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# On Fractional Operators Involving the Incomplete Mittag-Leffler Matrix Function and Its Applications

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**Abstract:** In this study, we derive multiple incomplete matrix Mittag-Leffler (ML) functions. We systematically investigate several properties of these incomplete matrix ML functions, which include some general properties and distinct representations of integral transforms. We further study the properties of the Riemann–Liouville fractional integrals and derivatives related to the incomplete matrix ML functions. Additionally, some interesting special cases of this work are highlighted. Finally, we establish the solution to the kinetic equations as an application.

**Keywords:** fractional calculus operators; incomplete hypergeometric matrix function; matrix functional calculus; Mittag-Leffler functions

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## 1. Introduction

Gosta Mittag-Leffler (ML), a Swedish mathematician, introduced the given function in 1903 [1].

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}. \tag{1}$$

The well-known Gamma function, represented by $\Gamma$, is involved in the definition of the ML function, $E_{\alpha}(z)$, for a complex variable $z$. The variable $\alpha$ in this case is non-negative. When $\alpha = 1$, this function corresponds to the standard exponential function, which is a straightforward extension of the exponential function. It is estimated between the hypergeometric function $\frac{1}{1-z}$ and the pure exponential function for $0 < \alpha < 1$. Its application in a number of disciplines, such as engineering, biology, chemistry, physics, and applied sciences, has increased its importance over the past 20 years. Whenever fractional order differential equations or fractional order integral equations are obtained, the ML function naturally appears. Understanding its characteristics and applications in a variety of domains has been made possible by its generalisation, as Wiman first investigated in 1905. $\mathbf{Re}(z)$ and $\mathbf{Im}(z)$ denote the real part and imaginary part, respectively, of a complex number $z$.

The two parametric Mittag-Leffler function is defined by [2]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{2}$$

where $\alpha, \beta \in \mathbb{C}$ with $\mathbf{Re}(\alpha) > 0$ and $\mathbf{Re}(\beta) > 0$.

In 1971, Prabhakar [3] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in the form of

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n \, z^n}{\Gamma(\alpha n + \beta) n!}, \tag{3}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ and $\mathbf{Re}(\alpha) > 0, \mathbf{Re}(\beta) > 0, \mathbf{Re}(\gamma) > 0$, where $(\gamma)_n$ is the Pochhammer symbol, as

$$(\gamma)_n = \begin{cases} \gamma(\gamma+1)...(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}, & n \geq 1, \\ 1, & n = 0. \end{cases} \tag{4}$$

Thus , it is clear that the following special cases holds.

Shukla and Prajapati [4] introduced the ML function generalized in 2007, as follows:

$$E^{\gamma,k}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk}\, z^n}{\Gamma(\alpha n + \beta) n!}, \tag{5}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ and $\mathbf{Re}(\alpha) > \max\{0, \mathbf{Re}(k) - 1\}, \mathbf{Re}(\beta) > 0, \mathbf{Re}(k) > 0$.

After that, many other authors investigated and explored several properties and applications of the generalized ML function in the solution of fractional order integral and fractional order differential equations (see, e.g. [5–9]). Later on, a detailed and comprehensive review was provided by Haubold et al. (see [10–12]).

The solutions for special matrix differential equations have become accessible in recent years due to their special function with matrix parameters. These are systems of differential equations as these matrix differential equations are each carried out by a corresponding scalar special function. A corresponding scalar special function may fulfil the system of equations that represent the other results for special matrix functions, including generating functions, series definitions, recurrence relations, etc. (see [13–18]).

Additionally, the ML function with matrix arguments is equally as useful as its scalar version, and it can be utilized successfully in a wide range of applications, such as control theory and other related fields, the efficient and stable solution of systems of fractional differential equations (FDEs), the estimation of the solution for particular multiterm FDEs, and other related fields (see [19,20]).

At the present, the generalized version allows for matrices as arguments and provides a powerful tool for solving fractional differential and integral equations in matrix form. In 2018, Garrappa et al. [21] computed the Mittag-Leffler function with matrix arguments,with some applications in fractional calculus, and in 2023, Pal et al. [22] introduced a special matrix analog of the four-parameter Mittag-Leffler function.

The main objective of this work is to develop various aspects of incomplete matrix ML functions, which include certain fundamental properties and various representations of integral transformations, and to analyse incomplete ML functions with matrix parameters by means of the incomplete Pochhammer symbol. We investigate some of the features of the Riemann–Liouville fractional integrals and derivatives associated with the incomplete matrix ML functions. We additionally highlight some interesting particular examples of our major results. In the conclusion, incomplete matrix ML functions are utilized as an application to address the kinetic equations.

This work follows the following outline: in Section 2, we look at some of the most common definitions and basic applications of matrix arguments in special functions. A matrix variation of the incomplete ML functions is presented and its convergence is investigated in Section 3. In Section 4, representations based on the Euler-Beta and Laplace transforms for the incomplete matrix ML functions are presented, in addition to some important theorems. Utilizing incomplete matrix ML functions, we acquire several interesting characteristics of the fractional calculus operators in Section 5. The implementation of ML functions in the fractional kinetic equation solution will be discussed in Section 6. Finally, some concluding remarks are collected in Section 7.

