

Article

The Properties of Structures with Two Planes of Symmetry

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Abstract: In the domain of civil engineering, the structures encountered usually present symmetries of different types. The causes that determine the use of these symmetries are diverse, starting from esthetic motivations but also dictated by practical reasons, such as the reduction in construction time and costs. These geometric symmetries lead to specific properties that, in certain situations, can help to simplify the calculation of these structures. They can be identified both in the static analysis and the deformability of the structure, as well as in the dynamic analysis in the study of vibrations. If these properties are used, it is possible to reduce the analysis time, and the designer can obtain a quick estimate of the behavior of the structure. Starting from these considerations, this work will determine some of the properties specific to the natural vibrations of certain structures with identical parts in their components (structures that present two planes of symmetry) and will demonstrate how these properties can contribute to reducing the time required for dynamic analysis. An example will be used to exemplify the presented methods. This work allows for further developments and makes possible the development of the existing finite element software by adding some modules to identify such situations by analyzing the input data and using the properties described in the newly introduced algorithms.

Keywords: structure; eigenfrequency; vibration; multiple symmetry; eigenmode



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1. Introduction

Symmetry occurs in the real world in almost all areas of life. In engineering, it often happens that the systems we operate with have identical parts or different types of symmetry.

In the field of civil engineering, symmetries have played an important role since the beginning of the development of this industry, being frequently encountered in all types of buildings and constructions. In modern constructions, symmetries have become a common element and are often used in structures with large roofs or other modern facilities. We can find them in warehouses, exhibition halls, stadiums, performance halls or those for sports competitions, train stations, airports, etc. The use of these symmetries from the design stage offers numerous advantages to builders [1]. From the designer's point of view, an analysis is performed on the substructures, reducing the design and calculation time [2]. For example, in static analysis, it has been known for a long time that for the resistance of materials, the strain and stress analysis can be performed; for the case of symmetrical structures, only half of the structure requires analysis while paying attention to the boundary conditions introduced. Obviously, the number of operations and the calculation time are reduced by more than half in these cases. Users of the finite element method (FEM) have long noticed these advantages and used them empirically in their applications.

To realize a building with symmetric parts, the number of different construction elements used decreases. Logistical and labor costs also decrease. An analysis of the vibrations of such a structure used in construction can be performed more easily, faster, and with a low computational effort [3,4]. So, in the end, the costs are low.

The information needed to define a system with symmetries is less. This implies a reduction in design effort and the volume of calculations required and facilitates static and dynamic analysis [5]. The advantages of using symmetries and the properties that certain systems with symmetries have were presented by a specialized study [6]. But in general, these observations focused more on the mathematical formalism that is simplified by the existence of symmetries and on the way in which they can obtain, with less effort, the equations of motion. Obviously, symmetries can appear in all industrial branches, such as automotive, aerospace, and transport engineering, where a product can have symmetrical, identical, or repetitive parts. The aeronautical industry has long noticed the existence of identical parts in the construction of an aircraft and uses in the design stage the condensation and analysis of some substructures of the main project, thus saving design time and cost. In this way, it is easier to approach the entire structure [7]. The analysis of free and forced vibrations of such a system will finally provide a system of differential equations. Solving it will give us a dynamic answer. Taking symmetries into account makes it easier to obtain the dynamic response [8].

The modern solutions offered by civil engineering for the coverage of large spaces, such as warehouses, exhibition halls, stadiums, concert halls, or other modern buildings, are layered spatial networks (two or three-layer configurations). They show a high degree of symmetry, both at the modular level and at the overall level. In [9], such an analysis is conducted using a group theoretical approach to the study of vibrations.

On a human level, symmetry provides a sense of harmony and balance and is sought after by human beings. It is expressed in the case of construction by the distribution of building parts and bodies and the surrounding spaces [10]. The concretization of the ideas of symmetry in a certain period of time is determined by the cultural influences and trends of the period. In the construction of structures, its role is well defined. Thus, it was found that a building with symmetries tends to be more stable in response to various anthropic activities or natural phenomena. Obviously, such a construction is more attractive to the eye and has esthetic qualities. But at the same time, it offers specific properties of the system that make the calculation easier [11]. There are studies concerning different aspects of the problem and the mathematical aspects are presented in [12–14]. The study of a multilevel structure is presented in [15]. The results concerning engineering aspects of symmetry can be found in [16].

In construction practice, different types of symmetries are used, such as bilateral symmetry, rotational symmetry, chiral symmetry, and axial symmetry [17–19]. This paper will deal with cases of bilateral symmetry, which is present in structures with identical halves along an axis. This type of symmetry is widely found in the construction of buildings (i.e., the Taj Mahal mosque). The axis of symmetry can be horizontal or vertical.

The concept of symmetry, widespread in all fields of human knowledge, is used in common practice in two senses. The first approach to symmetries refers to the esthetic aspect of an object, buildings, and cars. Proportionality and harmony are analyzed as human beings naturally and instinctively seek symmetry in nature. There is, however, a practical engineering aspect that uses geometric or other regularities to facilitate certain design, calculation, and product manufacturing activities, or to provide simple explanations in the description of a process/pattern of repetition in an activity or within a certain phenomenon. Simply explained, symmetry is equivalent to harmony in the dimensions, proportions, and arrangement of objects [20–28].

In this paper, the properties of the equations of motion of systems with multiple symmetries will be presented, properties that can lead to easing the effort required for the dynamic analysis of such structures. An application of the calculation of a chiller support structure will illustrate the results obtained.

2. Methods

Let us consider a mechanical system with elastic elements and damping subject to external loads. The equations of motion of such a structure have the following classical form [29,30]:

$$[M]\{\ddot{\Delta}\} + [C]\{\dot{\Delta}\} + [K]\{\Delta\} = \{F\}. \quad (1)$$

Let us now consider a mechanical system that can be subjected to vibrations and that is made up of two identical parts attached to a third (Figure 1).

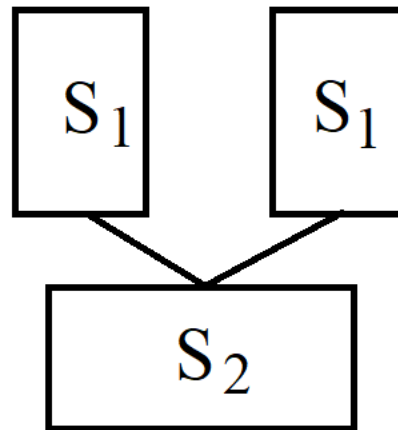


Figure 1. A mechanical system with two identical parts.

