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Some Generalized Neutrosophic Metric Spaces and Fixed Point Results with Applications

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Abstract: This paper contains several novel definitions including neutrosophic E_β metric space, neutrosophic quasi-S_β-metric space, neutrosophic pseudo-S_β-metric space, neutrosophic quasi-E-metric space and neutrosophic pseudo-E_β-metric space. Further, we present some generalized fixed point results with non-trivial examples and the decomposition theorem in the setting of the neutrosophic pseudo-E_β-metric space. Moreover, by using the main result, we examine the existence and uniqueness of the solution to an integral equation, a system of linear equations, and nonlinear fractional differential equations.

Keywords: neutrosophic metric space; neutrosophic sets; system of linear equations; existence and uniqueness

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1. Introduction

There are numerous applications for fixed points (FPs) in neutrosophic metric spaces in the fields of mathematics, computer science, economics, and engineering. By adding the idea of indeterminacy or uncertainty, neutrosophic metric space broadens the concept of metric spaces. The literature on mathematical analysis contains many generalizations of a metric space (MS). The well-known 2-MS concept was first proposed by Gahlar [1]; however, in fact, MS is continuous while 2-MS is not. Following that, Dhage [2] presented the notion of D-MS; however, Mustafa and Sims [3] clarified that several of D-MS's topological characteristics were false, and they provided the concept of G-MS. According to Jleli and Samet [4], most of the FP theorems in the context of the G-MS can be easily verified by using MS and quasi MS. Authors in [5] presented the notion of S-MS, in 2012, and demonstrated several FP theorems in the context of complete S-MS. Bakhtin [6] introduced b-MS, in 1989, by multiplying the right side of triangle inequality by a real value $\varpi \geq 1$. In b-MS, it becomes MS if we choose $\varpi = 1$. Subsequently, Sedghi et al. [7] combined the concepts of b-MS and S-MS to introduce the idea of S_β-MS; however, S_β-MS is not continuous.

Zadeh [8] presented the notion of fuzzy logic. As compared to the notion of traditional logic, fuzzy logic attributes a number to an element inside the interval [0, 1], even though certain numbers are not contained within the set. Uncertainty, a crucial component of actual difficulty, has assisted Zadeh in learning fuzzy set (FS) theories in order to cope with the problem of indefinites. For a variety of processes, including one that incorporates the use of fuzzy logic, the theory is viewed as an FP in the fuzzy metric space (FMS). Eventually, Heilpern [9] and Zadeh's results developed the fuzzy mapping idea and a theorem based on FPs for fuzzy contraction mapping in linear MS, representing a fuzzy generalized version of Banach's contraction concept. If the distance between the elements is not an exact integer, imprecision is introduced in the concept of FMSs given by Kaleva and Saikkala [10]. Following that, the notation of an FMS was introduced, first by Kramosil and Michalek [11] and then by George and Veeramani [12]. The definition and various characteristics of fuzzy b-MS were established by Nadaban [13]. Malviya [14] used contraction mappings to demonstrate several FP results and presented the ideas of N-FMS and pseudo N-FMS. The notion of N_b-FMS and its topological features were demonstrated with several FP theorems for contraction mappings by Fernandez et al. [15]. Using the ideas of intuitionistic FSs, continuous t-norm (CtN), and continuous t-conorm (CtCN), Park [16] introduced intuitionistic fuzzy metric spaces (IFMSs), in 2004, as a generalization of FMSs. Many scholars [17–22] then turned their attention to IFMS generalizations and developed FP results for contraction mappings. Several FP theorems for contraction mappings under random conditions were proven by Ionescu et al. [23] and Mehmood et al. [24] in their work on IFMSs and extended b-metric spaces. The concept of neutrosophic metric space (NMS) was introduced and various results were proven by Murat and Necip [25]. Ishtiaq et al. [26] provided the notion of intuitionistic fuzzy N_b metric space and proved numerous FP results. See [27–29] for more details related to this study.

The generalizations of NMSs increase the area of mathematical analysis by providing tools for understanding complicated structures and systems, as well as insights into abstract algebraic and topological features. They are critical for building the theoretical foundations of many fields of mathematics, as well as applying mathematical principles to a wide range of scientific and real-world situations. Motivated by [25,26], we present numerous notions including neutrosophic E_β metric space (NNbMS), neutrosophic quasi-S_β-metric space (NQSbMS), neutrosophic pseudo-S_β-metric space (NPSbMS), neutrosophic quasi-E-metric space (NQNMS) and neutrosophic pseudo-E_β-metric space (NPNbMS). We prove several FP theorems and decomposition theorems in the context of NNBMS. At the end, we apply the main result and find the existence and uniqueness of an integral equation (IE), a system of linear equations (SLEs) and nonlinear fractional differential equations (FDE).

2. Preliminaries

We include several definitions from existing literature to support the main study.

Definition 1 ([22])**.** *Assume that* $* : I^3 \to I \ (I = [0,1])$ *be a mapping.* $*$ *is said to be a CtN if it fulfills the following axioms:*

(i) $*(\gamma, 1, 1) = \gamma,\ *(0, 0, 0) = 0;$
(ii) $*(\gamma, \beta, \pi) = *(\gamma, \pi, \beta) = *(\beta, \pi, \gamma);$
(iii) ** is continuous;*
(iv) $*(\gamma_1, \beta_1, \pi_1) \geq *(\gamma_2, \beta_2, \pi_2)$ *for* $\gamma_1 \geq \gamma_2,\ \beta_1 \geq \beta_2, \pi_1 \geq \pi_2.$

$\gamma * \beta * \pi = \gamma\beta\pi$ *is a to be product CtN and* $\gamma * \beta * \pi = min\{\gamma, \beta, \pi\}$ *is a minimum CtN.*

Definition 2 ([22])**.** *Assume that* $\diamond : I^3 \to I \ (I = [0,1])$ *be a mapping.* $\diamond$ *is called a CtCN if it satisfies the conditions below:*