## 2. Basic Definitions and Preliminaries

Consider $\mathbb{C}^h$, which denotes a complex vector space of dimension $h$, and let $\mathbb{C}^{h\times h}$ be the set of all square complex matrices of order $h$. For any matrix $T$ in $\mathbb{C}^{h\times h}$, $\sigma(T)$ represents the spectrum of $T$, which is the set containing all eigenvalues of $T$, referred to as $T$, then

$$a(T) = \max\{\mathbf{Re}(z) : z \in \sigma(T)\}, \quad b(T) = \min\{\mathbf{Re}(z) : z \in \sigma(T)\}, \tag{6}$$

where $a(-T) = -b(T)$ and $a(T)$ represent the spectral abscissa of $T$. The square matrix $T$ becomes positive stable if and only if $b(T) > 0$, and we refer to it as [16]

$$\|e^{tT}\| \leq e^{ta(T)} \sum_{s=0}^{h-1} \frac{(\|T\|h^{\frac{1}{2}}t)^s}{s!}; \quad h \geq 1; \quad and\ t \in R \tag{7}$$

and

$$\|n^T\| = \|e^{(\ln n)T}\| \leq n^{a(T)} \sum_{s=0}^{h-1} \frac{(\|T\|h^{\frac{1}{2}}\ln n)^s}{s!}; \quad h \geq 1; \quad and\ t \in R. \tag{8}$$

The Beta matrix function is described similarly (see, e.g. [23–25]), where $T$ and $U$ are positive stable and commuting matrices in $\mathbb{C}^{h\times h}$, so that the matrices $T + nI$, $U + nI$, and $T + U + nI$ are invertible for any integer $n \geq 0$. Now

$$\mathbb{B}(T,U) = \int_0^1 t^{T-I}(1-t)^{U-I}dt = \Gamma(T)\Gamma(U)[\Gamma(T+U)]^{-1}, \tag{9}$$

where

$$\Gamma(T) = \int_0^\infty e^{-t}t^{T-I}dt, \quad t^{T-I} = \exp((T-I)\ln t), \tag{10}$$

is the matrix function of Gamma (see, for example, [26]). In addition, if for all integers $n \geq 0$, we have

$$T + nI \quad \text{is an invertible matrix}, \tag{11}$$

then, $\Gamma(T)$ is proven to be invertible and its inverse is denoted by $\Gamma^{-1}(T)$. This leads to the emergence of the Pochhammer symbol with a matrix argument, as detailed in references such as [27].

$$(T)_n = \begin{cases} T(T+I)\ldots(T+(n-1)I) = \Gamma(T+nI)\,\Gamma^{-1}(T), & n \geq 1, \\ I, & n = 0. \end{cases} \tag{12}$$

As described in Abdalla's work [24], the Gamma matrix functions can be decomposed into two incomplete Gamma matrix functions.

$$\gamma(T,z) = \int_0^z e^{-t}t^{T-I}dt, \tag{13}$$

$$\Gamma(T,z) = \int_z^\infty e^{-t}t^{T-I}dt \tag{14}$$

and

$$\Gamma(T) = \gamma(T,z) + \Gamma(T,z). \tag{15}$$

where $z \in \mathbb{C}$ and $T$ are a positive stable matrix in $\mathbb{C}^{h\times h}$.

Consider that $T$ represents any stable positive matrix within a complex system $\mathbb{C}^{h\times h}$. In this context, for every non-negative $x$, the Pochhammer incomplete matrix $(T;x)_m$, along with its complement $[T;x]_m$, as discussed in [28], is described as follows:

$$(T;x)_m = \gamma(T+mI,x)\,\Gamma^{-1}(T) \tag{16}$$

and

$$[T;x]_m = \Gamma(T+mI,x)\,\Gamma^{-1}(T), \tag{17}$$

satisfy the decomposition formula, respectively.

$$(T;x)_m + [T;x]_m = (T)_m\,. \tag{18}$$

The Laplace transform of the original $\phi$(t) is [29]

$$\bar{\phi}(h) = \mathcal{L}[\phi(t)](h) = \int_0^\infty e^{-ht}\,\phi(t)\,dt, \qquad Re(h) > 0, \tag{19}$$

where $\bar{\phi}(h)$ represents the Laplace transform of $\phi(t)$.

Moreover, the Euler-Beta transform (see [29]) of the function $f(z)$ is given by

$$\beta\{f(z);a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1}f(z)\,dz. \tag{20}$$

**Definition 1** ([22])**.** *Let $T$ and $U$ be two positive stable matrices in $\mathbb{C}^{h\times h}$, such that $U$ satisfys the condition* (11) *. Then, the matrix ML function is as follows:*

$$\begin{aligned} E^{\beta,T}_{\alpha,U}(z) &= \Gamma^{-1}(T)\sum_{m=0}^{\infty}\Gamma(T+\beta mI)\,\Gamma^{-1}(U+\alpha mI)\,\frac{z^m}{m!} \\ &= \sum_{m=0}^{\infty}(T)_{\beta m}\,\Gamma^{-1}(U+\alpha mI)\,\frac{z^m}{m!}, \end{aligned} \tag{21}$$

*where $\alpha, \beta \in \mathbf{R}^+$ .*

## 3. Incomplete Matrix ML Functions

In this section, we discuss a matrix variant of the incomplete ML functions and develop its convergence for $|z| = 1$, and we demonstrate some properties and differential relations.

