symmetry

MDPI

Article

Refinement of a Lyapunov-Type Inequality for a Fractional Differential Equation

Hongying Xiao [1], Zhaofeng Li [2], Yuanyuan Zhang [2] and Xiaoyou Liu [3,*]

1 Faculty of Science, Yibin University, Yibin 644000, China; 2020080001@yibinu.edu.cn
2 Department of Mathematics, China Three Gorges University, Yichang 443002, China; lizhaofeng@ctgu.edu.cn (Z.L.); zhangyuanyuan@ctgu.edu.cn (Y.Z.)
3 School of Mathematics and Computing Sciences, Hunan University of Science and Technology, Xiangtan 411201, China
* Correspondence: xiaoyouliu@hnust.edu.cn

Abstract: In this paper, we focus on a fractional differential equation ${}_0^C D^\alpha u(t) + q(t)u(t) = 0$ with boundary value conditions $u(0) = \delta u(1), u'(0) = \gamma u'(1)$. The paper begins by pointing out the inadequacies of the study conducted by Ma and Yangin establishing Lyapunov-type inequalities. It then discusses the properties of its Green's function and investigates extremum problems related to several linear functions. Finally, thorough classification and analysis of various cases for parameters δ and γ are conducted. As a result, a comprehensive solution corresponding to the Lyapunov-type inequality is obtained.

Keywords: Lyapunov inequality; green function; fractional differential equations; boundary value problem (BVP)

check for updates

Citation: Xiao, H.; Li, Z.; Zhang, Y.; Liu, X. Refinement of a Lyapunov-Type Inequality for a Fractional Differential Equation. *Symmetry* **2024**, *16*, 941. https://doi.org/10.3390/sym16080941

Academic Editors: Włodzimierz Fechner and Jacek Chudziak

Received: 14 May 2024
Revised: 13 June 2024
Accepted: 12 July 2024
Published: 23 July 2024

1. Introduction

If $q(t)$ is a real, continuous function such that $y''(t) + q(t)y(t) = 0$ has a nontrivial solution $y(t)$ for $t \in (a, b)$ satisfying $y(a) = 0, y(b) = 0$, then the following inequality holds:

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{1}$$

This result was provided in [1] by the Russian mathematician Lyapunov in the year 1893, marking the inaugural work in this field. Inspired by this, researchers have derived Lyapunov-type inequalities for higher order BVPs, extending the applicability of this result (see survey [2]).

On the other hand, the nonlocality of fractional differential operators has been utilized to represent a number of real-world situations, often by changing an ordinary derivative in a differential equation to a fractional one (see, e.g., [3]). This results in the emerging of Lyapunov-type inequalities related to the fractional BVP. For a comprehensive review, please refer to references [4–6].

Researchers have employed various methods to derive the corresponding Lyapunov inequalities for various types of fractional differential equations and various types of boundary conditions: fractional differential equations under mixed boundary conditions [7], fractional differential equations under Robin boundary conditions [8], a Caputo fractional differential equation under a boundary condition involving the Caputo fractional derivative [9], a Hadamard fractional differential equation under Sturm–Liouville boundary conditions [10], a class of fractional boundary value problems [11], new Hartman–Wintner-type inequalities for a class of nonlocal fractional boundary value problems [12], a class of fractional differential equations under fractional boundary conditions with the Katugampola derivative [13], boundary value problems involving generalized Caputo fractional derivatives that unite the Caputo and Caputo–Hadamrad fractional derivatives [14], Hilfer–Katugampola fractional differential equations [15], and a p-Laplacian eigenvalue boundary value problem involving both right Caputo and left Riemann–Liouville types fractional derivatives [16]. New achievements on the theory and applications of fractional order differential equations can be referred to [17–19].