(i) $\diamond(\gamma, 0, 0) = \gamma,\ \diamond(0, 0, 0) = 0;$
(ii) $\diamond(\gamma, \beta, \pi) = \diamond(\gamma, \pi, \beta) = \diamond(\beta, \pi, \gamma);$
(iii) $\diamond$ *is continuous;*
(iv) $\diamond(\gamma_1, \beta_1, \pi_1) \geq \diamond(\gamma_2, \beta_2, \pi_2)$ *for* $\gamma_1 \geq \gamma_2,\ \beta_1 \geq \beta_2, \pi_1 \geq \pi_2.$

$\gamma \diamond \beta \diamond \pi = max\{\gamma, \beta, \pi\}$ *is an example of a maximum CtCN.*

Definition 3 ([14])**.** *A triple* $(\mathbf{Z}, E, *)$ *is an NFMS with an FS E on* $\mathbf{Z}^3 \times (0, +\infty)$ *and* $*$ *is a CtN if it fulfills the axioms below for all* $z, \varkappa, \nu, \gamma \in \mathbf{Z}$ *and* $\varsigma, \boldsymbol{\varpi}, \boldsymbol{\sigma} > 0:$

(a) $E(z, \varkappa, \nu, \boldsymbol{\sigma}) > 0$,
(b) $E(z, \varkappa, \nu, \boldsymbol{\sigma}) = 1$ *if and only if* $z = \varkappa = \nu$,
(c) $E(z, \varkappa, \nu, \varsigma + \boldsymbol{\varpi} + \boldsymbol{\sigma}) \geq E(z, z, \gamma, \varsigma) * E(\varkappa, \varkappa, \gamma, \boldsymbol{\varpi}) * E(\nu, \nu, \gamma, \boldsymbol{\sigma})$,
(d) $E(z, \varkappa, \nu, .) : (0, +\infty) \to (0, 1]$ *is a continuous function (CF).*

Definition 4 ([7]). *Let Z be an arbitrary set that is not empty and* $\boldsymbol{\zeta} \geq 1$ *be a positive real number. Then, a mapping* $S_\beta : \mathbf{Z}^3 \to [0, +\infty)$ *is said to be* S_β*-metric if it satisfies for all* $z, \varkappa, \nu, \gamma \in \mathbf{Z}$:

(S1) $S_\beta(z, \varkappa, \nu) = 0$ *if* $z = \varkappa = \nu$,
(S2) $S_\beta(z, \varkappa, \nu) \leq \boldsymbol{\zeta}\left[S_\beta(z, z, \gamma) + S_\beta(\varkappa, \varkappa, \gamma) + S_\beta(\nu, \nu, \gamma)\right]$.

Then, $(Z,\ S_\beta)$ *is an* S_β*- MS.*

Definition 5 ([16]). *Let* $(\mathbf{Z}, E, \Theta, *, \diamond)$ *be an IFMS if* $\mathbf{Z} \neq \varnothing$ *is an arbitrary set,* $*$ *is a CtN,* $\diamond$ *is a CtCN, and E,* Θ *are FSs on* $\mathbf{Z}^2 \times (0, +\infty)$ *fulfilling the axioms which are given below for all* $z, \varkappa, \nu \in \mathbf{Z}$ *and* $\boldsymbol{\varpi}, \boldsymbol{\sigma} > 0$:

(i) $E(z, \varkappa, \boldsymbol{\sigma}) + \Theta(z, \varkappa, \boldsymbol{\sigma}) \leq 1$,
(ii) $E(z, \varkappa, \boldsymbol{\sigma}) > 0$,
(iii) $E(z, \varkappa, \sigma) = 1$ *if and only if* $z = \varkappa$,
(iv) $E(z, \varkappa, \varpi + \boldsymbol{\sigma}) \geq E(z, \nu, \boldsymbol{\varpi}) * E(\nu, \varkappa, \boldsymbol{\varpi})$,
(v) $E(z, \varkappa, .) : (0, +\infty) \to (0, 1]$ *is a CF,*
(vi) $\Theta(z, \varkappa, \boldsymbol{\sigma}) > 0$,
(vii) $\Theta(z, \varkappa, \boldsymbol{\sigma}) = 0$ *if and only if* $z = \varkappa$,
(viii) $\Theta(z, \varkappa, \boldsymbol{\varpi} + \boldsymbol{\sigma}) \leq \Theta(z, \nu, \boldsymbol{\varpi}) \diamond \Theta(\nu, \varkappa, \boldsymbol{\varpi})$,
(ix) $\Theta(z, \varkappa, .) : (0, +\infty) \to (0, 1]$ *is a CF.*

Definition 6 ([26]). *Let a six tuple* $(Z, E, \Theta, *, \diamond, \boldsymbol{\zeta})$ *be an IFNbMS if* $\mathbf{Z} \neq \varnothing$ *is an arbitrary set,* $\boldsymbol{\zeta} \geq 1$ *is a real number,* $*$ *is a CtN,* $\diamond$ *is a CtCN, E and* Θ *are FSs on* $\mathbf{Z}^3 \times (0, +\infty)$ *fulfills the following conditions for all* $z, \varkappa, \nu, \gamma \in \mathbf{Z}$ *and* $\varsigma, \boldsymbol{\varpi}, \boldsymbol{\sigma} > 0$:

(i) $E(z, \varkappa, \nu, \boldsymbol{\sigma}) + \Theta(z, \varkappa, \nu, \boldsymbol{\sigma}) \leq 1$,
(ii) $E(z, \varkappa, \nu, \boldsymbol{\sigma}) > 0$,
(iii) $E(z, \varkappa, \nu, \boldsymbol{\sigma}) = 1$ *if and only if* $z = \varkappa = \nu$,
(iv) $E(z, \varkappa, \nu, \boldsymbol{\zeta}(\varsigma + \boldsymbol{\varpi} + \boldsymbol{\sigma})) \geq E(z, z, \gamma, \varsigma) * E(\varkappa, \varkappa, \gamma, \boldsymbol{\varpi}) * E(\nu, \nu, \gamma, \boldsymbol{\sigma})$,
(v) $E(z, \varkappa, \nu, .) : (0, +\infty) \to (0, 1]$ *is a CF,*
(vi) $\Theta(z, \varkappa, \nu, \boldsymbol{\sigma}) > 0$,
(vii) $\Theta(z, \varkappa, \nu, \boldsymbol{\sigma}) = 0$ *if and only if* $z = \varkappa = \nu$,
(viii) $\Theta(z, \varkappa, \nu, \boldsymbol{\zeta}(\varsigma + \boldsymbol{\varpi} + \boldsymbol{\sigma})) \leq \Theta(z, z, \gamma, \varsigma) \diamond \Theta(\varkappa, \varkappa, \gamma, \boldsymbol{\varpi}) \diamond \Theta(\nu, \nu, \gamma, \boldsymbol{\sigma})$,
(ix) $\Theta(z, \varkappa, \nu, .) : (0, +\infty) \to (0, 1]$ *is a CF.*

