	0.0500	0.1632	0.4202	0.7668	0.9687	0.9996	1.0000	1.0000
4	25	0.05	0.0500	0.0905	0.1643	0.2930	0.4969	0.7566	0.9566	0.9998
		0.075	0.0500	0.0887	0.1582	0.2790	0.4725	0.7285	0.9439	0.9997
		0.1	0.0500	0.0869	0.1523	0.2654	0.4483	0.6990	0.9286	0.9993
	50	0.05	0.0500	0.1197	0.2646	0.5093	0.7952	0.9685	0.9996	1.0000
		0.075	0.0500	0.1165	0.2533	0.4865	0.7707	0.9594	0.9993	1.0000
		0.1	0.0500	0.1134	0.2423	0.4640	0.7448	0.9482	0.9988	1.0000
	75	0.05	0.0500	0.1461	0.3574	0.6733	0.9249	0.9966	1.0000	1.0000
		0.075	0.0500	0.1416	0.3417	0.6481	0.9098	0.9950	1.0000	1.0000
		0.1	0.0500	0.1372	0.3263	0.6225	0.8927	0.9926	1.0000	1.0000
	100	0.05	0.0500	0.1710	0.4422	0.7883	0.9739	0.9997	1.0000	1.0000
		0.075	0.0500	0.1653	0.4231	0.7650	0.9663	0.9994	1.0000	1.0000
		0.1	0.0500	0.1597	0.4043	0.7406	0.9571	0.9990	1.0000	1.0000
6	25	0.05	0.0500	0.0897	0.1610	0.2844	0.4801	0.7353	0.9461	0.9997
		0.075	0.0500	0.0868	0.1516	0.2629	0.4421	0.6889	0.9215	0.9991
		0.1	0.0500	0.0841	0.1428	0.2429	0.4056	0.6405	0.8899	0.9976
	50	0.05	0.0500	0.1180	0.2578	0.4940	0.7770	0.9611	0.9993	1.0000
		0.075	0.0500	0.1131	0.2404	0.4583	0.7361	0.9432	0.9984	1.0000
		0.1	0.0500	0.1084	0.2241	0.4240	0.6929	0.9198	0.9965	1.0000
	75	0.05	0.0500	0.1436	0.3476	0.6558	0.9134	0.9952	1.0000	1.0000
		0.075	0.0500	0.1367	0.3232	0.6153	0.8862	0.9913	1.0000	1.0000
		0.1	0.0500	0.1302	0.3003	0.5747	0.8543	0.9849	0.9999	1.0000
	100	0.05	0.0500	0.1678	0.4300	0.7718	0.9680	0.9995	1.0000	1.0000
		0.075	0.0500	0.1589	0.4002	0.7332	0.9533	0.9988	1.0000	1.0000
		0.1	0.0500	0.1506	0.3717	0.6926	0.9341	0.9974	1.0000	1.0000

Table A3. The power $h(c_1)$ at $d = 2$, $\alpha = 0.1$.