**Definition 2.** *Suppose $T$ and $U$ are positive stable matrices in $\mathbb{C}^{h\times h}$, such that $U$ satisfies condition* (11)*; we can now define a matrix incomplete ML functions as*

$$\begin{aligned} \mathfrak{E}^{\beta,T}_{\alpha,U}(z) &= \sum_{n=0}^{\infty}\Gamma^{-1}(U+n\alpha I)\,(T;x)_{\beta n}\,\frac{z^n}{n!} \\ &= \Gamma^{-1}(T)\sum_{n=0}^{\infty}\gamma(T+n\beta I,x)\,\Gamma^{-1}(U+n\alpha I)\,\frac{z^n}{n!} \end{aligned} \tag{22}$$

and

$$\begin{aligned} \mathbb{E}^{\beta,T}_{\alpha,U}(Z) &= \sum_{n=0}^{\infty}\Gamma^{-1}(U+n\alpha I)\,[T;x]_{\beta n}\,\frac{z^n}{n!} \\ &= \Gamma^{-1}(T)\sum_{n=0}^{\infty}\Gamma(T+n\beta I,x)\,\Gamma^{-1}(U+n\alpha I)\,\frac{z^n}{n!}, \end{aligned} \tag{23}$$

where $\alpha, \beta$ , x$\in \mathbf{R}^+$, and we can find that

$$\mathfrak{E}^{\beta,T}_{\alpha,U}(z) + \mathbb{E}^{\beta,T}_{\alpha,U}(Z) = E^{\beta,T}_{\alpha,U}(z). \tag{24}$$

**Remark 1.**

*(i)* *For $x = 0$ in (22) and (23), we find the definition in (21)*

*(ii)* *By using (22) and (23), we find that*

$$\mathfrak{E}^{\beta,T}_{\alpha,U}(z) + \mathbb{E}^{\beta,T}_{\alpha,U}(Z) = E^{\beta,T}_{\alpha,U}(z). \tag{25}$$

**Theorem 1.** *Suppose $T$ and $U$ are two positive constants if the following relation holds $a(U) > b(T)$ in $\mathbb{C}^{h\times h}$, then, the matrix of incomplete ML functions (22) converges completely for the given $|z| = 1$.*

**Proof.** Consider that relation $a(U) > b(T)$ holds. Under this assumption, there exists a positive number $\lambda$, such as

$$a(U) - b(T) = 2\lambda. \tag{26}$$

Now, we can write

$$\begin{aligned} & n^{1+\lambda}\left[\Gamma^{-1}(T)\;\gamma(T+\beta nI,x)\Gamma^{-1}(U+\alpha nI)\frac{1}{n!}\right] \\ &= \frac{n^{\lambda}}{(n-1)!}\left(\frac{n^{-T}\,\Gamma^{-1}(T)\;\gamma(T+\beta nI,x)}{(n-1)!}\right)n^{T}\times\left(n^{-U}(n-1)!\,\Gamma^{-1}(U+\alpha nI)\right)n^{U}. \end{aligned} \tag{27}$$

From (8) and $a(-U) = -b(U)$, we have

$$\begin{aligned} \left\|n^{T}\right\|\left\|n^{-U}\right\| &\le n^{b(T)-a(U)}\left\{\sum_{u=0}^{r-1}\frac{\left(\|U\|r^{1/2}\ln n\right)^{u}}{u!}\right\}\times\left\{\sum_{u=0}^{r-1}\frac{\left(\|T\|r^{1/2}\ln n\right)^{u}}{u!}\right\} \\ &\le n^{-2\lambda}\left\{\sum_{u=0}^{r-1}\frac{\left(\max\{\|U\|,\|T\|\}r^{1/2}\ln n\right)^{u}}{u!}\right\}^{2}. \end{aligned} \tag{28}$$

Letting $n \to \infty$ in (27) and (28), for $|z| = 1$, we have

$$\begin{aligned} &\lim_{n\to\infty} n^{1+\lambda}\left\|\Gamma^{-1}(T)\;\gamma(T+\beta nI,x)\;\Gamma^{-1}(U+\alpha nI)\frac{1}{n!}\right\| \\ &\le \lim_{n\to\infty}\frac{n^{\lambda}}{(n-1)!}\left\|\frac{n^{-T}\,\Gamma^{-1}(T)\;\gamma(T+\beta nI,x)}{(n-1)!}\right\|\left\|n^{T}\right\|\left\|n^{U}(n-1)!\,\Gamma^{-1}(U+\alpha nI)\right\|\left\|n^{-U}\right\| \\ &\le \lim_{n\to\infty}\frac{n^{-\lambda}}{(n-1)!}\left\|\frac{n^{-T}\,\Gamma^{-1}(T)\;\gamma(T+\beta nI,x)}{(n-1)!}\right\|\left\|n^{U}(n-1)!\,\Gamma^{-1}(U+\alpha nI)\right\| \\ &\times\left\{\sum_{u=0}^{r-1}\frac{\left(\max\{\|U\|,\|T\|\}r^{1/2}\ln n\right)^{u}}{u!}\right\}^{2} = 0, \end{aligned} \tag{29}$$

on $|z| = 1$ the series (22) is convergent absolutely. □

**Theorem 2.** *Assume T and U are positive stable matrices in $\mathbb{C}^{h\times h}$ and $U+nI$ is invertible for every integer $n \geq 0$. The incomplete matrix ML function defined by (23), with $\alpha, \beta > 0$, satisfies*

$$\mathbb{E}^{\beta,T}_{\alpha,U}(z) = U\,\mathbb{E}^{\beta,T}_{\alpha,U+I}(z) + \alpha z \frac{d}{dz}\,\mathbb{E}^{\beta,T}_{\alpha,U+I}(z). \tag{30}$$

