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# New Fixed Point Theorems for Generalized Meir–Keeler Type Nonlinear Mappings with Applications to Fixed Point Theory

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**Abstract:** In this paper, we investigate new fixed point theorems for generalized Meir–Keeler type nonlinear mappings satisfying the condition **(DH)**. As applications, we obtain many new fixed point theorems which generalize and improve several results available in the corresponding literature. An example is provided to illustrate and support our main results.

**Keywords:** Meir–Keeler type mapping; simultaneous generalization; Banach contraction principle; Kannan's fixed point theorem; Chatterjea's fixed point theorem; Meir–Keeler's fixed point theorem

**MSC:** 47H10; 54H25

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## 1. Introduction and Preliminaries

Since the pioneering establishment of the famous Banach contraction principle [1] and Brouwer fixed point theorem [2–5], fixed point theory and its applications have developed rapidly in the past one hundred years, and have been applied to study its uses in nonlinear analysis, economics, game theory, integral differential equations, optimization theory, dynamic system theory, signal and image processing and other related fields of applied mathematics. For more details, we refer the reader to the research monographs and papers [2–20] and the references quoted therein.

Let $A$ be a selfmapping from a metric space $(X, d)$ into itself. A point $z \in X$ is called a *fixed point* of $A$ if $Az = z$. Let us recall the concepts of Meir–Keeler contraction and $L$-function.

**Definition 1** (see [20,21]). *A selfmapping $A$ on $X$ is said to be a Meir–Keeler contraction if the condition **(MK)** holds, where*

***(MK)*** *for each $\beta > 0$, there exists $\varkappa = \varkappa(\beta) > 0$ such that for $x, y \in X$,*

$$d(x,y) \in [\beta, \beta + \varkappa) \quad \text{implies} \quad d(Ax, Ay) < \beta.$$

**Definition 2** (see [15,20]). *A function $\tau : [0, \infty) \to [0, \infty)$ is called an $L$-function if $\tau(0) = 0$, $\tau(t) > 0$ for $t > 0$, and for every $x > 0$, there exists $c > 0$ such that $\tau(t) \leq x$ for all $t \in [x, x + c]$.*

In [15], Lim used $L$-functions to characterize Meir–Keeler contractions.

**Theorem 1** (see [15]). *$A$ is a Meir–Keeler contraction if and only if there exists an (nondecreasing, right continuous) $L$-function $\tau$, such that*

$$d(Tx, Ty) < \tau(d(x,y)) \quad \text{for all } x, y \in X \text{ with } x \neq y.$$

In 1969, Meir and Keeler established an interesting fixed point theorem (the so-called Meir–Keeler's fixed point theorem) as follows:

**Theorem 2** (see [21]). *Let $(X, d)$ be a complete metric space and $A : X \to X$ be a Meir–Keeler contraction. Then $A$ admits a unique fixed point in $X$.*

It is worth noting that Meir–Keeler's fixed point theorem is a real generalization of the Banach contraction principle (see, e.g., [15,16,18,21,22]). Several authors have studied various types of generalized Meir–Keeler contractions to establish new Meir–Keeler-type fixed point theorems. For a more comprehensive understanding of the advances in the Meir–Keeler's fixed point theorem, interested readers are encouraged to consult the remarkable monographs and papers [2,3,5,6,8–10,12,13,15,17,21].

The main purpose of this work is to establish new fixed point theorems for generalized Meir–Keeler type nonlinear mappings and their applications to fixed point theory. The paper is divided into four sections. In Section 2, we first establish a fixed point theorem for generalized Meir–Keeler type nonlinear mappings, satisfying the condition **(DH)** (see Theorem 3 below). As applications to fixed point theory, we obtain many new fixed point theorems in Section 3. An example (see Example 1) is given to illustrate that our new fixed point theorem (see Theorem 8) is a real simultaneous generalization of Banach contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem. The paper concludes by summarizing the results achieved and outlining future research directions in Section 4.

## 2. New Fixed Point Theorem for Generalized Meir–Keeler Type Mappings

The following theorem is one of the main results of this paper.

**Theorem 3.** *Let $(X, d)$ be a metric space and $A : X \to X$ be a selfmapping. Define a mapping $U : X \times X \to [0, \infty)$ by*

$$U(x, y) = \max_{1 \le i \le 10} f_i(x, y)$$

*where*

$$f_1(x, y) = d(x, y),$$
$$f_2(x, y) = \frac{d(x, Ax) + d(y, Ay)}{2},$$
$$f_3(x, y) = \frac{d(x, Ay) + d(y, Ax)}{2},$$
$$f_4(x, y) = \frac{2d(x, Ax) + d(y, Ax)}{3},$$
$$f_5(x, y) = \frac{2d(x, Ax) + d(y, Ay)}{3},$$
$$f_6(x, y) = \frac{d(x, y) + 2d(y, Ay)}{3},$$
$$f_7(x, y) = \frac{2d(x, Ax) + d(y, Ax) + d(y, Ay)}{4},$$
$$f_8(x, y) = \frac{2d(x, y) + d(x, Ax) + d(y, Ax)}{4},$$
$$f_9(x, y) = \frac{d(x, Ax) + d(x, Ay) + 2d(y, Ax)}{4},$$