			c_1							
m	n	p	0.8	0.825	0.85	0.875	0.9	0.925	0.95	0.975
2	25	0.05	0.1000	0.1697	0.2855	0.4628	0.6918	0.9011	0.9927	1.0000
		0.075	0.1000	0.1687	0.2825	0.4568	0.6839	0.8955	0.9917	1.0000
		0.1	0.1000	0.1677	0.2794	0.4508	0.6758	0.8895	0.9906	1.0000
	50	0.05	0.1000	0.2125	0.4109	0.6789	0.9065	0.9926	1.0000	1.0000
		0.075	0.1000	0.2108	0.4060	0.6715	0.9014	0.9917	1.0000	1.0000
		0.1	0.1000	0.2091	0.4011	0.6639	0.8959	0.9907	1.0000	1.0000
	75	0.05	0.1000	0.2490	0.5132	0.8111	0.9728	0.9995	1.0000	1.0000
		0.075	0.1000	0.2468	0.5071	0.8044	0.9705	0.9994	1.0000	1.0000
		0.1	0.1000	0.2445	0.5009	0.7974	0.9680	0.9993	1.0000	1.0000
	100	0.05	0.1000	0.2821	0.5981	0.8900	0.9923	1.0000	1.0000	1.0000
		0.075	0.1000	0.2793	0.5913	0.8847	0.9913	1.0000	1.0000	1.0000
		0.1	0.1000	0.2766	0.5844	0.8790	0.9903	0.9999	1.0000	1.0000
4	25	0.05	0.1000	0.1718	0.2897	0.4671	0.6926	0.8982	0.9916	1.0000
		0.075	0.1000	0.1690	0.2813	0.4510	0.6712	0.8823	0.9884	1.0000
		0.1	0.1000	0.1661	0.2731	0.4351	0.6493	0.8648	0.9842	1.0000
	50	0.05	0.1000	0.2154	0.4160	0.6819	0.9055	0.9919	1.0000	1.0000
		0.075	0.1000	0.2107	0.4026	0.6617	0.8910	0.9890	0.9999	1.0000
		0.1	0.1000	0.2061	0.3894	0.6411	0.8751	0.9852	0.9999	1.0000
	75	0.05	0.1000	0.2525	0.5187	0.8129	0.9720	0.9994	1.0000	1.0000
		0.075	0.1000	0.2463	0.5020	0.7943	0.9652	0.9990	1.0000	1.0000
		0.1	0.1000	0.2402	0.4854	0.7748	0.9571	0.9985	1.0000	1.0000
	100	0.05	0.1000	0.2861	0.6036	0.8910	0.9919	1.0000	1.0000	1.0000
		0.075	0.1000	0.2785	0.5851	0.8761	0.9891	0.9999	1.0000	1.0000
		0.1	0.1000	0.2710	0.5665	0.8599	0.9855	0.9999	1.0000	1.0000
6	25	0.05	0.1000	0.1703	0.2846	0.4560	0.6762	0.8850	0.9887	1.0000
		0.075	0.1000	0.1658	0.2715	0.4308	0.6418	0.8571	0.9818	1.0000
		0.1	0.1000	0.1615	0.2592	0.4067	0.6072	0.8260	0.9719	0.9999
	50	0.05	0.1000	0.2127	0.4073	0.6673	0.8939	0.9894	0.9999	1.0000
		0.075	0.1000	0.2054	0.3863	0.6348	0.8687	0.9832	0.9998	1.0000
		0.1	0.1000	0.1985	0.3664	0.6022	0.8403	0.9745	0.9996	1.0000
	75	0.05	0.1000	0.2488	0.5075	0.7992	0.9665	0.9991	1.0000	1.0000
		0.075	0.1000	0.2391	0.4813	0.7683	0.9536	0.9982	1.0000	1.0000
		0.1	0.1000	0.2300	0.4559	0.7359	0.9373	0.9965	1.0000	1.0000
	100	0.05	0.1000	0.2815	0.5910	0.8799	0.9896	0.9999	1.0000	1.0000
		0.075	0.1000	0.2697	0.5616	0.8542	0.9838	0.9998	1.0000	1.0000
		0.1	0.1000	0.2584	0.5326	0.8259	0.9758	0.9995	1.0000	1.0000

Table A4. The power $h(c_1)$ at $d = 3$, $\alpha = 0.01$.