Here, $E(z, \varkappa, \nu, \boldsymbol{\sigma})$ *is said to be a membership function and* $\Theta(z, \varkappa, \nu, \boldsymbol{\sigma})$ *non-membership function of* $z, \varkappa$ *and* ν *concerning* $\boldsymbol{\sigma}$.

Definition 7 ([25]). *A sextuple* $(\mathbf{Z}, E, \Theta, R, *, \diamond)$ *is called an NMS if* $\mathbf{Z} \neq \varnothing$ *is an arbitrary set,* $*$ *is a CtN,* $\diamond$ *is a CtCN, E,* Θ *and R are NSs on* $\mathbf{Z}^3 \times (0, +\infty)$ *fulfills the following conditions for all* $z, \varkappa, \nu, \gamma \in \mathbf{Z}$ *and* $\varsigma, \boldsymbol{\varpi}, \boldsymbol{\sigma} > 0$:

(i) $E(z, \varkappa, \nu, \boldsymbol{\sigma}) + \Theta(z, \varkappa, \nu, \boldsymbol{\sigma}) + R(z, \varkappa, \nu, \boldsymbol{\sigma}) \leq 3$,
(ii) $E(z, \varkappa, \nu, \boldsymbol{\sigma}) > 0$,
(iii) $E(z, \varkappa, \nu, \boldsymbol{\sigma}) = 1$ *if and only if* $z = \varkappa = \nu$,
(iv) $E(z, \varkappa, \nu, (\varsigma + \varpi + \boldsymbol{\sigma})) \geq E(z, z, \gamma, \varsigma) * E(\varkappa, \varkappa, \gamma, \boldsymbol{\varpi}) * E(\nu, \nu, \gamma, \boldsymbol{\sigma})$,
(v) $E(z, \varkappa, \nu, .) : (0, +\infty) \to (0, 1]$ *is a CF,*
(vi) $\Theta(z, \varkappa, \nu, \boldsymbol{\sigma}) > 0$,
(vii) $\Theta(z, \varkappa, \nu, \boldsymbol{\sigma}) = 0$ *if and only if* $z = \varkappa = \nu$,
(viii) $\Theta(z, \varkappa, \nu, (\varsigma + \boldsymbol{\varpi} + \boldsymbol{\sigma})) \leq \Theta(z, z, \gamma, \varsigma) \diamond \Theta(\varkappa, \varkappa, \gamma, \boldsymbol{\varpi}) \diamond \Theta(\nu, \nu, \gamma, \boldsymbol{\sigma})$,
(ix) $\Theta(z, \varkappa, \nu, .) : (0, +\infty) \to (0, 1]$ *is a CF.*
(x) $R(z, \varkappa, \nu, \boldsymbol{\sigma}) > 0$,

(xi) $R(z,\varkappa,\nu,\boldsymbol{\sigma})=0$ *if and only if* $z=\varkappa=\nu$,
(xii) $R(z,\varkappa,\nu,(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma}))\leq R(z,z,\gamma,\varsigma)\diamond\Theta(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi})\diamond R(\nu,\nu,\gamma,\boldsymbol{\sigma})$,
(xiii) $R(z,\varkappa,\nu,.):(0,+\infty)\rightarrow(0,1]$ *is a CF.*

Definition 8 ([26]). *A sextuple* $(\mathbf{Z},E_{\boldsymbol{\varpi}},\Theta_{\boldsymbol{\varpi}},*,\diamond,\boldsymbol{\zeta})$ *is an IFQSbMS if* $\mathbf{Z}\neq\varnothing$ *is an arbitrary set,* $\boldsymbol{\zeta}\geq 1$ *is a real number,* $*$ *is a CtN,* $\diamond$ *is a CtCN,* $E_{\boldsymbol{\varpi}}$ *and* $\Theta_{\boldsymbol{\varpi}}$ *are FSs on* $\mathbf{Z}^3\times[0,+\infty)$ *satisfies the axioms below for all* $z,\varkappa,\nu,\gamma\in\mathbf{Z}$ *and* $\varsigma,\boldsymbol{\varpi},\boldsymbol{\sigma}>0:$

(a) $E_{\boldsymbol{\varpi}}(z,\varkappa,\nu,\boldsymbol{\sigma})+\Theta_{\boldsymbol{\varpi}}(z,\varkappa,\nu,\boldsymbol{\sigma})\leq 1$,
(b) $E_{\boldsymbol{\varpi}}(z,\varkappa,\nu,\boldsymbol{\sigma})\geq 0$,
(c) $E_{\boldsymbol{\varpi}}(z,\varkappa,\nu,\boldsymbol{\sigma})=E_{\boldsymbol{\varpi}}(P\{z,\varkappa,\nu,\boldsymbol{\sigma}\})=1$ *if and only if* $z=\varkappa=\nu$, *where P is permutation,*
(d) $E_{\boldsymbol{\varpi}}(z,\varkappa,\nu,\boldsymbol{\zeta}(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma}))\geq E_{\boldsymbol{\varpi}}(z,z,\gamma,\varsigma)*E_{\boldsymbol{\varpi}}(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi})*E_{\boldsymbol{\varpi}}(\nu,\nu,\gamma,\boldsymbol{\sigma})$,
(e) $E_{\boldsymbol{\varpi}}(z,\varkappa,\nu,.):(0,+\infty)\rightarrow(0,1]$ *is a CF,*
(f) $\Theta_{\boldsymbol{\varpi}}(z,\varkappa,\nu,\boldsymbol{\sigma})\geq 0$,
(g) $\Theta_{\boldsymbol{\varpi}}(z,\varkappa,\nu,\boldsymbol{\sigma})=\Theta_{\boldsymbol{\varpi}}(P\{z,\varkappa,\nu,\boldsymbol{\sigma}\})=0$ *if and only if* $z=\varkappa=\nu$, *where P is permutation,*
(h) $\Theta_{\boldsymbol{\varpi}}(z,\varkappa,\nu,\boldsymbol{\zeta}(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma}))\leq\Theta_{\boldsymbol{\varpi}}(z,z,\gamma,\varsigma)\diamond\Theta_{\boldsymbol{\varpi}}(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi})\diamond\Theta_{\boldsymbol{\varpi}}(\nu,\nu,\gamma,\boldsymbol{\sigma})$,
(i) $\Theta_{\boldsymbol{\varpi}}(z,\varkappa,\nu,.):(0,+\infty)\rightarrow(0,1]$ *is a CF.*