**Proof.** By using (23), we find that

$$\begin{aligned} U\,\mathbb{E}^{\beta,T}_{\alpha,U+I}(z) + \alpha\, z\frac{d}{dz}\,\mathbb{E}^{\beta,T}_{\alpha,U+I}(z) &= U\,\mathbb{E}^{\beta,T}_{\alpha,U+I}(z) + \alpha\, z\frac{d}{dz}\sum_{n=0}^{\infty}\Gamma^{-1}(U+(n\alpha+1)I)\;[T;x]_{\beta n}\,\frac{z^n}{n!} \\ &= U\,\mathbb{E}^{\beta,T}_{\alpha,U+I}(z) + \sum_{n=0}^{\infty}\Gamma^{-1}(U+(n\alpha+1)I)\;[T;x]_{\beta n}\;(\alpha n I + U - U)\,\frac{z^n}{n!} \\ &= \mathbb{E}^{\beta,T}_{\alpha,U}(z). \end{aligned} \tag{31}$$

The Theorem 2 proof has become complete. □

**Remark 2.** *For $x=0$ in (30), we have*

$$E^{\beta,T}_{\alpha,U}(z) = U E^{\beta,T}_{\alpha,U+I}(z) + \alpha z \frac{d}{dz}\,E^{\beta,T}_{\alpha,U+I}(z). \tag{32}$$

**Theorem 3.** *Suppose $U, T$ in $\mathbb{C}^{h\times h}$ is a positive stable matrix, such that U satisfies a condition (11). Then, the derivative of matrix of the incomplete ML function can be represented by*

*(i)* $\left(\frac{d}{dz}\right)^m \mathbb{E}^{\beta,T}_{\alpha,U}(z) = (T)_{\beta m}\,\mathbb{E}^{\beta,T+\beta m I}_{\alpha,U+m\alpha I}(z),$

*(ii)* $\left(\frac{d}{dz}\right)^m \left[z^{U-I}\mathbb{E}^{\beta,T}_{\alpha,U}(wz^{\alpha})\right] = z^{U-(m+I)I}\,\mathbb{E}^{\beta,T}_{\alpha,U-mI}\,(wz^{\alpha}).$

**Proof.** (i) By applying term-wise differentiation $m$ times, (23) gives

$$\begin{aligned} \left(\frac{d}{dz}\right)^m \mathbb{E}^{\beta,T}_{\alpha,U}(z) &= \sum_{n=m}^{\infty}\Gamma^{-1}(U+n\alpha I)\;[T;x]_{\beta n}\,\frac{z^{n-m}}{(n-m)!} \\ &= \sum_{n=0}^{\infty}\Gamma^{-1}(U+(\alpha(n+m)I)\;[T;x]_{\beta(n+m)}\,\frac{z^n}{n!} \\ &= (T)_{\beta m}\sum_{n=0}^{\infty}\Gamma^{-1}(U+\alpha(n+m)I)\;[T+\beta m I;x]_{\beta n}\,\frac{z^n}{n!} \\ &= (T)_{\beta m}\,\mathbb{E}^{\beta,T+\beta m I}_{\alpha,U+m\alpha I}(z). \end{aligned}$$

(ii) By using (23), we obtain

$$\left(\frac{d}{dz}\right)^m \left[z^{U-I}\mathbb{E}^{\beta,T}_{\alpha,U}(wz^{\alpha})\right] = \sum_{n=0}^{\infty}\Gamma^{-1}(U+\alpha n I)\;[T;x]_{\beta n}\,\frac{w^n}{n!}\left(\frac{d}{dz}\right)^m\left[z^{U+(\alpha n-1)I}\right].$$

Differentiating term by term under the sign of summation, we find that

$$\begin{aligned} \left(\frac{d}{dz}\right)^m \left[z^{U-I}\,\mathbb{E}^{\beta,T}_{\alpha,U}(wz^{\alpha})\right] &= \sum_{n=0}^{\infty}\Gamma^{-1}(U+(\alpha n-m)I)\;[T;x]_{\beta n}\,\frac{w^n}{n!}\left[z^{U+(\alpha n-m-1)I}\right] \\ &= z^{U-(m+I)I}\,\mathbb{E}^{\beta,T}_{\alpha,U-mI}.(wz^{\alpha}). \end{aligned}$$

This complete the proof. □

**Remark 3.**

*(i)* *Taking* $\alpha$ *and* $\beta = 1$ *in (ii), we obtain*

$$\left(\frac{d}{dz}\right)^m \left[z^{U-I}\, {}_1\Gamma_1\left[[T;x];U;wz\right]\right] = \Gamma^{-1}(U-mI)\,\Gamma(U)\,z^{U-(m+I)I}\,{}_1\Gamma_1\left[[T;x];U-mI;wz\right]. \tag{33}$$

*(ii)* *Using* $x = 0$ *in Theorem 3, we obtain*

*(a)* $\left(\frac{d}{dz}\right)^m E^{\beta,T}_{\alpha,U}(z) = (T)_{\beta m}\, E^{\beta,T+kmI}_{\alpha,U+m\alpha I}(z)$.

*(b)* $\left(\frac{d}{dz}\right)^m \left[z^{U-I} E^{\beta,T}_{\alpha,U}(wz^\alpha)\right] = z^{U-(m+I)I} E^{\beta,T}_{\alpha,U-mI}(wz^\alpha)$.

