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Precise Wigner–Weyl Calculus for the Honeycomb Lattice

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Abstract: In this paper, we propose a precise Wigner–Weyl calculus for the models defined on the honeycomb lattice. We construct two symbols of operators: the $\mathscr{B}$ symbol, which is similar to the one introduced by F. Buot, and the W (or, Weyl) symbol. The latter possesses the set of useful properties. These identities allow us to use it in physical applications. In particular, we derive topological expression for the Hall conductivity through the Wigner-transformed Green function. This expression may be used for the description of the systems with artificial honeycomb lattice, when magnetic flux through the lattice cell is of the order of elementary quantum of magnetic flux. It is worth mentioning that, in the present paper, we do not consider the effect of interactions.

Keywords: Wigner–Weyl calculus; honeycomb lattice; artificial lattice; quantum Hall effect

1. Introduction

Wigner–Weyl calculus in its original form was proposed by H. Groenewold [1] and J. Moyal [2]. It was designed to replace operator formulation of quantum mechanics by phase space formulation, where the basic notion is Weyl symbol of operator, i.e., function in phase space that carries all necessary information about the operator itself. Here, the ideas of H. Weyl [3] and E. Wigner [4] have been used, which gave to this calculus its present name. Instead of the non-commutative operator product in the Wigner–Weyl formalism, the non-commutative Moyal product is used [5,6]. The calculus had several applications to quantum mechanics [7,8] and quantum field theory [9–15].

Originally Wigner–Weyl formalism has been proposed for the systems defined in continuum space. It is well known, however, that lattice regularization is necessary for the self-consistent definition of quantum field theory. Moreover, in solid-state physics, the tight-binding models give reasonable description of collective excitations. Therefore, the field theory systems defined on the lattice represent an important domain of condensed matter physics. The attempts to define Wigner–Weyl calculus for the lattice models have been undertaken long time ago, starting from the works of Schwinger [16]. Important ideas in this direction have been proposed by Buot [15,17,18]. The corresponding constructions have been proposed by several authors, including Wooters [19], Leonhardt [20], Kasperowitz [21], and Ligabó [22]; see also [23–30] and references therein. It is worth mentioning also that the large chapter of pure mathematics called deformational quantization is based on Wigner–Weyl calculus [31–34].

With the purpose of being applied to non-dissipative transport phenomena, the so-called approximate lattice Wigner–Weyl calculus was suggested [35]. This version of Wigner–Weyl calculus may be applied to the lattice systems with weak inhomogeneity, caused by slowly varying external fields. In practice, this formalism can be used for the consideration of any lattice-regularized continuous field theory, and can be applied to the solid-state systems in the presence of elastic deformations, weak disorder, and magnetic fields much smaller than 10^5 Tesla (at this value of magnetic field, the magnetic flux through the lattice cell of the typical crystal lattice becomes of the order of elementary magnetic flux). The latter requirement seems—at first glance—to be always fulfilled, because the maximal values of magnetic fields accessed in laboratories do not exceed 100 Tesla. With

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the aid of approximate Wigner–Weyl calculus, the conductivities of several non-dissipative transport effects have been expressed through the topological invariants [36–42]. Essentially non-homogeneous systems within this methodology have been considered in [35,43–45].

In the present paper, we concentrate on the quantum Hall effect (QHE). Its topological description for the idealized system of non-interacting electrons in the presence of constant external magnetic field has been proposed in [46]. The QHE conductivity in such systems is proportional to the so-called TKNN invariant, expressed through the integral of Berry curvature over the occupied energy levels. The TKNN invariant is robust to the smooth modification of the one-particle Hamiltonian [47–52]. However, the application of the TKNN invariant is limited to unphysically idealized systems, without interactions or disorder.

The alternative topological description of the QHE is given in terms of Green functions. First, the intrinsic anomalous QHE (existing without external magnetic field) in homogeneous topological insulators without interactions [53–55] has been given. The corresponding expression for the conductivity is given by

$$\sigma_H = \frac{e^2}{h}\mathcal{N},$$

where

$$\mathcal{N} = -\frac{\epsilon_{ijk}}{3!\,4\pi^2}\int d^3p\,\mathrm{tr}\left[G(p)\frac{\partial G^{-1}(p)}{\partial p_i}\frac{\partial G(p)}{\partial p_j}\frac{\partial G^{-1}(p)}{\partial p_k}\right]. \tag{1}$$

Here, ϵ_{ijk} is the totally anti-symmetric tensor with $\epsilon_{123} = 1$, while p_i are the components of 3-momentum (in 2 + 1D momentum space of the two-dimensional system). In this expression, $G(p)$ is the two-point Green function in momentum space. The advantage of Equation (1) is that, contrary to the TKNN invariant, it may be extended to the systems with interactions. Then, the two-point Green function is taken with the interaction corrections [44,56,57]. Notice that the role of interaction corrections to the QHE conductivity was considered long time ago, well before the mentioned description with the aid of Equation (1) [58–61].

Equation (1) solves the problem with interaction corrections to the QHE in topological insulators. However, strictly speaking, it cannot be used for the description of ordinary QHE in the presence of external magnetic field, and, more widely, for the QHE in the non-homogeneous systems. The extension of this expression to the non-homogeneous systems has been given in [35]. For the tight-binding model of a two-dimensional lattice system the Hall conductivity averaged over the system area is $\sigma_H = \frac{\mathcal{N}}{2\pi}\frac{e^2}{\hbar}$, with

$$\begin{aligned}\mathcal{N} \;=\; & -\frac{T\epsilon_{ijk}}{|\mathbf{A}|\,3!\,4\pi^2}\int d^3x\int_{\mathcal{M}} d^3p\;\mathrm{tr}\,G_{\mathcal{C}}(x,p)\star \\ & \star\frac{\partial Q_{\mathcal{C}}(x,p)}{\partial p_i}\star\frac{\partial G_{\mathcal{C}}(x,p)}{\partial p_j}\star\frac{\partial Q_{\mathcal{C}}(x,p)}{\partial p_k}\end{aligned} \tag{2}$$

Here, the integral is over momentum space $\mathcal{M}$, while $T \to 0$ is temperature, $|\mathbf{A}| \to \infty$ is the total area of the system, and $G_{\mathcal{C}}(x,p)$ is the Wigner transformation of the two-point Green function $\hat{G}$. $Q_{\mathcal{C}}(x,p)$ is the lattice Weyl symbol of operator $\hat{Q}$. Here, $\hat{Q}$ is the operator inverse to the Green function. The Moyal product is denoted by $\star$. As mentioned above, here, the approximate version of the lattice Wigner–Weyl calculus has been used, which allows to deal with realistic magnetic fields much smaller than 10^5 Tesla. It is worth mentioning that, by definition, the Weyl symbol is defined as a function of real valued coordinates, not only for the discrete lattice points. In [44,62], it was proven that Equation (2) remains valid in the presence of interactions (taken into account perturbatively) if bare non-interacting Green function is replaced by the complete interacting one.

Equation (2) is to be modified for the artificial lattices or in cases of strong inhomogeneities. These are the systems where the so-called Hofstadter butterfly appears. Extension of Equation (2) to such systems has been given in [43], where the consideration was limited by the tight-binding models defined on the infinite rectangular lattice. The corresponding version of lattice Wigner–Weyl calculus was called "precise" because the corresponding Weyl symbol of an operator satisfies the basic identities of continuous Wigner–Weyl calculus precisely. Within this formalism, the expression for the QHE conductivity has been derived, where Equation (2) is replaced by

$$\begin{aligned} \mathcal{N} &= -\frac{\epsilon_{ijk}}{|\mathbf{A}|\,3!\,4\pi^2}\frac{|\mathcal{V}^{(2)}|}{2^2}\sum_{\vec{x}\in\mathscr{D}}\int_{\mathcal{M}} d^3p\mathrm{Tr}\Big[G_W(x,p)\star \\ &\quad \star\frac{\partial Q_W(x,p)}{\partial p_i}\star\frac{\partial G_W(x,p)}{\partial p_j}\star\frac{\partial Q_W(x,p)}{\partial p_k}\Big]. \end{aligned} \tag{3}$$

Here, $x = (\tau, \vec{x})$, $\vec{x}$ is the point in space, and τ is imaginary time that belongs to the interval between 0 and $1/T \to \infty$. Wigner transformation of the Green function G_W and the Weyl symbol of Dirac operator Q_W are defined within the precise Wigner–Weyl calculus and do not depend on τ. $\mathcal{V}^{(2)}$ is the area of the elementary lattice cell, and that above $|\mathbf{A}| \to \infty$ is the system area. By $\mathscr{D}$, we denote the extended lattice, in which the extra lattice sites are added with the half-integer coordinates (assuming the coordinates of the original lattice are integer). Notice that, although we expect that in the presence of interactions (taken into account perturbatively), Equation (3) remains valid (just like for the case of approximate Wigner–Weyl calculus), the direct proof of this statement is still not given.

Recall that Equation (3) has been derived specifically for the systems defined on the rectangular lattice. In practice, however, the systems with large magnetic flux through the lattice cell have been obtained with the artificial lattices that do not have the rectangular form (see, for example, [63] and references therein). In particular, the artificial lattices may have the honeycomb form [64]. Therefore, in the present paper, we extend the precise Wigner–Weyl calculus to the systems defined on the honeycomb lattices. We found that it is possible to overcome technical difficulties specific for the honeycomb lattice, and we arrive at the Wigner–Weyl calculus with the definition of the Weyl symbol that obeys the basic properties of the continuous Wigner–Weyl calculus.

- Star product identity:

$$(AB)_W(x,p) = A_W(x,p)\star B_W(x,p) \tag{4}$$

- First trace identity:

$$\mathrm{Tr}A_W = \mathrm{tr}\hat{A} \tag{5}$$

  Here, by Tr, we understand the trace of the Weyl symbol, while tr is the trace of an operator itself.
- Second trace identity:

$$\mathrm{Tr}[A_W(x,p)B_W(x,p)] = \mathrm{Tr}[A_W(x,p)\star B_W(x,p)] \tag{6}$$

- Weyl symbol of the identity operator:

$$\left(\hat{1}\right)_W(x,p) = 1 \tag{7}$$

Here the Moyal product, as defined below, serves as the star product mentioned in property (4):

$$\star \equiv \star_{x,p} = e^{\frac{i}{2}\left(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_x\right)}. \tag{8}$$

Here, the left arrow means that the derivative acts on the expression standing left to the star, while the right arrow points out that the derivative acts to the right. Using the designed precise Wigner–Weyl calculus, we derive expression for the Hall conductivity of the *non-interacting* system defined on the honeycomb lattice. We rely on the unification of the lattice Wigner–Weyl calculus with the Keldysh technique. Here, we follow the methodology developed in [45,65–68].

In the present paper, we do not consider interactions either perturbatively or non-perturbatively. However, the very form of the obtained below expression for QHE conductivity through a topological invariant admits extension to the case, when both interactions and disorder are present. The disorder can be taken into account immediately, without additional efforts, because the given expression contains the Wigner-transformed Green functions. One should simply write down expression for $\hat{Q}$ (inverse of the Green function) in the presence of disorder, and the proposed topological invariant gives the answer for integer QHE conductivity. As for the interactions, one can extend the results of [44] to the case of the present precise Wigner–Weyl calculus. Then, supposedly, the expression presented in the present paper will remain at work with the non-interacting Green function replaced by the interacting one (if the interactions are taken into account perturbatively). We suppose that this might be performed relatively easily. However, this is still the work to be performed, and is out of the scope of the present paper. The more difficult problem is to take into account interactions non-perturbatively. That is what should give the true microscopic explanation of the fractional QHE. Furthermore, this problem is also out of the scope of the present paper. Nevertheless, in spite of the absence of the consideration of interactions, the derivation of expression for the Hall conductivity through the Wigner-transformed Green functions seems to us an important step towards the complete understanding of the problem with the interactions taken into account both perturbatively and non-perturbatively.

2. Statement of the Main Results

2.1. Definition of Weyl Symbol and Its Properties

We are considering the honeycomb lattice of mono-layer graphene. The physical lattice vectors of graphene are defined as follows:

$$\begin{aligned}\vec{l}_1 &= \frac{\ell}{2}\left(3\hat{x} + \sqrt{3}\hat{y}\right)\\ \vec{l}_2 &= \frac{\ell}{2}\left(3\hat{x} - \sqrt{3}\hat{y}\right)\\ \vec{\ell} &= \ell\hat{x},\end{aligned} \tag{9}$$

where the last vector is the basis vector. These are the reciprocal lattice vectors:

$$\begin{aligned}\vec{g}_1 &= \frac{2\pi}{3\ell}\left(\hat{k}_x + \sqrt{3}\hat{k}_y\right)\\ \vec{g}_2 &= \frac{2\pi}{3\ell}\left(\hat{k}_x - \sqrt{3}\hat{k}_y\right).\end{aligned} \tag{10}$$

The physical lattice is defined by the following:

$$\mathcal{O} \equiv \begin{Bmatrix} 2c_1^1\vec{l}_1 + 2c_2^1\vec{l}_2, & c_{1,2}^1 \in \mathbb{Z} \\ 2\vec{\ell} + 2c_1^2\vec{l}_1 + 2c_2^2\vec{l}_2, & c_{1,2}^2 \in \mathbb{Z} \end{Bmatrix}. \tag{11}$$

The physical lattice's first Brillouin zone is given by:

$$\mathcal{M} = \left\{ \frac{1}{2}m_1\vec{f}_1 + \frac{1}{2}m_2\vec{f}_2, \begin{cases} m_1 \in (-1/2, 1/2] \\ m_2 \in (-1/4, 1/4] \end{cases} \right\}. \tag{12}$$

The extended lattice, which is the union of the physical and auxiliary lattices, is:

$$\mathfrak{D} \equiv \begin{Bmatrix} c_1^1\vec{l}_1 + c_2^1\vec{l}_2, & c_{1,2}^1 \in \mathbb{Z} \\ -\vec{\ell} + c_1^2\vec{l}_1 + c_2^2\vec{l}_2, & c_{1,2}^2 \in \mathbb{Z} \end{Bmatrix} = \mathscr{O} \cup \mathscr{O}'. \quad (13)$$

The extended lattice's first Brillouin zone is given by:

$$\mathfrak{M} = \left\{ m_1\vec{f}_1 + m_2\vec{f}_2, \begin{Bmatrix} m_1 \in (-1/2, 1/2] \\ m_2 \in (-1/4, 1/4] \end{Bmatrix} \right\}. \quad (14)$$

For $x \in \mathfrak{D}$, $p \in \mathscr{M}$, we define Weyl symbol of operator $\hat{A}$, as follows

$$A_W(x,p) \equiv \int_{\mathscr{M}} d^2q e^{2ixq} \langle p+q|\hat{A}|p-q\rangle \times \left(1 + e^{-2il_1q} + e^{-2il_2q} + e^{-2i(l_1+l_2)q}\right). \quad (15)$$

Here, $|p\rangle = \frac{1}{\sqrt{|\mathscr{M}|}} \sum_{x\in\mathscr{O}} e^{ixp}|x\rangle$ is vector of momentum space expressed through the eigenvectors of the coordinates $|x\rangle$. This expression is naturally extended to continuous values of x.

We formulate the following properties of the Weyl symbol:

1. Trace property:

$$\mathrm{Tr}\,\hat{A} = \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} \frac{d^2p}{|\mathfrak{M}|} A_W(x,p) \quad (16)$$

2. Second trace identity:

$$\mathrm{Tr}\hat{A}\hat{B} = \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} \frac{d^2p}{|\mathfrak{M}|} A_W(x,p) B_W(x,p) \quad (17)$$

3. Star property:

$$(\hat{A}\hat{B})_W(x,p)\Big|_{p\in\mathcal{M},x\in\mathfrak{D}} = A_W(p,q) e^{\frac{i}{2}(\overleftarrow{\partial_q}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_q})} B_W(p,q) \quad (18)$$

4. Weyl symbol of unity:

$$1_W(x,p)\Big|_{p\in\mathcal{M},x\in\mathfrak{D}} = 1 \quad (19)$$

5. Star product without differentiation:

$$A_W(x,p) \star B_W(x,p)\Big|_{x\in\mathscr{D}} = \sum_{z,\bar{z}\in\mathscr{D}} \int \frac{dp'}{|\mathfrak{M}|} \frac{d\bar{p}'}{|\mathfrak{M}|} e^{2ip'(\bar{z}-x)+2i\bar{p}'(x-z)} A_W(z, p-p') B_W(\bar{z}, p-\bar{p}') \quad (20)$$

2.2. Quantum Hall Effect in Condensed Matter System Defined on Honeycomb Lattice

We consider an inhomogeneous system defined on the honeycomb lattice $\mathscr{O}$. Time remains continuous. The fermionic field Φ is defined on lattice sites. We assume that the system remains non-interacting. The Keldysh Green function is defined as the following matrix:

$$\hat{G}(t,x|t',x') = -i\begin{pmatrix} \langle \mathfrak{T}\Phi(t,x)\Phi^+(t',x')\rangle & -\langle \Phi^+(t',x')\Phi(t,x)\rangle \\ \langle \Phi(t,x)\Phi^+(t',x')\rangle & \langle \tilde{\mathfrak{T}}\Phi(t,x)\Phi^+(t',x')\rangle \end{pmatrix}. \quad (21)$$

Here, Heisenberg fermionic field operator Φ depends on time t and spatial coordinates x. By $\mathfrak{T}$, we denote the time ordering, while $\tilde{\mathfrak{T}}$ is anti-time ordering.

The $2+1$-dimensional vectors (with space and time components) are denoted by large Latin letters. Correspondingly, in momentum space, $A(P_1,P_2) = \langle P_1|\hat{A}|P_2\rangle$. The space components of momentum belong to the Brillouin zone, while its time component (frequency) is real-valued. We then define the Weyl symbol of an operator $\hat{A}$ as the mixture of the lattice Weyl symbol and Wigner transformation, with respect to the frequency component:

$$A_W(X|P) = 2\int dP^0 \int_{\mathscr{M}} d^2\vec{Q}\, e^{-2iX^\mu Q_\mu} A(P+Q,P-Q) \Big(1+e^{-2il_1\vec{Q}}+e^{-2il_2\vec{Q}}+e^{-2i(l_1+l_2)\vec{Q}}\Big). \quad (22)$$

Here, $\mu = 0,1,2$ is the space–time index, while $2+1$ momentum is denoted by $P^\mu = (P^0,\vec{p})$, and $P_\mu = (P^0,-\vec{p})$. Here, $\vec{p}$ is spatial momentum with 2 components. The Keldysh Green function is an operator inverse to $\hat{\mathbf{Q}}$. The Weyl symbol of the Keldysh Green function $\hat{\mathbf{G}}$ is denoted by $\hat{G}_W$, while the Weyl symbol of Keldysh $\hat{\mathbf{Q}}$ is $\hat{Q}_W$.

We obtain the following results for the dynamics of the lattice system written in terms of the Weyl symbols of the operators.

1. Weyl symbols $\hat{G}_W$ and $\hat{Q}_W$ obey Groenewold equation

$$\hat{Q}_W * \hat{G}_W = 1_W. \quad (23)$$

Here, the Moyal product $*$ is defined as

$$(A*B)(X|P) = A(X|P)\, e^{-i(\overleftarrow{\partial}_{X^\mu}\overrightarrow{\partial}_{P_\mu} - \overleftarrow{\partial}_{P_\mu}\overrightarrow{\partial}_{X^\mu})/2} B(X|P). \quad (24)$$

It is worth mentioning that, for the complete description of the system, we need the values of the Weyl symbols defined on spatial phase space $\mathfrak{D}\otimes\mathscr{M}$. For such values of spatial momenta and coordinates, the Weyl symbol of unity is equal to 1.