The outline of this paper is as follows. In Section 2, we begin by providing definitions of fractional differentiation and fractional integration and point out the unresolved issue in the study conducted by Ma and Yang [20] while establishing Lyapunov-type inequalities. Next, we identify the connection between the maximal value problem of a Green function and that of the absolute values of several linear functions. In Section 3, we present the main results of the paper and provide their proofs. In Section 4, we highlight both the value and limitations of the approach used in discussing extremum problems in this paper. We provide an outlook for future research on Lyapunov inequalities for the other BVP and welcome other researchers to introduce new methods to obtain more refined results.

2. Preliminary and Lemmas

Below are the definitions for fractional differentiation and fractional integration.

Definition 1. *The gamma function $\Gamma(x)$ is a mathematical function extending the concept of factorial to complex and real numbers:*

$$\Gamma(z) = \int_a^\infty t^{z-1} e^{-t} dt, \quad Re(z) > 0.$$

Definition 2 ([3])**.** *Let $\alpha > 0$ and $y(t)$ be a real function defined on $[a, b]$. The Riemann–Liouville fractional integral of order α for $y(t)$ is defined by*

$$({}_aI^\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds, \qquad t \in [a, b]$$

where Γ is the gamma function.

Definition 3 ([3])**.** *Let $\alpha > 0$ and $y(t)$ be a real function defined on $[a, b]$. The Caputo fractional derivative of order α for $y(t)$ is defined by*

$${}_a^CD^\alpha y(t) = \left(aI_t^{n-\alpha} y^{(n)} \right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{y^{(n)}(s)}{(t-s)^{1+\alpha-n}} ds, t \in [a, b]$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Ma and Yang considered the following fractional differential equation in [20]:

$$\begin{cases} {}_0^CD^\alpha u(t) + q(t)u(t) = 0, \quad t \in (0, 1), \\ u(0) = \delta u(1), \quad u'(0) = \gamma u'(1), \end{cases} \tag{2}$$

where $\delta > 0, \gamma > 0, \alpha \in (1, 2)$, $q(t) \in L(0, 1)$ is not identically zero on any compact subinterval of $(0, 1)$.

Lemma 1 ([20])**.** *A function $u(t)$ is a solution of BVP (2) if and only if*

$$u(t) = \int_0^1 G(t; s) q(s) u(s) ds$$

where the Green function $G(t; s)$ is given as

$$G(t;s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-\alpha)\frac{\delta\gamma(1-t)+\gamma t}{(1-\gamma)(1-\delta)}(1-s)^{\alpha-2} - \frac{\delta}{1-\delta}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ (1-\alpha)\frac{\delta\gamma(1-t)+\gamma t}{(1-\gamma)(1-\delta)}(1-s)^{\alpha-2} - \frac{\delta}{1-\delta}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases}$$

Moreover, if $u(t)$ is nontrivial, then

$$1 < \int_0^1 \max_{0\le t\le 1} |G(t;s)||q(s)|ds. \tag{3}$$

To obtain a Lyapunov inequality, one need to solve the problem $\max_{0\le t\le 1}|G(t;s)|$. However, the study conducted by Ma and Yang [20] only considered partial cases for parameters $\delta, \gamma > 0$.

Theorem 1 ([20])**.** *Suppose BVP (2) has a nonzero solution u(t), we conclude that*

(i) *If $\delta \in (0,1)$ and $\gamma \in (0,1)$, then*

$$\int_0^1 (1-s)^{\alpha-2}[\gamma(\alpha-1) + (1-\gamma)(1-s)]|q(s)|ds > \Gamma(\alpha)(1-\delta)(1-\gamma)$$

(ii) *If $\delta \in (1,+\infty)$ and $\gamma \in (0,1)$, then*

$$\int_0^1 (1-s)^{\alpha-2}[\gamma(\alpha-1) + (1-\gamma)(1-s)]|q(s)|ds > \frac{\Gamma(\alpha)(\delta-1)(1-\gamma)}{\delta}$$

(iii) *If $\delta \in (0,1)$ and $\gamma \in \left(1, 1+\frac{(\alpha-1)\delta}{2-\alpha}\right]$, then*

$$\int_0^1 (1-s)^{\alpha-2}[\gamma(\alpha-1) - (\gamma-1)(1-s)]|q(s)|ds > \Gamma(\alpha)(1-\delta)(\gamma-1)$$