			c_1							
m	n	p	0.8	0.825	0.85	0.875	0.9	0.925	0.95	0.975
2	25	0.05	0.0100	0.0209	0.0449	0.0983	0.2139	0.4415	0.7791	0.9916
		0.075	0.0100	0.0207	0.0441	0.0957	0.2074	0.4289	0.7656	0.9899
		0.1	0.0100	0.0205	0.0432	0.0931	0.2009	0.4162	0.7514	0.9880
	50	0.05	0.0100	0.0305	0.0888	0.2339	0.5119	0.8385	0.9897	1.0000
		0.075	0.0100	0.0301	0.0867	0.2275	0.4998	0.8281	0.9881	1.0000
		0.1	0.0100	0.0296	0.0847	0.2211	0.4874	0.8171	0.9862	1.0000
	75	0.05	0.0100	0.0400	0.1389	0.3801	0.7354	0.9635	0.9997	1.0000
		0.075	0.0100	0.0394	0.1354	0.3706	0.7236	0.9595	0.9996	1.0000
		0.1	0.0100	0.0387	0.1320	0.3610	0.7113	0.9551	0.9995	1.0000
	100	0.05	0.0100	0.0497	0.1931	0.5162	0.8670	0.9926	1.0000	1.0000
		0.075	0.0100	0.0489	0.1882	0.5049	0.8583	0.9914	1.0000	1.0000
		0.1	0.0100	0.0480	0.1833	0.4934	0.8491	0.9901	1.0000	1.0000
4	25	0.05	0.0100	0.0211	0.0453	0.0983	0.2112	0.4311	0.7625	0.9888
		0.075	0.0100	0.0205	0.0430	0.0914	0.1941	0.3980	0.7245	0.9825
		0.1	0.0100	0.0199	0.0408	0.0849	0.1781	0.3658	0.6838	0.9733
	50	0.05	0.0100	0.0306	0.0888	0.2312	0.5020	0.8258	0.9870	1.0000
		0.075	0.0100	0.0295	0.0833	0.2144	0.4697	0.7960	0.9811	1.0000
		0.1	0.0100	0.0285	0.0781	0.1985	0.4377	0.7634	0.9731	1.0000
	75	0.05	0.0100	0.0402	0.1383	0.3746	0.7237	0.9579	0.9995	1.0000
		0.075	0.0100	0.0385	0.1291	0.3493	0.6910	0.9452	0.9991	1.0000
		0.1	0.0100	0.0368	0.1205	0.3249	0.6569	0.9296	0.9984	1.0000
	100	0.05	0.0100	0.0499	0.1917	0.5084	0.8575	0.9908	1.0000	1.0000
		0.075	0.0100	0.0475	0.1788	0.4781	0.8324	0.9867	1.0000	1.0000
		0.1	0.0100	0.0453	0.1665	0.4481	0.8046	0.9811	0.9999	1.0000
6	25	0.05	0.0100	0.0207	0.0436	0.0929	0.1970	0.4020	0.7274	0.9826
		0.075	0.0100	0.0198	0.0401	0.0827	0.1720	0.3518	0.6628	0.9669
		0.1	0.0100	0.0190	0.0369	0.0737	0.1497	0.3052	0.5943	0.9414
	50	0.05	0.0100	0.0298	0.0845	0.2173	0.4737	0.7983	0.9813	1.0000
		0.075	0.0100	0.0281	0.0764	0.1923	0.4235	0.7462	0.9676	1.0000
		0.1	0.0100	0.0266	0.0690	0.1698	0.3757	0.6890	0.9471	0.9998
	75	0.05	0.0100	0.0389	0.1309	0.3531	0.6944	0.9459	0.9991	1.0000
		0.075	0.0100	0.0363	0.1174	0.3148	0.6404	0.9203	0.9978	1.0000
		0.1	0.0100	0.0340	0.1052	0.2793	0.5846	0.8872	0.9951	1.0000
	100	0.05	0.0100	0.0481	0.1812	0.4823	0.8347	0.9869	1.0000	1.0000
		0.075	0.0100	0.0446	0.1619	0.4351	0.7902	0.9774	0.9999	1.0000
		0.1	0.0100	0.0414	0.1446	0.3899	0.7404	0.9629	0.9996	1.0000

Table A5. The power $h(c_1)$ at $d = 3$, $\alpha = 0.05$.