Here, $E_{\boldsymbol{\varpi}}(z,\varkappa,\nu,\boldsymbol{\sigma})$ *is said to be a membership function and* $\Theta_{\boldsymbol{\varpi}}(z,\varkappa,\nu,\boldsymbol{\sigma})$ *is non-membership function of* $z,\varkappa$ *and* ν *concerning* $\boldsymbol{\sigma}$.

Definition 9 ([26]). *A sextuple* $(\mathbf{Z},E_E,\Theta_E,*,\diamond)$ *is an IFQNMS if* $\mathbf{Z}\neq\varnothing$ *is an arbitrary set,* $*$ *is a CtN,* $\diamond$ *is a CtCN, and* E_E, Θ_E *are FSs on* $\mathbf{Z}^3\times(0,+\infty)$ *and fulfills the below conditions for all* $z,\varkappa,\nu,\gamma\in\mathbf{Z}$ *and* $\varsigma,\boldsymbol{\varpi},\boldsymbol{\sigma}>0:$

(a) $E_E(z,\varkappa,\nu,\sigma)+\Theta_E(z,\varkappa,\nu,\boldsymbol{\sigma})\leq 1$,
(b) $E_E(z,\varkappa,\nu,\boldsymbol{\sigma})>0$,
(c) $E_E(z,\varkappa,\nu,\boldsymbol{\sigma})=E_E(P\{z,\varkappa,\nu,\boldsymbol{\sigma}\})=1$ *if and only if* $z=\varkappa=\nu$, *where P is permutation,*
(d) $E_E(z,\varkappa,\nu,\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma})\geq E_E(z,z,\gamma,\varsigma)*E_E(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi})*E_E(\nu,\nu,\gamma,\boldsymbol{\sigma})$,
(e) $E_E(z,\varkappa,\nu,.):(0,+\infty)\rightarrow(0,1]$ *is a CF,*
(f) $\Theta_E(z,\varkappa,\nu,\boldsymbol{\sigma})>0$,
(g) $\Theta_E(z,\varkappa,\nu,\boldsymbol{\sigma})=\Theta_E(P\{z,\varkappa,\nu,\boldsymbol{\sigma}\})=0$ *if and only if* $z=\varkappa=\nu$, *where P is permutation,*
(h) $\Theta_E(z,\varkappa,\nu,\varsigma+\varpi+\boldsymbol{\sigma})\leq\Theta_E(z,z,\gamma,\varsigma)\diamond\Theta_E(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi})\diamond\Theta_E(\nu,\nu,\gamma,\boldsymbol{\sigma})$,
(i) $\Theta_E(z,\varkappa,\nu,.):(0,+\infty)\rightarrow(0,1]$ *is a CF.*

Definition 10 ([26]). *A sextuple* $(\mathbf{Z},E_q,\Theta_q,*,\diamond,\boldsymbol{\zeta})$ *is an IFPNbMS if* $\mathbf{Z}\neq\varnothing$ *is an arbitrary set,* $\boldsymbol{\zeta}\geq 1$ *is a real number,* $*$ *is a CtN,* $\diamond$ *is a CtCN and* E_q, Θ_q *are FSs on* $\mathbf{Z}^3\times(0,+\infty)$ *and satisfies the below axioms for all* $z,\varkappa,\nu,\gamma\in\mathbf{Z}$ *and* $\varsigma,\boldsymbol{\varpi},\boldsymbol{\sigma}>0:$

(i) $E_q(z,\varkappa,\nu,\boldsymbol{\sigma})+\Theta_q(z,\varkappa,\nu,\boldsymbol{\sigma})\leq 1$,
(ii) $E_q(z,\varkappa,\nu,\boldsymbol{\sigma})>0$,
(iii) $E_q(z,\varkappa,\nu,\boldsymbol{\sigma})=1$ *if and only if* $z=\varkappa=\nu$,
(iv) $E_q(z,\varkappa,\nu,\boldsymbol{\zeta}(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma}))\geq E_q(z,z,\gamma,\varsigma)*E_q(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi})*E_q(\nu,\nu,\gamma,\boldsymbol{\sigma})$,
(v) $E_q(z,\varkappa,\nu,.):(0,+\infty)\rightarrow(0,1]$ *is a CF,*
(vi) $\Theta_q(z,\varkappa,\nu,\boldsymbol{\sigma})>0$,
(vii) $\Theta_q(z,\varkappa,\nu,\boldsymbol{\sigma})=0$ *if and only if* $z=\varkappa=\nu$,
(viii) $\Theta_q(z,\varkappa,\nu,\boldsymbol{\zeta}(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma}))\leq\Theta_q(z,z,\gamma,\varsigma)\diamond\Theta_q(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi})\diamond\Theta_q(\nu,\nu,\gamma,\boldsymbol{\sigma})$,
(ix) $\Theta_q(z,\varkappa,\nu,.):(0,+\infty)\rightarrow(0,1]$ *is a CF.*

3. Neutrosophic N_b Metric Space

The concept of NNbMS is introduced and some non-trivial examples are given in this section.