*and*

$$f_{10}(x, y) = \frac{2d(x, y) + d(x, Ax) + d(y, Ax) + d(y, Ay)}{5},$$

*for $x, y \in X$. Suppose that*

***(DH)*** *for each $\beta > 0$, there exists $\varkappa = \varkappa(\beta) > 0$ such that for $x, y \in X$,*

$$U(x, y) \in [\beta, \beta + \varkappa) \quad \text{implies} \quad d(Ax, Ay) < \beta.$$

*Given $z \in X$. Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence defined by $c_1 = z$ , $c_{n+1} = Ac_n$ for all $n \in \mathbb{N}$. If $c_{n+1} \neq c_n$ for all $n \in \mathbb{N}$, then $\{c_n\}_{n\in\mathbb{N}}$ is Cauchy in $X$.*

**Proof.** Since $c_{n+1} \neq c_n$ for all $n \in \mathbb{N}$, we obtain

$$U(c_n, c_{n+1}) \geq f_1(c_n, c_{n+1}) = d(c_n, c_{n+1}) > 0, \quad \forall\, n \in \mathbb{N}.$$

For any $n \in \mathbb{N}$, the following hold:

- $f_1(c_n, c_{n+1}) = d(c_n, c_{n+1})$,
- $f_2(c_n, c_{n+1}) = \frac{d(c_n,c_{n+1})+d(c_{n+1},c_{n+2})}{2}$,
- $f_3(c_n, c_{n+1}) = \frac{d(c_n,c_{n+2})}{2} \leq \frac{d(c_n,c_{n+1})+d(c_{n+1},c_{n+2})}{2}$,
- $f_4(c_n, c_{n+1}) = \frac{2d(c_n,c_{n+1})}{3}$,
- $f_5(c_n, c_{n+1}) = \frac{2d(c_n,c_{n+1})+d(c_{n+1},c_{n+2})}{3}$,
- $f_6(c_n, c_{n+1}) = \frac{d(c_n,c_{n+1})+2d(c_{n+1},c_{n+2})}{3}$,
- $f_7(c_n, c_{n+1}) = \frac{2d(c_n,c_{n+1})+d(c_{n+1},c_{n+2})}{4}$,
- $f_8(c_n, c_{n+1}) = \frac{3d(c_n,c_{n+1})}{4}$,
- $f_9(c_n, c_{n+1}) = \frac{d(c_n,c_{n+1})+d(c_n,c_{n+2})}{4} \leq \frac{2d(c_n,c_{n+1})+d(c_{n+1},c_{n+2})}{4}$,
- $f_{10}(c_n, c_{n+1}) = \frac{3d(c_n,c_{n+1})+d(c_{n+1},c_{n+2})}{5}$.

Suppose that there exists $k \in \mathbb{N}$, such that $d(c_k, c_{k+1}) \leq d(c_{k+1}, c_{k+2})$. Hence, through the above, we obtain

$$U(c_k, c_{k+1}) = \max_{1\leq i\leq 10} f_i(c_k, c_{k+1}) \leq d(c_{k+1}, c_{k+2}).$$

For $\gamma := U(c_k, c_{k+1}) > 0$, through condition **(DH)**, we acquire

$$d(c_{k+1}, c_{k+2}) = d(Ac_k, Ac_{k+1}) < \gamma = U(c_k, c_{k+1}) \leq d(c_{k+1}, c_{k+2}),$$

which leads to a contradiction. Therefore, it must be $d(c_{n+1}, c_{n+2}) < d(c_n, c_{n+1})$ for all $n \in \mathbb{N}$. Consequently, we arrive at

$$U(c_n, c_{n+1}) = \max_{1\leq i\leq 10} f_i(c_n, c_{n+1}) = d(c_n, c_{n+1}) \qquad \text{for all } n \in \mathbb{N}. \tag{1}$$

Since $\{d(c_{n+1}, c_n)\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence in $[0, \infty)$, we deduce that

$$\ell := \lim_{n\to\infty} d(c_{n+1}, c_n) = \inf_{n\in\mathbb{N}} d(c_{n+1}, c_n) \qquad \text{exists.} \tag{2}$$

We now need to prove $\ell = 0$. Suppose on the contrary that $\ell > 0$. For $\delta > 0$, by using (1) and (2), we have

$$\ell \leq U(c_{p+1}, x_p) = d(c_{p+1}, c_p) < \ell + \delta \qquad \text{for some } p \in \mathbb{N}.$$