The equations of motion for the systems (left, right) and the system taken independently are as follows:

$$[M_1]\{\ddot{\Delta}_{1l}\} + [C_1]\{\dot{\Delta}_{1l}\} + [K_1]\{\Delta_{1l}\} = \{F_1\}, \quad (2)$$

$$[M_1]\{\ddot{\Delta}_{1r}\} + [C_1]\{\dot{\Delta}_{1r}\} + [K_1]\{\Delta_{1r}\} = \{F_1\}, \quad (3)$$

$$[M_2]\{\ddot{\Delta}_2\} + [C_2]\{\dot{\Delta}_2\} + [K_2]\{\Delta_2\} = \{F_2\}. \quad (4)$$

A more detailed explanation of the writing of the motion equations is provided in [31]. The vectors $\{F_1\}$ and $\{F_2\}$ represent the external forces applied to the systems S_1 and S_2 . For the entire system, if the links between the three mechanical systems are also taken into account, the equations of motion become as follows:

$$\begin{bmatrix} M_{11} & 0 & M_{12} \\ 0 & M_{11} & M_{12} \\ M_{12}^T & M_{12}^T & M_{22} \end{bmatrix} \begin{Bmatrix} \ddot{\Delta}_{1l} \\ \ddot{\Delta}_{1r} \\ \ddot{\Delta}_2 \end{Bmatrix} + \begin{bmatrix} C_{11} & 0 & C_{12} \\ 0 & C_{11} & C_{12} \\ C_{12}^T & C_{12}^T & C_{22} \end{bmatrix} \begin{Bmatrix} \Delta_{1l} \\ \Delta_{1r} \\ \Delta_2 \end{Bmatrix} + \begin{bmatrix} K_{11} & 0 & K_{12} \\ 0 & K_{11} & K_{12} \\ K_{12}^T & K_{12}^T & K_{22} \end{bmatrix} \begin{Bmatrix} \Delta_{1l} \\ \Delta_{1r} \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_1 \\ F_2 \end{Bmatrix}, \quad (5)$$

where:

M_{11} —the inertial matrix for the subsystems S_1 ;

M_{12} —the inertial coupling matrix between the two identical parts S_1 and the system S_2 ;

M_{22} —the inertial matrix for the subsystem S_2 ;

K_{11} —the stiffness matrix for the subsystems S_1 ;

K_{12} —the stiffness coupling matrix between the two identical parts S_1 and the system S_2 ;

K_{22} —the stiffness matrix for the subsystem S_2 ;

C_{11} —the damping matrix for the subsystems S_1 ;

C_{12} —the damping coupling matrix between the two identical parts S_1 and the system S_2 ;

C_{22} —the damping matrix for the subsystem S_2 ;

Δ_{1l} —the vector of independent coordinates for the subsystem S_1 on the left side;

Δ_{1r} —the vector of independent coordinates for the subsystem S_1 on the right side;

Δ_2 —the vector of independent coordinates for the subsystem S_2 .

The system of equations of undamped free vibrations is as follows:

$$\begin{bmatrix} M_{11} & 0 & M_{12} \\ 0 & M_{11} & M_{12} \\ M_{12}^T & M_{12}^T & M_{22} \end{bmatrix} \begin{Bmatrix} \ddot{\Delta}_{1l} \\ \ddot{\Delta}_{1r} \\ \ddot{\Delta}_2 \end{Bmatrix} + \begin{bmatrix} K_{11} & 0 & K_{12} \\ 0 & K_{11} & K_{12} \\ K_{12}^T & K_{12}^T & K_{22} \end{bmatrix} \begin{Bmatrix} \Delta_{1l} \\ \Delta_{1r} \\ \Delta_2 \end{Bmatrix} = \{0\}. \quad (6)$$

If we denote with p the eigenvalue, considering Equation (6), solving the eigenvalue problem comes down to solving the following characteristic equation:

$$\begin{vmatrix} K_{11} - p^2 M_{11} & 0 & K_{12} - p^2 M_{12} \\ 0 & K_{11} - p^2 M_{11} & K_{12} - p^2 M_{12} \\ K_{12}^T - p^2 M_{12}^T & K_{12}^T - p^2 M_{12}^T & K_{22} - p^2 M_{22} \end{vmatrix} = 0. \quad (7)$$

For the substructure S_1 , the eigenvalue problem comes back to solving the following characteristic equation:

$$\det([K_{11}] - p^2[M_{11}]) = \{0\}. \quad (8)$$

In the following, fundamental results regarding such systems are presented in a unitary form [32,33]. The main result regarding such systems is the following.

Let us consider the most general case with a matrix made up of matrices with complex coefficients of size n , denoted as A and $Z = O_n$; the square matrix is C , and the matrices B and L are chosen so that the matrix M is square:

$$M = \begin{pmatrix} A & Z & B \\ Z & A & B \\ L & L & C \end{pmatrix}. \quad (9)$$

In this case, $\det(M)$ is divisible by $\det(A)$. A demonstration of the result is presented in [34]. So, really, in our case, having the S_2 system with two identical systems S_1 in the component shows that the eigenvalues of one of the identical systems are found among the eigenvalues of the whole system. Then, in the next step, we have two identical systems $S_1 + S_2$, and the result is repeated. From this result, the following conclusions are immediately obtained in such cases:

P1. The eigenvalues for a half structure are also the eigenvalues for the whole structure. (This result was argued in the sentence above.)

P2. For identical eigenvalues for half of the structure and for the whole structure, the eigenmodes are of the following form:

$$\Phi = \begin{Bmatrix} \Phi_1 \\ -\Phi_1 \\ 0 \end{Bmatrix}, \quad (10)$$

(The eigenmodes corresponding to the identical parts are skew-symmetric.)

P3. For the other eigenvalues, the eigenvectors are of the following form:

$$\Phi = \begin{Bmatrix} \Phi_1 \\ \Phi_1 \\ \Phi_2 \end{Bmatrix}, \quad (11)$$

(For the other eigenvalues of the system, the components of the eigenmodes corresponding to the identical parts are identical).

In the following, the presented properties will be expanded for systems that present two levels of bilateral symmetry (Figure 2).