## 4. Some Integral Transforms of Matrix Incomplete ML Function

We establish the Euler-Beta and Laplace transform representations for the incomplete matrix ML functions in this section. Utilizing incomplete Fox–Wright matrix functions, we first established incomplete matrix ML functions as follows:

**Definition 3.** *Let* $T_P$ *and* $U_q$ *be positive stable matrices in* $\mathbb{C}^{h\times h}$*, such that* $U_q + nI$ *is invertible for every integer* $n \geq 0$*; thus, we can defne the incomplete Fox–Wright matrix function as follows*

$$ {}_p\psi_q^{(\gamma)}\left[\begin{array}{l}(T_1,\beta_1;x),(T_2,\beta_2)....(T_P,\beta_P)\\(U_1,\alpha_1)....(U_q,\alpha_q)\end{array};z\right]$$

$$= \sum_{n\geq 0}\gamma(T_1+n\beta_1 I;x)\prod_{l=2}^{p}\Gamma(T_l+n\beta_l I)\prod_{j=1}^{q}\Gamma^{-1}(U_j+n\alpha_l I)\frac{z^n}{n!},$$

*where p and q are finite positive integers.*

**Definition 4.** *Suppose* $T$*,* $U$ *in* $\mathbb{C}^{h\times h}$ *is positive stable matrices, as* $U$ *satisfies Equation (11). Using the incomplete Fox–Wright matrix, we can generate the incomplete matrix ML functions as follows:*

$$\mathfrak{E}^{\beta,T}_{\alpha,U}(z) = \Gamma^{-1}(T)\,{}_1\psi_1^{(\gamma)}\left[\begin{array}{l}(T,\beta;x)\\(U,\alpha)\end{array};z\right] \tag{34}$$

and

$$\mathbb{E}^{\beta,T}_{\alpha,U}(z) = \Gamma^{-1}(T)\,{}_1\psi_1^{(\gamma)}\left[\begin{array}{l}(T,\beta;x)\\(U,\alpha)\end{array};z\right]. \tag{35}$$

**Theorem 4.** *Suppose* $A$*,* $T$*, and* $U$ *in* $\mathbb{C}^{h\times h}$ *are positive stable matrices, such that* $U$ *holds in Equation (11). Then, the Laplace transform representation of the incomplete matrix ML function satisfies*

$$\mathcal{L}\left\{z^{A-I}\,\mathbb{E}^{\beta,T}_{\alpha,U}(wz^m)\right\} = s^{-A}\,\Gamma^{-1}(T)\,{}_2\psi_1^{(\Gamma)}\left[\begin{array}{l}(T,\beta;x),(A,m)\\(U,\alpha)\end{array};ws^{-m}\right]. \tag{36}$$

**Proof.** From (19) of the Laplace transform and utilizing Equation (23), we obtain

$$\begin{aligned}\mathcal{L}\Big\{z^{A-I}\,\mathbb{E}_{\alpha,U}^{\beta,T}(wz^m)\Big\} &= \int_0^\infty z^{A-I}\,e^{-sz}\,\mathbb{E}_{\alpha,U}^{\beta,T}(wz^m)dz \\ &= \int_0^\infty z^{A-I}\,e^{-sz}\left(\sum_{n=0}^\infty \Gamma^{-1}(U+\alpha nI)\,[T;x]_{\beta m}\,\frac{\omega^n z^{mn}}{n!}\right)dz \\ &= \sum_{n=0}^\infty \Gamma^{-1}(U+\alpha nI)\,[T;x]_{\beta m}\,\frac{\omega^n}{n!}\int_0^\infty z^{A+(mn-1)I}\,e^{-sz}dz \\ &= \sum_{n=0}^\infty \Gamma^{-1}(U+\alpha nI)\,[T;x]_{\beta m}\,\Big[s^{A+mnI}\Big]^{-1}\,\Gamma(A+mnI)\,\frac{\omega^n}{n!} \\ &= \Gamma^{-1}(T)\,s^{-A}\,{}_2\psi_1^{(\Gamma)}\left[\begin{array}{l}(T,\beta;x),(A,m)\\(U,\alpha)\end{array};ws^{-m}\right]. \end{aligned} \tag{37}$$

Thus, the proof is completed. □

**Remark 4.**

*(i)* *If we put $A=U,\beta=1$, and $m=\alpha$ in Equation (36), we have*

$$\int_0^\infty z^{U-I}e^{-sz}E_{\alpha,U}^{1,T}(wz^\alpha) = s^{-U}\,{}_1\Gamma_0\Big[(T;x);-;\frac{w}{s^\alpha}\Big].$$

*(ii)* *If putting $x=0$ in Equation (36), we find that*

$$\mathcal{L}\Big\{z^{A-I}E_{\alpha,U}^{\beta,T}(wz^m)\Big\} = \Gamma^{-1}(T)\,s^{-A}\,{}_2\psi_1^{(\Gamma)}\left[\begin{array}{l}(T,\beta),(A,m)\\(U,\alpha)\end{array};\frac{w}{s^m}\right].$$

**Theorem 5.** *The Euler-Beta transform repression of incomplete matrix ML function is given as follows:*

$$\beta\Big\{\mathbb{E}_{\alpha,U}^{\beta,T}(wz^m);A,B\Big\} = \Gamma^{-1}(T)\Gamma(B)\,{}_2\psi_2^{(\Gamma)}\left[\begin{array}{l}(T,\beta;x),(A,m);\\(U,\alpha),(A+B,m);\end{array}w\right], \tag{38}$$