(iv) *If $\delta \in (1,+\infty)$ and $\gamma \in \left(1, \frac{1}{2-\alpha}\right]$, then*

$$\int_0^1 (1-s)^{\alpha-2}\left\{\delta\gamma(\alpha-1) - \left[\delta(\gamma-1) - \gamma(2-\alpha)(\delta-1)\left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{2-\alpha}}\right](1-s)\right\}|q(s)|ds$$
$$> \Gamma(\alpha)(\delta-1)(\gamma-1). \tag{4}$$

This paper aims to provide a comprehensive solution to the problem $\max_{0\le t\le 1}|G(t;s)|$ for parameters $\delta, \gamma > 0$. Additionally, we aim to derive the corresponding Lyapunov-type inequalities. Since the case $0 < \gamma < 1$ has already been discussed in Theorem 1, we need only to conduct our work under the next assumptions:

$$1 < \alpha < 2, \gamma > 1. \tag{5}$$

2.1. Properties of the Green Function

To address the maximum problem $\max_{0\le t\le 1}|G(t;s)|$, we will initially fix s within the interval $[0,1]$ and examine the monotonicity of the function $G(t;s)$ by analyzing its derivative $G'_t(t;s)$.

We compute the derivative of the above Green function as

$$G'_t(t;s) = \begin{cases} \dfrac{(\alpha-1)}{\Gamma(\alpha)}\left[\frac{\gamma}{\gamma-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-2}\right], & 0 \le s < t \le 1, \\ \dfrac{\gamma(\alpha-1)(1-s)^{\alpha-2}}{\Gamma(\alpha)(\gamma-1)}, & 0 \le t < s \le 1. \end{cases} \tag{6}$$

Lemma 2. *Assume that relation (5) holds, and fix $s \in [0,1]$. Define*

$$s^* = s + \left(\frac{\gamma}{\gamma-1}\right)^{\frac{1}{\alpha-2}}(1-s). \tag{7}$$

Then, $0 \le s^* \le 1$ and the Green function $G(t;s)$ is increasing in $t \in [0,s], [s^*,1]$, and decreasing in $[s,s^*]$.

Proof. It is straightforward to verify that $s \le s^* \le 1$. Define $\psi(t) = G'_t(t;s), t \in (s,1]$. One can check that $\psi(s^*) = 0$ and $\psi(t)$ is monotonous at $(s,1]$, which implies that

$$\begin{aligned} &\psi(t) < 0, \quad s < t < s^*, \\ &\psi(t) > 0, \quad s^* < t < 1. \end{aligned}$$

Thus, we have

$$\begin{aligned} &G'_t(t;s) > 0, \quad 0 < t < s, \\ &G'_t(t;s) < 0, \quad s < t < s^*, \\ &G'_t(t;s) > 0, \quad s^* < t < 1. \end{aligned}$$

□

Proposition 1. *Assume that Relation (5) holds, and that $s \in [0,1]$, then*

$$\max_{0 \le t \le 1} |G(t;s)| = \frac{1}{\Gamma(\alpha)|1-\delta|}(1-s)^{\alpha-2} \max_{1 \leqslant i \leqslant 4}\{|f_i(s)|\} \tag{8}$$

where

$$\begin{aligned} f_1(s) &= \left[\frac{\gamma(1-\alpha)}{1-\gamma} + s - 1\right]\delta, \\ f_2(s) &= \frac{(1-\alpha)\delta\gamma + (1-\alpha)\gamma(1-\delta)s}{1-\gamma} + \delta(s-1), \\ f_3(s) &= \frac{\delta\gamma(\alpha-1)}{\gamma-1} + \left[(2-\alpha)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\alpha-1}{\alpha-2}}(1-\delta) + \delta\right](s-1) + \frac{\gamma(\alpha-1)(1-\delta)}{\gamma-1}s, \\ f_4(s) &= \frac{1}{\delta}f_1(s). \end{aligned} \tag{9}$$