Hence, the condition **(DH)** yields

$$d(c_{p+2}, c_{p+1}) < \ell = \inf_{n\in\mathbb{N}} d(c_{n+1}, c_n) \leq d(c_{p+2}, c_{p+1}),$$

a contradiction. So we conclude that

$$\ell = \inf_{n\in\mathbb{N}} d(c_{n+1}, c_n) = \lim_{n\to\infty} d(c_{n+1}, c_n) = 0. \tag{3}$$

We shall demonstrate that $\{c_n\}_{n\in\mathbb{N}}$ is Cauchy in $X$. Given $\beta > 0$, take $\zeta > 0$ satisfying $\beta > 3\zeta$. Using **(DH)**, there exists $0 < \varkappa(\zeta)$, satisfying the following implication:

$$U(x,y) \in [\zeta, \zeta + \varkappa(\zeta)) \quad \Longrightarrow \quad d(Ax, Ay) < \zeta. \tag{4}$$

Choose $\varkappa' = \min\{1, \zeta, \varkappa(\zeta)\}$. Clearly, (4) also holds if $\varkappa(\zeta)$ is replaced with $\varkappa'$. From (3), there exists $j_0 \in \mathbb{N}$ such that

$$d(c_{n+1}, c_n) < \frac{\varkappa'}{6}, \ \forall\, n \geq j_0. \tag{5}$$

Let

$$\mathcal{W} = \left\{a \in \mathbb{N} : a \geq j_0 \text{ and } d(c_a, c_{j_0}) < \zeta + \frac{\varkappa'}{2}\right\}.$$

Clearly, $j_0 \in \mathcal{W}$. So $\mathcal{W} \neq \emptyset$. We want to prove that $b \in \mathcal{W}$ implies $b+1 \in \mathcal{W}$. Let $b \in \mathcal{W}$ be given. Thus $b \geq j_0$ and

$$d(c_b, c_{j_0}) < \zeta + \frac{\varkappa'}{2}.$$

If $b = j_0$, then, using (5), we obtain $b+1 \in \mathcal{W}$. If $b > j_0$, we need to use the following two possible cases to verify $b+1 \in \mathcal{W}$:

**Case (i).** Assume that $\zeta \leq d(c_b, c_{j_0}) < \zeta + \frac{\delta'}{2}$. Since

$$f_1(c_b, c_{j_0}) = d(c_b, c_{j_0}) < \zeta + \frac{\varkappa'}{2} < \zeta + \varkappa',$$

$$\begin{aligned} f_2(c_b, c_{j_0}) &= \frac{1}{2}\big(d(c_b, c_{b+1}) + d(c_{j_0}, c_{j_0+1})\big) \\ &< \frac{1}{2}\left(\frac{\varkappa'}{6} + \frac{\varkappa'}{6}\right) \\ &< \zeta + \varkappa', \end{aligned}$$

$$\begin{aligned} f_3(c_b, c_{j_0}) &= \frac{1}{2}\big(d(c_b, c_{j_0+1}) + d(c_{j_0}, c_{b+1})\big) \\ &\leq \frac{1}{2}\big(2d(c_b, c_{j_0}) + d(c_{j_0}, c_{j_0+1}) + d(c_b, c_{b+1})\big) \\ &< \frac{1}{2}\left(2\left(\zeta + \frac{\varkappa'}{2}\right) + \frac{\varkappa'}{6} + \frac{\varkappa'}{6}\right) \\ &< \zeta + \varkappa', \end{aligned}$$

$$\begin{aligned} f_4(c_b, c_{j_0}) &= \frac{1}{3}\big(2d(c_b, c_{b+1}) + d(c_{j_0}, c_{b+1})\big) \\ &\leq \frac{1}{3}\big(3d(c_b, c_{b+1}) + d(c_{j_0}, c_b)\big) \\ &< \frac{1}{3}\left(\frac{1}{2}\varkappa' + \zeta + \frac{\varkappa'}{2}\right) \\ &< \zeta + \varkappa', \end{aligned}$$

$$
\begin{aligned}
f_5(c_b, c_{j_0}) &= \frac{1}{3}\big(2d(c_b, c_{b+1}) + d(c_{j_0}, c_{j_0+1})\big) \\
&< \frac{1}{3}(\frac{1}{3}\varkappa' + \frac{\varkappa'}{6}) \\
&< \zeta + \varkappa',
\end{aligned}
$$

$$
\begin{aligned}
f_6(c_b, c_{j_0}) &= \frac{1}{3}\big(d(c_b, c_{j_0}) + 2d(c_{j_0}, c_{j_0+1})\big) \\
&< \frac{1}{3}\left(\zeta + \frac{\varkappa'}{2} + \frac{\varkappa'}{3}\right) \\
&= \frac{1}{3}\zeta + \frac{5}{18}\varkappa' \\
&< \zeta + \varkappa',
\end{aligned}
$$