Definition 11. *A septuple* $(\mathbf{Z},E,\Theta,R,*,\diamond,\boldsymbol{\zeta})$ *is said to ba a NNbMS if* $\mathbf{Z}\neq\varnothing$ *is an arbitrary set,* $\boldsymbol{\zeta}\geq 1$ *is a real number,* $*$ *is a CtN,* $\diamond$ *is a CtCN,* E, Θ *and R are NSs on* $\mathbf{Z}^3\times(0,+\infty)$ *satisfying the axioms below for all* $z,\varkappa,\nu,\gamma\in\mathbf{Z}$ *and* $\varsigma,\boldsymbol{\varpi},\boldsymbol{\sigma}>0:$

i. $E(z,\varkappa,\nu,\boldsymbol{\sigma})+\Theta(z,\varkappa,\nu,\boldsymbol{\sigma})+R(z,\varkappa,\nu,\boldsymbol{\sigma})\leq 3,$
ii. $E(z,\varkappa,\nu,\boldsymbol{\sigma})>0,$
iii. $E(z,\varkappa,\nu,\boldsymbol{\sigma})=1$ *if and only if* $z=\varkappa=\nu,$
iv. $E(z,\varkappa,\nu,\boldsymbol{\zeta}(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma}))\geq E(z,z,\gamma,\varsigma)*E(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi})*E(\nu,\nu,\gamma,\boldsymbol{\sigma}),$
v. $E(z,\varkappa,\nu,.):(0,+\infty)\rightarrow(0,1]$ *is a CF,*
vi. $\Theta(z,\varkappa,\nu,\boldsymbol{\sigma})>0,$
vii. $\Theta(z,\varkappa,\nu,\boldsymbol{\sigma})=0$ *if* $z=\varkappa=\nu,$
viii. $\Theta(z,\varkappa,\nu,\boldsymbol{\zeta}(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma}))\leq\Theta(z,z,\gamma,\varsigma)\diamond\Theta(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi})\diamond\Theta(\nu,\nu,\gamma,\boldsymbol{\sigma}),$
ix. $\Theta(z,\varkappa,\nu,.):(0,+\infty)\rightarrow(0,1]$ *is a CF.*
x. $R(z,\varkappa,\nu,\boldsymbol{\sigma})>0,$
xi. $R(z,\varkappa,\nu,\boldsymbol{\sigma})=0$ *if and only if* $z=\varkappa=\nu,$
xii. $R(z,\varkappa,\nu,\boldsymbol{\zeta}(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma}))\leq R(z,z,\gamma,\varsigma)\diamond R(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi})\diamond R(\nu,\nu,\gamma,\boldsymbol{\sigma}),$
xiii. $R(z,\varkappa,\nu,.):(0,+\infty)\rightarrow(0,1]$ *is a CF.*

In this, membership, non-membership and neutral functions of $z,\varkappa$ *and* ν *concerning* $\boldsymbol{\sigma}$ *are* $E(z,\varkappa,\nu,\boldsymbol{\sigma})$, $\Theta(z,\varkappa,\nu,\boldsymbol{\sigma})$ *and* $R(z,\varkappa,\nu,\boldsymbol{\sigma})$.

Remark 1. *The definition of NNMS can be obtained by considering the* $\boldsymbol{\zeta}=1$ *in Definition 11.*

Example 1. *Let* $\mathbf{Z}=\mathbb{R}$ *and* E, Θ *and* R *are the functions on* $\mathbf{Z}^3\times(0,+\infty)$ *defined by*

$$E(z,\varkappa,\nu,\boldsymbol{\sigma})=\frac{\boldsymbol{\sigma}}{\boldsymbol{\sigma}+\boldsymbol{\zeta}[|z-\nu|+|z+\nu-2\varkappa|]^2}$$

$$\Theta(z,\varkappa,\nu,\boldsymbol{\sigma})=\frac{\boldsymbol{\zeta}[|z-\nu|+|z+\nu-2\varkappa|]^2}{\boldsymbol{\sigma}+\boldsymbol{\zeta}[|z-\nu|+|z+\nu-2\varkappa|]^2},$$

$$R(z,\varkappa,\nu,\boldsymbol{\sigma})=\frac{\boldsymbol{\zeta}[|z-\nu|+|z+\nu-2\varkappa|]^2}{\boldsymbol{\sigma}}.$$

for all $z,\varkappa,\nu\in\mathbf{Z}$ *and* $\boldsymbol{\sigma}>0$. *Therefore* $(\mathbf{Z},E,\Theta,R,*,\diamond,\boldsymbol{\zeta})$ *is an NNbMS with CtN* $\gamma*\beta*\pi=\gamma\beta\pi$, *CtCN* $\gamma\diamond\beta\diamond\pi=max\{\gamma,\beta,\pi\}$ *and constant* $\boldsymbol{\zeta}\geq 1$. *Figures 1–3 show the graphical behavior of* E,Θ *and* R, *respectively.*

Proof. We verify (iv) and (viii), others are easy to prove. We can write

$$\begin{aligned}
&E(z,z,\gamma,\varsigma)*E(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi})*E(\nu,\nu,\gamma,\boldsymbol{\sigma})\\
&=\frac{\varsigma}{\varsigma+\boldsymbol{\zeta}[|z-\gamma|+|z+\gamma-2z|]^2}\cdot\frac{\boldsymbol{\varpi}}{\boldsymbol{\varpi}+\boldsymbol{\zeta}[|\varkappa-\gamma|+|\varkappa+\gamma-2\varkappa|]^2}\cdot\frac{\boldsymbol{\sigma}}{\boldsymbol{\sigma}+\boldsymbol{\zeta}[|\nu-\gamma|+|\nu+\gamma-2\nu|]^2}\\
&=\frac{1}{1+\frac{\boldsymbol{\zeta}[|z-\gamma|+|z+\gamma-2z|]^2}{\varsigma}}\cdot\frac{1}{1+\frac{\boldsymbol{\zeta}[|\varkappa-\gamma|+|\varkappa+\gamma-2\varkappa|]^2}{\boldsymbol{\varpi}}}\cdot\frac{1}{1+\frac{\boldsymbol{\zeta}[|\nu-\gamma|+|\nu+\gamma-2\nu|]^2}{\boldsymbol{\sigma}}}\\
&\leq\frac{1}{1+\frac{\boldsymbol{\zeta}[|z-\gamma|+|z+\gamma-2z|]^2}{(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma})}}\cdot\frac{1}{1+\frac{\boldsymbol{\zeta}[|\varkappa-\gamma|+|\varkappa+\gamma-2\varkappa|]^2}{(\varsigma+\varpi+\sigma)}}\cdot\frac{1}{1+\frac{\boldsymbol{\zeta}[|\nu-\gamma|+|\nu+\gamma-2\nu|]^2}{(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma})}}\\
&\leq\frac{1}{1+\frac{\boldsymbol{\zeta}[|z-\gamma|+|z+\gamma-2z|]^2+\boldsymbol{\zeta}[|\varkappa-\gamma|+|\varkappa+\gamma-2\varkappa|]^2+\boldsymbol{\zeta}[|\nu-\gamma|+|\nu+\gamma-2\nu|]^2}{(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma})}}\\
&\leq\frac{1}{1+\frac{[|z-\nu|+|z+\nu-2\varkappa|]^2}{\boldsymbol{\zeta}(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma})}}\\
&\leq\frac{\boldsymbol{\zeta}(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma})}{\boldsymbol{\zeta}(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma})+[|z-\nu|+|z+\nu-2\varkappa|]^2}\\
&=E(z,\varkappa,\nu,\boldsymbol{\zeta}(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma})).
\end{aligned}$$