Proof. It follows from Lemma 2 that

$$\max_{0 \le t \le 1} |G(t;s)| = \max\{|G(0;s)|, |G(s;s)|, |G(s^*;s)|, |G(1;s)|\}.$$

We can easily check that, for any $s \in [0,1]$,

$$\begin{aligned} G(0;s) &= \frac{1}{\Gamma(\alpha)(1-\delta)}(1-s)^{\alpha-2}f_1(s), \\ G(s;s) &= \frac{1}{\Gamma(\alpha)(1-\delta)}(1-s)^{\alpha-2}f_2(s), \\ G(s^*;s) &= \frac{1}{\Gamma(\alpha)(1-\delta)}(1-s)^{\alpha-2}f_3(s), \\ G(1;s) &= \frac{1}{\Gamma(\alpha)(1-\delta)}(1-s)^{\alpha-2}f_4(s). \end{aligned} \tag{10}$$

Thus, we obtain Relation (8). □

2.2. A Study on Linear Functions $f_i(s)$

In this section, we will study functions $f_i(s)$ defined in (9) and obtain a solution to the problem $\max\limits_{1\leqslant i\leqslant 4}\{|f_i(s)|\}, s \in [0,1]$.

Lemma 3. *Assume that relation (5) holds and $f_i(s)$ is given in (9).*

(1) Assume that $\delta > 1$, then

$$f_1(s) \geq f_2(s), f_2(s) \leq f_3(s), f_3(s) \geq f_4(s), \forall\, s \in [0,1].$$

(2) Assume that $0 < \delta < 1$, then

$$f_1(s) \leq f_2(s), f_2(s) \geq f_3(s), f_3(s) \leq f_4(s), \forall\, s \in [0,1].$$

Proof. It follows from Lemma 2 that $G(0;s) \leq G(s;s) \geq G(s^*;s) \leq G(0;s)$ for any $s \in [0,1]$. Combining this with relation (10), we can accomplish this proof. □

In the following, we point out that the values of linear function $f_i(s)$ at $s = 0,1$ are:

$$\begin{aligned} &f_1(0) = f_2(0), && f_1(1) = \delta f_2(1), \\ &f_2(0) = \frac{\delta[(2-\alpha)\gamma-1]}{1-\gamma}, && f_2(1) = \frac{(1-\alpha)\gamma}{1-\gamma}, \\ &f_3(0) = \frac{\delta[(2-\alpha)\gamma-1]}{1-\gamma} + (2-\alpha)(\delta-1)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\alpha-1}{\alpha-2}}, && f_3(1) = f_2(1), \\ &f_4(0) = \frac{f_2(0)}{\delta}, && f_4(1) = f_2(1). \end{aligned} \tag{11}$$

Equation (11) contains eight parameters, and it is not difficult to discuss the positive or negative signs of the seven parameters excluding $f_3(0)$. Regarding the positive or negative sign of $f_3(0)$, we provide the following results.

Lemma 4. *Assume that $1 < \alpha < 2$. Define $\theta(t) = (2-\alpha)\, t^{\frac{\alpha-1}{\alpha-2}} + (\alpha-1)t - 1$, then*

$$\theta(t) > 0, \forall t > 1.$$

Proof. Compute the derivative as $\theta'(t) = (\alpha-1)\left(1 - t^{\frac{1}{\alpha-2}}\right) > 0, t > 1$. Thus, $\theta(t)$ is monotonously increasing on $[1,\infty]$, $\theta(t) > \theta(1) = 0, \forall t > 1$. □

Lemma 5. *Assume that relation (5) holds, and define*

$$\delta^* = \frac{(2-\alpha)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\alpha-1}{\alpha-2}}}{\frac{(2-\alpha)\gamma-1}{1-\gamma} + (2-\alpha)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\alpha-1}{\alpha-2}}}. \tag{12}$$

We conclude that

(1) If $(2-\alpha)\gamma < 1$, then $\delta^* \in (0,1)$;
(2) If $(2-\alpha)\gamma > 1$, then $\delta^* \in (1,\infty)$;
(3) If $0 < \delta < \delta^*$, then $f_3(0) < 0$;
(4) If $\delta > \delta^*$, then $f_3(0) > 0$.