$$
\begin{aligned}
f_7(c_b, c_{j_0}) &= \frac{1}{4}\big(2d(c_b, c_{b+1}) + d(c_{j_0}, c_{b+1}) + d(c_{j_0}, c_{j_0+1})\big) \\
&\leq \frac{1}{4}\big(3d(c_b, c_{b+1}) + d(c_{j_0}, c_b) + d(c_{j_0}, c_{j_0+1})\big) \\
&< \frac{1}{4}\left(\frac{1}{2}\varkappa' + \zeta + \frac{\varkappa'}{2} + \frac{\varkappa'}{6}\right) \\
&< \zeta + \varkappa',
\end{aligned}
$$

$$
\begin{aligned}
f_8(c_b, c_{j_0}) &= \frac{1}{4}\big(2d(c_b, c_{j_0}) + d(c_b, c_{b+1}) + d(c_{j_0}, c_{b+1})\big) \\
&\leq \frac{1}{4}\big(3d(c_b, c_{j_0}) + 2d(c_b, c_{b+1})\big) \\
&< \frac{1}{4}\left(3(\zeta + \frac{\varkappa'}{2}) + \frac{\varkappa'}{3}\right) \\
&< \zeta + \varkappa',
\end{aligned}
$$

$$
\begin{aligned}
f_9(c_b, c_{j_0}) &= \frac{1}{4}\big(d(c_b, c_{b+1}) + d(c_b, c_{j_0+1}) + 2d(c_{j_0}, c_{b+1})\big) \\
&\leq \frac{1}{4}\big(2d(c_b, c_{b+1}) + 2d(c_b, c_{j_0}) + d(c_{j_0}, c_{j_0+1})\big) \\
&< \frac{1}{4}\left(\frac{\varkappa'}{3} + 2\left(\zeta + \frac{\varkappa'}{2}\right) + \frac{\varkappa'}{6}\right) \\
&= \frac{1}{2}\zeta + \frac{3}{8}\varkappa' \\
&< \zeta + \varkappa',
\end{aligned}
$$

and

$$
\begin{aligned}
f_{10}(c_b, c_{j_0}) &= \frac{1}{5}\big(2d(c_b, c_{j_0}) + d(c_b, c_{b+1}) + d(c_{j_0}, c_{b+1}) + d(c_{j_0}, c_{j_0+1})\big) \\
&\leq \frac{1}{5}\big(3d(c_b, \psi_{j_0}) + 2d(c_b, c_{b+1}) + d(c_{j_0}, c_{j_0+1})\big) \\
&< \frac{1}{5}\left(3(\zeta + \frac{\varkappa'}{2}) + \frac{\varkappa'}{3} + \frac{\varkappa'}{6}\right) \\
&< \zeta + \varkappa',
\end{aligned}
$$

we obtain

$$\zeta \leq d(c_b, c_{j_0}) \leq U(c_b, c_{j_0}) < \zeta + \varkappa'.$$

By virtue of **(DH)**, we have

$$d\big(c_{b+1}, c_{j_0+1}\big) = d\big(Ac_b, Ac_{j_0}\big) < \zeta. \tag{6}$$

Combining (5) with (6) reveals

$$\begin{aligned} d\big(c_{b+1}, c_{j_0}\big) &\leq d\big(c_{b+1}, c_{j_0+1}\big) + d\big(c_{j_0+1}, c_{j_0}\big) \\ &< \zeta + \frac{\varkappa'}{6} < \zeta + \frac{\varkappa'}{2}, \end{aligned}$$

which implies $b+1 \in \mathcal{W}$.

**Case (ii).** Assume that $\varphi(\psi_b, \psi_{j_0}) < \zeta$. Then we have

$$\begin{aligned} \varphi\big(\psi_{b+1}, \psi_{j_0}\big) &\leq \varphi(\psi_{b+1}, \psi_b) + \varphi(\psi_b, \psi_{j_0}) \\ &< \zeta + \frac{\varkappa'}{6} < \zeta + \frac{\varkappa'}{2}, \end{aligned}$$

which means that $b+1 \in \mathcal{W}$.

Consequently, from Cases (i) and (ii), we show that $b \in \mathcal{W} \Longrightarrow b+1 \in \mathcal{W}$. Hence, the finite induction principle implies

$$\mathcal{W} = \{b \in \mathbb{N} : b \geq j_0\}$$

and

$$d(c_b, c_{j_0}) < \zeta + \frac{\varkappa'}{2} \quad \text{for all } b \geq j_0.$$

For $m, n \in \mathbb{N}$ with $m \geq n \geq j_0$, the inequality (2.6) yields

$$d(c_m, c_n) \leq d(c_m, c_{j_0}) + d(c_{j_0}, c_n) < 2\zeta + \varkappa' \leq 3\zeta < \beta,$$

which concludes that $\{c_n\}_{n\in\mathbb{N}}$ is Cauchy in $X$. The proof is completed. □

Now, we establish the following new fixed point theorem for generalized Meir–Keeler type nonlinear mappings, satisfying condition **(DH)**.