*where $A$,$B$, $T$, and $U$ are positive stable matrices in $\mathbb{C}^{h\times h}$, such that $U+kI$ is invertible for all $k\geq 0$.*

**Proof.** From Equation (20) of the Euler-Beta transform, one can easily obtain

$$\begin{aligned}\beta\Big\{\mathbb{E}_{\alpha,U}^{\beta,T}(wz^m);A,B\Big\} &= \int_0^\infty z^{A-I}(1-z)^{B-I}\mathbb{E}_{\alpha,U}^{\beta,T}(\omega z^m)dz \\ &= \int_0^1 z^{A-I}(1-z)^{B-I}\left(\sum_{n=0}^\infty \Gamma^{-1}(U+n\alpha I)\,[T;x]_{\beta n}\,\frac{w^n z^{mn}}{n!}\right) \\ &= \sum_{n=0}^\infty \Gamma^{-1}(U+n\alpha I)\,[T;x]_{\beta n}\,\frac{w^n}{n!}\left(\int_0^1 z^{A+(nm-1)I}(1-z)^{B-I}dz\right) \\ &= \sum_{n=0}^\infty \Gamma^{-1}(U+n\alpha I)\,[T;x]_{\beta n}\,\Gamma^{-1}(A+B+nmI)\,\Gamma(B)\,\Gamma(A+nm)\,\frac{w^n}{n!}, \end{aligned}$$

which, in view of the definitions of ${}_2\psi_2^{(\Gamma)}$ in Definition 3, provide the proper representation (38). □

**Remark 5.**

*(i)* *If putting $A = U$ and $\alpha = m$ in (36), we obtain*

$$\int_0^\infty z^{U-I}(1-z)^{B-I} E_{m,U}^{\beta,T}(wz^m)dz = \Gamma(B)\, E_{m,U+B}^{\beta,T}(w).$$

## 5. Fractional Calculus Operators with Incomplete Matrix ML Functions

We derive several interesting features of the fractional calculus operators with respect to incomplete matrix ML functions in this section.

For an operator of fractional order $\mu$ and $x > 0$, the integral and derivatives of Riemann–Liouville with respect to $\mathbf{Re}(\mu) > 0$ are presented in the subsequent form (see [13,30])

$$(\mathbf{I}_a^\mu f)(x) = {}_0D_t^{-\mu}[f(t)] = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)dt. \tag{39}$$

Moreover,

$$\mathbf{D}_a^\mu f(x) = \mathbf{I}_a^{n-\mu} \mathbf{D}^n f(x),\ \mathbf{D} = \frac{d}{dx}. \tag{40}$$

Bakhet [13] studied the fractional order integrals and derivatives using the operators (39) and (40) as follows

**Definition 5.** *Let $T$ be a stable positive matrix in $\mathbb{C}^{h\times h}$ with the properties $\mathbf{Re}(\mu) > 0$ and $\mu \in \mathbb{C}$. The fractional integrals of order $\mu$ in the Riemann–Liouville sense is defined as*

$$\mathbf{I}^\mu(x^T) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1}\, t^T dt. \tag{41}$$

**Lemma 1.** *Let $T$ to be a positive stable matrix in $\mathbb{C}^{h\times h}$, such that $\mathbf{Re}(\mu) > 0$. Then, the Riemann–Liouville integrals fractional of order $\mu$ can be written as*

$$\mathbf{I}^\mu(x^{T-I}) = \Gamma(T)\Gamma^{-1}(T+\mu I)\, x^{T+(\mu-1)I}. \tag{42}$$

**Theorem 6.** *Suppose $T$ and $U$ are a positive stable matrices in $\mathbb{C}^{h\times h}$ and $\mathbf{Re}(\mu) > 0$; then, the fractional integration of incomplete matrix ML function can be denoted as*

$$\mathbf{I}^\mu\left[x^{U-I}\mathbb{E}_{\alpha,U}^{\beta,T}(wx^\alpha)\right] = x^{U+(\mu-1)I}\, \mathbb{E}_{\alpha,U+\mu I}^{\beta,T}(wx^\alpha). \tag{43}$$

**Proof.** From Definition (5) and (23),we obtain

$$\begin{aligned} \mathbf{I}^\mu\left[x^{U-I}\, \mathbb{E}_{\alpha,U}^{T,k}(wx^\alpha)\right] &= \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1}\, t^{U-I}\, \mathbb{E}_{\alpha,U}^{T,k}(wx^\alpha)dt \\ &= \sum_{m=0}^{\infty} [T;x]_{\beta m}\, \Gamma^{-1}(U+\alpha m I)\, \frac{\beta^m}{m!} \times \mathbf{I}^\mu\left(x^{U+(\alpha m-1)I}\right). \end{aligned}$$

By using Lemma 1, we obtain

$$
\begin{aligned}
&\mathbf{I}^{\mu}\left[x^{U-I}\,\mathbb{E}_{\alpha,U}^{\beta,T}(wx^{\alpha})\right] \\
&= \sum_{m=0}^{\infty} [T;x]_{\beta m}\,\Gamma^{-1}(U+\alpha mI)\,\frac{\beta^m}{m!} \\
&\quad \times x^{U+(\alpha m+\mu-1)I}\,\Gamma(U+\alpha mI)\,\Gamma^{-1}(U+(\alpha m+\mu I)) \\
&= \sum_{m=0}^{\infty} [T;x]_{\beta m}\,\Gamma^{-1}(U+(\alpha m+\mu I))\,\frac{\beta^m}{m!}\times x^{U+(\alpha m+\mu-1)I}.
\end{aligned}
$$