Now, let

$$\begin{aligned}
\zeta[|z-\nu|+|z+\nu-2\varkappa|]^2 &= \zeta[|z-\nu|+|z+\nu-2\varkappa|]^2 \\
&\max\left\{\begin{array}{c}\frac{\zeta[|z-\gamma|+|z+\gamma-2z|]^2}{\zeta[|z-\gamma|+|z+\gamma-2z|]^2}, \frac{\zeta[|\varkappa-\gamma|+|\varkappa+\gamma-2\varkappa|]^2}{\zeta[|\varkappa-\gamma|+|\varkappa+\gamma-2\varkappa|]^2}, \\ \frac{\zeta[|\nu-\gamma|+|\nu+\gamma-2\nu|]^2}{\zeta[|\nu-\gamma|+|\nu+\gamma-2\nu|]^2}\end{array}\right\} \\
&\leq \left[(\varsigma+\varpi+\sigma)+\zeta[|z-\nu|+|z+\nu-2\varkappa|]^2\right] \\
&\max\left\{\begin{array}{c}\frac{\zeta[|z-\gamma|+|z+\gamma-2z|]^2}{\varsigma+\zeta[|z-\gamma|+|z+\gamma-2z|]^2}, \frac{\zeta[|\varkappa-\gamma|+|\varkappa+\gamma-2\varkappa|]^2}{\varpi+\zeta[|\varkappa-\gamma|+|\varkappa+\gamma-2\varkappa|]^2}, \\ \frac{\zeta[|\nu-\gamma|+|\nu+\gamma-2\nu|]^2}{\sigma+\zeta[|\nu-\gamma|+|\nu+\gamma-2\nu|]^2}\end{array}\right\}, \\
&\frac{\zeta[|z-\nu|+|z+\nu-2\varkappa|]^2}{(\varsigma+\varpi+\sigma)+\zeta[|z-\nu|+|z+\nu-2\varkappa|]^2} \\
&\leq \max\left\{\begin{array}{c}\frac{\zeta[|z-\gamma|+|z+\gamma-2z|]^2}{\varsigma+\zeta[|z-\gamma|+|z+\gamma-2z|]^2}, \frac{\zeta[|\varkappa-\gamma|+|\varkappa+\gamma-2\varkappa|]^2}{\varpi+\zeta[|\varkappa-\gamma|+|\varkappa+\gamma-2\varkappa|]^2}, \\ \frac{\zeta[|\nu-\gamma|+|\nu+\gamma-2\nu|]^2}{\sigma+\zeta[|\nu-\gamma|+|\nu+\gamma-2\nu|]^2}\end{array}\right\} \\
\Theta(z,\varkappa,\nu,\zeta(\varsigma+\varpi+\sigma)) &\leq \Theta(z,z,\gamma,\varsigma)\diamond\Theta(\varkappa,\varkappa,\gamma,\varpi)\diamond\Theta(\nu,\nu,\gamma,\sigma).
\end{aligned}$$

and

$$\begin{aligned}
\zeta[|z-\nu|+|z+\nu-2\varkappa|]^2 &= \zeta[|z-\nu|+|z+\nu-2\varkappa|]^2 \\
\max\Big\{\zeta[|z-\gamma|+|z+\gamma-2z|]^2, &\zeta[|\varkappa-\gamma|+|\varkappa+\gamma-2\varkappa|]^2, \zeta[|\nu-\gamma|+|\nu+\gamma-2\nu|]^2\Big\} \\
&\leq \left[(\varsigma+\varpi+\sigma)+\zeta[|z-\nu|+|z+\nu-2\varkappa|]^2\right] \\
&\max\left\{\begin{array}{c}\frac{\zeta[|z-\gamma|+|z+\gamma-2z|]^2}{\varsigma}, \frac{\zeta[|\varkappa-\gamma|+|\varkappa+\gamma-2\varkappa|]^2}{\varpi}, \\ \frac{\zeta[|\nu-\gamma|+|\nu+\gamma-2\nu|]^2}{\sigma}\end{array}\right\}, \\
&\frac{\zeta[|z-\nu|+|z+\nu-2\varkappa|]^2}{(\varsigma+\varpi+\sigma)} \\
&\leq \max\left\{\begin{array}{c}\frac{\zeta[|z-\gamma|+|z+\gamma-2z|]^2}{\varsigma}, \frac{\zeta[|\varkappa-\gamma|+|\varkappa+\gamma-2\varkappa|]^2}{\varpi}, \\ \frac{\zeta[|\nu-\gamma|+|\nu+\gamma-2\nu|]^2}{\sigma}\end{array}\right\} \\
R(z,\varkappa,\nu,\zeta(\varsigma+\varpi+\sigma)) &\leq R(z,z,\gamma,\varsigma)\diamond R(\varkappa,\varkappa,\gamma,\varpi)\diamond R(\nu,\nu,\gamma,\sigma).
\end{aligned}$$