Proof. Define a linear function

$$h(\delta) = \frac{\delta[(2-\alpha)\gamma - 1]}{1-\gamma} + (2-\alpha)(\delta-1)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\alpha-1}{\alpha-2}},$$

then $h(\delta^*) = 0$, $h'(\delta) = (2-\alpha)\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\alpha-1}{\alpha-2}} + \frac{(\alpha-1)\gamma}{\gamma-1} - 1$. By using Lemma 4, we obtain that $h'(\delta) = \theta\left(\frac{\gamma}{\gamma-1}\right) > 0$. Thus, $h(t)$ is monotonously increasing on $[0,\infty]$. Noting $h(0) < 0, h(1) = \frac{(2-\alpha)\gamma-1}{1-\gamma}$, we obtain that $\delta^* \in (0,1)$ if $(2-\alpha)\gamma < 1$ and $\delta^* \in (1,\infty)$ if $(2-\alpha)\gamma > 1$. Furthermore, we obtain that $h(\delta) < 0$ if $0 < \delta < \delta^*$ and $h(\delta) > 0$ if $\delta > \delta^*$. Combining this with relation (11), we accomplish this proof. □

Next, we will study maximal value problem $\max\limits_{1\le i\le 4}|f_i(0)|$ and $\max\limits_{1\le i\le 4}|f_i(1)|$, respectively.

Proposition 2. *Assume that Relation (5) holds, with $(2-\alpha)\gamma > 1$. Let δ^* be given in (12), then $\delta^* > 1$ and*

$$\max_{1\le i\le 4}|f_i(0)| = \begin{cases} -f_3(0), & 0 < \delta < 1, \\ -f_2(0), & 1 < \delta < \delta^*, \end{cases}$$

$$\max_{1\le i\le 4}|f_i(0)| \le f_3(0) - f_2(0),\ \delta > \delta^*.$$

Proof. Combining the assumptions with Lemma 3, Lemma 5, and Relation (11), we can verify that $\delta^* > 1$ and

$$0 > f_2(0) = f_1(0) \ge f_4(0) \ge f_3(0), \forall\, 0 < \delta < 1,$$
$$0 > f_3(0) \ge f_4(0) \ge f_2(0) = f_1(0), \forall\, 1 < \delta < \delta^*,$$
$$f_3(0) > 0 > f_4(0) \ge f_2(0) = f_1(0), \forall\, \delta > \delta^*.$$

Thus, it suffices to point out $\max\limits_{1\le i\le 4}|f_i(0)| = \max\{f_3(0), -f_2(0)\} \le f_3(0) - f_2(0)$ if $\delta > \delta^*$. □

Proposition 3. *Assume that Relation (5) holds, with $(2-\alpha)\gamma < 1$. Let δ^* be given in (12), then $0 < \delta^* < 1$ and*

$$\max_{1\le i\le 4}|f_i(0)| = \begin{cases} f_4(0), & \delta^* < \delta < 1, \\ f_3(0), & \delta > 1. \end{cases}$$

$$\max_{1\le i\le 4}|f_i(0)| \le f_4(0) - f_3(0), 0 < \delta < \delta^*.$$

Proof. Similar to the proof of the last Proposition, we conclude from the above assumptions with Lemma 3, Lemma 5, and Relation (11) that $0 < \delta^* < 1$ and

$$f_4(0) \geq f_2(0) = f_1(0) > 0 \geq f_3(0), \forall\, 0 < \delta < \delta^* < 1,$$
$$f_4(0) \geq f_2(0) = f_1(0) \geq f_3(0) > 0, \forall\, \delta^* < \delta < 1,$$
$$f_3(0) \geq f_2(0) = f_1(0) \geq f_4(0) > 0, \forall\, \delta > 1.$$