**Theorem 4.** *Let $(X, d)$ be a complete metric space. Let $A$, $\{f_i\}_{i=1}^{10}$ and $U$ be the same as in Theorem 3. Assuming that the condition* **(DH)** *holds, then $A$ admits a unique fixed point in $X$.*

**Proof.** Let $z \in X$ be given. Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence defined by $c_1 = z$ , $c_{n+1} = Ac_n$ for all $n \in \mathbb{N}$. In order to verify that $A$ has a fixed point in $X$, we consider two separate cases below:

**Case 1.** Assume that $c_\alpha = c_{\alpha+1} = Ac_\alpha$ for some $\alpha \in \mathbb{N}$. Therefore $c_\alpha$ is a fixed point of $A$.

**Case 2.** Assume that $c_{n+1} \neq c_n$ for all $n \in \mathbb{N}$. By applying Theorem 3, $\{c_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $X$. Therefore the completeness of $X$ guarantees that $c_n \to s$ as $n \to \infty$ for some $s \in X$. We now show that $s \in \mathcal{F}(A)$ (the set of fixed points of $A$). For any $n \in \mathbb{N}$, straightforward computation yields

- $f_1(c_n, s) = d(c_n, s)$,
- $f_2(c_n, s) = \frac{d(c_n, c_{n+1}) + d(s, As)}{2}$,
- $f_3(c_n, s) = \frac{d(c_n, As) + d(s, c_{n+1})}{2}$,
- $f_4(c_n, s) = \frac{2d(c_n, c_{n+1}) + \varphi(s, c_{n+1})}{3}$,
- $f_5(c_n, s) = \frac{2d(c_n, c_{n+1}) + d(s, As)}{3}$,

- $f_6(c_n, s) = \frac{d(c_n,s)+2d(s,As)}{3}$,
- $f_7(c_n, s) = \frac{2d(c_n,c_{n+1})+d(s,c_{n+1})+d(s,As)}{4}$,
- $f_8(c_n, s) = \frac{2d(c_n,s)+d(c_n,c_{n+1})+d(s,c_{n+1})}{4}$,
- $f_9(c_n, s) = \frac{d(c_n,c_{n+1})+d(c_n,As)+2d(s,c_{n+1})}{4}$,
- $f_{10}(c_n, s) = \frac{2d(c_n,s)+d(c_n,c_{n+1})+d(s,c_{n+1})+d(s,As)}{5}$.

Since $c_{n+1} \neq c_n$ for all $n \in \mathbb{N}$, we know $d(c_n, c_{n+1}) > 0$. So

$$U(c_n, s) \geq f_1(c_n, s) = \frac{d(c_n, c_{n+1}) + d(s, As)}{2} > 0 \quad \text{for all } n \in \mathbb{N}.$$

Using **(DH)**, we obtain

$$d(c_{n+1}, As) < U(c_n, s) \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

Since $c_n \to s$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(c_{n+1}, As) = d(s, As)$$

and

$$\lim_{n\to\infty} U(c_n, s) = \lim_{n\to\infty} \max_{1\leq i\leq 10} f_i(c_n, s) = \frac{2}{3} d(s, As).$$

Therefore, using (7), we conclude

$$d(s, As) = \lim_{n\to\infty} d(c_{n+1}, As) \leq \lim_{n\to\infty} U(c_n, s) = \frac{2}{3} d(s, As)$$

which implies that $d(s, As) = 0$. Thus, we show $As = s$ and hence $s \in \mathcal{F}(A)$. Finally, we claim that $\mathcal{F}(A) = \{s\}$. Suppose there exists $w \in \mathcal{F}(A)$ with $w \neq s$. Then, $d(w, s) > 0$. Since

$$f_2(w, s) = f_5(w, s) = 0,$$

$$f_1(w, s) = f_3(w, s) = d(w, s),$$

$$f_6(w, s) = f_4(w, s) = \frac{1}{3} d(w, s),$$

$$f_7(w, s) = \frac{1}{4} d(w, s)$$

$$f_8(w, s) = f_9(w, s) = \frac{3}{4} d(w, s),$$

and

$$f_{10}(w, s) = \frac{3}{5} d(w, s),$$

we obtain

$$U(w, s) = \max_{1\leq i\leq 10} f_i(w, s) = d(w, s) > 0.$$

So, by virtue of **(DH)**, we have

$$d(w, s) = d(Aw, As) < U(w, s) = d(w, s),$$

which is a contradiction. Accordingly, $\mathcal{F}(A) = \{s\}$. Therefore, we prove that $A$ admits a unique fixed point $s$ in $X$. The proof is completed. □

In this article, we cannot directly apply Theorem 4 to prove Meir–Keeler's fixed point theorem. Concerning conditions **(MK)** and **(DH)**, we would like to propose the following open problems:

- **Open problem 1.** Is the condition **(DH)** a real generalization of the condition **(MK)**? Or are these two conditions independent?
- **Open problem 2.** Is Theorem 4 a real generalization of Meir–Keeler's fixed point theorem? Or are these two theorems independent?