			c_1							
m	n	p	0.8	0.825	0.85	0.875	0.9	0.925	0.95	0.975
2	25	0.05	0.0500	0.0959	0.1817	0.3312	0.5575	0.8149	0.9753	1.0000
		0.075	0.0500	0.0952	0.1793	0.3259	0.5490	0.8068	0.9728	0.9999
		0.1	0.0500	0.0945	0.1769	0.3205	0.5404	0.7984	0.9701	0.9999
	50	0.05	0.0500	0.1253	0.2825	0.5395	0.8194	0.9740	0.9997	1.0000
		0.075	0.0500	0.1241	0.2783	0.5315	0.8118	0.9715	0.9996	1.0000
		0.1	0.0500	0.1228	0.2740	0.5235	0.8038	0.9689	0.9995	1.0000
	75	0.05	0.0500	0.1515	0.3729	0.6901	0.9294	0.9965	1.0000	1.0000
		0.075	0.0500	0.1498	0.3672	0.6817	0.9247	0.9960	1.0000	1.0000
		0.1	0.0500	0.1482	0.3615	0.6730	0.9198	0.9954	1.0000	1.0000
	100	0.05	0.0500	0.1761	0.4540	0.7942	0.9728	0.9995	1.0000	1.0000
		0.075	0.0500	0.1740	0.4472	0.7865	0.9703	0.9994	1.0000	1.0000
		0.1	0.0500	0.1719	0.4403	0.7785	0.9677	0.9993	1.0000	1.0000
4	25	0.05	0.0500	0.0959	0.1808	0.3270	0.5474	0.8020	0.9702	0.9999
		0.075	0.0500	0.0940	0.1744	0.3130	0.5248	0.7794	0.9622	0.9999
		0.1	0.0500	0.0921	0.1683	0.2993	0.5022	0.7555	0.9524	0.9997
	50	0.05	0.0500	0.1251	0.2802	0.5319	0.8090	0.9696	0.9995	1.0000
		0.075	0.0500	0.1219	0.2689	0.5106	0.7877	0.9618	0.9992	1.0000
		0.1	0.0500	0.1187	0.2580	0.4893	0.7650	0.9525	0.9987	1.0000
	75	0.05	0.0500	0.1512	0.3694	0.6814	0.9227	0.9955	1.0000	1.0000
		0.075	0.0500	0.1467	0.3541	0.6584	0.9089	0.9937	1.0000	1.0000
		0.1	0.0500	0.1423	0.3393	0.6348	0.8935	0.9912	1.0000	1.0000
	100	0.05	0.0500	0.1756	0.4494	0.7860	0.9691	0.9993	1.0000	1.0000
		0.075	0.0500	0.1699	0.4312	0.7645	0.9615	0.9990	1.0000	1.0000
		0.1	0.0500	0.1644	0.4133	0.7420	0.9525	0.9984	1.0000	1.0000
6	25	0.05	0.0500	0.0944	0.1755	0.3147	0.5265	0.7800	0.9620	0.9998
		0.075	0.0500	0.0915	0.1660	0.2933	0.4910	0.7419	0.9458	0.9996
		0.1	0.0500	0.0887	0.1570	0.2732	0.4565	0.7016	0.9250	0.9990
	50	0.05	0.0500	0.1225	0.2707	0.5128	0.7889	0.9618	0.9992	1.0000
		0.075	0.0500	0.1176	0.2536	0.4794	0.7529	0.9465	0.9982	1.0000
		0.1	0.0500	0.1129	0.2375	0.4471	0.7148	0.9271	0.9966	1.0000
	75	0.05	0.0500	0.1475	0.3563	0.6606	0.9096	0.9937	1.0000	1.0000
		0.075	0.0500	0.1407	0.3330	0.6234	0.8847	0.9894	0.9999	1.0000
		0.1	0.0500	0.1342	0.3109	0.5860	0.8560	0.9830	0.9999	1.0000
	100	0.05	0.0500	0.1710	0.4338	0.7665	0.9619	0.9990	1.0000	1.0000
		0.075	0.0500	0.1623	0.4055	0.7307	0.9471	0.9979	1.0000	1.0000
		0.1	0.0500	0.1542	0.3785	0.6934	0.9285	0.9961	1.0000	1.0000

Table A6. The power $h(c_1)$ at $d = 3$, $\alpha = 0.1$.