Thus, it suffices to point out $\max\limits_{1\leq i\leq 4} |f_i(0)| = \max\{f_4(0), -f_3(0)\} \leq f_4(0) - f_3(0)$ if $0 < \delta < \delta^*$. □

Proposition 4. *Assuming that Relation (5) holds, we conclude that*

$$\max_{1\leq i\leq 4} |f_i(1)| = \begin{cases} f_2(1), & 0 < \delta < 1, \\ f_1(1), & \delta > 1. \end{cases}$$

Proof. As a direct application of Relation (11), we obtain that $f_1(1) > f_2(1) = f_3(1) = f_4(1) > 0$ if $\delta > 1$ and $f_2(1) = f_3(1) = f_4(1) > f_1(1) > 0$ if $0 < \delta < 1$. □

3. Main Results

In comparison with the study conducted by Ma and Yang [20], we present our main results as follows.

Theorem 2. *Assume that $u(t)$ is a nontrivial solution of the BVP (2) and Relation (5) holds with $(2-\alpha)\gamma > 1$. Define δ^* as in (12), and define*

$$M = M(\alpha, \delta, \gamma) = \frac{\delta[(2-\alpha)\gamma - 1]}{1-\gamma} + (2-\alpha)(\delta - 1)\left(\frac{\gamma}{\gamma - 1}\right)^{\frac{\alpha-1}{\alpha-2}}. \tag{13}$$

Then, we have $\delta^* > 1$ and

(1) If $0 < \delta < 1$, then

$$\Gamma(\alpha)(1-\delta) < \frac{(1-\alpha)\gamma}{1-\gamma}\int_0^1 (1-s)^{\alpha-1}|q(s)|ds - M\int_0^1 s(1-s)^{\alpha-2}|q(s)|ds$$

(2) If $1 < \delta < \delta^*$, then

$$\Gamma(\alpha)(\delta - 1) < \frac{\delta(1-\alpha)\gamma}{1-\gamma}\int_0^1 (1-s)^{\alpha-1}|q(s)|ds - \frac{\delta[(2-\alpha)\gamma - 1]}{1-\gamma}\int_0^1 s(1-s)^{\alpha-2}|q(s)|ds \tag{14}$$

(3) If $\delta > \delta^*$, then

$$\Gamma(\alpha)(\delta - 1) < \frac{\delta(1-\alpha)\gamma}{1-\gamma}\int_0^1 (1-s)^{\alpha-1}|q(s)|ds + \left\{M - \frac{\delta[(2-\alpha)\gamma - 1]}{1-\gamma}\right\}\int_0^1 s(1-s)^{\alpha-2}|q(s)|ds$$

Proof. It follows from Lemma 5 that $\delta^* > 1$. Let $f_i(s)$ be defined in (9). Since each f_i is linear, we conclude that, for any $s \in [0,1]$,

$$\max_{1\leq i\leq 4} |f_i(s)| \leq y(s) = s\max_{1\leq i\leq 4} |f_i(0)| + (1-s)\max_{1\leq i\leq 4} |f_i(1)|$$

where $y(s)$ is a line through $(0, \max\limits_{1\leq i\leq 4} |f_i(0)|)$ $(1, \max\limits_{1\leq i\leq 4} |f_i(1)|)$.