## 3. Applications to Fixed Point Theory

In this section, unless otherwise specified, we shall assume that $(X, d)$ is a complete metric space and $A : X \to X$ is a selfmapping.

We first recall the Banach contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem as follows:

**Theorem 5** (Banach contraction principle [1])**.** *Suppose that there exists $\lambda \in [0, 1)$ such that*

$$d(Ax, Ay) \leq \lambda d(x, y) \text{ for all } x, y \in X.$$

*Then $A$ has a unique fixed point in $X$.*

**Theorem 6** (Kannan's fixed point theorem [23])**.** *Suppose that there exists $\lambda \in \left[0, \frac{1}{2}\right)$ such that*

$$d(Ax, Ay) \leq \lambda(d(x, Ax) + d(y, Ay)) \text{ for all } x, y \in X.$$

*Then $A$ has a unique fixed point in $X$.*

**Theorem 7** (Chatterjea's fixed point theorem [24])**.** *Suppose that there exists $\lambda \in \left[0, \frac{1}{2}\right)$ such that*

$$d(Ax, Ay) \leq \lambda(d(x, Ay) + d(y, Ax)) \text{ for all } x, y \in X.$$

*Then $A$ has a unique fixed point in $X$.*

By virtue of Theorem 4, we present the following simultaneous generalization of the Banach contraction principle, Chatterjea's fixed point theorem, Kannan's fixed point theorem and some known fixed point theorems in the literature.

**Theorem 8.** *Let $U$ be the same as in Theorem 3. Suppose that there exists $\rho \in [0, 1)$ such that*

$$d(Ax, Ay) \leq \rho U(x, y) \quad \text{for all } x, y \in X. \tag{8}$$

*Then $A$ admits a unique fixed point in $X$.*

**Proof.** Let $\beta > 0$ be given. Take $\kappa \in (\rho, 1)$ and define

$$\varkappa(\beta) = \beta\left(\frac{1}{\kappa} - 1\right).$$

If $\beta \leq U(x, y) < \beta + \varkappa(\beta)$, then, using (8), we have

$$d(Ax, Ay) \leq \rho U(x, y) < \kappa(\beta + \varkappa(\beta)) = \beta.$$

Hence, we verify that the condition **(DH)** holds. By applying Theorem 4, $A$ admits a unique fixed point in $X$. □

Here, we give an example to illustrate that Theorem 8 is a real simultaneous generalization of Banach contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem.

**Example 1.** *Let $X = [0,2]$ with the metric $d(x,y) = |x-y|$ for $x,y \in X$. Then $(X,d)$ is a complete metric space. Define a mapping $A : X \to X$ by*

$$Ax = \begin{cases} 1\,, & \text{if } 0 \le x < 2, \\ 0\,, & \text{if } x = 2. \end{cases}$$

*Obviously, 1 is the unique fixed point of A. It is worth noting the following facts:*

(a) *Since $d(A(1), A(2)) = 1 > \lambda d(1,2)$ for any $\lambda \in [0,1)$, T is not a contraction. Hence, the Banach contraction principle is not applicable here.*

(b) *Since $d(A(1), A(2)) = 1$ and $d(1, A(1)) + d(2, A(2)) = 2$, we have*

$$d(A(1), A(2)) > \lambda(d(1, A(1)) + d(2, A(2))) \quad \text{for any } \lambda \in \left[0, \frac{1}{2}\right).$$

*Hence, Kannan's fixed point theorem is not applicable here.*

(c) *Since $d(A(1), A(2)) = 1$ and $d(1, A(2)) + d(2, A(1)) = 2$, we have*

$$d(A(1), A(2)) > \lambda(d(1, A(2)) + d(2, A(1))) \quad \text{for any } \lambda \in \left[0, \frac{1}{2}\right).$$

*Hence, Chatterjea's fixed point theorem is not applicable here.*

*We now claim that $d(Ax, Ay) \le \frac{4}{5}U(x,y)$ for all $x, y \in X$. In order to verify this fact, we consider the following four possible cases:*

***Case 1.*** *For $x, y \in [0,2)$, we have $d(Ax, Ay) = 0 \le \frac{4}{5}U(x,y)$.*

***Case 2.*** *For $x \in [0,2)$ and $y = 2$, we have $d(Ax, Ay) = 1$. Since*

$$f_6(x,y) = \frac{d(x,y) + 2d(y, Ay)}{3} = \frac{|x-2| + 4}{3},$$

*we obtain $\frac{4}{3} < f_6(x,y) \le 2$. Hence*

$$d(Ax, Ay) = 1 < \frac{4}{5} \times \frac{4}{3} < \frac{4}{5} f_6(x,y) \le \frac{4}{5} U(x,y).$$