Hence the Theorem 6 is proved. □

**Corollary 1.** *Suppose $x=0$ is substituted in Theorem 6; then, the fractional integration of the Mittag-Leffler matrix function will be*

$$\mathbf{I}^{\mu}\left[x^{U-I}\,\mathrm{E}_{\alpha,U}^{\beta,T}(\kappa x^{\alpha})\right] = x^{U+(\mu-1)I}\mathrm{E}_{\alpha,U+\mu I}^{\beta,T}(\kappa x^{\alpha}),$$

*where* $\mathbf{Re}(\mu)>0$.

**Theorem 7.** *Suppose $U$ and $T$ are a positive stable matrices in $\mathbb{C}^{h\times h}$ and $0<\mathbf{Re}(\mu)$. Then, the fractional derivative of the incomplete matrix ML function denotes*

$$\mathbf{D}^{\mu}\left[x^{U-I}\mathbb{E}_{\alpha,U}^{\beta,T}(wx^{\alpha})\right] = x^{U-(\mu+1)I}\,\mathbb{E}_{\alpha,U-\mu I}^{\beta,T}(wx^{\alpha}). \tag{44}$$

**Proof.** By using (40), we find

$$\mathbf{D}^{\mu}\left[x^{U-I}\mathbb{E}_{\alpha,U}^{\beta,T}(wx^{\alpha})\right] = \left(\frac{d}{dx}\right)^{n}\left[\mathbf{I}^{n-\mu}\left([x^{U-I}\mathbb{E}_{\alpha,U}^{\beta,T}(wx^{\alpha})\right)\right].$$

Using Theorem 6 gives

$$\mathbf{D}^{\mu}\left[x^{U-I}\,\mathbb{E}_{\alpha,U}^{\beta,T}(wx^{\alpha})\right] = \left(\frac{d}{dx}\right)^{n}\left[x^{U-(n-\mu-1)I}\,\mathbb{E}_{\alpha,U-(n-\mu)I}^{\beta,T}(wx^{\alpha})\right].$$

By using (ii) in Theorem 7, this immediately yields the desired proof. □

**Corollary 2.** *When putting $x=0$ in Theorem (7), we find the fractional derivative of matrix ML function as*

$$D^{\mu}\left[x^{U-I}E_{\alpha,U}^{\beta,T}(\kappa x^{\alpha})\right] = x^{U-(\mu+1)I}E_{\alpha,U-\mu I}^{\beta,T}(\kappa x^{\alpha}).$$

*where,* $\mathbf{Re}(\mu)>0$.

## 6. Application to the Solution of Fractional Kinetic Equation

The resolution of fractional kinetic equations (FKEs) has attracted significant attention from researchers in a number of applied scientific domains, which include engineering, dynamical systems, physics, and control systems. Its ability to support the development of mathematical models for an extensive variety of physical procedures and mathematical physics applications is the reason that it has attracted this attention. Kinetic equations (KEs) are crucial for mathematical physics and natural science as they explain the continuity of matter's motion in a variety of astrophysical situations. As stated in references like [12,31], recent research has discovered various fractional calculus operators that enable extending and generalizing FKEs. Haubold and Mathai established a functional differential equation in [12], which relates the rate of reaction change, decomposition rate, and production rate. It appears as follows:

$$\frac{dN}{dt} = -d(N_t)+p(N_t), \tag{45}$$

where $N = N(t)$ represents the reaction rate, $d = d(N)$ represents the destruction rate , $p = p(N)$ denotes the production rate, and $N_t$ satisfies the function defined by $N_t(t^*) = N(t - t^*), t^* > 0$.

The differential equation given below is a particular case of (45) when spatial fluctuations or inhomogeneities in the value $N(t)$ are ignored.

$$\frac{dN}{dt} = -m_i N_i(t), \tag{46}$$

The initial conditions specified for this differential equations are $N_i(t = 0) = N_0$, where $i$ represents the number of density of species at time $t = 0$, $m_i > 0$. If term $i$ is ignored and the specific kinetic Equation (46) is integrated, the following relation will be obtained

$$N(t) - N_0 = -m\,{}_0D_t^{-1}N(t). \tag{47}$$

Within this context, $m_0$ is a constant value, while $D_t^{-1}$ stands for the Riemann–Liouville integral operator with a degree of $\mu = 1$. The fractional kinetic equation (FKE) is redefined by Haubold and Mathai as follows, as detailed in [12].

$$N(t) - N_0 = -m^{\mu}\,{}_0D_t^{-\mu}N(t). \tag{48}$$

where ${}_0D_t^{-\mu}$ is given in (39).

In light of this, the solution for $N(t)$ can be expressed as follows:

$$N(t) = N_0 \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(\mu r + 1)} (mt)^{\mu r} = N_0\, E_{\mu}(-m^{\mu}t^{\mu}), \tag{49}$$

where $E_{\mu}(-m^{\mu}t^{\mu})$ denotes the ML function (see [32,33]).