			c_1								
m	n	p	0.8	0.825	0.85	0.875	0.9	0.925	0.95	0.975
2	25	0.05	0.1000	0.1793	0.3106	0.5041	0.7354	0.9235	0.9950	1.0000
		0.075	0.1000	0.1783	0.3075	0.4984	0.7286	0.9193	0.9944	1.0000
		0.1	0.1000	0.1772	0.3044	0.4926	0.7216	0.9149	0.9937	1.0000
	50	0.05	0.1000	0.2222	0.4328	0.7015	0.9144	0.9926	1.0000	1.0000
		0.075	0.1000	0.2205	0.4280	0.6947	0.9100	0.9918	1.0000	1.0000
		0.1	0.1000	0.2187	0.4232	0.6878	0.9054	0.9909	0.9999	1.0000
	75	0.05	0.1000	0.2584	0.5300	0.8192	0.9717	0.9992	1.0000	1.0000
		0.075	0.1000	0.2561	0.5241	0.8131	0.9696	0.9991	1.0000	1.0000
		0.1	0.1000	0.2538	0.5182	0.8067	0.9672	0.9990	1.0000	1.0000
	100	0.05	0.1000	0.2909	0.6094	0.8900	0.9905	0.9999	1.0000	1.0000
		0.075	0.1000	0.2881	0.6029	0.8850	0.9895	0.9999	1.0000	1.0000
		0.1	0.1000	0.2853	0.5964	0.8798	0.9885	0.9999	1.0000	1.0000
4	25	0.05	0.1000	0.1789	0.3081	0.4972	0.7245	0.9150	0.9935	1.0000
		0.075	0.1000	0.1760	0.2998	0.4820	0.7058	0.9026	0.9913	1.0000
		0.1	0.1000	0.1732	0.2917	0.4669	0.6866	0.8890	0.9885	1.0000
	50	0.05	0.1000	0.2214	0.4288	0.6931	0.9071	0.9908	0.9999	1.0000
		0.075	0.1000	0.2168	0.4159	0.6746	0.8943	0.9881	0.9999	1.0000
		0.1	0.1000	0.2123	0.4033	0.6558	0.8803	0.9846	0.9998	1.0000
	75	0.05	0.1000	0.2573	0.5249	0.8114	0.9681	0.9990	1.0000	1.0000
		0.075	0.1000	0.2512	0.5091	0.7943	0.9614	0.9985	1.0000	1.0000
		0.1	0.1000	0.2453	0.4935	0.7764	0.9536	0.9978	1.0000	1.0000
	100	0.05	0.1000	0.2896	0.6037	0.8836	0.9888	0.9999	1.0000	1.0000
		0.075	0.1000	0.2822	0.5863	0.8694	0.9856	0.9998	1.0000	1.0000
		0.1	0.1000	0.2749	0.5688	0.8542	0.9817	0.9997	1.0000	1.0000
6	25	0.05	0.1000	0.1765	0.3008	0.4831	0.7062	0.9022	0.9911	1.0000
		0.075	0.1000	0.1721	0.2881	0.4594	0.6760	0.8803	0.9864	1.0000
		0.1	0.1000	0.1678	0.2760	0.4365	0.6453	0.8557	0.9797	1.0000
	50	0.05	0.1000	0.2176	0.4174	0.6758	0.8945	0.9880	0.9999	1.0000
		0.075	0.1000	0.2105	0.3975	0.6461	0.8721	0.9822	0.9998	1.0000
		0.1	0.1000	0.2037	0.3784	0.6163	0.8471	0.9743	0.9995	1.0000
	75	0.05	0.1000	0.2522	0.5109	0.7954	0.9615	0.9984	1.0000	1.0000
		0.075	0.1000	0.2429	0.4863	0.7670	0.9488	0.9972	1.0000	1.0000
		0.1	0.1000	0.2340	0.4624	0.7372	0.9333	0.9953	1.0000	1.0000
	100	0.05	0.1000	0.2834	0.5883	0.8704	0.9857	0.9998	1.0000	1.0000
		0.075	0.1000	0.2720	0.5607	0.8460	0.9792	0.9996	1.0000	1.0000
		0.1	0.1000	0.2611	0.5336	0.8195	0.9706	0.9991	1.0000	1.0000

Table A7. The power $h(c_1)$ at $d = 4$, $\alpha = 0.01$.