Combining this with Relations (3) and (8), we conclude that if $u(t)$ is a nontrivial solution of Problem (2), then

$$1 < \int_0^1 \max_{0\le t\le 1} |G(t;s)||q(s)|ds \tag{15}$$

$$< \int_0^1 \frac{|q(s)|(1-s)^{\alpha-2}}{\Gamma(\alpha)|1-\delta|}\left[s \max_{1\le i\le 4} |f_i(0)| + (1-s) \max_{1\le i\le 4} |f_i(1)|\right]ds. \tag{16}$$

Thus,

$$\Gamma(\alpha)|1-\delta| < \max_{1\le i\le 4} |f_i(0)| \int_0^1 s(1-s)^{\alpha-2}|q(s)|ds + \max_{1\le i\le 4} |f_i(1)| \int_0^1 (1-s)^{\alpha-1}|q(s)|ds \tag{17}$$

Theorem 2 follow from Propositions 2–4 and Relations (11) and (17). □

Theorem 3. *Assume that $u(t)$ is a nontrivial solution of the BVP (2) and that Relation (5) holds with $(2-\alpha)\gamma < 1$. Let δ^*, M be defined in (12) and (13), respectively. Then, we have $0 < \delta^* < 1$ and*

(1) If $0 < \delta < \delta^*$, then

$$\Gamma(\alpha)(1-\delta) < \frac{(1-\alpha)\gamma}{1-\gamma} \int_0^1 (1-s)^{\alpha-1}|q(s)|ds + \left[\frac{(2-\alpha)\gamma-1}{1-\gamma} - M\right] \int_0^1 s(1-s)^{\alpha-2}|q(s)|ds$$

(2) If $\delta^* < \delta < 1$, then

$$\Gamma(\alpha)(1-\delta) < \frac{(1-\alpha)\gamma}{1-\gamma} \int_0^1 (1-s)^{\alpha-1}|q(s)|ds + \frac{(2-\alpha)\gamma-1}{1-\gamma} \int_0^1 s(1-s)^{\alpha-2}|q(s)|ds$$

(3) If $\delta > 1$, then

$$\Gamma(\alpha)(\delta-1) < \frac{\delta(1-\alpha)\gamma}{1-\gamma} \int_0^1 (1-s)^{\alpha-1}|q(s)|ds + M\int_0^1 s(1-s)^{\alpha-2}|q(s)|ds. \tag{18}$$

Proof. This result can be proved by the same approaches as to the previous theorem. □

Finally, we compare the results given by Theorems 2 and 3 and those in the paper by Ma and Yang [20]. Both studies follow a similar approach: they first seek to find the maximum value of the function $|G(t;s)|$. Next, they combine the results with Inequality (3) to derive the Lyapunov inequalities that the function $q(t)$ must satisfy. The difference is that the work conducted by Ma and Yang [20] discusses the maximum value of $|G(t;s)|$ only under certain conditions for the parameters δ and γ, thus obtaining the Lyapunov inequalities under these assumptions. In contrast, through a detailed discussion of several linear functions f_i defined in this paper, we study the maximum value of $|G(t;s)|$ for all cases of the parameters δ and γ, (with necessary amplifications of $|G(t;s)|$ when needed), thereby obtaining the Lyapunov inequalities for any parameter.

4. Conclusions

This paper identifies the connection between the maximal value problem of a Green function and that of the absolute values of several linear functions. Ultimately, it provides a comprehensive solution to the unresolved problem highlighted in the study conducted by Ma and Yang [20]. The author believes that the approach of discussing linear functions has significant value and can be potentially helpful in establishing Lyapunov-type inequalities for other BVP in the future. However, it should be noted that in Propositions 2 and 3, we employed necessary amplifications while solving certain extremum problems. The author welcomes other researchers to employ alternative techniques to obtain more refined research results.

Author Contributions: All authors contributed equally to this paper: Conceptualization, H.X.; validation, Z.L. and Y.Z.; data curation, X.L.; writing—original draft preparation, H.X.; writing—review and editing, X.L. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the specialized research fund of YiBin University (Grant No. 412-2021QH027).