2. We express the DC conductivity (in units of $e^2/\hbar$, averaged over the lattice area) of the two-dimensional *non-interacting* systems as

$$\sigma^{ij} = \frac{1}{4}\frac{1}{|\mathfrak{D}|}\int \frac{dP^0}{2\pi}\int_{\mathscr{M}}\frac{d^2\vec{P}}{(2\pi)^2}\sum_{x\in\mathfrak{D}} \mathrm{tr}\Big(\partial_{\pi_i}\hat{Q}_W\Big[\hat{G}_W \star \partial_{\pi_{[0}}\hat{Q}_W \star \partial_{\pi_{j]}}\hat{G}_W\Big]\Big)^< + \mathrm{c.c.} \quad (25)$$

through the lesser component of expression that contains the Wigner-transformed Keldysh Green function $G(X|\pi)$ and its inverse Q that obey Groenewold equation $Q\star G = 1$. We denote $\pi = P - A$. Here, π^μ is a $2+1$-dimensional vector, similar to P^μ. By $<$, we denote the lesser component of the Keldysh matrix. Here, by $|\mathfrak{D}|$, we denote the number of lattice points in the extended lattice $\mathfrak{D}$. It is assumed here that this number is large but remains finite. Using the representation of the Keldysh Green function of Equation (21), the lesser component in the above representation may be rewritten explicitly as

$$\sigma^{ij} = \frac{1}{4}\frac{1}{|\mathfrak{D}|}\int \frac{dP^0}{2\pi}\int_{\mathscr{M}}\frac{d^2\vec{P}}{(2\pi)^2}\sum_{x\in\mathfrak{D}} \mathrm{tr}\Big(\gamma^<\partial_{\pi_i}\hat{Q}_W\Big[\hat{G}_W \star \partial_{\pi_{[0}}\hat{Q}_W \star \partial_{\pi_{j]}}\hat{G}_W\Big]\Big) + \mathrm{c.c.} \quad (26)$$

where trace is taken over the Keldysh components as well as over the internal indices, while

$$\gamma^< = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Here, $(\dots)_{[0}(\dots)_{j]} = (\dots)_0(\dots)_j - (\dots)_j(\dots)_0$ means anti-symmetrization.

3. We show that the above expression for the conductivity (in units of $e^2/\hbar$, averaged over the system area) is reduced to the following expression in the case of the equilibrium system at zero temperature:

$$\bar{\sigma}^{ij} = \frac{\mathcal{N}}{2\pi}\epsilon^{ij},$$

where

$$\begin{aligned}\mathcal{N} \quad = \quad & \frac{1}{24\,\pi^2}\epsilon^{\mu\nu\rho}\frac{1}{|\mathfrak{D}|}\int dP^0 \int_{\mathscr{M}} d^2\vec{P} \sum_{x\in\mathfrak{D}} \mathrm{tr}\,\Big(\partial_{\Pi^\mu}\hat{Q}_W^{\mathrm{M}} \star \\ & \star\hat{G}_W^{\mathrm{M}} \star \partial_{\Pi^\nu}\hat{Q}_W^{\mathrm{M}} \star \hat{G}_W^{\mathrm{M}} \star \partial_{\Pi^\rho}\hat{Q}_W^{\mathrm{M}} \star \hat{G}_W^{\mathrm{M}}\Big).\end{aligned} \quad (27)$$

Here, the $\hat{G}_W^M$ is the Weyl symbol of the Matsubara Green function, while $\hat{Q}_W^M$ is the Weyl symbol of its inverse. Momentum space is a Euclidean one; its points are denoted by Π_i. Π^3 is the Matsubara frequency.

4. One can check that Equation (27) is a topological invariant. For that, we need the system to have been in thermal equilibrium originally, and need the thermal equilibrium to correspond to zero temperature. Moreover, we need the Hamiltonian to not depend on time. The value of the average conductivity $\bar{\sigma}^{ij}$ is then robust to smooth variations of the system. The sum over x is important for the topological invariance of this quantity.

3. Wigner–Weyl Calculus of Felix Buot (1D)

The precise Wigner–Weyl calculus for lattice models, inspired by the original construction of F. Buot, is excerpted here. Notice that, strictly speaking, the Buot symbol of an operator differs from the original definition given by F. Buot. We call it the Buot symbol to distinguish it from the construction that uses the extended lattice. We restrict ourselves to a simple one-dimensional physical lattice $\mathscr{O}$ with a one-dimensional first Brillouin zone $\mathscr{M}$.

3.1. The Hilbert Space

The following definition applies to the physical lattice, which is represented by the symbol $\mathscr{O}$:

$$\mathscr{O} \equiv \{2\ell k,\ k \in \mathbb{Z}\}, \quad (28)$$

where 2ℓ stands for the lattice spacing. Its first Brillouin zone is denoted by the symbol $\mathscr{M}$ and has the following definition:

$$\mathscr{M} = \Big(-\frac{\pi}{2\ell}, \frac{\pi}{2\ell}\Big]. \quad (29)$$

Furthermore, the auxiliary lattice, which is a translation of the physical lattice by half the lattice spacing, is defined as follows:

$$\mathscr{O}' \equiv \{\ell(2k+1), k \in \mathbb{Z}\} = \mathscr{O} + \ell. \quad (30)$$

Consequently, it is possible to define the extended lattice, represented by the symbol $\mathfrak{D}$, which comprises both the auxiliary lattice and the physical lattice, as follows:

$$\mathfrak{D} \equiv \{\ell k,\ k \in \mathbb{Z}\} = \mathscr{O} \cup \mathscr{O}'. \quad (31)$$

Here, ℓ stands for the lattice spacing, and $\mathfrak{M}$ is the definition of the first Brillouin zone of the extended lattice:

$$\mathfrak{M} = \left(-\frac{\pi}{\ell}, \frac{\pi}{\ell}\right]. \tag{32}$$

The characteristics of the physical states are described by the following definitions:

$$\hat{1}_{\mathscr{O}} = \sum_{x \in \mathscr{O}} |x\rangle\langle x| = \int_{\mathscr{M}} dp|p\rangle\langle p| \qquad \langle x|p\rangle = \frac{1}{\sqrt{|\mathscr{M}|}} e^{ixp} \tag{33}$$

$$\langle p|q\rangle = \delta^{\left[\frac{\pi}{\ell}\right]}(p-q) \qquad \langle x|y\rangle = \delta_{x,y} \tag{34}$$

$$|p\rangle = \frac{1}{\sqrt{|\mathscr{M}|}} \sum_{x \in \mathscr{O}} e^{ixp}|x\rangle. \tag{35}$$

The latter being the Fourier transform. Here, by $\delta^{\left[\frac{\pi}{\ell}\right]}(p-q)$, we denote the delta function of an argument-defined modulo, $\frac{\pi}{\ell}$.

3.2. The $\mathscr{B}$ Symbol

The Buot transformation of a function $B(p,q)$ (which is assumed to be equal to matrix elements of an operator B), where $p, q \in \mathscr{M}$, is defined as follows:

$$\begin{aligned} B_{\mathscr{B}}(x,p) &\equiv \frac{1}{2}\int_{\mathfrak{M}} dq e^{ixq} B\left(p+\frac{q}{2}, p-\frac{q}{2}\right) \\ &= \int_{\mathscr{M}} dq e^{2ixq} B(p+q, p-q). \end{aligned} \tag{36}$$

This expression is also considered as the definition of the Buot $\mathscr{B}$ symbol of the given operator.

In addition, the inverse transformation of a function $Q_{\mathscr{B}}(x,p)$ is given by:

$$Q(p,q) = \frac{1}{|\mathscr{M}|} \sum_{x \in \mathfrak{D}} e^{-i(p-q)x} Q_{\mathscr{B}}\left(x, \frac{p+q}{2}\right). \tag{37}$$

Using the Fourier transform approach (35) and the $\mathscr{B}$ symbol definition (36), the following can be demonstrated

$$\begin{aligned} &\frac{1}{|\mathscr{M}|}\sum_{x\in\mathfrak{D}} e^{-ikx} Q_{\mathscr{B}}(x,p) = \frac{1}{|\mathscr{M}|}\sum_{x\in\mathfrak{D}} e^{-ikx}\int_{\mathscr{M}} dq e^{2ixq} \\ &\times Q(p+q,p-q) \\ &= \frac{1}{|\mathscr{M}|}\sum_{x\in\mathfrak{D}} e^{2ix\left(q-\frac{k}{2}\right)}\int_{\mathscr{M}} Q(p+q,p-q)dq \\ &= \frac{1}{|\mathscr{M}|}\sum_{n\in\mathbb{Z}} e^{in\ell(2q-k)}\int_{\mathscr{M}} Q(p+q,p-q)dq \\ &= \frac{2\pi}{|\mathscr{M}|}\sum_{n\in\mathbb{Z}} \delta[\ell(2q-k)-2\pi n]\int_{\mathscr{M}} Q(p+q,p-q)dq \\ &= \frac{2\pi}{|\mathscr{M}|}\frac{1}{2\ell}\sum_{n\in\mathbb{Z}} \delta\left(q-\frac{k}{2}-\frac{\pi}{\ell}n\right)\int_{\mathscr{M}} Q(p+q,p-q)dq \\ &= \int_{\mathscr{M}} dq\delta^{\left[\frac{\pi}{\ell}\right]}\left(q-\frac{k}{2}\right)Q(p+q,p-q) \\ &= Q\left(p+\frac{k}{2}, p-\frac{k}{2}\right). \end{aligned}$$

■

3.3. Moyal Product

It is possible to show that the $\mathscr{B}$ symbol of an operator obeys the star product identity stated in Section 1—(4) through the definition of the $\mathscr{B}$ symbol in (36):

$$
\begin{aligned}
&(AB)_{\mathscr{B}}(x,p) = \frac{1}{2}\int_{\mathfrak{M}} d\mathcal{P} e^{ix\mathcal{P}} \left\langle p + \frac{\mathcal{P}}{2} \middle| \hat{A}\hat{B} \middle| p - \frac{\mathcal{P}}{2} \right\rangle \\
&= \frac{1}{2}\int_{\mathfrak{M}} d\mathcal{P} e^{ix\mathcal{P}} \int_{\mathscr{M}} d\mathcal{R} \left\langle p + \frac{\mathcal{P}}{2} \middle| \hat{A} \middle| \mathcal{R} \right\rangle \left\langle \mathcal{R} \middle| \hat{B} \middle| p - \frac{\mathcal{P}}{2} \right\rangle \\
&= \frac{1}{4}\int_{\mathfrak{M}} d\mathcal{P} d\mathcal{K} e^{ix\mathcal{P}} \left\langle p + \frac{\mathcal{P}}{2} \middle| \hat{A} \middle| p - \frac{\mathcal{K}}{2} \right\rangle \\
&\times \left\langle p - \frac{\mathcal{K}}{2} \middle| \hat{B} \middle| p - \frac{\mathcal{P}}{2} \right\rangle \\
&= \frac{2}{2\times 4}\int_{\mathfrak{M}} dq dk e^{ix(q+k)} \left\langle p + \frac{q}{2} + \frac{k}{2} \middle| \hat{A} \middle| p - \frac{q}{2} + \frac{k}{2} \right\rangle \\
&\times \left\langle p - \frac{q}{2} + \frac{k}{2} \middle| \hat{B} \middle| p - \frac{q}{2} - \frac{k}{2} \right\rangle \\
&= \frac{1}{4}\int_{\mathfrak{M}} dq dk e^{ixq} \left\langle p + \frac{q}{2} \middle| \hat{A} \middle| p - \frac{q}{2} \right\rangle e^{\frac{k}{2}\overleftarrow{\partial}_p - \frac{q}{2}\overrightarrow{\partial}_p} \\
&\times e^{ixk} \left\langle p + \frac{k}{2} \middle| \hat{B} \middle| p - \frac{k}{2} \right\rangle \\
&= \frac{1}{2}\int_{\mathfrak{M}} dq e^{ixq} \left\langle p + \frac{q}{2} \middle| \hat{A} \middle| p - \frac{q}{2} \right\rangle e^{\frac{i}{2}\left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right)} \\
&\times \frac{1}{2}\int_{\mathfrak{M}} dk e^{ixk} \left\langle p + \frac{k}{2} \middle| \hat{B} \middle| p - \frac{k}{2} \right\rangle. \qquad \blacksquare
\end{aligned}
$$

The factor 2 in the fifth line is the Jacobian, generated by changing the variables $\mathcal{P} = q + k$ and $\mathcal{K} = q - k$, where $q, k \in \mathfrak{M}$. Additionally, changing the variables transforms the integration region contained within $\mathfrak{M} \times \mathfrak{M}$ from a square to a rhomboid, as shown in Figure 1. Due to the periodicity of the integral over the rhomboid form, the factor $1/2$ appears in the fifth line when the integration region is transformed back from a rhomboid to a square.

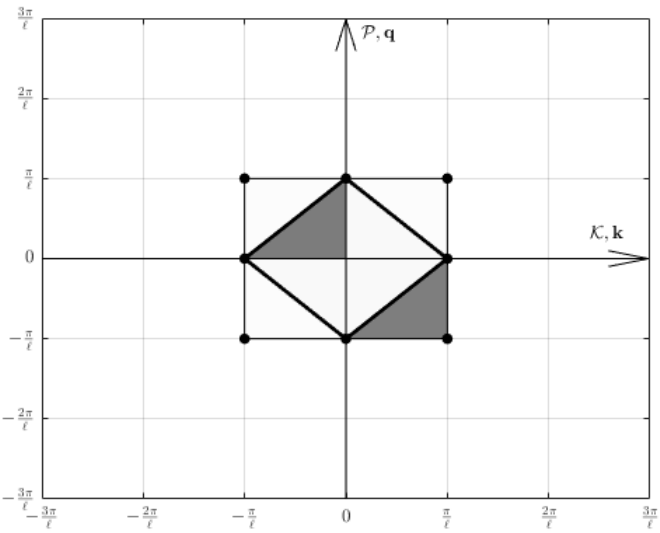

Figure 1. The integral over a square is transformed into a rhombus; the contributions from shaded areas are equal.

3.4. Trace and Its Properties

The definitions of the traces of the physical and extended lattices are as follows:

$$\begin{aligned}
\mathrm{Tr}_{\mathscr{O}} A_{\mathscr{B}} &\equiv \sum_{x\in\mathscr{O}} \int_{\mathscr{M}} \frac{dp}{|\mathscr{M}|} A_{\mathscr{B}}(x,p) \\
\mathrm{Tr}_{\mathfrak{D}} A_{\mathscr{B}} &\equiv \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} \frac{dp}{|\mathscr{M}|} A_{\mathscr{B}}(x,p).
\end{aligned} \tag{38}$$

Both adhere to the first trace identity stated in Section 1—(5). As proof, consider the following:

$$\begin{aligned}
&\mathrm{Tr}_{\mathfrak{D}} A_{\mathscr{B}} \equiv \frac{1}{|\mathscr{M}|} \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} dp A_{\mathscr{B}}(x,p) \\
&= \frac{1}{|\mathscr{M}|} \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} dpdq e^{2ixq} \langle p+q|\hat{A}|p-q\rangle \\
&= \frac{1}{|\mathscr{M}|} \sum_{x\in\mathfrak{D}} \sum_{x,y\in\mathscr{O}} \int_{\mathscr{M}} dpdq e^{2ixq} \langle p+q|z\rangle\langle z|\hat{A}|y\rangle\langle y|p-q\rangle \\
&= \frac{1}{|\mathscr{M}|^2} \sum_{x\in\mathfrak{D}} \sum_{z,y\in\mathscr{O}} \int_{\mathscr{M}} dpdq e^{i(2x-z-y)q} e^{i(y-z)p} \langle z|\hat{A}|y\rangle \\
&= \sum_{x\in\mathfrak{D}} \sum_{y,z\in\mathscr{O}} \delta_{2x,y+z}\delta_{y,z} \langle z|\hat{A}|y\rangle \\
&= \sum_{x\in\mathscr{O}} \langle x|\hat{A}|x\rangle = \mathrm{tr}\hat{A}. \qquad \blacksquare
\end{aligned}$$

The same can be demonstrated for the physical lattice, but only the extended lattice accommodates the second trace identity specified in Section 1—(6). The proof:

$$\begin{aligned}
&\mathrm{Tr}_{\mathfrak{D}}(A_{\mathscr{B}} \star B_{\mathscr{B}}) \equiv \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} \frac{dp}{|\mathscr{M}|} A_{\mathscr{B}}(x,p) \star B_{\mathscr{B}}(x,p) \\
&= \frac{1}{|\mathscr{M}|} \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} dpdqdk e^{2ixq} \langle p+q|\hat{A}|p-q\rangle \\
&\times e^{\frac{i}{2}\left(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}\right)} e^{2ixk} \langle p+k|\hat{B}|p-k\rangle \\
&= \frac{1}{|\mathscr{M}|^3} \sum_{x\in\mathfrak{D}} \sum_{y,z,\bar{y},\bar{z}\in\mathscr{O}} \int_{\mathscr{M}} dpdqdk e^{i(2x-y-z)q+i(z-y)p} \\
&\times \langle y|\hat{A}|z\rangle e^{\frac{i}{2}\left(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}\right)} e^{i(2x-\bar{y}-\bar{z})k+i(\bar{z}-\bar{y})p} \langle \bar{y}|\hat{B}|\bar{z}\rangle \\
&= \frac{1}{|\mathscr{M}|^3} \sum_{x\in\mathfrak{D}} \sum_{y,z,\bar{y},\bar{z}\in\mathscr{O}} \int_{\mathscr{M}} dpdqdk e^{i(2x-y-z)q+i(z-y)p} \\
&\times \langle y|\hat{A}|z\rangle e^{i(\bar{y}-\bar{z})q+i(z-y)k} e^{i(2x-\bar{y}-\bar{z})k+i(\bar{z}-\bar{y})p} \langle \bar{y}|\hat{B}|\bar{z}\rangle \\
&= \sum_{x\in\mathfrak{D}} \sum_{y,z,\bar{y},\bar{z}\in\mathscr{O}} \delta_{2x+\bar{y}-\bar{z},y+z} \langle y|\hat{A}|z\rangle \delta_{y-z,\bar{z}-\bar{y}} \\
&\times \delta_{2x-y+z,\bar{y}+\bar{z}} \langle \bar{y}|\hat{B}|\bar{z}\rangle \\
&= \sum_{x\in\mathscr{O}} \langle x|\hat{A}\hat{B}|x\rangle = \mathrm{tr}\hat{A}\hat{B}. \qquad \blacksquare
\end{aligned}$$

3.5. The $\mathscr{B}$ Symbol of the Identity Operator

The identity operator's $\mathscr{B}$ symbol is given by:

$$(\hat{1})_{\mathscr{B}} = \frac{1}{2}\left[1 + \cos\frac{\pi}{\ell}x\right], \tag{39}$$

where $x \in \mathfrak{D}$. Contrary to how it is shown in Section 1—(7), the identity operator's $\mathscr{B}$ symbol fluctuates and is not necessarily unitary. The following provides proof for this:

$$
\begin{aligned}
(\hat{1})_{\mathscr{B}} &= \int_{\mathscr{M}} dq e^{2ixq} \langle p+q|p-q\rangle \\
&= \int_{-\frac{\pi}{2\ell}}^{\frac{\pi}{2\ell}} dq e^{2ixq} \delta^{[\frac{\pi}{\ell}]}(2q) = \frac{1}{2}\int_{-\frac{\pi}{2\ell}}^{\frac{\pi}{2\ell}} dq e^{2ixq} \delta^{[\frac{\pi}{2\ell}]}(q) \\
&= \frac{1}{2}\int_{-\frac{\pi}{2\ell}}^{\frac{\pi}{2\ell}} dq e^{2ixq} \left[\delta(q) + \frac{1}{2}\delta\left(q-\frac{\pi}{2\ell}\right) + \frac{1}{2}\delta\left(q+\frac{\pi}{2\ell}\right)\right] \\
&= \frac{1}{2} + \frac{1}{4}e^{i\frac{\pi}{\ell}x} + \frac{1}{4}e^{-i\frac{\pi}{\ell}x} = \frac{1}{2} + \frac{1}{2}\cos\frac{\pi}{\ell}x. \qquad \blacksquare
\end{aligned}
$$

Because only half of each function exists inside the integration region $\mathscr{M}$, the shifted Dirac delta functions, enclosed in square parentheses in the third line, are factored by $1/2$.