			c_1							
m	n	p	0.8	0.825	0.85	0.875	0.9	0.925	0.95	0.975
2	25	0.05	0.0100	0.0224	0.0509	0.1154	0.2530	0.5077	0.8361	0.9961
		0.075	0.0100	0.0222	0.0500	0.1125	0.2461	0.4957	0.8259	0.9953
		0.1	0.0100	0.0220	0.0490	0.1096	0.2393	0.4836	0.8151	0.9945
	50	0.05	0.0100	0.0323	0.0968	0.2549	0.5427	0.8560	0.9910	1.0000
		0.075	0.0100	0.0319	0.0947	0.2484	0.5313	0.8472	0.9897	1.0000
		0.1	0.0100	0.0314	0.0925	0.2419	0.5197	0.8378	0.9882	1.0000
	75	0.05	0.0100	0.0420	0.1477	0.3974	0.7468	0.9637	0.9996	1.0000
		0.075	0.0100	0.0413	0.1442	0.3882	0.7360	0.9601	0.9995	1.0000
		0.1	0.0100	0.0407	0.1407	0.3788	0.7248	0.9561	0.9994	1.0000
	100	0.05	0.0100	0.0518	0.2015	0.5264	0.8667	0.9913	1.0000	1.0000
		0.075	0.0100	0.0510	0.1966	0.5156	0.8586	0.9901	1.0000	1.0000
		0.1	0.0100	0.0501	0.1917	0.5047	0.8501	0.9887	1.0000	1.0000
4	25	0.05	0.0100	0.0223	0.0501	0.1120	0.2430	0.4866	0.8146	0.9941
		0.075	0.0100	0.0217	0.0476	0.1047	0.2254	0.4551	0.7845	0.9910
		0.1	0.0100	0.0211	0.0453	0.0978	0.2087	0.4241	0.7518	0.9865
	50	0.05	0.0100	0.0320	0.0945	0.2460	0.5237	0.8383	0.9879	1.0000
		0.075	0.0100	0.0308	0.0890	0.2294	0.4933	0.8122	0.9829	1.0000
		0.1	0.0100	0.0297	0.0837	0.2135	0.4631	0.7837	0.9764	1.0000
	75	0.05	0.0100	0.0414	0.1436	0.3839	0.7277	0.9560	0.9993	1.0000
		0.075	0.0100	0.0397	0.1346	0.3598	0.6976	0.9441	0.9989	1.0000
		0.1	0.0100	0.0381	0.1260	0.3363	0.6660	0.9298	0.9981	1.0000
	100	0.05	0.0100	0.0510	0.1956	0.5101	0.8519	0.9886	1.0000	1.0000
		0.075	0.0100	0.0487	0.1830	0.4815	0.8281	0.9842	0.9999	1.0000
		0.1	0.0100	0.0465	0.1710	0.4531	0.8020	0.9784	0.9999	1.0000
6	25	0.05	0.0100	0.0218	0.0479	0.1052	0.2261	0.4552	0.7835	0.9907
		0.075	0.0100	0.0209	0.0443	0.0945	0.2002	0.4068	0.7310	0.9827
		0.1	0.0100	0.0200	0.0409	0.0849	0.1768	0.3607	0.6738	0.9696
	50	0.05	0.0100	0.0310	0.0895	0.2302	0.4938	0.8116	0.9826	1.0000
		0.075	0.0100	0.0293	0.0812	0.2056	0.4465	0.7658	0.9713	1.0000
		0.1	0.0100	0.0277	0.0738	0.1831	0.4009	0.7152	0.9549	0.9999
	75	0.05	0.0100	0.0399	0.1353	0.3608	0.6977	0.9436	0.9988	1.0000
		0.075	0.0100	0.0374	0.1219	0.3241	0.6477	0.9199	0.9974	1.0000
		0.1	0.0100	0.0351	0.1099	0.2899	0.5960	0.8898	0.9948	1.0000
	100	0.05	0.0100	0.0490	0.1839	0.4825	0.8280	0.9840	0.9999	1.0000
		0.075	0.0100	0.0455	0.1653	0.4379	0.7861	0.9740	0.9998	1.0000
		0.1	0.0100	0.0424	0.1483	0.3952	0.7396	0.9597	0.9994	1.0000

Table A8. The power $h(c_1)$ at $d = 4$, $\alpha = 0.05$.