$$[M_{11}]\{\ddot{\Delta}_{1l}\} + [K_{11}]\{\Delta_{1l}\} = \{0\}; \quad (12)$$

for the left side and for the right half, the equation is as follows:

$$[M_{11}]\{\ddot{\Delta}_{1r}\} + [K_{11}]\{\Delta_{1r}\} = \{0\}. \quad (13)$$

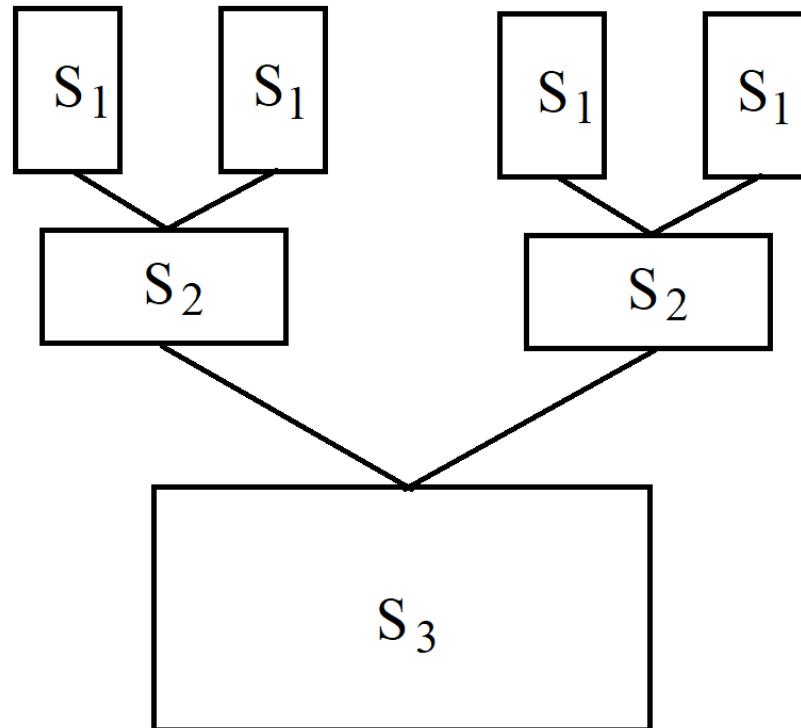


Figure 2. A mechanical system with two levels of symmetry.

If we consider the structure formed by two such substructures and if we neglect the structural damping and other types of proportional damping that may occur, then the equations of motion of the free vibrations are as follows:

$$\begin{bmatrix} M_{22}^* & 0 & M_{23}^* \\ 0 & M_{22}^* & M_{23}^* \\ M_{23}^{*T} & M_{23}^{*T} & M_{33} \end{bmatrix} \begin{Bmatrix} \ddot{\Delta}_l \\ \ddot{\Delta}_r \\ \ddot{\Delta}_3 \end{Bmatrix} + \begin{bmatrix} K_{22}^* & 0 & M_{23}^* \\ 0 & K_{22}^* & M_{23}^* \\ K_{23}^{*T} & K_{23}^{*T} & K_{33} \end{bmatrix} \begin{Bmatrix} \Delta_l \\ \Delta_r \\ \Delta_3 \end{Bmatrix} = \{0\}, \quad (14)$$

where, if we also take into account the second level of symmetry, existing on the left and right sides of the mechanical system, the following is noted:

$$[M_{22}^*] = \begin{bmatrix} M_{11} & 0 & M_{12} \\ 0 & M_{11} & M_{12} \\ M_{12}^T & M_{12}^T & M_{22} \end{bmatrix}; \quad [K_{22}^*] = \begin{bmatrix} K_{11} & 0 & K_{12} \\ 0 & K_{11} & K_{12} \\ K_{12}^T & K_{12}^T & K_{22} \end{bmatrix}. \quad (15)$$

The modeling of the two systems was performed in this paper using the finite element method (FEM). The program used for the calculation was ABAQUS.

After a reordering of the equations so that the quantities associated with the systems S_1 , S_2 , and S_3 appear in the first positions, the motion equations are of the form of Equation (1), where the matrices that intervene in the equations are as follows:

$$[M] = \begin{bmatrix} M_{11} & 0 & 0 & 0 & M_{12} & 0 & 0 \\ 0 & M_{11} & 0 & 0 & M_{12} & 0 & 0 \\ 0 & 0 & M_{11} & 0 & 0 & M_{12} & 0 \\ 0 & 0 & 0 & M_{11} & 0 & M_{12} & 0 \\ M_{12}^T & M_{12}^T & 0 & 0 & M_{22} & 0 & 0 \\ 0 & 0 & M_{12}^T & M_{12}^T & 0 & M_{22} & M_{23} \\ 0 & 0 & 0 & 0 & 0 & M_{23}^T & M_{33} \end{bmatrix}; \quad (16)$$

$$[K] = \begin{bmatrix} K_{11} & 0 & 0 & 0 & K_{12} & 0 & 0 \\ 0 & K_{11} & 0 & 0 & K_{12} & 0 & 0 \\ 0 & 0 & K_{11} & 0 & 0 & K_{12} & 0 \\ 0 & 0 & 0 & K_{11} & 0 & K_{12} & 0 \\ K_{12}^T & K_{12}^T & 0 & 0 & K_{22} & 0 & 0 \\ 0 & 0 & K_{12}^T & K_{12}^T & 0 & K_{22} & K_{23} \\ 0 & 0 & 0 & 0 & 0 & K_{23}^T & K_{33} \end{bmatrix}; \quad (17)$$

$$[C] = \begin{bmatrix} C_{11} & 0 & 0 & 0 & C_{12} & 0 & 0 \\ 0 & C_{11} & 0 & 0 & C_{12} & 0 & 0 \\ 0 & 0 & C_{11} & 0 & 0 & C_{12} & 0 \\ C_{12}^T & C_{12}^T & 0 & 0 & C_{22} & 0 & 0 \\ 0 & 0 & C_{12,1}^T & C_{12,1} & 0 & C_{22} & C_{23} \\ 0 & 0 & 0 & 0 & 0 & C_{23}^T & C_{33} \end{bmatrix}. \quad (18)$$

The characteristic equation for this mechanical system is as follows:

$$\left(\begin{bmatrix} K_{11} & 0 & 0 & 0 & K_{12} & 0 & 0 \\ 0 & K_{11} & 0 & 0 & K_{12} & 0 & 0 \\ 0 & 0 & K_{11} & 0 & 0 & K_{12} & 0 \\ 0 & 0 & 0 & K_{11} & 0 & K_{12} & 0 \\ K_{12}^T & K_{12}^T & 0 & 0 & K_{22} & 0 & 0 \\ 0 & 0 & K_{12}^T & K_{12}^T & 0 & K_{22} & K_{23} \\ 0 & 0 & 0 & 0 & 0 & K_{23}^T & K_{33} \end{bmatrix} - p^2 \begin{bmatrix} M_{11} & 0 & 0 & 0 & M_{12} & 0 & 0 \\ 0 & M_{11} & 0 & 0 & M_{12} & 0 & 0 \\ 0 & 0 & M_{11} & 0 & 0 & M_{12} & 0 \\ 0 & 0 & 0 & M_{11} & 0 & M_{12} & 0 \\ M_{12}^T & M_{12}^T & 0 & 0 & M_{22} & 0 & 0 \\ 0 & 0 & M_{12}^T & M_{12}^T & 0 & M_{22} & M_{23} \\ 0 & 0 & 0 & 0 & 0 & M_{23}^T & M_{33} \end{bmatrix} \right) \begin{Bmatrix} \Phi_{1ss} \\ \Phi_{1sd} \\ \Phi_{1ds} \\ \Phi_{1dd} \\ \Phi_{2s} \\ \Phi_{2d} \\ \Phi_3 \end{Bmatrix} = \{0\} \quad (19)$$