4. Wigner–Weyl Calculus of Felix Buot in Graphene

In the following, we will sometimes refer to the honeycomb lattice as to the lattice of graphene. However, all our considerations are applicable to any tight-binding models with honeycomb lattices, including those with artificial lattices. Moreover, we do not require that those models have only the nearest-neighbor hoppings.

It is possible to think of the honeycomb lattice as the union of two Bravais lattices, but it is not one. However, it transforms into a Bravais lattice when the basis is set to two atoms (each lattice point represents one atom). The primitive physical lattice vectors of are defined as Equation (9), where the last vector is the representation of the basis. The reciprocal lattice vectors are given by Equation (10).

Furthermore, the reciprocal lattice vectors' sum and difference are given by:

$$
\begin{aligned}
\vec{f}_1 &= \vec{g}_1 + \vec{g}_2 = \frac{4\pi}{3\ell}\hat{k}_x \\
\vec{f}_2 &= \vec{g}_1 - \vec{g}_2 = \frac{4\pi}{\sqrt{3}\ell}\hat{k}_y.
\end{aligned} \tag{40}
$$

The aforementioned vectors are necessary in the application of the formalism in a graphene lattice.

4.1. The Hilbert Space (Physical Properties)

The physical lattice is defined by Equation (11) (see Figure 2). The physical lattice's first Brillouin zone is given by Equation (12) (see Figures 3 and 4).

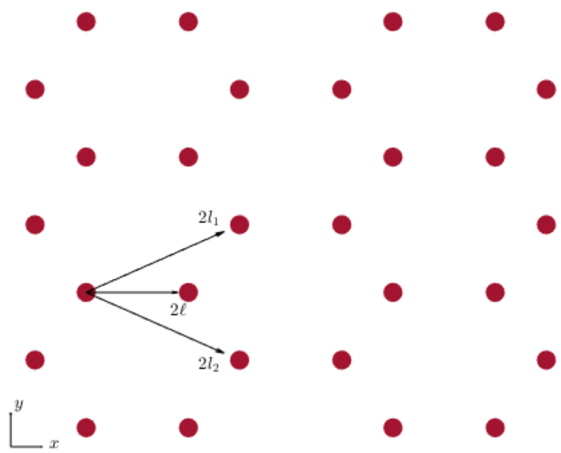

Figure 2. An illustration of the physical lattice $\mathscr{O}$.

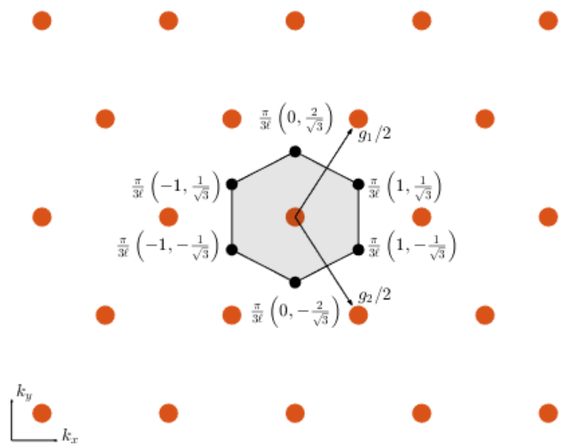

Figure 3. The first Brillouin zone and the reciprocal lattice of $\mathscr{O}$.

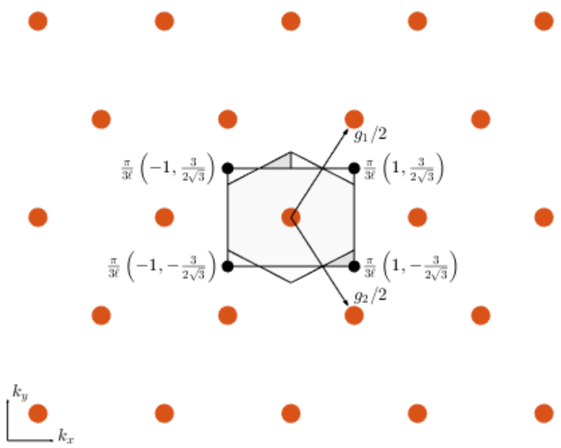

Figure 4. The translation of the first Brillouin zone; it is denoted by the symbol $\mathscr{M}$ and can be seen in (12); the contributions from shaded areas are equal.

The auxiliary lattice, denoted by the symbol $\mathscr{O}'$ and comprised of three distinct translations of the physical lattice by half of its spacing, is further defined as follows (see Figure 5):

$$\mathscr{O}' \equiv \left\{ \begin{array}{l} \left(2c_1^1 + 1\right)\vec{l}_1 + 2c_2^1\vec{l}_2 \\ 2\vec{\ell} + \left(2c_1^1 + 1\right)\vec{l}_1 + 2c_2^1\vec{l}_2 \\ 2c_1^2\vec{l}_1 + \left(2c_2^2 + 1\right)\vec{l}_2 \\ 2\vec{\ell} + 2c_1^2\vec{l}_1 + \left(2c_2^2 + 1\right)\vec{l}_2 \\ \left(2c_1^3 + 1\right)\vec{l}_1 + \left(2c_2^3 + 1\right)\vec{l}_2 \\ 2\vec{\ell} + \left(2c_1^3 + 1\right)\vec{l}_1 + \left(2c_2^3 + 1\right)\vec{l}_2 \end{array} \right\}, \tag{41}$$

where $c_{1,2}^1, c_{1,2}^2, c_{1,2}^3 \in \mathbb{Z}$. This leads to the definition of the extended lattice, which is the union of the physical and auxiliary lattices (Equation (13)). The extended lattice's first Brillouin zone is given by Equation (14) (see Figures 6 and 7).

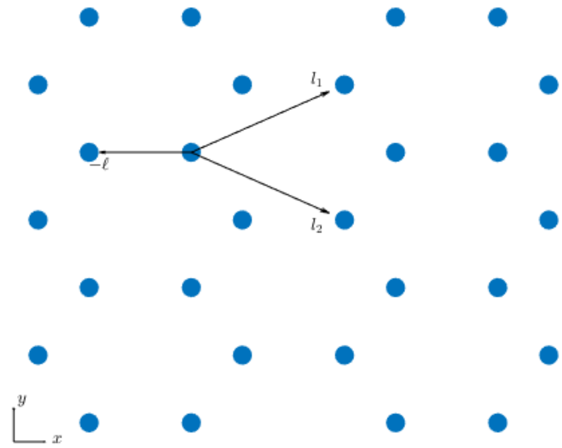

Figure 5. An illustration of the extended lattice $\mathfrak{D}$.

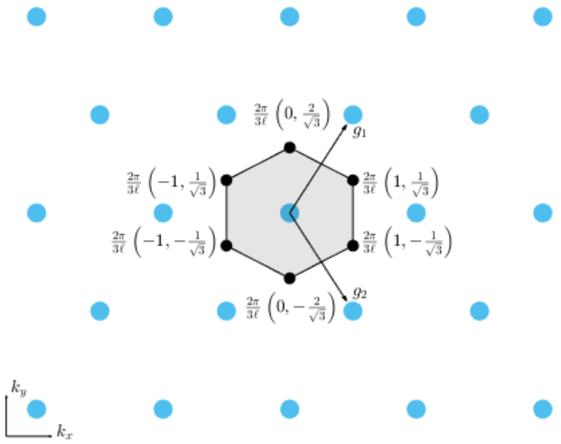

Figure 6. The first Brillouin zone and the reciprocal lattice of $\mathfrak{D}$.

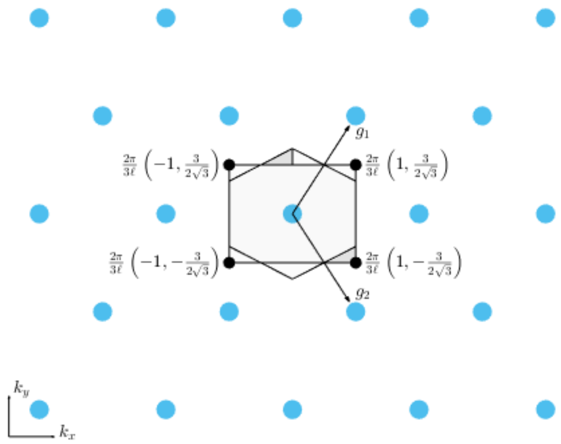

Figure 7. The translation of the first Brillouin zone; it is denoted by the symbol $\mathfrak{M}$ and can be seen in Equation (14); the contributions from shaded areas are equal.

The definitions that describe the characteristics of the physical states repeat those of Equations (33) and (35) (defined on the physical lattice $\mathscr{O}$), while Equation (34) is substituted by:

$$\langle p|q\rangle = \delta^{\left[\left(\frac{1}{2}\vec{g}_1, \frac{1}{2}\vec{g}_2\right)\right]}(p-q) \qquad \langle x|y\rangle = \delta_{x,y}. \tag{42}$$

4.2. $\mathscr{B}$ Symbol

The following definition applies to the two-dimensional $\mathscr{B}$ symbol of an operator $\hat{B}$:

$$\begin{aligned} B_{\mathscr{B}}(x,p) &\equiv \int_{\mathscr{M}} d^2q e^{2ixq} \langle p+q|\hat{B}|p-q\rangle \\ &= \int_{\mathscr{M}} d^2q e^{2ixq} B(p+q,p-q) \\ &= \frac{1}{4}\int_{\mathfrak{M}} d^2q e^{ixq} B\Big(p+\frac{q}{2},p-\frac{q}{2}\Big). \end{aligned} \tag{43}$$

In contrast to Section 3.2—(36), a $1/4$ factor is produced when a second dimension is added to the definition of an operator's $\mathscr{B}$ symbol. Moreover, the following is the equation for the inverse transformation, or $Q(p,q)$, of the function $Q_{\mathscr{B}}(x,p)$:

$$Q(p,q) = \frac{1}{|\mathscr{M}|}\sum_{x\in\mathfrak{D}} e^{-i(p-q)x} Q_{\mathscr{B}}\Big(x,\frac{p+q}{2}\Big). \tag{44}$$

The following symbols have to be defined in order to show that the inverse transformation holds true in this two-dimensional case:

$$\begin{aligned} \delta^{[a]}(q) &= \sum_{n\in\mathbb{Z}} \delta[q-an] \\ \tilde{\delta}^{[a]}(q) &= \sum_{n\in\mathbb{Z}} \delta[q-a\tilde{n}], \end{aligned} \tag{45}$$

where $\tilde{n} = n - 1/2$. Here, $\delta^{[a]}(q)$ is again the delta function of an argument defined modulo a. At the same time, $\tilde{\delta}^{[a]}(q)$ is the similar delta function, which vanishes when the argument differs from $a/2$ modulo a. These Dirac functions δ and $\tilde{\delta}$ denote, respectively, even and odd shifts in steps of $a/2$. Therefore, the inverse transformation can be demonstrated as follows:

$$\begin{aligned} &\frac{1}{|\mathscr{M}|}\sum_{x\in\mathfrak{D}} e^{2ix\left(q-\frac{k}{2}\right)} = \frac{1}{|\mathscr{M}|}\sum_{c_1,c_2\in\mathbb{Z}} e^{i\frac{3\ell}{2}(c_1+c_2)(2q_x-k_x)} \\ &\times e^{i\frac{\sqrt{3}\ell}{2}(c_1-c_2)(2q_y-k_y)} \\ &= \frac{1}{|\mathscr{M}|}\sum_{d_1,d_2\in\mathbb{Z}} e^{i\frac{3\ell}{2}(2q_x-k_x)d_1} e^{i\frac{\sqrt{3}\ell}{2}(2q_y-k_y)d_2} \\ &\times \frac{1}{2}\Big(1+e^{i\pi(d_1+d_2)}\Big) \\ &= \frac{1}{2|\mathscr{M}|}\sum_{d_1,d_2\in\mathbb{Z}} e^{i\frac{3\ell}{2}(2q_x-k_x)d_1} e^{i\frac{\sqrt{3}\ell}{2}(2q_y-k_y)d_2} \\ &+ \frac{1}{2|\mathscr{M}|}\sum_{d_1,d_2\in\mathbb{Z}} e^{i\left[\frac{3\ell}{2}(2q_x-k_x)+\pi\right]d_1} e^{i\left[\frac{\sqrt{3}\ell}{2}(2q_y-k_y)+\pi\right]d_2} \\ &= \sum_{n\in\mathbb{Z}} \delta\left[\Big(q_x-\frac{k_x}{2}\Big)-\frac{2\pi}{3\ell}n\right]\delta\left[\Big(q_y-\frac{k_y}{2}\Big)-\frac{2\pi}{\sqrt{3}\ell}n\right] \\ &+ \sum_{n\in\mathbb{Z}} \delta\left[\Big(q_x-\frac{k_x}{2}\Big)-\frac{2\pi}{3\ell}\tilde{n}\right]\delta\left[\Big(q_y-\frac{k_y}{2}\Big)-\frac{2\pi}{\sqrt{3}\ell}\tilde{n}\right] \\ &= \delta^{\left[\frac{2\pi}{3\ell}\right]}\Big(q_x-\frac{k_x}{2}\Big)\delta^{\left[\frac{2\pi}{\sqrt{3}\ell}\right]}\Big(q_y-\frac{k_y}{2}\Big) \\ &+ \tilde{\delta}^{\left[\frac{2\pi}{3\ell}\right]}\Big(q_x-\frac{k_x}{2}\Big)\tilde{\delta}^{\left[\frac{2\pi}{\sqrt{3}\ell}\right]}\Big(q_y-\frac{k_y}{2}\Big), \end{aligned}$$

such that

$$\frac{1}{|\mathscr{M}|}\sum_{x\in\mathfrak{D}} e^{-ikx}Q_{\mathscr{B}}(x,p) = \frac{1}{|\mathscr{M}|}\sum_{x\in\mathfrak{D}}\int_{\mathscr{M}} d^2q e^{2ix\left(q-\frac{k}{2}\right)}$$
$$\times Q(p+q,p-q)$$
$$= \int_{\mathscr{M}} d^2q\delta^{[\frac{1}{2}\vec{f}_1,\frac{1}{2}\vec{f}_2]}\left(q-\frac{k}{2}\right)Q(p+q,p-q)$$
$$+\int_{\mathscr{M}} d^2q\tilde{\delta}^{[\frac{1}{2}\vec{f}_1,\frac{1}{2}\vec{f}_2]}\left(q-\frac{k}{2}\right)Q(p+q,p-q)$$
$$= \int_{\mathscr{M}} d^2q\delta^{[(\frac{1}{2}\vec{g}_1,\frac{1}{2}\vec{g}_2)]}\left(q-\frac{k}{2}\right)Q(p+q,p-q)$$
$$= Q\left(p+\frac{k}{2},p-\frac{k}{2}\right). \qquad \blacksquare$$

In this expression $\delta^{[\vec{a}_1,\vec{a}_2]}(q)$ is the delta function of an argument with two components. This vector is defined as the modulo $\vec{a}_i$, $i=1,2$. The function $\tilde{\delta}^{[\vec{a}_1,\vec{a}_2]}(q)$ is the similar delta function, in which the argument is a defined modulo $\vec{a}_i$; however, this function vanishes when the argument differs from $\sum_i \vec{a}_i n_i/2$ with integer n_i. The Buot symbol of an operator can also be expressed as

$$\begin{aligned} B_{\mathscr{B}}(x,p) &\equiv \int_{\mathcal{M}} dq\, e^{2\mathrm{i}xq}\langle p+q|\hat{B}|p-q\rangle \\ &= \sum_{z,y\in\mathscr{O}}\int_{\mathscr{M}} dq\, e^{2\mathrm{i}xq}\langle p+q|z\rangle\langle z|\hat{B}|y\rangle\langle y|p-q\rangle \\ &= \frac{1}{|\mathscr{M}|}\sum_{z,y\in\mathscr{O}}\int_{\mathscr{M}} dq\, e^{2\mathrm{i}xq-\mathrm{i}(p+q)z+\mathrm{i}(p-q)y}\langle z|\hat{B}|y\rangle \\ &= \sum_{z,y\in\mathscr{O}} e^{-\mathrm{i}p(z-y)}\mathbf{d}(2x-z-y)\langle z|\hat{B}|y\rangle, \end{aligned} \tag{46}$$

where

$$\mathbf{d}(w) = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} dq\, e^{\mathrm{i}wq}.$$

Here, the integral is over the Brillouin zone. Function $B_{\mathscr{B}}(x,p)$ is defined by (46) for any real-valued x, not only for the values of $x\in\mathscr{O}$. Function $\mathbf{d}(w)$ is reduced to $\delta_{w,0}$ for $w\in\mathscr{O}$.

4.3. Moyal Product

The two-dimensional case is consistent with the definition of the star product in Section 1—(4) and the two-dimensional definition of the $\mathscr{B}$ symbol in (43). Consider the following as proof:

$$A_{\mathscr{B}}(x,p)\star B_{\mathscr{B}}(x,p) = \frac{1}{4}\int_{\mathfrak{M}} d^2q e^{ixq}\left\langle p+\frac{q}{2}|\hat{A}|p-\frac{q}{2}\right\rangle$$
$$\times e^{\frac{i}{2}\left(\overleftarrow{\partial_x}\overrightarrow{\partial_p}-\overleftarrow{\partial_p}\overrightarrow{\partial_x}\right)}\times\frac{1}{4}\int_{\mathfrak{M}} d^2k e^{ixk}\left\langle p+\frac{k}{2}|\hat{B}|p-\frac{k}{2}\right\rangle$$
$$= \frac{1}{4^2}\int_{\mathfrak{M}} d^2q d^2k e^{ix(q+k)}\left\langle p+\frac{q}{2}+\frac{k}{2}|\hat{A}|p-\frac{q}{2}+\frac{k}{2}\right\rangle$$
$$\times\left\langle p+\frac{k}{2}-\frac{q}{2}|\hat{B}|p-\frac{k}{2}-\frac{q}{2}\right\rangle$$
$$= \frac{2^2}{2^2\times 4^2}\int_{\mathfrak{M}} d^2\mathcal{P}d^2\mathcal{K}e^{ix\mathcal{P}}\left\langle p+\frac{\mathcal{P}}{2}|\hat{A}|p-\frac{\mathcal{K}}{2}\right\rangle$$
$$\times\left\langle p-\frac{\mathcal{K}}{2}|\hat{B}|p-\frac{\mathcal{P}}{2}\right\rangle$$
$$= \frac{1}{4}\int_{\mathfrak{M}} d^2\mathcal{P}e^{ix\mathcal{P}}\int_{\mathscr{M}} d^2\mathcal{K}\left\langle p+\frac{\mathcal{P}}{2}|\hat{A}|p-\mathcal{K}\right\rangle$$
$$\times\left\langle p-\mathcal{K}|\hat{B}|p-\frac{\mathcal{P}}{2}\right\rangle$$
$$= \frac{1}{4}\int_{\mathfrak{M}} d^2\mathcal{P}e^{ix\mathcal{P}}\left\langle p+\frac{\mathcal{P}}{2}|\hat{A}\hat{B}|p-\frac{\mathcal{P}}{2}\right\rangle. \qquad \blacksquare$$

The transition to the third line is possible via the Taylor expansion given below:

$$e^{a\partial_x} f(x) = \sum_{n=0}^{\infty} \frac{a^n}{n!} \partial_x^n f(x) = f(x+a). \tag{47}$$

The origin of the $1/2^2$ factor in the fifth line is the Jacobian generated by changing the variables $\mathcal{P} = q + k$ and $\mathcal{K} = q - k$, where $q, k \in \mathfrak{M}$. When the variables are changed, the integration region also changes from a four-dimensional square to a four-dimensional rhomboid. The x and y projections of the four-dimensional square and rhomboid regions are shown in Figures 8 and 9, respectively:

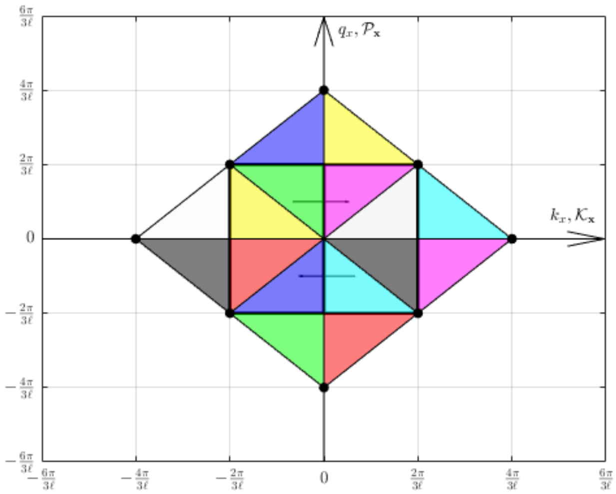

Figure 8. The x projections of the four-dimensional square and rhomboid regions; the contributions from colored areas are equal.