			c_1							
m	n	p	0.8	0.825	0.85	0.875	0.9	0.925	0.95	0.975
2	25	0.05	0.0500	0.0998	0.1938	0.3555	0.5906	0.8400	0.9808	1.0000
		0.075	0.0500	0.0991	0.1914	0.3503	0.5828	0.8333	0.9790	1.0000
		0.1	0.0500	0.0984	0.1890	0.3450	0.5748	0.8262	0.9770	1.0000
	50	0.05	0.0500	0.1290	0.2929	0.5535	0.8266	0.9741	0.9996	1.0000
		0.075	0.0500	0.1278	0.2888	0.5460	0.8197	0.9719	0.9995	1.0000
		0.1	0.0500	0.1266	0.2846	0.5383	0.8126	0.9696	0.9994	1.0000
	75	0.05	0.0500	0.1548	0.3801	0.6938	0.9271	0.9957	1.0000	1.0000
		0.075	0.0500	0.1532	0.3746	0.6858	0.9227	0.9951	1.0000	1.0000
		0.1	0.0500	0.1515	0.3690	0.6777	0.9180	0.9945	1.0000	1.0000
	100	0.05	0.0500	0.1789	0.4574	0.7910	0.9691	0.9993	1.0000	1.0000
		0.075	0.0500	0.1768	0.4508	0.7837	0.9666	0.9991	1.0000	1.0000
		0.1	0.0500	0.1747	0.4442	0.7761	0.9640	0.9990	1.0000	1.0000
4	25	0.05	0.0500	0.0990	0.1903	0.3464	0.5745	0.8240	0.9758	0.9999
		0.075	0.0500	0.0971	0.1841	0.3327	0.5535	0.8047	0.9697	0.9999
		0.1	0.0500	0.0953	0.1780	0.3194	0.5325	0.7842	0.9624	0.9998
	50	0.05	0.0500	0.1276	0.2868	0.5401	0.8122	0.9687	0.9994	1.0000
		0.075	0.0500	0.1244	0.2759	0.5200	0.7926	0.9615	0.9990	1.0000
		0.1	0.0500	0.1213	0.2654	0.5000	0.7720	0.9531	0.9985	1.0000
	75	0.05	0.0500	0.1528	0.3718	0.6795	0.9177	0.9943	1.0000	1.0000
		0.075	0.0500	0.1485	0.3573	0.6579	0.9045	0.9922	1.0000	1.0000
		0.1	0.0500	0.1442	0.3431	0.6358	0.8899	0.9896	0.9999	1.0000
	100	0.05	0.0500	0.1764	0.4475	0.7777	0.9638	0.9989	1.0000	1.0000
		0.075	0.0500	0.1709	0.4302	0.7573	0.9559	0.9984	1.0000	1.0000
		0.1	0.0500	0.1655	0.4132	0.7360	0.9467	0.9976	1.0000	1.0000
6	25	0.05	0.0500	0.0973	0.1843	0.3328	0.5528	0.8032	0.9690	0.9999
		0.075	0.0500	0.0944	0.1748	0.3120	0.5199	0.7705	0.9568	0.9998
		0.1	0.0500	0.0916	0.1659	0.2923	0.4875	0.7357	0.9413	0.9994
	50	0.05	0.0500	0.1246	0.2763	0.5200	0.7919	0.9609	0.9990	1.0000
		0.075	0.0500	0.1198	0.2598	0.4887	0.7590	0.9469	0.9981	1.0000
		0.1	0.0500	0.1152	0.2443	0.4581	0.7242	0.9296	0.9965	1.0000
	75	0.05	0.0500	0.1488	0.3578	0.6578	0.9040	0.9920	1.0000	1.0000
		0.075	0.0500	0.1421	0.3356	0.6230	0.8803	0.9875	0.9999	1.0000
		0.1	0.0500	0.1359	0.3145	0.5879	0.8533	0.9810	0.9998	1.0000
	100	0.05	0.0500	0.1713	0.4308	0.7572	0.9556	0.9983	1.0000	1.0000
		0.075	0.0500	0.1629	0.4041	0.7233	0.9405	0.9970	1.0000	1.0000
		0.1	0.0500	0.1551	0.3785	0.6881	0.9220	0.9948	1.0000	1.0000

Table A9. The power $h(c_1)$ at $d = 4$, $\alpha = 0.1$.