or

$$\left([K_{11}] - p^2[M_{11}] \right) \{ \Phi_{1ll} \} + \left([K_{12,1}] - p^2[M_{12,1}] \right) \{ \Phi_{2l} \} = 0, \quad (20)$$

$$\left([K_{11}] - p^2[M_{11}] \right) \{ \Phi_{1lr} \} + \left([K_{12,1}] - p^2[M_{12,1}] \right) \{ \Phi_{2l} \} = 0, \quad (21)$$

$$\left([K_{11}] - p^2[M_{11}] \right) \{ \Phi_{1rl} \} + \left([K_{12,1}] - p^2[M_{12,1}] \right) \{ \Phi_{2r} \} = 0, \quad (22)$$

$$\left([K_{11}] - p^2[M_{11}] \right) \{ \Phi_{1rr} \} + \left([K_{12}] - p^2[M_{12}] \right) \{ \Phi_{2r} \} = 0, \quad (23)$$

$$\left([K_{22}] - p^2[M_{22}] \right) \{ \Phi_{2l} \} + \left([K_{12}]^T - p^2[M_{12}]^T \right) (\{ \Phi_{1ll} \} + \{ \Phi_{1lr} \}) + \left([K_{23}] - p^2[M_{23}] \right) \{ \Phi_3 \} = 0, \quad (24)$$

$$\left([K_{22}] - p^2[M_{22}] \right) \{ \Phi_{2r} \} + \left([K_{12}]^T - p^2[M_{12}]^T \right) (\{ \Phi_{1rl} \} + \{ \Phi_{1rr} \}) + \left([K_{23}] - p^2[M_{23}] \right) \{ \Phi_3 \} = 0, \quad (25)$$

$$\left([K_{23}]^T - p^2[M_{23}]^T \right) (\{ \Phi_{2l} \} + \{ \Phi_{2r} \}) + \left([K_{33}] - p^2[M_{33}] \right) \{ \Phi_3 \} = 0. \quad (26)$$

For ease of calculations, we note the following:

$$\begin{aligned} [A_{11}] &= [K_{11}] - p^2[M_{11}]; [A_{22}] = [K_{22}] - p^2[M_{22}]; [A_{33}] = [K_{33}] - p^2[M_{33}]; \\ [A_{12}] &= [K_{12}] - p^2[M_{12}]; [A_{23}] = [K_{23}] - p^2[M_{23}]. \end{aligned} \quad (27)$$

With these notations, the systems of Equations (20)–(26) become as follows:

$$[A_{11}]\{\Phi_{1ll}\} + [A_{12}]\{\Phi_{2l}\} = 0, \quad (28)$$

$$[A_{11}]\{\Phi_{1lr}\} + [A_{12}]\{\Phi_{2l}\} = 0, \quad (29)$$

$$[A_{11}]\{\Phi_{1rl}\} + [A_{12}]\{\Phi_{2r}\} = 0, \quad (30)$$

$$[A_{11}]\{\Phi_{1rr}\} + [A_{12}]\{\Phi_{2r}\} = 0, \quad (31)$$

$$[A_{22}]\{\Phi_{2l}\} + [A_{12}](\{\Phi_{1ll}\} + \{\Phi_{1lr}\}) + [A_{23}]\{\Phi_3\} = 0, \quad (32)$$

$$[A_{22}]\{\Phi_{2r}\} + [A_{12}](\{\Phi_{1rl}\} + \{\Phi_{1rr}\}) + [A_{23}]\{\Phi_3\} = 0, \quad (33)$$

$$[A_{23}](\{\Phi_{2l}\} + \{\Phi_{2r}\}) + [A_{33}]\{\Phi_3\} = 0. \quad (34)$$

Theorem 1. *If an eigenvalue for the system is also an eigenvalue for S_1 , then the eigenvector associated with this eigenvalue is of the following form:*

$$\{\Phi\} = \begin{Bmatrix} \Phi_1 \\ -\Phi_1 \\ \Phi_1 \\ -\Phi_1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}.$$

Proof. If $\det[A_{11}] = 0$, it can be found $\{\Phi_{1ll}\}$ (eigenvector for system S_1) such that

$$[A_{11}]\{\Phi_{1ll}\} = 0. \quad (35)$$

In this case, the first system, Equation (27), becomes as follows:

$$[A_{12}]\{\Phi_{2l}\} = 0. \quad (36)$$

Adding Equation (27) with Equation (28) gives the following:

$$[A_{11}](\{\Phi_{1ll}\} + \{\Phi_{1lr}\}) + 2[A_{12}]\{\Phi_{2l}\} = 0. \quad (37)$$

If the conditions in (36) are taken into account, it results in the following:

$$[A_{11}](\{\Phi_{1ll}\} + \{\Phi_{1lr}\}) = 0, \quad (38)$$

from which the following is immediately obtained:

$$\{\Phi_{1ll}\} = -\{\Phi_{1lr}\} = \{\Phi_1\}. \quad (39)$$

A similar reasoning for Equations (29) and (30) leads us to the following relation:

$$\{\Phi_{1rl}\} = -\{\Phi_{1rr}\} = \{\Phi_1\}. \quad (40)$$

Equations (31) and (32) become

$$[A_{22}]\{\Phi_{2l}\} + [A_{23}]\{\Phi_3\} = 0, \quad (41)$$

$$[A_{22}]\{\Phi_{2r}\} + [A_{23}]\{\Phi_3\} = 0. \quad (42)$$

If Equation (42) is subtracted from Equation (41), we obtain the following:

$$[A_{22}](\{\Phi_{2l}\} - \{\Phi_{2r}\}) = 0. \quad (43)$$

Because, in general, when $\det[A_{22}] \neq 0$, it results in the following:

$$\{\Phi_{2l}\} = \{\Phi_{2r}\}. \quad (44)$$

Adding (1), (2), (3), and (4), we obtain the following:

$$[A_{11}](\{\Phi_{1ll}\} + \{\Phi_{1lr}\} + \{\Phi_{1rl}\} + \{\Phi_{1rr}\}) + 2[A_{12}](\{\Phi_{2l}\} + \{\Phi_{2r}\}) = 0. \quad (45)$$

where

$$\{\Phi_{2l}\} = \{\Phi_{2r}\} = 0. \quad (46)$$

Inserting this into Equation (33) and because, in general, $\det[A_{33}] \neq 0$, it is immediately obtained that

$$\{\Phi_3\} = 0. \quad (47)$$

Thus, the eigenmode of vibration will have the following form:

$$\{\Phi\} = \begin{Bmatrix} \Phi_{1ll} \\ \Phi_{1lr} \\ \Phi_{1rl} \\ \Phi_{1rr} \\ \Phi_{2l} \\ \Phi_{2r} \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} \Phi_1 \\ -\Phi_1 \\ \Phi_1 \\ -\Phi_1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \quad (48)$$