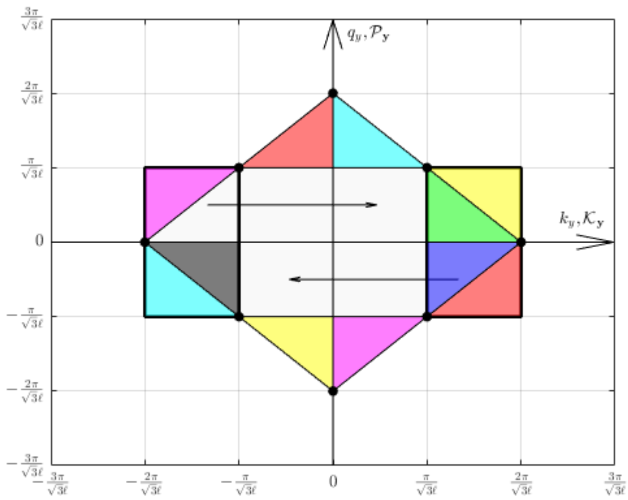

Figure 9. The y projections of the four-dimensional square and rhomboid regions; the contributions from colored areas are equal.

Due to the integral's periodic nature, with periods of $2\pi/3\ell$ in x and $2\pi/\sqrt{3}\ell$ in y, the four-dimensional rhomboid region can be converted back into the four-dimensional square region. The white, grey, green, and blue triangles are shifted twice by $4\pi/3\ell$ in x, whereas the red, yellow, cyan, and magenta triangles are shifted once by $\pm 2\pi/3\ell$ in x and once more by $\pm 2\pi/\sqrt{3}\ell$ in y. The black-outlined rectangles are eventually shifted once more in x and y (the shifts are denoted by the arrows). This shifts in x and y are linearly correlated, which means there are eight four-dimensional triangles that can be shifted into a single four-dimensional square, making the four-dimensional rhomboid region 2^2 times the size of the four-dimensional square region. Thus, the factor $1/2^2$ appears in the fifth line when the integration region is transformed back from a four-dimensional rhomboid to a four-dimensional square.

4.4. Trace and Its Properties

The definitions of the traces for $\mathscr{B}$ symbols of operators A and B on the physical and extended lattices are as follows:

$$\begin{aligned} \mathrm{Tr}_{\mathscr{O}} A_{\mathscr{B}} &\equiv \sum_{x\in\mathscr{O}} \int_{\mathscr{M}} \frac{d^2p}{|\mathscr{M}|} A_{\mathscr{B}}(x,p) \\ \mathrm{Tr}_{\mathfrak{D}} A_{\mathscr{B}} &\equiv \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} \frac{d^2p}{|\mathscr{M}|} A_{\mathscr{B}}(x,p). \end{aligned} \tag{48}$$

Both adhere to the first trace identity stated in Section 1—(5). As proof, consider the following:

$$\begin{aligned} &\mathrm{Tr}_{\mathfrak{D}} A_{\mathscr{B}} \equiv \frac{1}{|\mathscr{M}|} \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} d^2p A_{\mathscr{B}}(x,p) \\ &= \frac{1}{|\mathscr{M}|} \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} d^2p d^2q e^{2ixq} \langle p+q|\hat{A}|p-q\rangle \\ &= \frac{1}{|\mathscr{M}|} \sum_{x\in\mathfrak{D}} \sum_{x,y\in\mathscr{O}} \int_{\mathscr{M}} d^2p d^2q e^{2ixq} \langle p+q|z\rangle\langle z|\hat{A}|y\rangle\langle y|p-q\rangle \\ &= \frac{1}{|\mathscr{M}|^2} \sum_{x\in\mathfrak{D}} \sum_{z,y\in\mathscr{O}} \int_{\mathscr{M}} d^2p d^2q e^{i(2x-z-y)q} e^{i(y-z)p} \langle z|\hat{A}|y\rangle \\ &= \sum_{x\in\mathfrak{D}} \sum_{y,z\in\mathscr{O}} \delta_{2x,y+z}\delta_{y,z}\langle z|\hat{A}|y\rangle \\ &= \sum_{x\in\mathscr{O}} \langle x|\hat{A}|x\rangle = \mathrm{tr}\hat{A}. \qquad \blacksquare \end{aligned}$$

The same can be demonstrated for the physical lattice, but only the extended lattice accommodates the second trace identity specified in Section 1—(6). The proof:

$$\begin{aligned} &\mathrm{Tr}_{\mathfrak{D}}(A_{\mathscr{B}} \star B_{\mathscr{B}}) \equiv \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} \frac{d^2p}{|\mathscr{M}|} A_{\mathscr{B}}(x,p) \star B_{\mathscr{B}}(x,p) \\ &= \frac{1}{|\mathscr{M}|} \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} d^2p d^2q d^2k e^{2ixq} \langle p+q|\hat{A}|p-q\rangle \\ &\times e^{\frac{i}{2}\left(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}\right)} e^{2ixk} \langle p+k|\hat{B}|p-k\rangle \\ &= \frac{1}{|\mathscr{M}|^3} \sum_{x\in\mathfrak{D}} \sum_{y,z,\bar{y},\bar{z}\in\mathscr{O}} \int_{\mathscr{M}} d^2p d^2q d^2k e^{i(2x-y-z)q+i(z-y)p} \\ &\times \langle y|\hat{A}|z\rangle e^{\frac{i}{2}\left(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}\right)} e^{i(2x-\bar{y}-\bar{z})k+i(\bar{z}-\bar{y})p} \langle\bar{y}|\hat{B}|\bar{z}\rangle \\ &= \frac{1}{|\mathscr{M}|^3} \sum_{x\in\mathfrak{D}} \sum_{y,z,\bar{y},\bar{z}\in\mathscr{O}} \int_{\mathscr{M}} d^2p d^2q d^2k e^{i(2x-y-z)q+i(z-y)p} \\ &\times \langle y|\hat{A}|z\rangle e^{i(\bar{y}-\bar{z})q+i(z-y)k} e^{i(2x-\bar{y}-\bar{z})k+i(\bar{z}-\bar{y})p} \langle\bar{y}|\hat{B}|\bar{z}\rangle \\ &= \sum_{x\in\mathfrak{D}} \sum_{y,z,\bar{y},\bar{z}\in\mathscr{O}} \delta_{2x+\bar{y}-\bar{z},y+z} \langle y|\hat{A}|z\rangle \delta_{y-z,\bar{z}-\bar{y}} \\ &\times \delta_{2x-y+z,\bar{y}+\bar{z}} \langle\bar{y}|\hat{B}|\bar{z}\rangle \\ &= \sum_{x\in\mathscr{O}} \langle x|\hat{A}\hat{B}|x\rangle = \mathrm{tr}\hat{A}\hat{B}. \qquad \blacksquare \end{aligned}$$

4.5. $\mathscr{B}$ Symbol of the Identity Operator

The identity operator's two-dimensional $\mathscr{B}$ symbol is represented by:

$$(\hat{1})_{\mathscr{B}}(x,p) = \frac{1}{4}\Big[1 + \cos\pi c_1^1 + \cos\pi c_2^1 + \cos\pi\Big(c_1^1 + c_2^1\Big)\Big]. \tag{49}$$

Here, $x \in \mathfrak{D}$, $p \in \mathscr{M}$, and the extended lattice $\mathfrak{D}$ is defined by Equation (13). It is clear that Weyl symbol of identity operator is equal to unity if both c_1^1 and c_2^1 are even integers (i.e., $x \in \mathscr{O}$), but it is not equal to unity if either c_1^1 or c_2^1 is an odd integer. As a result, the function 1_W instead of being constant is fast oscillating. Take into account the following when validating (49):

$$\begin{aligned}
&(\hat{1})_{\mathscr{B}} = \int_{\mathscr{M}} d^2q e^{2ixq}\langle p+q|p-q\rangle \\
&= \int_{\mathscr{M}} d^2q e^{2ixq}\delta^{\left[\left(\frac{1}{2}\vec{g}_1,\frac{1}{2}\vec{g}_2\right)\right]}(2q) \\
&= \frac{1}{4}\int_{\mathscr{M}} d^2q e^{2ixq}\delta^{\left[\left(\frac{1}{4}\vec{g}_1,\frac{1}{4}\vec{g}_2\right)\right]}(q) \\
&= \frac{1}{4}\int_{-\frac{\pi}{3\ell}}^{\frac{\pi}{3\ell}} dq_x e^{2ixq_x}\delta(q_x)\int_{-\frac{\pi}{2\sqrt{3}\ell}}^{\frac{\pi}{2\sqrt{3}\ell}} dq_y e^{2iyq_y}\delta\big(q_y\big) \\
&+ \frac{1}{4}\int_{-\frac{\pi}{3\ell}}^{\frac{\pi}{3\ell}} dq_x e^{2ixq_x}\delta\Big(q_x \mp \frac{\pi}{6\ell}\Big)\int_{-\frac{\pi}{2\sqrt{3}\ell}}^{\frac{\pi}{2\sqrt{3}\ell}} dq_y e^{2iyq_y} \\
&\times\delta\Big(q_y \pm \frac{\pi}{2\sqrt{3}\ell}\Big) + \frac{1}{4}\int_{-\frac{\pi}{3\ell}}^{\frac{\pi}{3\ell}} dq_x e^{2ixq_x}\delta\Big(q_x \pm \frac{\pi}{6\ell}\Big) \\
&\times\int_{-\frac{\pi}{2\sqrt{3}\ell}}^{\frac{\pi}{2\sqrt{3}\ell}} dq_y e^{2iyq_y}\delta\Big(q_y \pm \frac{\pi}{2\sqrt{3}\ell}\Big) + \frac{1}{4}\int_{-\frac{\pi}{3\ell}}^{\frac{\pi}{3\ell}} dq_x e^{2ixq_x} \\
&\times\delta\Big(q_x \pm \frac{\pi}{3\ell}\Big)\int_{-\frac{\pi}{2\sqrt{3}\ell}}^{\frac{\pi}{2\sqrt{3}\ell}} dq_y e^{2iyq_y}\delta\big(q_y\big) \\
&= \frac{1}{4} + \frac{1}{8}e^{i\frac{\pi}{3\ell}x}e^{-i\frac{\pi}{\sqrt{3}\ell}y} + \frac{1}{8}e^{-i\frac{\pi}{3\ell}x}e^{i\frac{\pi}{\sqrt{3}\ell}y} + \frac{1}{8}e^{-i\frac{\pi}{3\ell}x}e^{-i\frac{\pi}{\sqrt{3}\ell}y} \\
&+ \frac{1}{8}e^{i\frac{\pi}{3\ell}x}e^{i\frac{\pi}{\sqrt{3}\ell}y} + \frac{1}{8}e^{-i\frac{2\pi}{3\ell}x} + \frac{1}{8}e^{i\frac{2\pi}{3\ell}x} \\
&= \frac{1}{4} + \frac{1}{2}\cos\frac{\pi}{3\ell}x\cos\frac{\pi}{\sqrt{3}\ell}y + \frac{1}{4}\cos\frac{2\pi}{3\ell}x,
\end{aligned}$$

where

$$\begin{aligned}
x &= \Big(c_1^1\vec{l}_1 + c_2^1\vec{l}_2\Big)_x = \frac{3\ell}{2}\Big(1^1 + c_2^1\Big) \\
y &= \Big(c_1^1\vec{l}_1 + c_2^1\vec{l}_2\Big)_y = \frac{\sqrt{3}\ell}{2}\Big(c_1^1 - c_2^1\Big)
\end{aligned} \tag{50}$$

These variables, x and y, are the linear projections of the extended lattice vectors (lattice $\mathfrak{D}$) onto the axes.

4.6. Star Product without Differentiation

Let us represent the star product of Buot symbols for $x \in \mathscr{D}$ through the matrix elements of $\hat{A}$ and $\hat{B}$:

$$
\begin{aligned}
& A_{\mathscr{B}}(x,p)\star B_{\mathscr{B}}(x,p) = \int_{\mathcal{M}} dkdq e^{\mathrm{i}2xk}\langle p+k|\hat{A}|p-k\rangle \\
& \star e^{2\mathrm{i}xq}\langle p+q|\hat{B}|p-q\rangle \\
= & \frac{1}{|\mathcal{M}|^2}\sum_{z_1,z_2,\bar{z}_1,\bar{z}_2\in\mathcal{O}}\int_{\mathcal{M}} dkdq e^{\mathrm{i}2xk-iz_1(p+k)+iz_2(p-k)}\langle z_1|\hat{A}|z_2\rangle \star \\
& e^{2\mathrm{i}xq-i\bar{z}_1(p+q)+i\bar{z}_2(p-q)}\langle \bar{z}_1|\hat{B}|\bar{z}_2\rangle \\
= & \frac{1}{|\mathcal{M}|^2}\sum_{z_1,z_2,\bar{z}_1,\bar{z}_2\in\mathcal{O}}\int_{\mathcal{M}} dkdq e^{\mathrm{i}(2x-z_1-z_2)k+\mathrm{i}p(-z_1+z_2)}\langle z_1|\hat{A}|z_2\rangle \star \\
& e^{\mathrm{i}(2x-\bar{z}_1-\bar{z}_2)q+\mathrm{i}p(-\bar{z}_1+\bar{z}_2)}\langle \bar{z}_1|\hat{B}|\bar{z}_2\rangle \\
= & \sum_{z_1,z_2,\bar{z}_1,\bar{z}_2\in\mathcal{O}} \delta_{2x,z_1+z_2} e^{ip(-z_1+z_2)}\langle z_1|\hat{A}|z_2\rangle \star \\
& e^{\mathrm{i}p(-\bar{z}_1+\bar{z}_2)}\delta_{2x,\bar{z}_1+\bar{z}_2}\langle \bar{z}_1|\hat{B}|\bar{z}_2\rangle \\
= & \sum_{z_1,z_2,\bar{z}_1,\bar{z}_2\in\mathcal{O}} \delta_{2x,z_1+z_2} e^{i(p-i\overrightarrow{\partial_x}/2)(-z_1+z_2)}\langle z_1|\hat{A}|z_2\rangle \\
& e^{\mathrm{i}(p+i\overleftarrow{\partial_x}/2)(-\bar{z}_1+\bar{z}_2)}\delta_{2x,\bar{z}_1+\bar{z}_2}\langle \bar{z}_1|\hat{B}|\bar{z}_2\rangle \\
= & \sum_{z_1,z_2,\bar{z}_1,\bar{z}_2\in\mathcal{O}} \delta_{2x+\bar{z}_1-\bar{z}_2,z_1+z_2} e^{ip(-z_1+z_2)}\langle z_1|\hat{A}|z_2\rangle \\
& e^{\mathrm{i}p(-\bar{z}_1+\bar{z}_2)}\delta_{2x-z_1+z_2,\bar{z}_1+\bar{z}_2}\langle \bar{z}_1|\hat{B}|\bar{z}_2\rangle.
\end{aligned} \tag{51}
$$

Next, using Equation (46) we obtain (for $x\in\mathscr{D}$):

$$
\begin{aligned}
A_{\mathscr{B}}(x,p)\star B_{\mathscr{B}}(x,p) = & \sum_{2z,2\bar{z},u,\bar{u}\in\mathcal{O}} \delta_{2x-\bar{u},2z}\delta_{2x+u,2\bar{z}} \\
& \int \frac{dp'}{|\mathcal{M}|}\frac{d\bar{p}'}{|\mathcal{M}|} e^{ip'u+i\bar{p}'\bar{u}} A_{\mathscr{B}}(z,p-p')B_{\mathscr{B}}(\bar{z},p-\bar{p}').
\end{aligned} \tag{52}
$$

One can see, that in order to define the star product of the symbols $A_{\mathscr{B}}(x,p)$ and $B_{\mathscr{B}}(x,p)$ for $x\in\mathscr{D}$ we do not need to know the values of these functions for all real values of x. It is enough to know the values of the Weyl symbols for $x\in\mathscr{D}$.

5. Precise Wigner–Weyl Calculus in Graphene

Here, the precise Wigner–Weyl formalism—which makes use of the extended lattice—is constructed. We use this precise Wigner–Weyl calculus on the extended lattice of graphene, utilizing both the physical lattice $\mathscr{O}$ and the extended lattice $\mathfrak{D}$, along with their respective first Brillouin zones $\mathscr{M}$ and $\mathfrak{M}$.

5.1. The Hilbert Space (Extended Properties)

This section is the same as Section 4.1, with the exception that the properties of the physical states that act on the physical lattice $\mathscr{O}$ are reconfigured to act on the extended lattice $\mathfrak{D}$. Consequently, the following definitions are given for the extended states' properties:

$$
\begin{aligned}
\hat{1}_{\mathfrak{D}} = \sum_{x\in\mathfrak{D}}|x\rangle\langle x| = \int_{\mathfrak{M}} d^2p|p\rangle\langle p| \qquad & \langle x|p\rangle = \frac{1}{\sqrt{|\mathfrak{M}|}}e^{ixp} \\
\langle p|q\rangle = \delta^{[(\vec{g}_1,\vec{g}_2)]}(p-q) \qquad & \langle x|y\rangle = \delta_{x,y}
\end{aligned} \tag{53}
$$

together with the Fourier decomposition, which is defined as follows:

$$
\begin{aligned}
|p\rangle &= \frac{1}{\sqrt{|\mathfrak{M}|}}\sum_{x\in\mathfrak{D}} e^{ixp}|x\rangle \\
&= \frac{1}{\sqrt{|\mathfrak{M}|}}\sum_{x\in\mathscr{O}} e^{ixp}|x\rangle + \frac{1}{\sqrt{|\mathfrak{M}|}}\sum_{x'\in\mathscr{O}'} e^{ix'p}|x'\rangle.
\end{aligned} \tag{54}
$$

Additionally, the following relationships hold true for the extended operators:

$$\begin{aligned}\langle x|\hat{Q}|y\rangle &= \langle x+l_1|\hat{Q}|y+l_1\rangle \\ &= \langle x+l_2|\hat{Q}|y+l_2\rangle \\ &= \langle x+l_1+l_2|\hat{Q}|y+l_1+l_2\rangle,\end{aligned} \tag{55}$$

where $x, y \in \mathscr{O}$, and the inter-lattice matrix elements vanish, such that

$$\langle x|\hat{Q}|y\rangle = \langle y|\hat{Q}|x\rangle = 0. \tag{56}$$

Here, $x \in \mathscr{O}$, and $y \in \mathscr{O}'$. Therefore, the matrix elements in the momentum space of such operators are given by:

$$\begin{aligned}\langle p|\hat{Q}|q\rangle &= \frac{1}{4}Q(p,q) \\ &\times \Big(1 + e^{il_1(q-p)} + e^{il_2(q-p)} + e^{il_{1,2}(q-p)}\Big),\end{aligned} \tag{57}$$

where $l_{1,2} = l_1 + l_2$. It is required to mention the following definition before verifying (57):

$$Q(p,q) = \frac{1}{|\mathscr{M}|}\sum_{x_1,x_2\in\mathscr{O}} \langle x_1|\hat{Q}|x_2\rangle e^{i(x_2 q - x_1 p)}. \tag{58}$$

As a result, it is now possible to demonstrate that (57) is true as follows:

$$\begin{aligned}\langle p|\hat{Q}|q\rangle &= \frac{1}{|\mathfrak{M}|}\sum_{x_1,x_2\in\mathscr{O}} \langle x_1|\hat{Q}|x_2\rangle e^{i(x_2 q - x_1 p)} \\ &+ \frac{1}{|\mathfrak{M}|}\sum_{x_1',x_2'\in\mathscr{O}'} \langle x_1'|\hat{Q}|x_2'\rangle e^{i(x_2' q - x_1' p)} \\ &= \frac{1}{|\mathfrak{M}|}\sum_{x_1,x_2\in\mathscr{O}} \langle x_1|\hat{Q}|x_2\rangle e^{i(x_2 q - x_1 p)} \\ &+ \frac{1}{|\mathfrak{M}|}\sum_{x_1,x_2\in\mathscr{O}} \langle x_1+l_1|\hat{Q}|x_2+l_1\rangle e^{i(x_2 q - x_1 p)} e^{il_1(q-p)} \\ &+ \frac{1}{|\mathfrak{M}|}\sum_{x_1,x_2\in\mathscr{O}} \langle x_1+l_2|\hat{Q}|x_2+l_2\rangle e^{i(x_2 q - x_1 p)} e^{il_2(q-p)} \\ &+ \frac{1}{|\mathfrak{M}|}\sum_{x_1,x_2\in\mathscr{O}} \langle x_1+l_1+l_2|\hat{Q}|x_2+l_1+l_2\rangle e^{i(x_2 q - x_1 p)} \\ &\times e^{i(l_1+l_2)(q-p)} \\ &= \frac{1}{|\mathfrak{M}|}\sum_{x_1,x_2\in\mathscr{O}} \langle x_1|\hat{Q}|x_2\rangle e^{i(x_2 q - x_1 p)} \\ &\times \Big(1 + e^{il_1(q-p)} + e^{il_2(q-p)} + e^{i(l_1+l_2)(q-p)}\Big),\end{aligned} \quad \blacksquare$$

where, in this two-dimensional case, $|\mathscr{M}| = |\mathfrak{M}|/4$. According to that definition, the matrix element $\langle p|\hat{Q}|q\rangle$ is periodic with periods of $2\pi/3\ell$ in x and $2\pi/\sqrt{3}\ell$ in y, whereas the function $Q(p,q)$ is periodic with periods of $\pi/3\ell$ in x and $\pi/\sqrt{3}\ell$ in y.