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors would like to express special thanks to Wang Jinrong and Qiu Xiaoling for their valuable suggestions during the process of writing.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

References

1. Liapunov, A.M. Probleme général de la stabilité du mouvement, (French Translation of a Russian paper dated 1893). *Ann. Fac. Sci. Univ. Toulouse* **1907**, *2*, 27–247.
2. Tiryaki, A. Recent developments of Lyapunov-type inequalities. *Adv. Dyn. Syst. Appl.* **2010**, *5*, 231–248.
3. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
4. Ntouyas, S.K.; Ahmad, B.; Horikis, T.P. Recent developments of Lyapunov-type inequalities for fractional differential equations. *Differ. Integral Inequalities* **2019**, *151*, 619–686.
5. Ntouyas, S.K.; Ahmad, B. Lyapunov-type inequalities for fractional differential equations: A survey. *Surv. Math. Appl.* **2021**, *16*, 43–93.
6. Ntouyas, S.K.; Ahmad, B.; Tariboon, J. A Survey on Recent Results on Lyapunov-Typ eInequalities for Fractional Differential Equations. *Fractal Fract.* **2022**, *6*, 273. [CrossRef]
7. Jleli, M.; Samet, B. Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions. *Math. Inequal. Appl.* **2015**, *18*, 443–451. [CrossRef]
8. Jleli, M.; Ragoub, L.; Samet, B. Lyapunov-type inequality for a fractional differential equation under a Robin boundary conditions. *J. Funct. Spaces* **2015**, *2015*, 468536. [CrossRef]
9. Rong, J.; Bai, C. Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions. *Adv. Differ. Equ.* **2015**, *2015*, 82. [CrossRef]
10. Wang, Y.; Liang, S.; Xia, C. A Lyapunov-type inequalities for a fractional differential equation under Sturm–Liouville boundary conditions. *Math. Inequal. Appl.* **2017**, *20*, 139–148. [CrossRef]
11. O'Regan, D.; Samet, B. Lyapunov-type inequalities for a class of fractional differential equations. *J. Inequal. Appl.* **2015**, *2015*, 247. [CrossRef]
12. Cabrera, I.; Sadarangani, K.; Samet, B. Hartman–Wintner-type inequalities for a class of nonlocal fractional boundary value problems. *Math. Methods Appl. Sci.* **2017**, *40*, 129–136. [CrossRef]
13. Lupinska, B.; Schmeidel, E. Analysis of some Katugampola fractional differential equations with fractional boundary conditions. *Math. Biosci. Eng.* **2021**, *18*, 7269–7279. [CrossRef] [PubMed]
14. Jarad, F.; Adjari, Y.; Abdeljawad, T.; Mallak, S.; Alrabaiah, H. Lyapunov type inequality in the frame of generalized Caputo derivative. *Discret. Contin. Dyn. Syst. Ser. S* **2021**, *14*, 2335–2355. [CrossRef]
15. Zhang, W.; Zhang, J.; Ni, J. New Lyapunov-type inequalities for fractional multi-point boundary value problems involving Hilfer–Katugampola fractional derivative. *AIMS Math.* **2022**, *7*, 1074–1094. [CrossRef]
16. Cabada, A.; Khaldi, R. Lyapunov-type inequality for higher order left and right fractional p-Laplacian problems. *Proyecciones* **2021**, *40*, 1031–1040. [CrossRef]
17. Li, G.; Zhang, Y.; Guan, Y.; Li, W. Stability analysis of multi-point boundary conditions for fractional differential equation with non-instantaneous integral impulse. *Math. Biosci. Eng.* **2023**, *20*, 7020–7041. [CrossRef] [PubMed]
18. Fan, H.; Rao, Y.; Shi, K.; Wen, H. Global synchronization of fractional-order multi-delay coupled neural networks with multi-link complicated structures via hybrid impulsive control. *Mathematics* **2023**, *11*, 3051. [CrossRef]
19. Fan, H.; Tang, J.; Shi, K.; Zhao, Y. Hybrid impulsive feedback control for drive-response synchronization of fractional-order multi-link Memristive neural networks with multi-delays. *Fractal Fract.* **2023**, *7*, 495. [CrossRef]
20. Ma, D.; Yang, Z. Lyapunov-type inequality and solution for a fractional differential equation. *J. Inequalities Appl.* **2020**, *2020*, 181. [CrossRef]