□

Theorem 2. *If the eigenpulsations are different from the eigenpulsations of the (S_1) system, then the eigenvectors corresponding to these eigenpulsations have the following particular form:*

$$\{\Phi\} = \begin{Bmatrix} \Phi_1 \\ \Phi_1 \\ \Phi_1 \\ \Phi_1 \\ \Phi_2 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix}. \quad (49)$$

Proof. Subtracting Equation (28) from Equation (27) yields

$$[A_{11}](\{\Phi_{1ll}\} - \{\Phi_{1lr}\}) = 0. \quad (50)$$

Whence, since $\det[A_{11}] \neq 0$, it results in the following:

$$\{\Phi_{1ll}\} - \{\Phi_{1lr}\} = 0. \quad (51)$$

So,

$$\{\Phi_{1ll}\} = \{\Phi_{1lr}\}. \quad (52)$$

Subtracting Equation (30) from Equation (29) yields

$$[A_{11}](\{\Phi_{1rl}\} - \{\Phi_{1rr}\}) = 0. \quad (53)$$

Its analogue is as follows:

$$\{\Phi_{1rl}\} = \{\Phi_{1rr}\}. \quad (54)$$

Eliminating $\{\Phi_{1lr}\}$ and $\{\Phi_{1rr}\}$ from Equations (27)–(33), the system can be reduced to the following:

$$[A_{11}]\{\Phi_{1ll}\} + [A_{12}]\{\Phi_{2l}\} = 0. \quad (55)$$

$$[A_{11}]\{\Phi_{1rl}\} + [A_{12}]\{\Phi_{2r}\} = 0. \quad (56)$$

$$[A_{22}]\{\Phi_{2l}\} + 2[A_{12}]\{\Phi_{1ll}\} + [A_{23}]\{\Phi_3\} = 0. \quad (57)$$

$$[A_{22}]\{\Phi_{2r}\} + 2[A_{12}]\{\Phi_{1rl}\} + [A_{23}]\{\Phi_3\} = 0. \quad (58)$$

$$[A_{23}](\{\Phi_{2l}\} + \{\Phi_{2r}\}) + [A_{33}]\{\Phi_3\} = 0. \quad (59)$$

From Equations (55) and (56), it follows, successively, that

$$\{\Phi_{1ll}\} = -[A_{11}]^{-1}[A_{12}]\{\Phi_{2l}\} \quad (60)$$

and

$$\{\Phi_{1rl}\} = -[A_{11}]^{-1}[A_{12}]\{\Phi_{2r}\}. \quad (61)$$

Inserting this into Equations (57) and (58), we obtain the following:

$$[A_{22}]\{\Phi_{2l}\} - 2[A_{12}][A_{11}]^{-1}[A_{12}]\{\Phi_{2l}\} + [A_{23}]\{\Phi_3\} = 0, \quad (62)$$

$$[A_{22}]\{\Phi_{2r}\} - 2[A_{12}][A_{11}]^{-1}[A_{12}]\{\Phi_{2r}\} + [A_{23}]\{\Phi_3\} = 0. \quad (63)$$

Subtracting Equations (62) and (63) gives

$$[A_{22}](\{\Phi_{2l}\} - \{\Phi_{2r}\}) - 2[A_{12}][A_{11}]^{-1}[A_{12}](\{\Phi_{2l}\} - \{\Phi_{2r}\}) = 0, \quad (64)$$

or

$$\left[[A_{22}] - 2[A_{12}][A_{11}]^{-1}[A_{12}] \right] (\{\Phi_{2l}\} - \{\Phi_{2r}\}) = 0, \quad (65)$$

wherefrom

$$\{\Phi_{2l}\} = \{\Phi_{2r}\} = \{\Phi_2\}. \quad (66)$$

Entering these results into Equations (60) and (61), you immediately obtain the following:

$$\{\Phi_{1ll}\} = \{\Phi_{1lr}\} = \{\Phi_{1rl}\} = \{\Phi_{1rr}\} = \{\Phi_1\}. \quad (67)$$

Then, the corresponding eigenvector is made by Equation (49). \square

3. Results

In what follows, we will study a metallic structure made up of a system of bars whose dimensions are shown below and which presents two planes of symmetry (Figure 3) realized to support a system of air conditioning. In what follows, a real structure, which is intended to support a chiller from a building complex, is analyzed from the point of view of the previously mentioned properties. The structure is made up of a series of four types of trusses welded together. The dimensions of the different parts of the structure are shown in Figure 3. The entire structure has 24 nodes connecting 56 bars. There are two median planes of the figure that connect two identical mirror substructures.

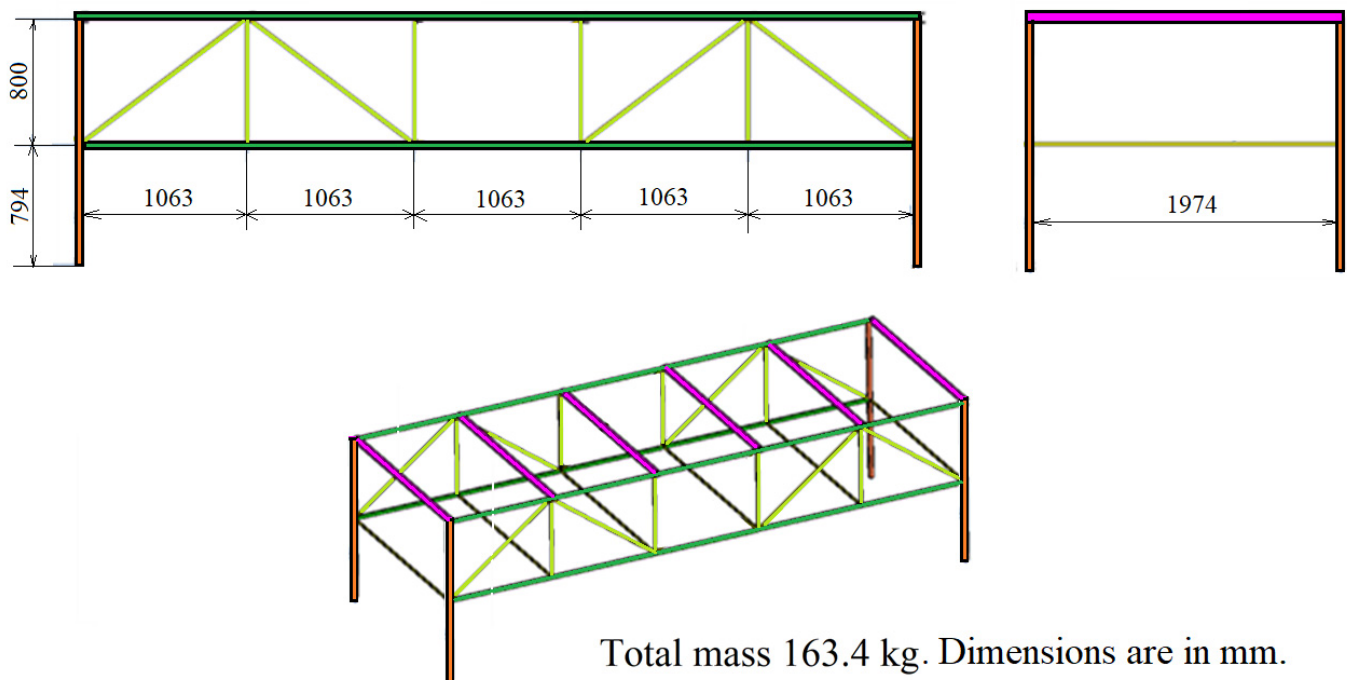


Figure 3. Metallic structure: geometry and dimensions. The different type of bars are represented in different colors.