5.2. W Symbol

The $\mathscr{B}$ symbol of an operator, defined on the extended lattice $\mathfrak{D}$, can be used to determine the W symbol of an operator, defined on the physical lattice $\mathscr{O}$. Take into account the following:

$$
\begin{aligned}
Q_W(x,p) &\equiv \int_{\mathfrak{M}} d^2q e^{2ixq} \langle p+q|\hat{Q}|p-q\rangle \\
&= \frac{1}{4}\int_{\mathfrak{M}} d^2q e^{2ixq} Q(p+q,p-q) \\
&\times \left(1+e^{-2il_1q}+e^{-2il_2q}+e^{-2i(l_1+l_2)q}\right) \\
&= \int_{\mathscr{M}} d^2q e^{2ixq} Q(p+q,p-q) \\
&\times \left(1+e^{-2il_1q}+e^{-2il_2q}+e^{-2i(l_1+l_2)q}\right),
\end{aligned} \tag{59}
$$

where $Q_W(x,p)$ is defined for any $x \in \mathbb{R}$, but due to the discrete values of $x \in \mathfrak{D}$ the integration is reduced from $\mathfrak{M}$ to $\mathscr{M}$. In addition, for brevity's sake, the expression in parentheses will henceforth be abbreviated as $f(q)$. Furthermore, the inverse transformation of a function $Q_W(x,p)$ is given by:

$$
Q(p,q) = \frac{f^{-1}\left(\frac{p-q}{2}\right)}{|\mathscr{M}|} \sum_{x\in\mathfrak{D}} e^{-i(p-q)x} Q_W\left(x, \frac{p+q}{2}\right). \tag{60}
$$

It is possible to validate (60) in a manner similar to how it was demonstrated in Section 4.2 by doing the following:

$$
\begin{aligned}
&\frac{1}{|\mathscr{M}|}\sum_{x\in\mathfrak{D}} e^{2ix\left(q-\frac{k}{2}\right)} = \frac{1}{|\mathscr{M}|}\sum_{c_1,c_2\in\mathbb{Z}} e^{i\frac{3\ell}{2}(c_1+c_2)(2q_x-k_x)} \\
&\times e^{i\frac{\sqrt{3}\ell}{2}(c_1-c_2)(2q_y-k_y)} \\
&= \frac{1}{|\mathscr{M}|}\sum_{d_1,d_2\in\mathbb{Z}} e^{i\frac{3\ell}{2}(2q_x-k_x)d_1} e^{i\frac{\sqrt{3}\ell}{2}(2q_y-k_y)d_2} \\
&\times \frac{1}{2}\left(1+e^{i\pi(d_1+d_2)}\right) \\
&= \frac{1}{2|\mathscr{M}|}\sum_{d_1,d_2\in\mathbb{Z}} e^{i\frac{3\ell}{2}(2q_x-k_x)d_1} e^{i\frac{\sqrt{3}\ell}{2}(2q_y-k_y)d_2} \\
&+ \frac{1}{2|\mathscr{M}|}\sum_{d_1,d_2\in\mathbb{Z}} e^{i\left[\frac{3\ell}{2}(2q_x-k_x)+\pi\right]d_1} e^{i\left[\frac{\sqrt{3}\ell}{2}(2q_y-k_y)+\pi\right]d_2} \\
&= \sum_{n\in\mathbb{Z}} \delta\left[\left(q_x-\frac{k_x}{2}\right)-\frac{2\pi}{3\ell}n\right]\delta\left[\left(q_y-\frac{k_y}{2}\right)-\frac{2\pi}{\sqrt{3}\ell}n\right] \\
&+ \sum_{n\in\mathbb{Z}} \delta\left[\left(q_x-\frac{k_x}{2}\right)-\frac{2\pi}{3\ell}\tilde{n}\right]\delta\left[\left(q_y-\frac{k_y}{2}\right)-\frac{2\pi}{\sqrt{3}\ell}\tilde{n}\right] \\
&= \delta^{\left[\frac{2\pi}{3\ell}\right]}\left(q_x-\frac{k_x}{2}\right)\delta^{\left[\frac{2\pi}{\sqrt{3}\ell}\right]}\left(q_y-\frac{k_y}{2}\right) \\
&+ \tilde{\delta}^{\left[\frac{2\pi}{3\ell}\right]}\left(q_x-\frac{k_x}{2}\right)\tilde{\delta}^{\left[\frac{2\pi}{\sqrt{3}\ell}\right]}\left(q_y-\frac{k_y}{2}\right),
\end{aligned}
$$

such that

$$
\begin{aligned}
&\frac{1}{|\mathscr{M}|}\sum_{x\in\mathfrak{D}} e^{-ikx} Q_W(x,p) = \frac{1}{|\mathscr{M}|}\sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} d^2q e^{2ix\left(q-\frac{k}{2}\right)} \\
&\times Q(p+q,p-q) f(q) \\
&= \int_{\mathscr{M}} d^2q \delta^{\left[\frac{1}{2}\vec{f}_1,\frac{1}{2}\vec{f}_2\right]}\left(q-\frac{k}{2}\right) Q(p+q,p-q) f(q) \\
&+ \int_{\mathscr{M}} d^2q \tilde{\delta}^{\left[\frac{1}{2}\vec{f}_1,\frac{1}{2}\vec{f}_2\right]}\left(q-\frac{k}{2}\right) Q(p+q,p-q) f(q) \\
&= \int_{\mathscr{M}} d^2q \delta^{\left[\left(\frac{1}{2}\vec{g}_1,\frac{1}{2}\vec{g}_2\right)\right]}\left(q-\frac{k}{2}\right) Q(p+q,p-q) f(q) \\
&= Q\left(p+\frac{k}{2}, p-\frac{k}{2}\right) f\left(\frac{k}{2}\right).
\end{aligned}
$$

■

The Buot symbol on the extended lattice may be expressed as

$$Q_W(x,p) = \sum_{z,y\in\mathscr{D}} e^{-\mathrm{i}p(z-y)}\mathbf{d}(2x-z-y)\langle z|\hat{Q}|y\rangle, \tag{61}$$

where

$$\mathbf{d}(w) = \frac{1}{|\mathfrak{M}|}\int_{\mathfrak{M}} dq\, e^{\mathrm{i}wq}.$$

The doubly extended lattice $\mathscr{S}$ may be defined as an extension of $\mathscr{D}$ in the same way as $\mathscr{D}$ is an extension of $\mathscr{O}$.

We should take into account the constraints of Equations (55) and (56). This results in

$$Q_W(x,p) = \sum_{z,y\in\mathscr{O};n_i=0,1} e^{-\mathrm{i}p(z-y)} \mathbf{d}(2x-z-y-2l_i n_i)\langle z|\hat{Q}|y\rangle, \tag{62}$$

and

$$Q_W(x,p)\Big|_{x\in\mathscr{S}\setminus\mathscr{D}} = 0. \tag{63}$$

5.3. Moyal Product

The two-dimensional case is consistent with the definition of the star product in Section 1—(4) and the definition of the W symbol in (59). Consider the following as proof:

$$\begin{aligned}
&A_W(x,p)\star B_W(x,p) = \int_{\mathscr{M}} d^2q e^{2ixq} f(q)A(p+q,p-q)\\
&\times e^{\frac{i}{2}(\overleftarrow{\partial}_x\overrightarrow{\partial}_p-\overleftarrow{\partial}_p\overrightarrow{\partial}_x)}\int_{\mathscr{M}} d^2k e^{2ixk} f(k)B(p+k,p-k)\\
&=\int_{\mathscr{M}} d^2q e^{2ixq} f(q)A(p+q+k,p-q+k)\\
&\times\int_{\mathscr{M}} d^2k e^{2ixk} f(k)B(p-q+k,p-q-k)\\
&=\frac{1}{2^2\times 2^2}\int_{\mathfrak{M}} d^2\mathcal{P}d^2\mathcal{K}e^{2ix\mathcal{P}} f\left(\frac{\mathcal{P}+\mathcal{K}}{2}\right)f\left(\frac{\mathcal{P}-\mathcal{K}}{2}\right)\\
&\times A(p+\mathcal{P},p-\mathcal{K})B(p-\mathcal{K},p-\mathcal{P})\\
&=\frac{1}{2^2\times 2^2}\int_{\mathfrak{M}} d^2\mathcal{P}d^2\mathcal{K}e^{2ix\mathcal{P}}\left[1+g\left(\frac{\mathcal{P}+\mathcal{K}}{2}\right)\right]\\
&\times\left[1+g\left(\frac{\mathcal{P}-\mathcal{K}}{2}\right)\right]A(p+\mathcal{P},p-\mathcal{K})B(p-\mathcal{K},p-\mathcal{P})\\
&=\frac{1}{2^2\times 2^2}\int_{\mathfrak{M}} d^2\mathcal{P}d^2\mathcal{K}e^{2ix\mathcal{P}}\left[1+g\left(\frac{\mathcal{P}+\mathcal{K}}{2}\right)+g\left(\frac{\mathcal{P}-\mathcal{K}}{2}\right)\right.\\
&\left.+g\left(\frac{\mathcal{P}+\mathcal{K}}{2}\right)g\left(\frac{\mathcal{P}-\mathcal{K}}{2}\right)\right]A(p+\mathcal{P},p-\mathcal{K})\\
&\times B(p-\mathcal{K},p-\mathcal{P})\\
&=\frac{4^2}{2^2\times 2^2}\int_{\mathscr{M}} d^2\mathcal{P}d^2\mathcal{K}e^{2ix\mathcal{P}} f(\mathcal{P})A(p+\mathcal{P},p-\mathcal{K})\\
&\times B(p-\mathcal{K},p-\mathcal{P}) = (AB)_W(x,p).
\end{aligned}$$

■

The two $1/2^2$ factors in the fifth line are produced by the Jacobian and the transition of the integration region from a four-dimensional rhomboid to a four-dimensional square $\mathfrak{M}\otimes\mathfrak{M}$. Due to the periodic nature of the functions $f((\mathcal{P}\pm\mathcal{K})/2)$, which have periods of $2\pi/3\ell$ in x and $2\pi/\sqrt{3}\ell$ in y, it is impossible to immediately reduce the integration region from $\mathfrak{M}\otimes\mathfrak{M}$ to $\mathscr{M}\otimes\mathscr{M}$. As a result, the following modification is implemented:

$$g(p) \equiv f(p) - 1 = e^{-2il_1 q} + e^{-2il_2 q} + e^{-2i(l_1+l_2)q}. \tag{64}$$

Therefore, after the expressions in square parentheses are multiplied and the integration region is reduced, the integrals, including the terms $g((\mathcal{P} \pm \mathcal{K})/2)$, cancel out because these functions change sign when shifted by $\pi/3\ell$ in x and/or by $\pi/\sqrt{3}\ell$ in y. Those shifts are demonstrated in Figures 10 and 11. Furthermore, canceling out are the integrals, including the terms 1 and $g((\mathcal{P}+\mathcal{K})/2) \times g((\mathcal{P}-\mathcal{K})/2)$, which do not add up to $f(\mathcal{P})$. Once $f(\mathcal{P})$ is all that is left, the integration region can be reduced from $\mathfrak{M} \otimes \mathfrak{M}$ to $\mathscr{M} \otimes \mathscr{M}$ because it is periodic, with periods of $\pi/3\ell$ in x and $\pi/\sqrt{3}\ell$ in y. The factor 4^2 in the final line is produced by this reduction.

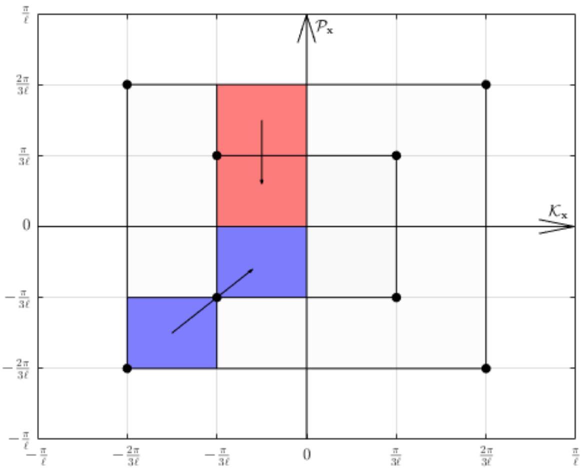

Figure 10. The four-dimensional squares $\mathfrak{M} \otimes \mathfrak{M}$ and $\mathscr{M} \otimes \mathscr{M}$'s x projections. Arrows depict the shifts: red shifts are negative, while blue shifts are positive.

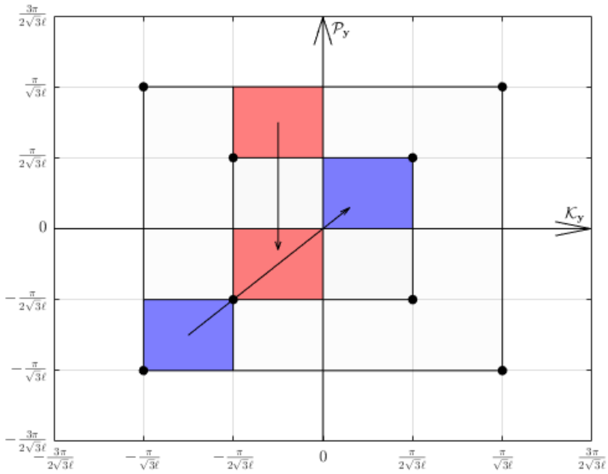

Figure 11. The four-dimensional squares $\mathfrak{M} \otimes \mathfrak{M}$ and $\mathscr{M} \otimes \mathscr{M}$'s y projections. Arrows depict the shifts: red shifts are negative, while blue shifts are positive.

5.4. Trace and Its Properties

The definitions of the traces of the physical and extended lattices are as follows:

$$\mathrm{Tr}_{\mathscr{O}} Q_W \equiv \sum_{x \in \mathscr{O}} \int_{\mathscr{M}} \frac{d^2 p}{|\mathscr{M}|} Q_W(x,p)$$
$$\mathrm{Tr}_{\mathfrak{D}} Q_W \equiv \sum_{x \in \mathfrak{D}} \int_{\mathscr{M}} \frac{d^2 p}{|\mathfrak{M}|} Q_W(x,p). \tag{65}$$

Both adhere to the first trace identity stated in Section 1—(5). As proof, consider the following:

$$\begin{aligned}
\mathrm{Tr}_{\mathscr{O}} Q_W &\equiv \sum_{x\in\mathscr{O}} \int_{\mathscr{M}} \frac{d^2p}{|\mathscr{M}|} Q_W(x,p) \\
&= \frac{1}{|\mathscr{M}|} \sum_{x\in\mathscr{O}} \int_{\mathscr{M}} d^2p d^2q e^{2ixq} Q(p+q,p-q) f(q) \\
&= \int_{\mathscr{M}} d^2p d^2q \delta^{[(\frac{1}{2}\vec{g}_1,\frac{1}{2}\vec{g}_2)]}(2q) Q(p+q,p-q) f(q) \\
&= \frac{1}{4} \int_{\mathscr{M}} d^2p d^2q \delta^{[(\frac{1}{4}\vec{g}_1,\frac{1}{4}\vec{g}_2)]}(q) Q(p+q,p-q) \\
&\times \left(1+e^{-2il_1q}+e^{-2il_2q}+e^{-2i(l_1+l_2)q}\right) \\
&= \frac{1}{4} \int_{\mathscr{M}} d^2p \int_{-\frac{\pi}{3\ell}}^{\frac{\pi}{3\ell}} dq_x \int_{-\frac{\pi}{2\sqrt{3}\ell}}^{\frac{\pi}{2\sqrt{3}\ell}} dq_y Q(p+q,p-q) \\
&\times \left(\delta(q_x)\delta(q_y) + \delta\left(q_x \mp \frac{\pi}{6\ell}\right)\delta\left(q_y \pm \frac{\pi}{2\sqrt{3}\ell}\right)\right. \\
&\left. + \delta\left(q_x \pm \frac{\pi}{6\ell}\right)\delta\left(q_y \pm \frac{\pi}{2\sqrt{3}\ell}\right) + \delta\left(q_x \pm \frac{\pi}{3\ell}\right)\delta(q_y)\right) \\
&\times \left(1+2e^{-i3\ell q_x}\cos\sqrt{3}\ell q_y + e^{-i6\ell q_x}\right) \\
&= \int_{\mathscr{M}} d^2p d^2q \delta(q) Q(p+q,p-q) \\
&= \int_{\mathscr{M}} d^2p\, Q(p,p) = \mathrm{tr}\hat{Q}. \qquad \blacksquare
\end{aligned}$$

The same can be demonstrated for the extended lattice. Therefore, both definitions of the traces in (65) obey the first trace identity as follows:

$$\mathrm{Tr}_{\mathscr{O}} Q_W = \mathrm{Tr}_{\mathfrak{D}} Q_W = \mathrm{tr}.\hat{Q} \tag{66}$$