Therefore, (iv) and (viii) are fulfilled. □

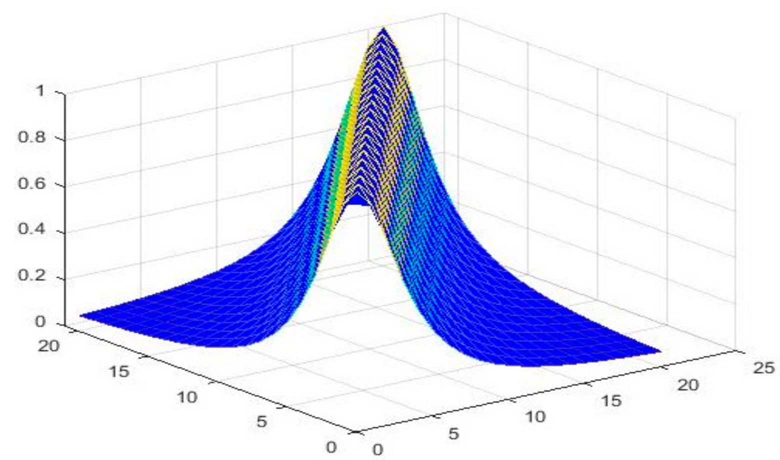

Figure 1. Demonstrating the performance of E for $\mathbf{Z}=[0,1]$, $\sigma=1$ and $\zeta=4$.

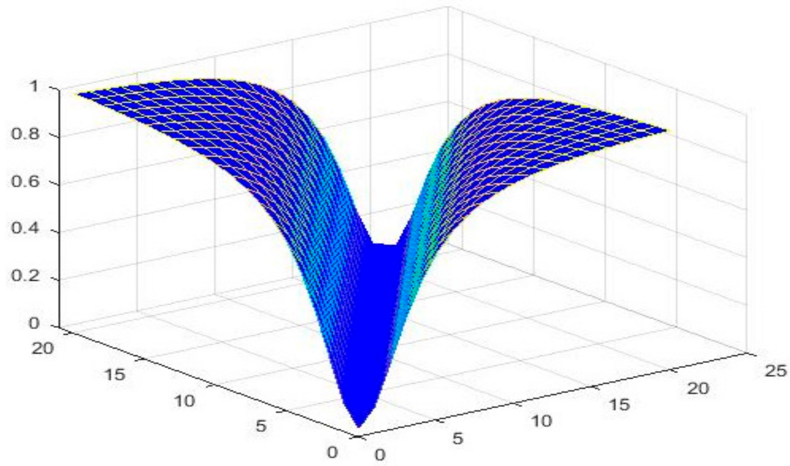

Figure 2. Demonstrating the performance of Θ for $\mathbf{Z} = [0,1]$, $\sigma = 1$ and $\zeta = 4$.

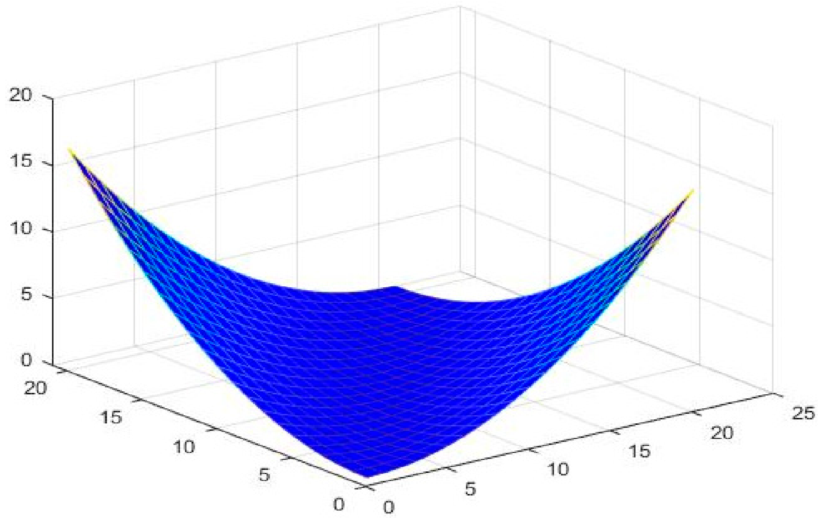

Figure 3. Demonstrating the performance of R for $\mathbf{Z} = [0,1]$, $\sigma = 1$ and $\zeta = 4$.

Definition 12. *Let* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \zeta)$ *be a NNbMS. Then* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \zeta)$ *is called symmetric if*

$$E(z, z, \varkappa, \sigma) = E(\varkappa, \varkappa, z, \sigma) \tag{1}$$

and

$$\Theta(z, z, \varkappa, \sigma) = \Theta(\varkappa, \varkappa, z, \sigma), \tag{2}$$

$$R(z, z, \varkappa, \sigma) = R(\varkappa, \varkappa, z, \sigma), \tag{3}$$

for all $z, \varkappa \in \mathbf{Z}$ *and* $\sigma > 0$.

Example 2. *Let* $\mathbf{Z} = \mathbb{R}$ *and* $E, \Theta,\ R$ *are the functions on* $\mathbf{Z}^3 \times (0, +\infty)$ *defined by*

$$E(z, \varkappa, \nu, \sigma) = \frac{\sigma}{\sigma + [|z - \varkappa| + |\varkappa - \nu| + |\nu - z|]^p}$$

$$\Theta(z, \varkappa, \nu, \sigma) = \frac{[|z - \varkappa| + |\varkappa - \nu| + |\nu - z|]^p}{\sigma + [|z - \varkappa| + |\varkappa - \nu| + |\nu - z|]^p},$$

and

$$R(z, \varkappa, \nu, \sigma) = \frac{[|z - \varkappa| + |\varkappa - \nu| + |\nu - z|]^p}{\sigma},$$

for all $z, \varkappa, \nu \in \mathbf{Z}$ *and* $\boldsymbol{\sigma} > 0$. *Then* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ *is a symmetric NNbMS with CtN* $\gamma * \beta * \pi = \gamma\beta\pi$, *CtCN* $\gamma \diamond \beta \diamond \pi = max\{\gamma, \beta, \pi\}$ *and constant* $\boldsymbol{\zeta} = 2^{2(p-1)}$. *Figures 4–6 show the graphical behavior of* E, Θ *and R, respectively.*