The dimensions are as follows:

- The lower sole and the upper sole of the lattice beams are made of hot-rolled square pipe with a section of 30×2 mm;
- The uprights—hot-rolled square pipe with a section of 20×2 mm;
- The diagonals—hot-rolled square pipe with a section of 20×2 mm;
- The uprights at the ends of the beams with lattices that also fulfill the role of pillars are made of hot-rolled square pipe with a section of 40×2.5 mm;
- The other elements are made of laminated profiles with the U65 section.

All elements were made of S235JR steel and assembled by welding performed on the entire contact surface of the parts.

The air conditioning unit was placed on the metal structure and was attached to it with threaded assemblies.

FEM is used to determine the natural frequencies and natural modes of vibration. For this system, a calculation of the natural frequencies and natural modes of vibration is made. It is noted that we are dealing with two types of modes: symmetric modes and antisymmetric modes. The two types of modes are due to the symmetry of the system, as was shown in the theoretical part.

Four cases will be analyzed in the following: the whole structure, then two substructures obtained by sectioning with a plane of symmetry the assembly, and finally, a quarter of the structure that can be obtained by sectioning with a plane of symmetry one of the two considered substructures (see Figure 4).

The eigenfrequencies are determined and presented for all four cases in Table 1.

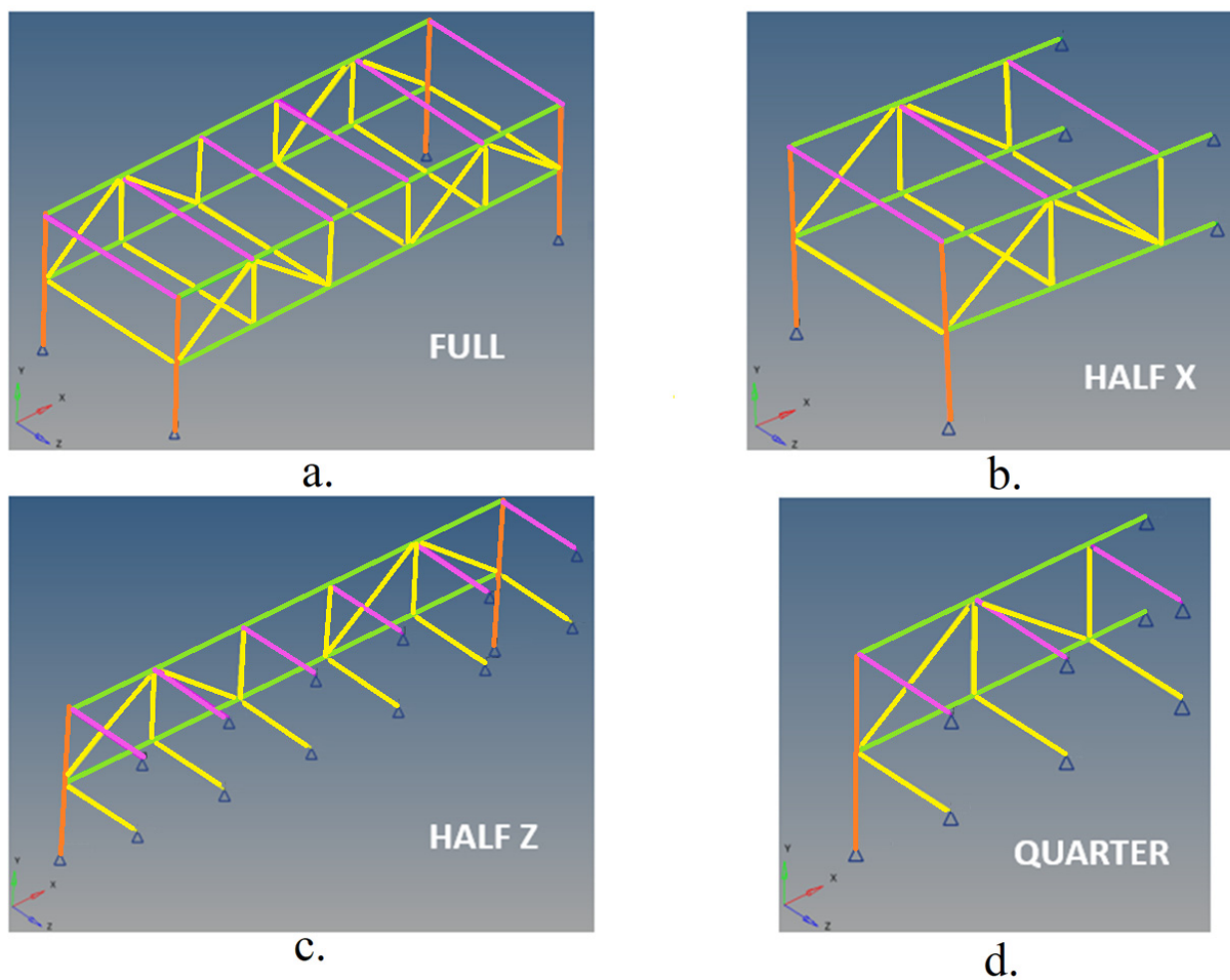


Figure 4. Four studied cases: (a) full structure; (b) symmetry plane is Oyz; (c) symmetry plane is Oxy; (d) quarter structure.

Table 1. The eigenfrequencies for the four considered cases.

FULL		HALF_X		HALF_Z		QUARTER	
Mode	Eigenfrequency [Hz]	Mode	Eigenfrequency [Hz]	Mode	Eigenfrequency [Hz]	Mode	Eigenfrequency [Hz]
1	4.90			1	4.90		
2	9.29						
3	10.74	1	10.74				
4	13.78			2	13.78		
5	16.37						
6	16.87			3	16.87		
7	18.81	2	18.81				
8	19.70						
9	22.60						
10	23.50			4	23.50		
11	26.11						
12	27.48	3	27.48	5	27.48	1	27.48

Table 1. Cont.