The second trace identity stated in Section 1—(6) is solely accommodated by the extended lattice, much like in Section 4.4. The proof is:

$$\begin{aligned}
\mathrm{Tr}_{\mathfrak{D}}(A_W B_W) &\equiv \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} \frac{d^2p}{|\mathfrak{M}|} A_W(x,p) B_W(x,p) \\
&= \sum_{x\in\mathfrak{D}} \int_{\mathscr{M}} \frac{d^2p}{|\mathfrak{M}|} \int_{\mathscr{M}} d^2q e^{2ixq} A(p-q,p+q) f(q) \\
&\times \int_{\mathscr{M}} d^2k e^{2ixk} B(p-k,p+k) f(k) \\
&= \frac{1}{4} \int_{\mathscr{M}} d^2p d^2q A(p-q,p+q) B(p+q,p-q) f(q) f(-q) \\
&= \frac{1}{4|\mathscr{M}|^2} \int_{\mathscr{M}} d^2p d^2q \sum_{x_{1,2}\in\mathscr{O}} \langle x_1|\hat{A}|x_2\rangle e^{i(x_2(p+q)-x_1(p-q))} \\
&\times \sum_{y_{1,2}\in\mathscr{O}} \langle y_1|\hat{B}|y_2\rangle e^{i(y_2(p+q)-y_1(p-q))} \\
&\times \left(4+2e^{\pm 2il_1q}+2e^{\pm 2il_2q}+e^{\pm 2i(l_1+l_2)q}+e^{\pm 2i(l_1-l_2)q}\right) \\
&= \frac{1}{4} \sum_{x_{1,2}\in\mathscr{O}} \langle x_1|\hat{A}|x_2\rangle \sum_{y_{1,2}\in\mathscr{O}} \langle y_1|\hat{B}|y_2\rangle \left[4\delta_{2x_1,2y_2}\delta_{2x_2,2y_1}\right. \\
&+ 2\delta_{2x_1,2y_2\pm 2l_1}\delta_{2x_2,2y_1\pm 2l_1} + 2\delta_{2x_1,2y_2\pm 2l_2}\delta_{2x_2,2y_2\pm 2l_2} \\
&+ \delta_{2x_1,2y_2\pm 2(l_1+l_2)}\delta_{2x_2,2y_1\pm 2(l_1+l_2)} \\
&\left. + \delta_{2x_1,2y_2\pm 2(l_1-l_2)}\delta_{2x_2,2y_1\pm 2(l_1-l_2)}\right] \\
&= \sum_{x_1\in\mathscr{O}} \langle x_1|\hat{A}\hat{B}|x_1\rangle = \mathrm{tr}\hat{A}\hat{B}. \qquad \blacksquare
\end{aligned}$$

All Kronecker terms enclosed in square parentheses disappear for $x_i, y_i \in \mathscr{O}$ save the first. Thus, a solution exists when $x_1 = y_2$ and $x_2 = y_1$. Following the example above, it can be simply demonstrated that:

$$\mathrm{Tr}_{\mathfrak{D}}(A_W B_W) = \mathrm{Tr}_{\mathfrak{D}}(A_W \star B_W) = \mathrm{tr}\hat{A}\hat{B}. \tag{67}$$

5.5. W Symbol of the Identity Operator

The identity operator's two-dimensional W symbol is represented by:

$$\begin{aligned}(\hat{1})_W(x,p)\Big|_{x\in\mathscr{D}} &= \frac{1}{4}\left[1+\cos 2\pi c_1^1 + \cos 2\pi c_2^1 + \cos 2\pi\left(c_1^1 + c_2^1\right)\right] \\ &= 1,\end{aligned} \tag{68}$$

where it is clear that for all values of c_1^1 and c_2^1, the identity operator is unitary. When validating (49), consider the following:

$$\begin{aligned}(\hat{1})_W &= \int_{\mathfrak{M}} d^2q e^{2ixq}\langle p+q|p-q\rangle \\ &= \frac{1}{4}\int_{\mathfrak{M}} d^2q e^{2ixq}\delta^{[(\vec{g}_1,\vec{g}_2)]}(2q)f(q) \\ &= \frac{1}{16}\int_{\mathfrak{M}} d^2q e^{2ixq}\delta^{\left[\left(\frac{1}{2}\vec{g}_1,\frac{1}{2}\vec{g}_2\right)\right]}(q) \\ &\times\left(1+e^{-2il_1q}+e^{-2il_2q}+e^{-2i(l_1+l_2)q}\right) \\ &= \frac{1}{16}\int_{-\frac{2\pi}{3\ell}}^{\frac{2\pi}{3\ell}} dq_x e^{2ixq_x}\int_{-\frac{\pi}{\sqrt{3}\ell}}^{\frac{\pi}{\sqrt{3}\ell}} dq_y e^{2iyq_y}\Big(\delta(q_x)\delta(q_y) \\ &+\delta\left(q_x \mp \frac{\pi}{3\ell}\right)\delta\left(q_y \pm \frac{\pi}{\sqrt{3}\ell}\right) \\ &+\delta\left(q_x \pm \frac{\pi}{3\ell}\right)\delta\left(q_y \pm \frac{\pi}{\sqrt{3}\ell}\right)+\delta\left(q_x \pm \frac{2\pi}{3\ell}\right)\delta(q_y)\Big) \\ &\times\left(1+2e^{-i3\ell q_x}\cos\sqrt{3}\ell q_y + e^{-i6\ell q_x}\right) \\ &= \frac{1}{4}+\frac{1}{8}e^{i\frac{2\pi}{3\ell}x}e^{-i\frac{2\pi}{\sqrt{3}\ell}y}+\frac{1}{8}e^{-i\frac{2\pi}{3\ell}x}e^{i\frac{2\pi}{\sqrt{3}\ell}y} \\ &+\frac{1}{8}e^{-i\frac{2\pi}{3\ell}x}e^{-i\frac{2\pi}{\sqrt{3}\ell}y}+\frac{1}{8}e^{i\frac{2\pi}{3\ell}x}e^{i\frac{2\pi}{\sqrt{3}\ell}y}+\frac{1}{8}e^{-i\frac{4\pi}{3\ell}x}+\frac{1}{8}e^{i\frac{4\pi}{3\ell}x} \\ &= \frac{1}{4}+\frac{1}{2}\cos\frac{2\pi}{3\ell}x\cos\frac{2\pi}{\sqrt{3}\ell}y+\frac{1}{4}\cos\frac{4\pi}{3\ell}x,\end{aligned}$$

where x and y are given by Equation (50). These variables, x and y, are the linear projections of the extended lattice vectors (lattice $\mathfrak{D}$) onto the axes.

5.6. Star Product without Differentiation

One can represent the star product of Weyl symbols for $x \in \mathscr{S}$ through the matrix elements of $\hat{A}$ and $\hat{B}$. Using Equation (46), we obtain (for $x \in \mathscr{S}$):

$$\begin{aligned}A_W(x,p)\star B_W(x,p) &= \sum_{2z,2\bar{z},u,\bar{u}\in\mathscr{D}}\delta_{2x-\bar{u},2z}\delta_{2x+u,2\bar{z}} \\ &\int\frac{dp'}{|\mathfrak{M}|}\frac{d\bar{p}'}{|\mathfrak{M}|}e^{ip'u+i\bar{p}'\bar{u}}A_W(z,p-p')B_W(\bar{z},p-\bar{p}') \\ = &\sum_{z,\bar{z}\in\mathscr{S}}\int\frac{dp'}{|\mathfrak{M}|}\frac{d\bar{p}'}{|\mathfrak{M}|} \\ &e^{2ip'(\bar{z}-x)+i\bar{p}'(x-z)}A_W(z,p-p')B_W(\bar{z},p-\bar{p}').\end{aligned} \tag{69}$$

One can see that, in order to define the star product of the symbols $A_W(x, p)$ and $B_W(x, p)$ for $x \in \mathscr{S}$, we do not need to know the values of these functions for all real values of x. It is enough to know the values of the Weyl symbols for $x \in \mathscr{S}$. Moreover, according to Equation (63), we have

$$A_W(x, p)\Big|_{x\in\mathscr{S}\setminus\mathscr{D}} = 0 \tag{70}$$

for the operators that obey Equation (56). As a result, for such operators all needed information is encoded in the Weyl symbols $A_W(x, p)$ and $B_W(x, p)$ for $x \in \mathscr{D}$:

$$A_W(x, p) \star B_W(x, p)\Big|_{x\in\mathscr{D}} = \sum_{z,\bar{z}\in\mathscr{D}} \int \frac{dp'}{|\mathfrak{M}|} \frac{d\bar{p}'}{|\mathfrak{M}|} e^{2ip'(\bar{z}-x)+2i\bar{p}'(x-z)} A_W(z, p - p')B_W(\bar{z}, p - \bar{p}'). \tag{71}$$

6. Dynamics of Systems Defined on Honeycomb Lattice, and Hall Conductivity

6.1. Keldysh Technique of Field Theory

In this section, we use for brevity the units with $\hbar = 1$ and absorb electric charge to the definition of the gauge field. As a result, the obtained expression for electric conductivity should be multiplied by $e^2/\hbar$ (which is pointed out explicitly if necessary).

Here, we follow the approach developed in [45,68]. We work in the relativistic system of units with $\hbar = 1$. Moreover, we absorb electric charge in the definition of electromagnetic field. In order to come back to the usual system of units expression for the conductivity to be obtained below should be multiplied by $e^2/\hbar$.

Let us consider the system defined on the honeycomb lattice $\mathcal{O}$. Time is not discretized and is continuous. Field theory Hamiltonian is denoted by $\hat{\mathcal{H}}$. An operator $O[\hat{\psi}, \hat{\bar{\psi}}]$ is a functional of fields $\hat{\psi}, \hat{\bar{\psi}}$. Operator O at time t is a function of ψ and $\bar{\psi}$ defined at the same time. The average of the corresponding quantity is given by

$$\langle O\rangle = \mathrm{tr}\left(\hat{\rho}(t_i)\, e^{-i\int_{t_i}^{t} \hat{\mathcal{H}}dt} O[\hat{\psi}, \hat{\bar{\psi}}] e^{-i\int_t^{t_f} \hat{\mathcal{H}}dt} e^{i\int_{t_i}^{t_f} \hat{\mathcal{H}}dt}\right).$$

Here, $t_i < t < t_f$, and $\hat{\rho}(t_i)$ is density matrix at t_i. Time ordering $\mathfrak{T}$ allows us to rewrite the above expression as

$$\langle O\rangle = \mathrm{tr}\left(\mathfrak{T}\left[\hat{\rho}(t_i)\, e^{-i\int_{t_i}^{t_f} \hat{\mathcal{H}}dt} O[\hat{\psi}, \hat{\bar{\psi}}]\right] e^{i\int_{t_i}^{t_f} \hat{\mathcal{H}}dt}\right).$$

For the considered lattice system, $\langle O\rangle$ is given by

$$\langle O\rangle = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, O[\psi, \bar{\psi}] \exp\left\{\mathrm{i}\int_C dt \sum_x \bar{\psi}(t, x)\hat{Q}\psi(t, x)\right\}.$$

Here, ψ and $\bar{\psi}$ are the Grassmann variables, x is a lattice point. Without interactions, $\hat{Q}$ is $\hat{Q} = i\partial_t - \hat{H}$, where $\hat{H}$ is the one-particle Hamiltonian. Integration over time is along the Keldysh contour C. The contour starts at the t_i, goes to t_f, and returns back from t_f to t_i.

The forward part of the contour carries fields $\bar{\psi}_-(t, x)$ and $\psi_-(t, x)$. The fields on the backward part are $\bar{\psi}_+(t, x)$ and $\psi_+(t, x)$.

The boundary conditions relate fields of the forward and backward parts of the Keldysh contour: $\bar{\psi}_-(t_f, x) = \bar{\psi}_+(t_f, x)$ and $\psi_-(t_f, x) = \psi_+(t_f, x)$. The integration measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ contains $\bar{\psi}_+(t_i, x)$, $\psi_+(t_i, x)$ and $\bar{\psi}_-(t_i, x)$, $\psi_-(t_i, x)$ and depends on initial density matrix $\hat{\rho}$:

$$\langle O\rangle = \int \frac{\mathcal{D}\bar{\psi}_\pm \mathcal{D}\psi_\pm}{\text{Det}\,(1+\rho)} O[\psi_+,\bar{\psi}_+] \exp\Big(\mathrm{i}\int_{t_i}^{t_f} dt \sum_x [\bar{\psi}_-(t,x)\hat{Q}\psi_-(t,x) - \bar{\psi}_+(t,x)\hat{Q}\psi_+(t,x)] - \sum_x \bar{\psi}_-(t_i,x)\rho\psi_+(t_i,x)\Big). \tag{72}$$

ρ is an operator in one-particle Hilbert space. Its eigenstates are $|\lambda_i\rangle$, the matrix elements enter expression $\frac{\langle\lambda_i|\rho|\lambda_i\rangle}{1+\langle\lambda_i|\rho|\lambda_i\rangle}$ for the probability that the one-particle state $|\lambda_i\rangle$ is occupied. At the same time, $\frac{1}{1+\langle\lambda_i|\rho|\lambda_i\rangle}$ is the probability that the same state is empty. Keldysh spinors are composed of

$$\Psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}, \tag{73}$$

The average of an operator O is

$$\langle O\rangle = \frac{1}{\text{Det}\,(1+\rho)} \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, O[\Psi,\bar{\Psi}] \exp\Big\{\mathrm{i}\int_{t_i}^{t_f} dt \sum_x \bar{\Psi}(t,x)\hat{\mathbf{Q}}\Psi(t,x)\Big\}. \tag{74}$$

Here, x is a two-dimensional vector. $\hat{\mathbf{Q}}$ is given in Keldysh representation as

$$\hat{\mathbf{Q}} = \begin{pmatrix} Q_{--} & Q_{-+} \\ Q_{+-} & Q_{++} \end{pmatrix}. \tag{75}$$

To calculate the components of this matrix, one should use continuum limit of lattice regularized expressions:

$$\begin{aligned} Q_{++} &= -\Big(\mathrm{i}\partial_t - \hat{H} - \mathrm{i}\epsilon\frac{1-\rho}{1+\rho}\Big), \\ Q_{--} &= \mathrm{i}\partial_t - \hat{H} + \mathrm{i}\epsilon\frac{1-\rho}{1+\rho}, \\ Q_{+-} &= -2\mathrm{i}\epsilon\frac{1}{1+\rho}, \\ Q_{-+} &= 2\mathrm{i}\epsilon\frac{\rho}{1+\rho}. \end{aligned} \tag{76}$$

Here, ρ gives rise to one-particle distribution $f = \rho(1+\rho)^{-1}$. For a distribution depending only on energy, $\rho = \rho(\hat{H})$ is a function of the one-particle Hamiltonian. (In particular, in cases of Fermi distribution with temperature T and chemical potential μ, we have $\rho(E) = e^{(E-\mu)/T}$.) The infinitely small parameter $\epsilon \to 0$ points out the way to avoid the singularities while calculating the inverse operators (see Section 5.1 of [69]).

The Green function $\hat{\mathbf{G}}$ is defined as

$$G_{\alpha_1\alpha_2}(t,x|t',x') = \int \frac{\mathcal{D}\bar{\Psi}\mathcal{D}\Psi}{\mathrm{i}\text{Det}\,(1+\rho)} \Psi_{\alpha_1}(t,x)\bar{\Psi}_{\alpha_2}(t',x') \exp\Big\{\mathrm{i}\int_{t_i}^{t_f} dt \sum_x \bar{\Psi}(t,x)\hat{\mathbf{Q}}\Psi(t,x)\Big\}. \tag{77}$$

Here, α is Keldysh spinor index (73). We have an equation

$$\hat{\mathbf{Q}}\hat{\mathbf{G}} = 1.$$

Components of $\hat{\mathbf{G}}$ obey

$$G_{--} + G_{++} - G_{-+} - G_{+-} = 0 \tag{78}$$

while

$$Q_{--} + Q_{++} + Q_{-+} + Q_{+-} = 0. \tag{79}$$

Another representation is related to the spinors defined above as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$$

$$\begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\psi}_- & \bar{\psi}_+ \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Green function in this representation is triangular

$$\begin{aligned} \hat{\mathbf{G}}^{(K)} &= -i\langle \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \otimes \begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 \end{pmatrix} \rangle \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} G^{\mathrm{R}} & G^{\mathrm{K}} \\ 0 & G^{\mathrm{A}} \end{pmatrix}. \end{aligned} \tag{80}$$

We introduced the above Keldysh, advanced, and retarded Green functions:

$$\begin{aligned} G^{\mathrm{K}} &= G^{-+} + G^{+-} = G^{--} + G^{++}, \\ G^{\mathrm{A}} &= G^{--} - G^{+-} = G^{-+} - G^{++}, \\ G^{\mathrm{R}} &= G^{--} - G^{-+} = G^{+-} - G^{++}. \end{aligned} \tag{81}$$

Another triangular representation will be used below

$$\begin{aligned} \hat{\mathbf{G}}^{(<)} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G^{\mathrm{R}} & G^{\mathrm{K}} \\ 0 & G^{\mathrm{A}} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} G^{\mathrm{R}} & 2G^{<} \\ 0 & G^{\mathrm{A}} \end{pmatrix}. \end{aligned} \tag{82}$$

It is expressed through the Green function defined by Equation (77) as

$$\hat{\mathbf{G}}^{(<)} = U\hat{\mathbf{G}}V, \tag{83}$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

and

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}.$$

In addition, we have

$$\begin{aligned} \hat{\mathbf{Q}}^{(<)} &= V^{-1}\hat{\mathbf{Q}}U^{-1} \\ &= \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} Q^{--} & Q^{-+} \\ Q^{+-} & Q^{++} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} Q^{--} + Q^{-+} & -2Q^{-+} \\ \frac{Q^{--}+Q^{+-}+Q^{-+}+Q^{++}}{2} & -Q^{-+} - Q^{++} \end{pmatrix} \\ &= \begin{pmatrix} Q^{\mathrm{R}} & 2Q^{<} \\ 0 & Q^{\mathrm{A}} \end{pmatrix}, \end{aligned} \tag{84}$$

Here, we denote

$$Q^{\rm R} = Q^{--} + Q^{-+}, \qquad Q^{\rm A} = -Q^{-+} - Q^{++}, \qquad Q^{<} = -Q^{-+}, \tag{85}$$

Then

$$G^{\rm A} = (Q^{\rm A})^{-1}, \qquad G^{\rm R} = (Q^{\rm A})^{-1}, \qquad G^{<} = -G^{\rm R} Q^{<} G^{\rm A}. \tag{86}$$

with

$$\begin{aligned} G^{\rm R} &= ({\rm i}\partial_t - \hat{H}e^{+\epsilon\partial_t})^{-1} = ({\rm i}\partial_t - \hat{H} + {\rm i}\epsilon)^{-1}, \\ G^{\rm A} &= ({\rm i}\partial_t - \hat{H}e^{-\epsilon\partial_t})^{-1} = ({\rm i}\partial_t - \hat{H} - {\rm i}\epsilon)^{-1}, \\ G^{<} &= (G^{\rm A} - G^{\rm R})\frac{\rho}{\rho+1}. \end{aligned} \tag{87}$$

$\hat{\mathbf{Q}}^{<}$ is inverse to $\hat{\mathbf{G}}^{<}$:

$$\begin{aligned} Q^{<} &= (Q^{\rm A} - Q^{\rm R})\frac{\rho}{\rho+1} = -2{\rm i}\epsilon\frac{\rho}{\rho+1}, \\ Q^{\rm R} &= {\rm i}\partial_t - \hat{H} + {\rm i}\epsilon, \\ Q^{\rm A} &= {\rm i}\partial_t - \hat{H} - {\rm i}\epsilon. \end{aligned} \tag{88}$$

(see [69,70]).

6.2. *Electric Conductivity and Wigner–Weyl Calculus*

Here, we adopt basic notions of Wigner–Weyl calculus [35,65] to the models defined on honeycomb lattice. The $2+1$-dimensional vectors are denoted by large Latin letters. For any operator, $\hat{A}$, its matrix elements in momentum space are denoted by $A(P_1, P_2) = \langle P_1|\hat{A}|P_2\rangle$. Correspondingly, the space components of momentum belong to the Brillouin zone while its time component (frequency) is real-valued. We then define Weyl symbol of an operator $\hat{A}$ as the mixture of lattice the Weyl symbol and the Wigner transformation, with respect to the frequency component via Equation (22), where $2+1$ D momentum is $P^\mu = (P^0, p)$, and $P_\mu = (P^0, -p)$. The Weyl symbol of Keldysh Green function $\hat{\mathbf{G}}$ is denoted by $\hat{G}$, while the Weyl symbol of the Keldysh operator $\hat{\mathbf{Q}}$ is $\hat{Q}$. The subscript W is omitted below for brevity.