***Case 3.*** *For $x = 2$ and $y \in [0,2)$, we have $d(Ax, Ay) = 1$. Since*

$$f_4(x,y) = \frac{2d(x, Ax) + d(y, Ax)}{3} = \frac{4+y}{3},$$

*we obtain $\frac{4}{3} \le f_4(x,y) < 2$. Hence*

$$d(Ax, Ay) = 1 < \frac{4}{5} \times \frac{4}{3} \le \frac{4}{5} f_4(x,y) \le \frac{4}{5} U(x,y).$$

***Case 4.*** *For $x = y = 2$, we have $d(Ax, Ay) = 0 \le \frac{4}{5}U(x,y)$.*

*Hence, by Cases 1, 2, 3 and 4, we prove that $d(Ax, Ay) \le \frac{4}{5}U(x,y)$ for all $x, y \in X$. Therefore, all the assumptions of Theorem 8 are satisfied. Applying Theorem 8, we also prove that A has a unique fixed point in X.*

The following result is a direct consequence of Theorem 8.

**Corollary 1.** *Let $\rho \in [0,1)$. Suppose that*

$$d(Ax, Ay) \le \rho \max\left\{ d(x,y), \frac{d(x, Ax) + d(y, Ay)}{2}, \frac{d(x, Ay) + d(y, Ax)}{2} \right\} \tag{9}$$

*for all $x, y \in X$. Then $A$ admits a unique fixed point in $X$.*

**Proof.** It is obvious that (9) implies

$$d(Ax, Ay) \leq \rho \max_{1 \leq i \leq 10} f_i(x, y) = \rho U(x, y) \quad \textit{for all } x, y \in X,$$

where $U(x, y) := \max_{1 \leq i \leq 10} f_i(x, y)$ for $x, y \in X$. Hence the desired conclusion follows immediately from Theorem 8. □

**Remark 1.** *Corollary 1 is also a simultaneous generalization of the Banach contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem.*

**Theorem 9.** *Let $\{f_i\}_{i=1}^{10}$ be the same as in Theorem 3. Suppose that there exists $\{\lambda_i\}_{i=1}^{10} \subseteq [0, \infty)$, satisfying $\sum_{i=1}^{10} \lambda_i < 1$, such that*

$$d(Ax, Ay) \leq \sum_{i=1}^{10} \lambda_i f_i(x, y) \quad \textit{for all } x, y \in X. \tag{10}$$

*Then $A$ admits a unique fixed point in $X$.*

**Proof.** Let $\rho := \sum_{i=1}^{10} \lambda_i$ and $U(x, y) := \max_{1 \leq i \leq 10} f_i(x, y)$ for $x, y \in X$. Then $\rho \in [0, 1)$. Since (10) yields

$$d(Ax, Ay) \leq \sum_{i=1}^{10} \lambda_i f_i(x, y) \leq \left( \sum_{i=1}^{10} \lambda_i \right) \max_{1 \leq i \leq 10} f_i(x, y) = \rho U(x, y) \quad \text{for all } x, y \in X,$$

the desired conclusion follows immediately from Theorem 8. □

**Theorem 10.** *Let $\{f_i\}_{i=1}^{10}$ be the same as in Theorem 3. Suppose that $A$ satisfies one of the following conditions:*

*(1) $d(Ax, Ay) \leq \frac{\rho}{n} \sum_{i=1}^{10} f_i(x, y)$ for all $x, y \in X$;*

*(2) $d(Ax, Ay) \leq \rho \sqrt[10]{\prod_{i=1}^{10} f_i(x, y)}$ for all $x, y \in X$, where $\prod_{i=1}^{10} f_i(x, y) := f_1(x, y) \times f_2(x, y) \times \cdots \times f_{10}(x, y)$ for $x, y \in X$.*

*Then $A$ admits a unique fixed point in $X$.*

**Proof.** By using the arithmetic mean–geometric mean (AM-GM) inequality, we obtain

$$\begin{aligned} d(Ax, Ay) &\leq \rho \sqrt[10]{\prod_{i=1}^{10} f_i(x, y)} \\ &\leq \frac{\rho}{n} \sum_{i=1}^{10} f_i(x, y) \\ &\leq \rho \max_{1 \leq i \leq 10} f_i(x, y) \\ &= \rho U(x, y) \qquad \text{for all } x, y \in X, \end{aligned}$$

where $U(x,y) := \max_{1\le i\le 10} f_i(x,y)$ for $x, y \in X$. Therefore, by using any condition and applying Theorem 8, we can prove the desired conclusion. □

Finally, applying Theorem 8, we can easily establish the following new fixed point theorems.

**Corollary 2.** *Assume that $\rho \in [0,1)$ and $A$ satisfies one of the following conditions:*

(1) $d(Ax,Ay) \le \rho \max\left\{\frac{d(x,Ax)+d(y,Ay)}{2}, \frac{2d(x,Ax)+d(y,Ay)}{3}\right\}$ *for all* $x,y \in X$;

(2) $d(Ax,Ay) \le \rho \max\left\{\frac{2d(x,Ax)+d(y,Ax)}{3}, \frac{2d(x,Ax)+d(y,Ax)+d(y,Ay)}{4}\right\}$ *for all* $x,y \in X$;

(3) $d(Ax,Ay) \le \rho \max\left\{\frac{d(x,Ax)+d(x,Ay)+2d(y,Ax)}{4}, \frac{2d(x,y)+d(x,Ax)+d(y,Ax)+d(y,Ay)}{5}\right\}$ *for all* $x,y \in X$.