FULL		HALF_X		HALF_Z		QUARTER	
A1		A2		A3		A4	
Mode	Eigenfrequency [Hz]	Mode	Eigenfrequency [Hz]	Mode	Eigenfrequency [Hz]	Mode	Eigenfrequency [Hz]
13	27.56						
14	29.22	4	29.22				
15	29.89	5	29.89	6	29.89	2	29.89
16	31.00	6	31.00	7	31.00	3	31.00
17	31.00	7	31.00				
18	31.56	8	31.56				
19	32.17	9	32.17	8	32.17	4	32.17
20	32.54			9	32.54		
21	32.85	10	32.85				
22	33.01	11	33.01	10	33.01	5	33.01
23	33.96	12	33.96	11	33.96	6	33.96
24	34.17	13	34.17				
25	34.26	14	34.26	12	34.26	7	34.26
26	34.26	15	34.26				
27	34.54	16	34.54				
28	35.52	17	35.52	13	35.52	8	35.52
29	36.54	18	36.54	14	36.54		
30	37.72	19	37.72			9	36.54
31	39.81	20	39.81	15	39.81	10	39.81
32	43.11						
33	46.96		46.96	16	46.96	11	46.96
34	47.21						
35	52.38						
36	52.56		52.56	17	52.56	12	52.56
37	52.78						
38	54.09			18	54.09		
39	61.29						
40	65.36		65.36	19	65.36	13	65.36
41	65.64			20	65.64		
42	67.96						
43	68.32						
44	70.33		70.33		70.33	14	70.33
45	70.42						
46	71.00						
47	71.74		71.74		71.74	15	71.74
48	72.23						
49	72.88						
50	73.72		73.72		73.72	16	73.72

In Table 1, the properties presented in Section 2 can be identified. So, if it is considered a half structure with the symmetry plane yOz (called Half X), all 20 eigenfrequencies calculated can be found among the eigenfrequencies of the whole structure. This property remains valid, too, for the second type of half structure (called Half Z) with the symmetry plan xOy . If it is considered now a quarter of the structure, at the second level of symmetry, it is easy to observe that the eigenfrequencies of the quarter structure can be found among the frequencies of the Half X, Half Z, and full structures. If the representation of the eigenmodes of vibration is analyzed, it is possible to see that Theorems 1 and 2 are respected. So, the calculations made validate our theoretical results.

Based on Table 1, a representation of the natural modes of vibration for the symmetric modes is presented in Table 2. In Table 2, the first column presents the eigenmodes of the entire system that have a counterpart in the parts of the structure considered separately. Columns 2, 3, and 4 present the coincident modes from the considered substructure parts.

Table 2. Symmetric modes of vibration. For each mode, the eigenfrequency was written (with red are represented the maximum values, with blue the minimum).

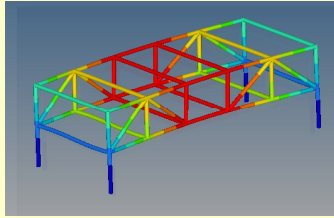
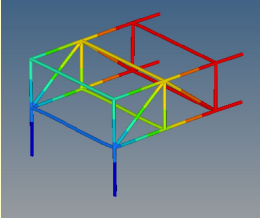
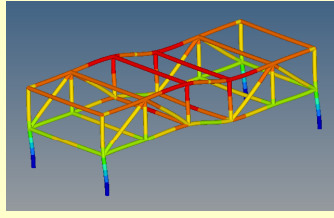
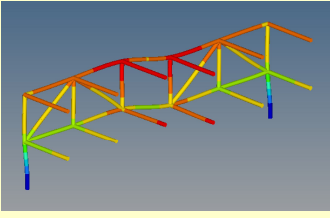
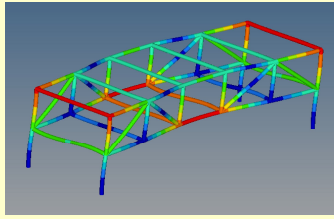
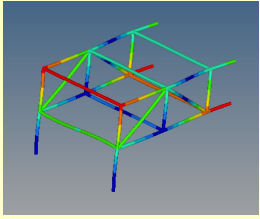
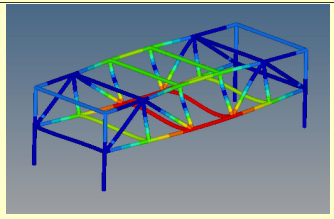
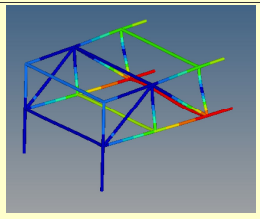
Full System	Type of Symmetry		
	Half System X	Half System Z	Quarter
 <p>Mode 1: 4.90 Hz</p>		 <p>Mode 1: 4.90 Hz</p>	
 <p>Mode 3: 10.74 Hz</p>	 <p>Mode 1: 10.74 Hz</p>		
 <p>Mode 4: $\nu = 13.78$ Hz</p>		 <p>Mode 2: $\nu = 13.78$ Hz</p>	
 <p>Mode 6: $\nu = 16.87$ Hz</p>		 <p>Mode 3: $\nu = 16.87$ Hz</p>	

Table 2. Cont.