$\hat{G}$ and $\hat{Q}$ obey the Groenewold equation (23). The Moyal product $*$ is written as Equation (24). Electromagnetic potential A corresponds to constant components of electric field with the field strength $\mathcal{F}^{\mu\nu}$. We choose the gauge, in which the spatial part of potential is proportional to time but does not depend on spatial coordinates.

6.3. *Gauge Transformation of Weyl Symbol*

The Weyl symbol of operator $\hat{A}$ may be represented also as

$$\begin{aligned} A_W(X|P) \quad = \quad & 2\int dZ^0 dY^0 \sum_{\vec{Z},\vec{Y}\in\mathcal{O};n_i=0,1} e^{{\rm i}(Z^\mu - Y^\mu)P_\mu} \langle Z|\,\hat{A}\,|Y\rangle \\ & \delta(2X^0 - Z^0 - Y^0)\mathbf{d}(2\vec{X} - \vec{Z} - \vec{Y} - 2l_i n_i) \end{aligned} \tag{89}$$

$U(1)$ gauge transformation acts as $|X\rangle \to e^{i\alpha(X)}|X\rangle$. As a result, the Weyl symbol of an operator $\hat{A}$ is transformed as

$$\begin{aligned} A_W(X|P) \quad \to \quad & 2\int dZ^0 dY^0 \sum_{\vec{Z},\vec{Y}\in\mathcal{O};n_i=0,1} e^{{\rm i}(Z^\mu - Y^\mu)P_\mu + i(\alpha(Y)-\alpha(Z))} \langle Z|\,\hat{A}\,|Y\rangle \\ & \delta(2X^0 - Z^0 - Y^0)\mathbf{d}(2\vec{X} - \vec{Z} - \vec{Y} - 2l_i n_i) \end{aligned}$$

Let us consider those gauge transformations, for which function α almost does not vary at the distances of the order of the correlation length λ characterizing operator $\hat{A}$, i.e., $|\lambda\partial\alpha| \ll 1$. We call these transformations "slow" (with respect to $\hat{A}$). For them, we obtain:

$$\begin{aligned} A_W(X|P) &\to 2\int dZ^0 dY^0 \sum_{\vec{Z},\vec{Y}\in\mathcal{O};n_i=0,1} e^{\mathrm{i}(Z^\mu - Y^\mu)(P_\mu - \partial_\mu\alpha(X))}\langle Z|\,\hat{A}\,|Y\rangle \\ & \quad \delta(2X^0 - Z^0 - Y^0)\mathbf{d}(2\vec{X} - \vec{Z} - \vec{Y} - 2l_i n_i) \\ &= A_W(X|P - \partial_\mu\alpha(X)) \end{aligned} \tag{90}$$

If operator $\hat{A}$ depends on the $U(1)$ gauge field A, then we may require that the gauge transformation of $\hat{A}$ should be compensated by the gauge transformation of field A. This occurs, for example, for Dirac operator $\hat{Q}$ due to gauge invariance of the whole model. Consideration of "slow" gauge transformation results in the requirement that Weyl symbol $A_W(x,p)$ depends on $A(x)$ through the functional dependence on $P - A(x)$, and gauge invariant quantities: field strength F_{ij} and its derivatives, provided that variation of $A(x)$ may be neglected at the distances of the order of λ, i.e., $|\lambda^2 F_{ij}| \ll 1$. As a result, for such $A(x)$ we may represent A_W as a series

$$\begin{aligned} A_W(X|P) &= A_W^{(0)}(X|P - A(x)) \\ &\quad + B_{(ij)W}^{(1)}(X|P - A(x))F_{ij}(X) \\ &\quad + A_{(ijk)W}^{(2)}(X|P - A(x))\partial_k F_{ij}(x) + \dots \end{aligned} \tag{91}$$

Here, dots denote the higher order terms in derivatives. This expansion is reasonable, i.e., the higher order terms are smaller than the lower order terms under the same condition $|\lambda^2 F_{ij}| \ll 1$.

In particular, for bare $\hat{Q}$, the correlation length λ is given by the lattice spacing, and we may use the above expansion for the fields A that vary slowly at the distance of the order of lattice spacing. Response to such fields gives electric conductivity.

Thus, in order to calculate conductivity we expand our expressions in powers of $\mathcal{F}^{\mu\nu}$ up to the linear term. We denote $\pi = P - A$. Here, π^μ is $2+1$-dimensional vector similar to P^μ. The Moyal product may be decomposed as

$$* = \star e^{-\mathrm{i}\mathcal{F}^{\mu\nu}\overleftarrow{\partial}_{\pi^\mu}\overrightarrow{\partial}_{\pi^\nu}/2}. \tag{92}$$

with

$$(A\star B)(X|\pi) = A(X|\pi)\, e^{-\mathrm{i}(\overleftarrow{\partial}_{X^\mu}\overrightarrow{\partial}_{\pi_\mu} - \overleftarrow{\partial}_{\pi_\mu}\overrightarrow{\partial}_{X^\mu})/2} B(X|\pi). \tag{93}$$

If the external field A does not depend on spatial coordinates, then this expression can be used.

Both $\hat{Q}$ and $\hat{G}$ are expanded in powers of $\mathcal{F}^{\mu\nu}$ up to the linear terms

$$\hat{Q} = \hat{Q}^{(0)} + \frac{1}{2}\mathcal{F}^{\mu\nu}\hat{Q}^{(1)}_{\mu\nu}, \quad \hat{G} = \hat{G}^{(0)} + \frac{1}{2}\mathcal{F}^{\mu\nu}\hat{G}^{(1)}_{\mu\nu}. \tag{94}$$

We omit below the superscript $^{(0)}$ for brevity. The Green function and its inverse are expressed through the one-particle Hamiltonian as

$$\begin{aligned} G^{\mathrm{R}} &= (\pi_0 - \hat{H}(\vec{\pi}, x) + \mathrm{i}\epsilon)^{-1}, \\ G^{\mathrm{A}} &= (\pi_0 - \hat{H}(\vec{\pi}, x) - \mathrm{i}\epsilon)^{-1}, \\ G^{<} &= (G^{\mathrm{A}} - G^{\mathrm{R}})f(\pi_0) = 2\pi i\delta(\pi_0 - \hat{H}(\vec{\pi}))f(\pi_0). \end{aligned} \tag{95}$$

$\hat{\mathbf{Q}}^<$ is inverse to $\hat{\mathbf{G}}^<$:

$$Q^< = (Q^{\mathrm{A}} - Q^{\mathrm{R}})f(\pi_0) = -2\mathrm{i}\epsilon f(\pi_0),$$
$$Q^{\mathrm{R}} = \pi_0 - \hat{H}(\vec{\pi}, x) + \mathrm{i}\epsilon, \tag{96}$$
$$Q^{\mathrm{A}} = \pi_0 - \hat{H}(\vec{\pi}, x) - \mathrm{i}\epsilon.$$

Groenewold equation can be written as

$$\left(\hat{Q} + \frac{1}{2}\mathcal{F}^{\mu\nu}\hat{Q}^{(1)}_{\mu\nu}\right) \star e^{-i\mathcal{F}^{\mu\nu}\overleftarrow{\partial}_{\pi^\mu}\overrightarrow{\partial}_{\pi^\nu}/2}\left(\hat{G} + \frac{1}{2}\mathcal{F}^{\mu\nu}\hat{G}^{(1)}_{\mu\nu}\right) = 1_W. \tag{97}$$

In the zeroth order in $\mathcal{F}$ it is reduced to $\hat{Q} \star \hat{G} = 1_W$ (in the following, we will write $\hat{G}$ instead of $\hat{G}^{(0)}$ if this will not lead to contradictions), while the first order gives $\hat{Q} \star \hat{G}^{(1)} + \hat{Q}^{(1)} \star \hat{G} - \mathrm{i}\hat{Q} \star \overleftarrow{\partial}_{\pi^\mu}\overrightarrow{\partial}_{\pi^\nu}\hat{G} = 0$.

At this point, we notice that, according to the properties of the Weyl symbols $Q_W(x,p)$ and $G(x,p)$, all needed information is encoded in their values at $x \in \mathscr{D}$. At these values $1_W(x,p) \equiv 1$. As a result, $\hat{G}^{(0)}(x,p)$ is smooth function of x and p as long as $\hat{Q}^{(0)} \equiv \hat{Q}$ is smooth function of x and p. Therefore, the expansion in powers of derivatives has sense. If we would use instead of the Weyl symbol $\hat{G}_W$ the $\mathscr{B}$ symbol $\hat{G}_{\mathscr{B}}$, then function $\hat{G}^{(0)}_{\mathscr{B}}(x,p)$ will be oscillating fast since $1_{\mathscr{B}}(x,p)$ is fast-oscillating. The derivative expansion of such functions is problematic, and most likely we cannot expand our expressions in powers of F_{ij} (although this question is to be analyzed more carefully).

We obtain

$$\hat{G}^{(1)}_{\mu\nu} = -\hat{G} \star \hat{Q}^{(1)}_{\mu\nu} \star \hat{G}$$
$$-\mathrm{i}\big(\hat{G} \star \partial_{\pi^\mu}\hat{Q} \star \hat{G} \star \partial_{\pi^\nu}\hat{Q} \star \hat{G} - (\mu \leftrightarrow \nu)\big)/2. \tag{98}$$

The above derivation is somehow similar to the derivation presented in [65]. The difference is that here we consider the lattice model. For the homogeneous system Weyl symbol $\hat{Q}$ does not depend on coordinate x. It may depend on P^0 and X^0 and on $p \in \mathcal{M}$. Electric current density is given by

$$\hat{j}^i = -\hat{\bar{\psi}}\frac{\partial \hat{Q}}{\partial p_i}\hat{\psi}, \quad i = 1, 2, \ldots D.$$

Recall that spatial components of momentum are $p^i = p_i = P^i = -P_i$.

$$\langle j^i(t,x)\rangle = -\frac{\mathrm{i}}{2}\mathrm{tr}\left[\hat{\mathbf{G}}\hat{\mathbf{v}}^i\right]. \tag{99}$$

The velocity operator is

$$\hat{\mathbf{v}}^i = \partial_{p_i}\begin{pmatrix} -Q^{--} & 0 \\ 0 & Q^{++} \end{pmatrix}.$$

For the non-uniform systems, the above expression for the electric current density is not valid. Nevertheless, response of the partition function to variation of electromagnetic potential gives expression for the electric current averaged over the whole area:

$$\langle J^i(t)\rangle = -\frac{\mathrm{i}}{2}\frac{1}{|\mathfrak{D}|}\int\frac{dP^0}{2\pi}\int_{\mathscr{M}}\frac{d^2\vec{P}}{(2\pi)^2}\sum_{x\in\mathfrak{D}}\mathrm{tr}\,\mathbf{G}(X|P)$$
$$\partial_i\begin{pmatrix} -Q^{--}(X|P) & 0 \\ 0 & Q^{++}(X|P) \end{pmatrix} \tag{100}$$

Here, we use the Weyl symbols of operators.

$$\hat{\mathbf{v}}^i = \partial_{p_i} \begin{pmatrix} -Q^{--}(P|X) & 0 \\ 0 & Q^{++}(P|X) \end{pmatrix}.$$

is the Weyl symbol of velocity operator.

The average current J may be expressed through the Keldysh Green function in the representation of Equation (82).

$$\begin{aligned} \hat{\mathbf{v}}_i^{(<)} &= \partial_{p_i} \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -Q^{--} & 0 \\ 0 & Q^{++} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \\ &= \partial_{p_i} \begin{pmatrix} -Q^{--} & 0 \\ \frac{-Q^{--}+Q^{++}}{2} & -Q^{++} \end{pmatrix} \end{aligned} \tag{101}$$

Using Equation (85) ($Q^{--} = Q^{\mathrm{R}} + Q^{<}$, $Q^{-+} = -Q^{<}$, $Q^{+-} = -Q^{\mathrm{R}} + Q^{\mathrm{A}} - Q^{<}$, and $Q^{++} = Q^{<} - Q^{\mathrm{A}}$) we represent current density as

$$\begin{aligned} \langle J^i \rangle = -\frac{\mathrm{i}}{2|\mathscr{O}|} \int \frac{dP^0}{2\pi} \mathrm{Tr}\Big[\hat{\mathbf{G}}\hat{\mathbf{v}}^i\Big] \\ = & -\frac{\mathrm{i}}{2|\mathscr{O}|} \int \frac{dP^0}{2\pi} \mathrm{Tr}\Big[\begin{pmatrix} G^{\mathrm{R}} & 2G^{<} \\ 0 & G^{\mathrm{A}} \end{pmatrix} \\ & \partial_{p_i} \begin{pmatrix} -Q^{\mathrm{R}} - Q^{<} & 0 \\ -\frac{Q^{\mathrm{R}}+Q^{\mathrm{A}}}{2} & -Q^{<} + Q^{\mathrm{A}} \end{pmatrix}\Big] \end{aligned} \tag{102}$$

$$\begin{aligned} = & \frac{\mathrm{i}}{2|\mathscr{O}|} \int \frac{dP^0}{2\pi} \mathrm{Tr}\Big(G^{\mathrm{R}} \partial_{p_i} Q^{\mathrm{R}} - G^{\mathrm{A}} \partial_{p_i} Q^{\mathrm{A}}\Big) \\ & + \frac{\mathrm{i}}{2|\mathscr{O}|} \int \frac{dP^0}{2\pi} \mathrm{Tr}\Big(G^{\mathrm{R}} \partial_{p_i} Q^{<} + G^{<} \partial_{p_i} Q^{\mathrm{A}}\Big) \\ & + \frac{\mathrm{i}}{2|\mathscr{O}|} \int \frac{dP^0}{2\pi} \mathrm{Tr}\Big(G^{\mathrm{A}} \partial_{p_i} Q^{<} + G^{<} \partial_{p_i} Q^{\mathrm{R}}\Big) \end{aligned} \tag{103}$$

The second term here is expressed through $\frac{\mathrm{i}}{2}\mathrm{Tr}\big(G\partial_{p_i} Q\big)^{<}$. We obtain

$$\begin{aligned} \langle J^i \rangle &= \frac{\mathrm{i}}{2|\mathscr{O}|} \int \frac{dP^0}{2\pi} \mathrm{Tr}\big(\hat{\mathbf{G}} \partial_{p_i} \hat{\mathbf{Q}}\big)^{\mathrm{R}} \\ & + \frac{\mathrm{i}}{2|\mathscr{O}|} \int \frac{dP^0}{2\pi} \mathrm{Tr}\big(\hat{\mathbf{G}} \partial_{p_i} \hat{\mathbf{Q}}\big)^{<} + \mathrm{c.c.} \end{aligned} \tag{104}$$

Electric current is given by

$$\begin{aligned} \langle J^i(t) \rangle = -\frac{\mathrm{i}}{2} \frac{1}{|\mathfrak{D}|} \int \frac{dP^0}{2\pi} \int_{\mathscr{M}} \frac{d^2\vec{P}}{(2\pi)^2} \sum_{x\in\mathfrak{D}} \mathrm{tr}\big(\hat{G}(\partial_{\pi_i}\hat{Q})\big)^{\mathrm{R}} \\ -\frac{\mathrm{i}}{2} \frac{1}{|\mathfrak{D}|} \int \frac{dP^0}{2\pi} \int_{\mathscr{M}} \frac{d^2\vec{P}}{(2\pi)^2} \sum_{x\in\mathfrak{D}} \mathrm{tr}\big(\hat{G}(\partial_{\pi_i}\hat{Q})\big)^{\mathrm{A}} \\ -\frac{\mathrm{i}}{2} \frac{1}{|\mathfrak{D}|} \int \frac{dP^0}{2\pi} \int_{\mathscr{M}} \frac{d^2\vec{P}}{(2\pi)^2} \sum_{x\in\mathfrak{D}} \mathrm{tr}\big(\hat{G}(\partial_{\pi_i}\hat{Q})\big)^{<} \\ -\frac{\mathrm{i}}{2} \frac{1}{|\mathfrak{D}|} \int \frac{dP^0}{2\pi} \int_{\mathscr{M}} \frac{d^2\vec{P}}{(2\pi)^2} \sum_{x\in\mathfrak{D}} \mathrm{tr}\big((\partial_{\pi_i}\hat{Q})\hat{G}\big)^{<}. \end{aligned}$$

The poles of G^{R} (G^{A}) are shifted out of the real axis of frequency ω. The integration is closed at infinity if we use lattice regularization of time. As a result, the sum of the first two terms vanishes in the above expression:

$$J^i(t) = -\frac{\mathrm{i}}{2}\frac{1}{|\mathfrak{D}|}\int\frac{dP^0}{2\pi}\int_{\mathcal{M}}\frac{d^2\vec{P}}{(2\pi)^2}\sum_{x\in\mathfrak{D}}\mathrm{tr}\big(\hat{G}(\partial_{\pi_i}\hat{Q})\big)^< - \frac{\mathrm{i}}{2}\frac{1}{|\mathfrak{D}|}\int\frac{dP^0}{2\pi}\int_{\mathcal{M}}\frac{d^2\vec{P}}{(2\pi)^2}\sum_{x\in\mathfrak{D}}\mathrm{tr}\big((\partial_{\pi_i}\hat{Q})\hat{G}\big)^<. \quad (105)$$

Using Equations (94)–(98), we calculate term in electric current proportional to the external electric field strength $\mathcal{F}^{\mu\nu}$:

$$J^i = -\frac{1}{4}\frac{1}{|\mathfrak{D}|}\int\frac{dP^0}{2\pi}\int_{\mathcal{M}}\frac{d^2\vec{P}}{(2\pi)^2}\sum_{x\in\mathfrak{D}}\mathrm{tr}\Big(\hat{G}\star\partial_{\pi^\mu}\hat{Q}\star\hat{G}\star\partial_{\pi^\nu}\hat{Q}\star\hat{G}\partial_{\pi_i}\hat{Q}\Big)^<\mathcal{F}^{\mu\nu} - \frac{1}{4}\frac{1}{|\mathfrak{D}|}\int\frac{dP^0}{2\pi}\int_{\mathcal{M}}\frac{d^2\vec{P}}{(2\pi)^2}\sum_{x\in\mathfrak{D}}\mathrm{tr}\Big(\partial_{\pi_i}\hat{Q}\hat{G}\star\partial_{\pi^\mu}\hat{Q}\star\hat{G}\star\partial_{\pi^\nu}\hat{Q}\star\hat{G}\Big)^<\mathcal{F}^{\mu\nu}. \quad (106)$$

Then

$$J^i = \sigma^{ij}\mathcal{F}_{0j},$$

where the conductivity tensor σ^{ij} may be given by Equation (25), where we restore index W for the Weyl symbols. The anti-symmetrization is denoted by $(\dots)_{[0}(\dots)_{j]} = (\dots)_0(\dots)_j - (\dots)_j(\dots)_0$. The asymmetric (Hall) part of conductivity is $\sigma_H^{ij} = (\sigma^{ij} - \sigma^{ji})/2$.

6.4. Equilibrium Limit of Hall Conductivity

According to the second trace identity, the star may be inserted between the two Weyl symbols standing under the trace. We obtain an expression for the average conductivity

$$\bar{\sigma}^{ij} = -\frac{1}{4}\frac{1}{|\mathfrak{D}|}\int\frac{dP^0}{2\pi}\int_{\mathcal{M}}\frac{d^2\vec{P}}{(2\pi)^2}\sum_{x\in\mathfrak{D}}\mathrm{tr}\Big(\partial_{\pi_i}\hat{Q}_W\star \star\hat{G}_W\star\partial_{\pi_{[0}}\hat{Q}_W\star\hat{G}_W\star\partial_{\pi_{j]}}\hat{Q}_W\star\hat{G}_W\Big)^< + \mathrm{c.c.} \quad (107)$$

We assume that $\hat{Q}$ does not depend on time. For the thermal distribution, we represent the integral as a sum over the Matsubara frequencies. By Π, we denote Euclidean 2 + 1-momentum, i.e., $\Pi^3 = \omega$ is Matsubara frequency, $\Pi^i = \pi^i$ for $i = 1,2$. $\partial_{\pi^0} = -\mathrm{i}\partial_{\Pi^3}$. We substitute $i\omega$ instead of π^0 and obtain the Matsubara Green function G^M.