			c_1							
m	n	p	0.8	0.825	0.85	0.875	0.9	0.925	0.95	0.975
2	25	0.05	0.1000	0.1844	0.3236	0.5243	0.7547	0.9320	0.9957	1.0000
		0.075	0.1000	0.1833	0.3205	0.5189	0.7486	0.9285	0.9952	1.0000
		0.1	0.1000	0.1822	0.3174	0.5134	0.7423	0.9249	0.9947	1.0000
	50	0.05	0.1000	0.2262	0.4409	0.7076	0.9146	0.9918	0.9999	1.0000
		0.075	0.1000	0.2245	0.4362	0.7013	0.9106	0.9911	0.9999	1.0000
		0.1	0.1000	0.2228	0.4315	0.6948	0.9063	0.9902	0.9999	1.0000
	75	0.05	0.1000	0.2613	0.5329	0.8169	0.9687	0.9989	1.0000	1.0000
		0.075	0.1000	0.2591	0.5273	0.8111	0.9666	0.9987	1.0000	1.0000
		0.1	0.1000	0.2568	0.5217	0.8051	0.9643	0.9986	1.0000	1.0000
	100	0.05	0.1000	0.2926	0.6080	0.8840	0.9882	0.9998	1.0000	1.0000
		0.075	0.1000	0.2900	0.6018	0.8792	0.9871	0.9998	1.0000	1.0000
		0.1	0.1000	0.2872	0.5956	0.8741	0.9859	0.9998	1.0000	1.0000
4	25	0.05	0.1000	0.1828	0.3181	0.5131	0.7402	0.9225	0.9942	1.0000
		0.075	0.1000	0.1800	0.3101	0.4988	0.7234	0.9120	0.9925	1.0000
		0.1	0.1000	0.1772	0.3022	0.4845	0.7061	0.9005	0.9903	1.0000
	50	0.05	0.1000	0.2238	0.4330	0.6949	0.9051	0.9897	0.9999	1.0000
		0.075	0.1000	0.2194	0.4207	0.6777	0.8932	0.9869	0.9999	1.0000
		0.1	0.1000	0.2150	0.4087	0.6602	0.8803	0.9836	0.9998	1.0000
	75	0.05	0.1000	0.2582	0.5235	0.8054	0.9637	0.9985	1.0000	1.0000
		0.075	0.1000	0.2524	0.5086	0.7892	0.9569	0.9978	1.0000	1.0000
		0.1	0.1000	0.2467	0.4938	0.7723	0.9492	0.9970	1.0000	1.0000
	100	0.05	0.1000	0.2890	0.5977	0.8744	0.9856	0.9998	1.0000	1.0000
		0.075	0.1000	0.2819	0.5812	0.8606	0.9821	0.9996	1.0000	1.0000
		0.1	0.1000	0.2749	0.5647	0.8459	0.9778	0.9994	1.0000	1.0000
6	25	0.05	0.1000	0.1801	0.3101	0.4982	0.7221	0.9108	0.9922	1.0000
		0.075	0.1000	0.1757	0.2978	0.4759	0.6948	0.8921	0.9885	1.0000
		0.1	0.1000	0.1716	0.2861	0.4543	0.6671	0.8714	0.9834	1.0000
	50	0.05	0.1000	0.2196	0.4208	0.6772	0.8924	0.9866	0.9998	1.0000
		0.075	0.1000	0.2127	0.4019	0.6495	0.8716	0.9809	0.9997	1.0000
		0.1	0.1000	0.2062	0.3838	0.6218	0.8487	0.9736	0.9994	1.0000
	75	0.05	0.1000	0.2527	0.5088	0.7888	0.9565	0.9978	1.0000	1.0000
		0.075	0.1000	0.2437	0.4855	0.7619	0.9438	0.9963	1.0000	1.0000
		0.1	0.1000	0.2352	0.4629	0.7340	0.9288	0.9941	1.0000	1.0000
	100	0.05	0.1000	0.2823	0.5815	0.8603	0.9819	0.9996	1.0000	1.0000
		0.075	0.1000	0.2714	0.5554	0.8367	0.9748	0.9992	1.0000	1.0000
		0.1	0.1000	0.2610	0.5298	0.8113	0.9657	0.9986	1.0000	1.0000

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