Furthermore, Saxena Kalla created an alternative FKE, as given in [31,34–36].

$$N(t) - N_0\, f(t) = -m^{\mu}\,{}_0D_t^{-\mu}N(t), \qquad m > 0,\ Re(\mu) > 0, \tag{50}$$

here, $m$ is a constant, $N(t)$ represents the species amount, which is initially $N_0 = N(0)$ at time level $t = 0$; however, $f$ is an integrable function at interval $(0, \infty)$.

As discussed in [34,37–41], a variety of research articles have appeared recently in this field utilising FKEs to solve different integral transforms, such as Fourier, Laplace, Sumudu, and Mellin transforms, including special functions and matrix functions.

In this theorem, we examined the solutions for FKEs that require the extension of incomplete matrix ML functions.

**Theorem 8.** *Consider $T$, $U$, and $R$ holds (11), where $T$, $U$, and $R$ are positive stable matrices in $\mathbb{C}^{h\times h}$. Then, for $\mathbf{Re}(v) > 0$ and $t, w \in \mathbb{C}$, the following generalized FK matrix equation of the incomplete matrix ML functions satisfies the following equation*

$$N(t)I - N_0\, \mathbb{E}_{\alpha,U}^{\beta,T}(wt) = -R^{v}\,{}_0D_t^{-v}N(t) \tag{51}$$

*has the solution*

$$N(t) = N_0 \sum_{n=0}^{\infty} [T; x]_{\beta n}\, \Gamma^{-1}(U + \alpha n I) \times (wt)^n\, E_{\nu,\beta+1}(-R^{\nu}t^{\nu}), \tag{52}$$

*where $E_{v,\beta+1}(-R^{v}t^{v})$ is called the generalized the Mittag–Leffler function (see [33]).*

**Proof.** By using the Laplace transform and using incomplete matrix ML function, we have

$$N^*(s)I = N_0 \left( \int_0^{\infty} e^{-st}\, \Gamma^{-1}(U + \alpha n I)[T; x]_{\beta n}\, (wt)^n dt \right) - R^{v}s^{-v}\, N^*(s), \tag{53}$$

where $N^*(s) = w\{N(t);s\}$ and

$$\left(1+\left(\frac{R}{s}\right)^v\right)N^*(s) = N_0 \sum_{n=0}^{\infty}[T;x]_{\beta n}\ \Gamma^{-1}(U+n\alpha I)\ \Gamma((n+1)I)\ \frac{w^n}{s^{n+1}},$$

we find that

$$\begin{aligned} N^*(s)I &= N_0 \sum_{n=0}^{\infty}[T;x]_{\beta n}\ \frac{w^n}{s^{n+1}};\Gamma^{-1}(U+n\alpha I)\left(1+\left(\frac{R}{s}\right)^v\right)^{-1} \\ &= N_0 \sum_{n=0}^{\infty}[T;x]_{1_n}\Gamma^{-1}(U+n\alpha I)\frac{w^n}{s^{n+1}} \times \sum_{h=0}^{\infty}(-1)^h\left(\frac{R}{s}\right)^{vh}. \end{aligned} \tag{54}$$

Employing the relation through the use of the inverse Laplace Transform

$$\mathcal{L}^{-1}(s^{-v},t) = \frac{t^{v-1}}{\Gamma(v)}, \quad (Re(v)>0),$$

we have,

$$\begin{aligned} N(t)I &= L^{-1}\{N^*(s),t\} \\ &= N_0 \sum_{n=0}^{\infty}[T;x]_{\beta n}\Gamma^{-1}(U+n\alpha I)\times(wt)^n \times \sum_{h=0}^{\infty}\frac{(-1)^h(Rt)^{\nu h}}{\Gamma(\nu h+\beta+1)} \\ &= N_0 \sum_{n=0}^{\infty}[T;x]_{\beta n}\Gamma^{-1}(U+n\alpha I)\times(wt)^n \times E_{\nu,n+1}(-R^v t^v). \end{aligned}$$

The proof is now completed. □

**Corollary 3.** *Consider $T$, $U$ and $R$ holds (11), where $T$, $U$, and $R$ are positive stable matrices in $\mathbb{C}^{h\times h}$. Then, for $\mathbf{Re}(v)>0$ and $t\in\mathbb{C}$, the generalized FK matrix equation of matrix ML functions satisfies the following equation*

$$N(t)I - N_0\ E^{\beta,T}_{\alpha,U}(wt) = -R^\nu\ {}_0D_t^{-v}N(t) \tag{55}$$

*has the solution*

$$N(t)I = N_0 \sum_{n=0}^{\infty}(T)_{\beta n}\ \Gamma^{-1}(U+\alpha n I)\times(wt)^n\ E_{\nu,\beta+1}(-R^\nu t^\nu). \tag{56}$$

## 7. Conclusions

In conclusion, this paper has explored an incomplete Mittag-Leffler (ML) function where matrix arguments were introduced. Some properties of these functions, such as functional relations, convergent, integral formulas, and integral representations were investigated and the properties of the Riemann–Liouville fractional integrals and derivatives related to the incomplete matrix ML function were studied. Additionally, some interesting special cases of this work were highlighted. Also, we established a solution to the kinetic equations involving the incomplete matrix ML function. Ultimately, these theoretical advancements find practical applications, particularly in theorem of the incomplete matrix ML functions across diverse types, such as the k-incomplete matrix ML function. This research opens avenues for further exploration and development within this intricate field of study. Certain applications to other research subjects and investigation regarding other properties of these newly introduced functions are left to the authors and the interested researchers for future study.

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