The system with the one-particle Hamiltonian $\hat{H}$ gives the real-time Green function

$$G(x_1, x_2, \omega) \equiv \langle x_1|(\omega - \hat{H})^{-1}|x_2\rangle$$

It produces advanced, retarded, or time-ordered Green functions when the integration contour in the plane of the complex ω is shifted in a specific way. The Feynman propagator is

$$G^{\mathrm{T}}(x, x', \omega) = \lim_{\eta\to 0} G(x, x', \omega + \mathrm{i}\eta\,\mathrm{sign}\,\omega). \quad (108)$$

The retarded Green function is

$$G^{\mathrm{R}}(x, x', \omega) = \lim_{\eta\to 0} G(x, x', \omega + \mathrm{i}\eta), \quad (109)$$

The advanced Green function is

$$G^{\mathrm{A}}(x, x', \omega) = \lim_{\eta\to 0} G(x, x', \omega - \mathrm{i}\eta). \quad (110)$$

The Matsubara Green function G^{M} is obtained as

$$G^{\mathrm{M}}(x, x', \omega_n) = G(x, x', \mathrm{i}\omega_n), \quad (111)$$

The latter may be written in terms of imaginary time τ:

$$G^{\mathrm{M}}(x,x',\tau) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty} e^{-\mathrm{i}\omega_n\tau} G(x,x',\mathrm{i}\omega_n). \tag{112}$$

$\omega_n = (2n+1)\pi/\beta$ is the Matsubara frequency, $\beta = 1/T$.

The conductivity averaged over the system area is (in the units of $e^2/\hbar$)

$$\bar{\sigma}^{ij} = \frac{\mathcal{N}}{2\pi}\epsilon^{ij},$$

where

$$\begin{aligned}\mathcal{N} &= 2\pi T\,\frac{1}{3!}\epsilon^{\mu\nu\rho}\frac{1}{|\mathfrak{D}|}\int_{\mathcal{M}}\frac{d^2\vec{P}}{(2\pi)^2}\sum_{x\in\mathfrak{D}}\sum_{\omega_n=2\pi T(n+1/2)} \\ &\quad \mathrm{tr}\Big(\partial_{\Pi^\mu}\hat{Q}^{\mathrm{M}}_W\star\hat{G}^{\mathrm{M}}_W\star\partial_{\Pi^\nu}\hat{Q}^{\mathrm{M}}_W\star\hat{G}^{\mathrm{M}}_W\star\partial_{\Pi^\rho}\hat{Q}^{\mathrm{M}}_W\star\hat{G}^{\mathrm{M}}_W\Big).\end{aligned} \tag{113}$$

Here, ϵ^{ij} and $\epsilon^{\mu\nu\rho}$ are anti-symmetric tensors. Q^M_W is the inverse to the Matsubara Green function G^M_W:

$$Q^M_W \star G^M_W = 1_W = 1$$

The sum over Matsubara frequencies is reduced to an integral for small temperatures (see Equation (27)), where the sum over x is important for the topological invariance of this quantity.

7. Conclusions

In the present paper, we extend the previously proposed construction of precise lattice Wigner–Weyl calculus [43] to the models defined on the honeycomb lattices. (Recall that, in [43], the rectangular lattices were considered.) Models with artificial honeycomb lattices realize the Hofstadter butterfly and the quantum Hall effect, with effectively large magnetic flux occurring through the lattice cell [64]. For such systems, the approximate lattice Wigner–Weyl calculus of [44] cannot be used because magnetic flux through the lattice cell appears to be of the order of the quantum of magnetic flux. Then, the present construction is inevitable if we are going to represent the QHE conductivity through the topological invariant composed of the Green functions. We derive the corresponding expression. It is manifestly topological invariant, which demonstrates that the QHE conductivity is robust to the smooth modifications of the system.

We started our consideration from the construction of the $\mathcal{B}$ symbol that realized the original ideas of F. Buot (F. Buot's original construction contains a few technical flaws. However, the very notion of such a construction that he offered seems to us to be so significant that we feel it is appropriate to give their name to our corrected construction). The $\mathcal{B}$ (or Buot) symbol obeys the basic properties of the continuous Wigner–Weyl calculus. However, we observe that the $\mathcal{B}$ symbol of the unity operator is a fast-oscillating function of coordinates. As a result, the Buot symbol of the Green function does not depend smoothly on coordinates, and the derivative expansion cannot be used in the expression for the electric current. In order to improve the situation, we propose the more involved construction with the new symbol of operator, which is here called the W symbol or the Weyl symbol. It obeys precisely the same properties as the Buot symbol, but, contrary to the latter, the Weyl symbol of the unity operator depends smoothly on coordinates. This allows us to apply derivative expansion to the corresponding expression for electric current. This expansion leads us finally to the expression for the Hall conductivity $\sigma_H = \frac{e^2}{h}\mathcal{N}$, where $\mathcal{N}$ is given by Equation (27).

Thus, we have similar constructions of lattice Wigner–Weyl calculus for the rectangular and honeycomb lattices. It would be instructive to extend these constructions to the lattices of arbitrary form. Additionally, it would be important to consider the effects of the interactions. We expect that the latter, being taken into account perturbatively, cannot change the form of Equation (27), in which the interacting Green function should replace the bare one. At a first look, the consideration of [44] (given for the case of approximate Wigner–Weyl calculus) might be extended to the case of the precise Wigner–Weyl calculus and the above-given expression for the QHE conductivity. This is, however, remains to be completed and is outside the scope of the present paper.

The important challenge is to understand the topological nature of fractional QHE. We suppose that the latter is a completely non-perturbative phenomenon. It is not clear at the present moment how the topological expression of Equations (7) and (27) is replaced by e^2/h times the fractional number. We do not seek to present a solution to the problem that occurs when taking into account interactions non-perturbatively, nor do we seek to explain the topological quantization of fractional QHE. A discussion of the relation of topological invariants to fractional QHE is given in [71], where the expression through the multi-particle Green functions is given. The consideration of these issues, however, remains outside the scope of the present paper.

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References

1. Groenewold, H.J. On the Principles of elementary quantum mechanics. *Physica* **1946**, *12*, 405–460. [CrossRef]
2. Moyal, J.E. Quantum mechanics as a statistical theory. *Math. Proc. Camb. Philos. Soc.* **1949**, *45*, 99–124. [CrossRef]
3. Weyl, H. Quantenmechanik und Gruppentheorie. *Z. Phys.* **1927**, *46*, 1–46. [CrossRef]
4. Wigner, E.P. On the quantum correction for thermodynamic equilibrium. *Phys. Rev* **1932**, *40*, 749–759. [CrossRef]
5. Ali, S.T.; Englis, M. Quantization Methods: A Guide for Physicists and Analysts. *Rev. Math. Phys.* **2005**, *17*, 391–490. [CrossRef]
6. Berezin, F.A.; Shubin, M.A. *Colloquia Mathematica Societatis Janos Bolyai*; North-Holland: Amsterdam, The Netherland, 1972; p. 21.
7. Curtright, T.L.; Zachos, C.K. Quantum Mechanics in Phase Space. *Asia Pac. Phys. Newsl.* **2012**, *1*, 37. [CrossRef]
8. Zachos, C.; Fairlie, D.; Curtright, T. *Quantum Mechanics in Phase Space*; World Scientific: Singapore, 2005.
9. Cohen, L. Generalized Phase-Space Distribution Functions. *J. Math. Phys.* **1966**, *7*, 781. [CrossRef]
10. Agarwal, G.S.; Wolf, E. Calculus for Functions of Noncommuting Operators and General Phase-Space Methods in Quantum Mechanics. I. Mapping Theorems and Ordering of Functions of Noncommuting Operators. *Phys. Rev. D* **1970**, *2*, 2161–2186. [CrossRef]
11. Sudarshan, E.C.G. Sudarshan Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams. *Phys. Rev. Lett.* **1963**, *10*, 277–279. [CrossRef]
12. Glauber, R.J. Coherent and Incoherent States of the Radiation Field. *Phys. Rev.* **1963**, *131*, 2766–2788. [CrossRef]
13. Husimi, K. Some Formal Properties of the Density Matrix. *Proc. Phys. Math. Soc. Jpn.* **1940**, *22*, 264–314.
14. Cahill, K.E.; Glauber, R.J. Glauber Ordered Expansions in Boson Amplitude Operators. *Phys. Rev.* **1969**, *177*, 1882–1902. [CrossRef]
15. Buot, F.A. *Nonequilibrium Quantum Transport Physics in Nanosystems*; World Scientific: Singapore, 2009.
16. Schwinger, J. Unitary operator bases. *Proc. Natl. Acad. Sci. USA* **1960**, *46*, 570–579. [CrossRef]
17. Buot, F.A. Method for calculating TrH^n in solid-state theory. *Phys. Rev. B* **1974**, *10*, 3700. [CrossRef]
18. Buot, F.A. Quantum Superfield Theory and Lattice Weyl Transform in Nonequilibrium Quantum Transport Physics. *Quantum Matter* **2013**, *2*, 247–288. [CrossRef]
19. Wootters, W.K. A Wigner-function formulation of finite-state quantum mechanics. *Ann. Phys.* **1987**, *176*, 1–21. [CrossRef]
20. Leonhardt, U. Quantum-State Tomography and Discrete Wigner Function. *Phys. Rev. Lett.* **1995**, *74*, 4101–4105. [CrossRef]
21. Kasperkovitz, P.; Peev, M. Wigner–Weyl Formalisms for Toroidal Geometries. *Ann. Phys.* **1994**, *230*, 21–51. [CrossRef]
22. Ligabò, M. Torus as phase space: Weyl quantization, dequantization, and Wigner formalism. *J. Math. Phys.* **2016**, *57*, 082110. [CrossRef]
23. Björk, G.; Klimov, A.B.; Sánchez-Soto, L.L. Chapter 7 The discrete Wigner function. *Prog. Opt.* **2008**, *51*, 469–516. [CrossRef]
24. Galetti, D.; de Toledo Piza, A. An extended Weyl-Wigner transformation for special finite spaces. *Phys. A Stat. Mech. Its Appl.* **1988**, *149*, 267–282. [CrossRef]

25. Cohendet, O.; Combe, P.; Sirugue, M.; Sirugue-Collin, M. A stochastic treatment of the dynamics of an integer spin. *J. Phys. Math. Gen.* **1988**, *21*, 2875. [CrossRef]
26. Opatrný, T.; Miranowicz, A.; Bajer, J. Coherent states in finite-dimensional Hilbert space and their Wigner representation. *J. Mod. Opt.* **1996**, *43*, 417–432. [CrossRef]
27. Opatrný, T.; Welsch, D.G.; Bužek, V. Parametrized discrete phase-space functions. *Phys. Rev. A* **1996**, *53*, 3822–3835. [CrossRef] [PubMed]
28. Rivas, A.; De Almeida, A.O. The Weyl representation on the torus. *Ann. Phys.* **1999**, *276*, 223–256. [CrossRef]
29. Mukunda, N.; Arvind, A.; Chaturvedi, S.; Simon, R. Wigner distributions and quantum mechanics on Lie groups: The case of the regular representation. *J. Math. Phys.* **2004**, *45*, 114–148. [CrossRef]
30. Chaturvedi, S.; Ercolessi, E.; Marmo, G.; Morandi, G.; Mukunda, N.; Simon, R. Wigner distributions for finite dimensional quantum systems: An algebraic approach. *Pramana* **2005**, *65*, 981–993. [CrossRef]
31. Bayen, F.; Flato, M.; Fronsdal, C.; Lichnerowicz, A.; Sternheimer, D. Deformation theory and quantization. I. Deformations of symplectic structures. *Ann. Phys.* **1978**, *111*, 61–110. [CrossRef]
32. Kontsevich, M. Deformation Quantization of Poisson Manifolds. *Lett. Math. Phys.* **2003**, *66*, 157–216. [CrossRef]
33. Felder, G.; Shoikhet, B. Deformation Quantization with Traces. *Lett. Math. Phys.* **2000**, *53*, 75–86. [CrossRef]
34. Kupriyanov, V.G.; Vassilevich, D.V. Star products made (somewhat) easier. *Eur. Phys. J. C* **2008**, *58*, 627–637. [CrossRef]
35. Zubkov, M.A.; Wu, X. Topological invariant in terms of the Green functions for the Quantum Hall Effect in the presence of varying magnetic field. *arXiv* **2019**, arXiv:1901.06661.
36. Zubkov, M.A.; Khaidukov, Z.V. Topology of the momentum space, Wigner transformations, and a chiral anomaly in lattice models. *JETP Lett.* **2017**, *106*, 166. [CrossRef]
37. Chernodub, M.N.; Zubkov, M.A. Scale magnetic effect in Quantum Electrodynamics and the Wigner–Weyl formalism. *Phys. Rev. D* **2017**, *96*, 056006. [CrossRef]
38. Khaidukov, Z.V.; Zubkov, M.A. Chiral Separation Effect in lattice regularization. *Phys. Rev. D* **2017**, *95*, 074502. [CrossRef]
39. Zubkov, M.A. Momentum space topology of QCD. *Ann. Phys.* **2018**, *393*, 264. [CrossRef]
40. Zubkov, M.A. Absence of equilibrium chiral magnetic effect. *Phys. Rev. D* **2016**, *93*, 105036. [CrossRef]
41. Zubkov, M.A. Wigner transformation, momentum space topology, and anomalous transport. *Ann. Phys.* **2016**, *373*, 298. [CrossRef]
42. Chernodub, M.N. Anomalous Transport Due to the Conformal Anomaly. *Phys. Rev. Lett.* **2016**, *117*, 141601. [CrossRef]
43. Fialkovsky, I.; Zubkov, M. Precise Wigner–Weyl calculus for lattice models. *Nucl. Phys. B* **2020**, *954*, 114999. [CrossRef]
44. Zhang, C.; Zubkov, M. Influence of interactions on integer quantum Hall effect. *Ann. Phys.* **2022**, *444*, 169016. [CrossRef]
45. Banerjee, C.; Fialkovsky, I.V.; Lewkowicz, M.; Zhang, C.X.; Zubkov, M.A. Wigner–Weyl calculus in Keldysh technique. *J. Comput. Electron.* **2021**, *20*, 2255–2283. [CrossRef]
46. Thouless, D.J.; Kohmoto, M.; Nightingale, M.P.; den Nijs, M. Quantized Hall Conductance in a Two-Dimensional Periodic Potential. *Phys. Rev. Lett.* **1982**, *49*, 405–408. [CrossRef]
47. Avron, J.E.; Seiler, R.; Simon, B. Homotopy and Quantization in Condensed Matter Physics. *Phys. Rev. Lett.* **1983**, *51*, 51–53. [CrossRef]
48. Fradkin, E. *Field Theories of Condensed Matter Physics*; Addison Wesley Publishing Company: Boston, MA, USA, 1991.
49. Hatsugai, Y. Topological aspects of the quantum Hall effect. *J. Phys. Condens. Matter* **1997**, *9*, 2507–2549. [CrossRef]
50. Qi, X.L.; Hughes, T.L.; Zhang, S.C. Topological field theory of time-reversal invariant insulators. *Phys. Rev. B* **2008**, *78*, 195424. [CrossRef]
51. Kaufmann, R.M.; Li, D.; Wehefritz-Kaufmann, B. Notes on topological insulators. *Rev. Math. Phys.* **2016**, *28*, 1630003, [CrossRef]
52. Tong, D. Lectures on the Quantum Hall Effect. *arXiv* **2016**, arXiv:1606.06687.
53. Ishikawa, K.; Matsuyama, T. Magnetic field induced multi component QED in three-dimensions and quantum Hall effect. *Z. Phys. C* **1986**, *33*, 41–45. [CrossRef]
54. Volovik, G.E. An analog of the quantum Hall effect in a superfluid 3He film. *JETP* **1988**, *67*, 9.
55. Volovik, G.E. *The Universe in a Helium Droplet*; Clarendon Press: Oxford, UK, 2003.
56. Coleman, S.; Hill, B. No more corrections to the topological mass term in QED_3. *Phys. Lett. B* **1985**, *159*, 184. [CrossRef]
57. Lee, T. The absence of radiative corrections from higher-order loops to the topological mass in (2+1)-dimensional electrodynamics. *Phys. Lett. B* **1986**, *171*, 247. [CrossRef]
58. Kubo, R.; Hasegawa, H.; Hashitsume, N. Quantum Theory of Galvanomagnetic Effect. I. Basic Considerations. *J. Phys. Soc. Jpn.* **1959**, *14*, 56–74. [CrossRef]
59. Niu, Q.; Thouless, D.J.; Wu, Y.S. Quantized Hall conductance as a topological invariant. *Phys. Rev. B* **1985**, *31*, 3372. [CrossRef]
60. Altshuler, B.L.; Khmel'nitzkii, D.; Larkin, A.I.; Lee, P.A. Magnetoresistance and Hall effect in a disordered two-dimensional electron gas. *Phys. Rev. B* **1980**, *22*, 5142. [CrossRef]
61. Altshuler, B.L.; Aronov, A.G. *Electron-Electron Inter-Action in Disordered Systems*; Efros, A.L., Pollak, M., Eds.; Elsevier/North Holland: Amsterdam, The Netherlands, 1985.
62. Zhang, C.X.; Zubkov, M.A. Hall conductivity as the topological invariant in phase space in the presence of interactions and non-uniform magnetic field. *JETP Lett.* **2019**, 110, 487–494. [CrossRef]
63. Dean, C.R.; Wang, L.; Maher, P.; Forsythe, C.; Ghahari, F.; Gao, Y.; Katoch, J.; Ishigami, M.; Moon, P.; Koshino, M.; et al. Hofstadter's butterfly and the fractal quantum Hall effect in moiré superlattices. *Nature* **2013**, *497*, 598–602. [CrossRef]

64. Polini, M.; Guinea, F.; Lewenstein, M.; Manoharan, H.C.; Pellegrini, V. Artificial honeycomb lattices for electrons, atoms and photons. *Nat. Nanotechnol.* **2013**, *8*, 625–633. [CrossRef] [PubMed]
65. Onoda, S.; Sugimoto, N.; Nagaosa, N. Theory of non-equilibirum states driven by constant electromagnetic fields: Non-commutative quantum mechanics in the Keldysh formalism. *Theor. Phys.* **2006**, *116*, 61. [CrossRef]
66. Onoda, S.; Sugimoto, N.; Nagaosa, N. Intrinsic Versus Extrinsic Anomalous Hall Effect in Ferromagnets. *Phys. Rev. Lett.* **2006**, *97*, 2. [CrossRef]
67. Sugimoto, N.; Onoda, S.; Nagaosa, N. Gauge covariant formulation of Wigner representation through deformation quantization—Application to Keldysh formalism with electromagnetic field. *Prog. Theor. Phys.* **2007**, *117*, 415. [CrossRef]
68. Onoda, S.; Sugimoto, N.; Nagaosa, N. Quantum transport theory of anomalous electric, thermoelectric, and thermal Hall effects in ferromagnets. *Phys. Rev. B* **2008**, *77*, 3. [CrossRef]
69. Kamenev, A. Many-body theory of non-equilibrium systems. *arXiv* **2004**, arXiv:cond-mat/0412296.
70. Kamenev, A. *Field Theory of Non-Equilibrium Systems*; Cambridge University Press: Cambridge, UK, 2011.
71. Miller, J.; Zubkov, M. Topological Quantization of Fractional Quantum Hall Conductivity. *Symmetry* **2022**, *14*, 2095. [CrossRef]