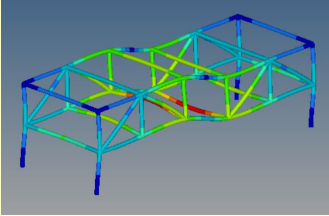
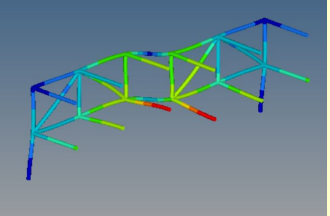
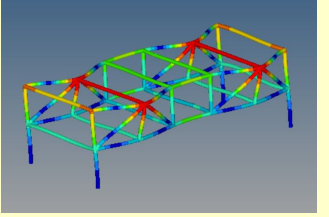
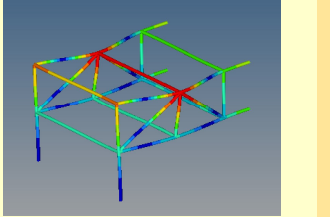
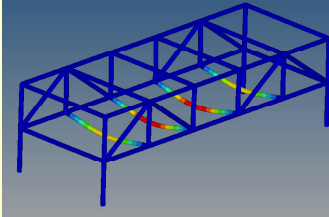
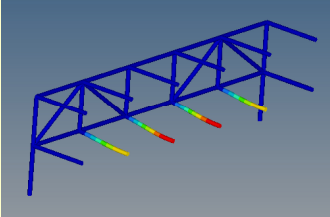
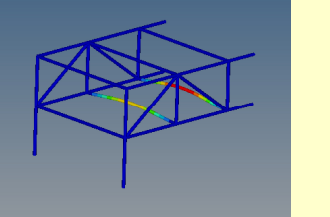
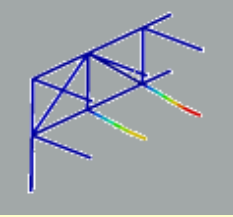
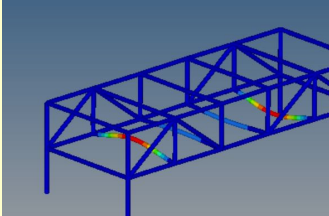
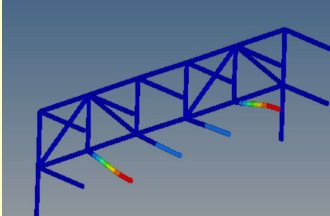
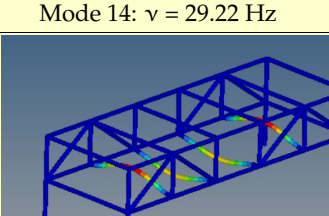
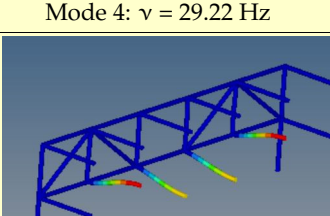
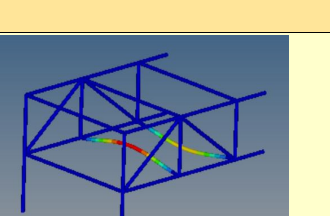
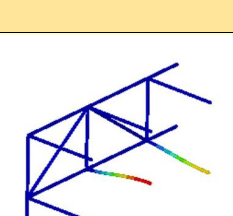
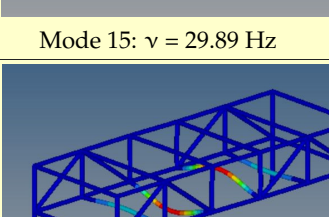
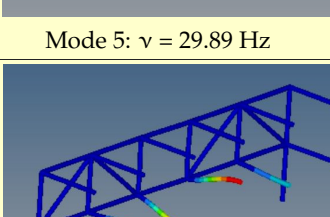
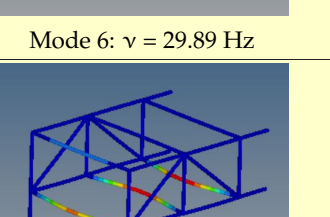
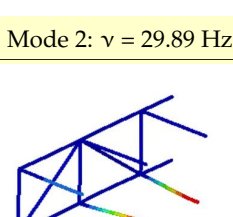
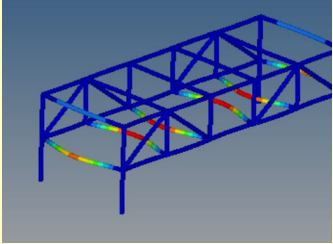
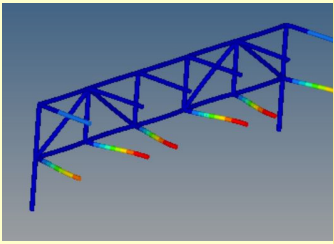
Type of Symmetry			
Full System	Half System X	Half System Z	Quarter
 Mode 7: $\nu = 18.81$ Hz	 Mode 2: $\nu = 18.81$ Hz		
 Mode 10: $\nu = 23.50$ Hz		 Mode 4: $\nu = 23.50$ Hz	
 Mode 12: $\nu = 27.48$ Hz	 Mode 3: $\nu = 27.48$ Hz	 Mode 5: $\nu = 27.48$ Hz	 Mode 1: $\nu = 27.48$ Hz
 Mode 14: $\nu = 29.22$ Hz	 Mode 4: $\nu = 29.22$ Hz		
 Mode 15: $\nu = 29.89$ Hz	 Mode 5: $\nu = 29.89$ Hz	 Mode 6: $\nu = 29.89$ Hz	 Mode 2: $\nu = 29.89$ Hz
 Mode 16: $\nu = 31.00$ Hz	 Mode 6: $\nu = 31.00$ Hz	 Mode 7: $\nu = 31.00$ Hz	 Mode 3: $\nu = 31.00$ Hz

Table 2. Cont.

Type of Symmetry			
Full System	Half System X	Half System Z	Quarter
			
Mode 17: $\nu = 31.00$ Hz	Mode 7: $\nu = 31.00$ Hz		

4. Discussion

In this paper, the FEM was used for the calculation of the natural frequencies and natural modes of vibration for a real structure of a cooling installation made in the physics department. The studied mechanical system is a system made up of four types of standardized bars rigidly connected to each other by welding. For the presented and physically realized system, a calculation of the natural vibration frequencies and natural vibration modes was made. During the analysis, it was found that we are dealing with two types of vibration modes: symmetric modes and skew-symmetric modes. The two types of modes are in accordance with the properties shown. Experimental results that confirm similar calculations made can be found in [35].

The types of finite elements used, the mechanical properties of the elements used, and the way of joining between elements can lead to (generally insignificant) variations in the value of the natural frequencies. However, these do not qualitatively influence the presented results. The accuracy of the numerical calculations can be rated as excellent, because when calculating the frequencies on the entire structure and on the component substructures, we obtain, in most cases, up to six significant figures.

5. Conclusions

Symmetries mean that the different parts of a construction are identical. The information required for the identification of the respective part is reduced in this way, and as a consequence, there is an economic aspect to the analysis, design, and manufacture of the respective part. Logistic expenses also decrease due to having a smaller number of different parts. The design stage of such structures becomes simpler, the necessary calculations are reduced in volume, and the costs necessary to complete such projects decrease. For the field of study of a vibrating structure, which is the object of this work, the advantages that appear are significant. It is possible to reduce the calculation effort, with the consequence of reducing the necessary time and costs. It is possible to perform the calculation on halves of the structure, eliminate the known eigenvalues from the characteristic equation, and determine the unknown ones from a simpler equation, with less effort.

In the present work, several properties were presented that refer to systems with multiple symmetries. We presented and justified some of the important vibration properties that symmetrical mechanical systems have. Only a symmetrical part of the system can be analyzed (which means reduced calculations in the first phase), and then, the eigenfrequencies thus determined are removed from the general equations of the system's vibrations. Then, a reduced system is analyzed to determine the other eigenfrequencies. So, in the end, there is a significant saving of time, potentially leading to cost savings on several levels.

In the end, the obtained results could be used by software developers to identify systems that present symmetries and to modify the calculation algorithm so that these symmetries can be used to reduce the necessary calculation time. For structural applications, where the calculation time is high, the advantages offered can be significant.

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Conflicts of Interest: The authors declare no conflicts of interest.

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