*Then $A$ admits a unique fixed point in $X$.*

**Corollary 3.** *Assume that $\rho \in [0,1)$ and $A$ satisfies one of the following conditions:*

(1) $d(Ax,Ay) \le \rho \max\left\{\frac{2d(x,Ax)+d(y,Ax)}{3}, \frac{2d(x,Ax)+d(y,Ay)}{3}, \frac{d(x,y)+2d(y,Ay)}{3}\right\}$ *for all* $x,y \in X$;

(2) $d(Ax,Ay) \le \rho \max\left\{\frac{2d(x,Ax)+d(y,Ax)+d(y,Ay)}{4}, \frac{2d(x,y)+d(x,Ax)+d(y,Ax)}{4}, \frac{d(x,Ax)+d(x,Ay)+2d(y,Ax)}{4}\right\}$ *for all* $x,y \in X$;

(3) $d(Ax,Ay) \le \rho \max\left\{d(x,y), \frac{2d(x,Ax)+d(y,Ay)}{3}, \frac{2d(x,y)+d(x,Ax)+d(y,Ax)+d(y,Ay)}{5}\right\}$ *for all* $x,y \in X$.

*Then $A$ admits a unique fixed point in $X$.*

**Corollary 4.** *Assume that $\rho \in [0,1)$ and $A$ satisfies one of the following conditions:*

(1) $d(Ax,Ay) \le \rho \max\left\{d(x,y), \frac{d(x,Ax)+d(y,Ay)}{2}, \frac{d(x,Ay)+d(y,Ax)}{2}, \frac{d(x,y)+2d(y,Ay)}{3}\right\}$ *for all* $x,y \in X$;

(2) $d(Ax,Ay) \le \rho \max\left\{\frac{d(x,Ay)+d(y,Ax)}{2}, \frac{2d(x,Ax)+d(y,Ax)}{3}, \frac{2d(x,y)+d(x,Ax)+d(y,Ax)}{4}, \frac{2d(x,y)+d(x,Ax)+d(y,Ax)+d(y,Ay)}{5}\right\}$ *for all* $x,y \in X$;

(3) $d(Ax,Ay) \le \rho \max\left\{d(x,y), \frac{d(x,y)+2d(y,Ay)}{3}, \frac{d(x,Ax)+d(x,Ay)+2d(y,Ax)}{4}, \frac{2d(x,y)+d(x,Ax)+d(y,Ax)+d(y,Ay)}{5}\right\}$ *for all* $x,y \in X$.

*Then $A$ admits a unique fixed point in $X$.*

## 4. Conclusions

In this paper, we establish the main result about fixed point theorems for generalized Meir–Keeler type nonlinear mappings as follows:

- (See Theorem 4):
  Let $(X,d)$ be a complete metric space and $A : X \to X$ be a selfmapping. Define a mapping $U : X \times X \to [0,\infty)$ by

$$U(x,y) = \max_{1\le i\le 10} f_i(x,y)$$

where

$$f_1(x,y) = d(x,y),$$

$$f_2(x,y) = \frac{d(x,Ax)+d(y,Ay)}{2},$$

$$f_3(x,y) = \frac{d(x,Ay) + d(y,Ax)}{2},$$
$$f_4(x,y) = \frac{2d(x,Ax) + d(y,Ax)}{3},$$
$$f_5(x,y) = \frac{2d(x,Ax) + d(y,Ay)}{3},$$
$$f_6(x,y) = \frac{d(x,y) + 2d(y,Ay)}{3},$$
$$f_7(x,y) = \frac{2d(x,Ax) + d(y,Ax) + d(y,Ay)}{4},$$
$$f_8(x,y) = \frac{2d(x,y) + d(x,Ax) + d(y,Ax)}{4},$$
$$f_9(x,y) = \frac{d(x,Ax) + d(x,Ay) + 2d(y,Ax)}{4},$$

and

$$f_{10}(x,y) = \frac{2d(x,y) + d(x,Ax) + d(y,Ax) + d(y,Ay)}{5},$$

for $x, y \in X$. Suppose that

**(DH)** for each $\beta > 0$, there exists $\varkappa = \varkappa(\beta) > 0$ such that for $x, y \in X$,

$$U(x,y) \in [\beta, \beta + \varkappa) \quad \textit{implies} \quad d(Ax, Ay) < \beta.$$

Then $A$ admits a unique fixed point in $X$.

As applications, some new fixed point theorems are presented in Section 3. Our new results will assist us in obtaining novel fixed point theorems for other generalized types of Meir–Keeler type nonlinear mappings as well as their proof techniques in future research.

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