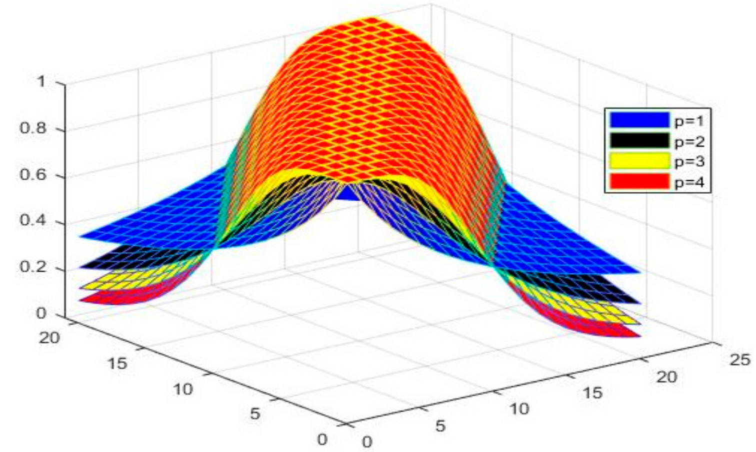

Figure 4. Shows the performance of E for $\mathbf{Z} = [0, 1]$, $\boldsymbol{\sigma} = 1$.

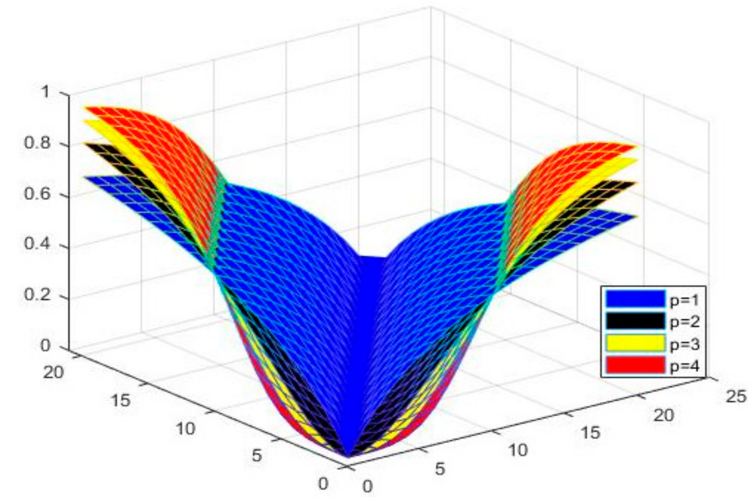

Figure 5. Shows the performance of Θ for $\mathbf{Z} = [0, 1]$, $\boldsymbol{\sigma} = 1$.

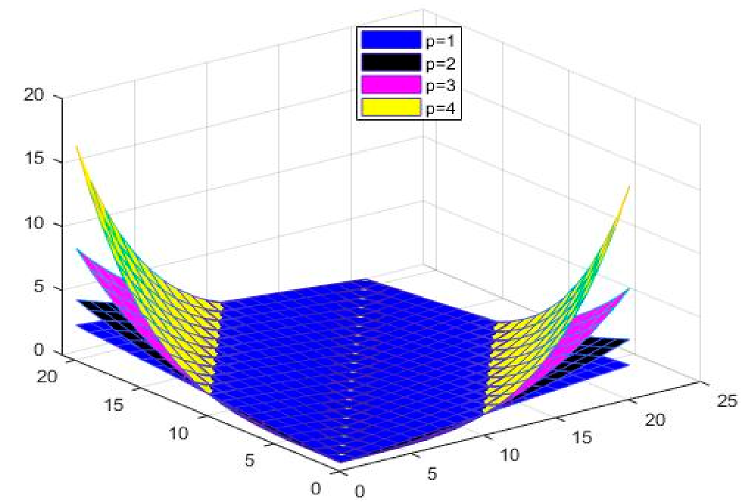

Figure 6. Shows the performance of R for $\mathbf{Z} = [0, 1]$, $\boldsymbol{\sigma} = 1$.

4. Generalized Definitions

In this section, we present some generalized notions and different non-trivial examples.

Definition 13. *A septuple* $(\mathbf{Z}, E_{\boldsymbol{\varpi}}, \Theta_{\boldsymbol{\varpi}}, R_{\boldsymbol{\varpi}}, *, \diamond, \boldsymbol{\zeta})$ *is an NQSbMS if* $\mathbf{Z} \neq \varnothing$ *is an arbitrary set,* $*$ *is a CtN,* $\diamond$ *is a CtCN,* $\boldsymbol{\zeta} \geq 1$ *is a positive real number,* $E_{\boldsymbol{\varpi}}$, $\Theta_{\boldsymbol{\varpi}}$ *and* $R_{\boldsymbol{\varpi}}$ *are NSs on* $\mathbf{Z}^3 \times [0, +\infty)$ *and satisfies the conditions below for all* $z, \varkappa, \nu, \gamma \in \mathbf{Z}$ *and* $\varsigma, \boldsymbol{\varpi}, \boldsymbol{\sigma} > 0$:

i. $E_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\sigma}) + \Theta_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\sigma}) + R_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\sigma}) \leq 3$,
ii. $E_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\sigma}) \geq 0$,
iii. $E_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\sigma}) = E_{\boldsymbol{\varpi}}(P\{z, \varkappa, \nu, \boldsymbol{\sigma}\}) = 1$ *if and only if* $z = \varkappa = \nu$, *where P is permutation,*
iv. $E_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\zeta}(\varsigma + \boldsymbol{\varpi} + \boldsymbol{\sigma})) \geq E_{\boldsymbol{\varpi}}(z, z, \gamma, \varsigma) * E_{\boldsymbol{\varpi}}(\varkappa, \varkappa, \gamma, \boldsymbol{\varpi}) * E_{\boldsymbol{\varpi}}(\nu, \nu, \gamma, \boldsymbol{\sigma})$,
v. $E_{\boldsymbol{\varpi}}(z, \varkappa, \nu, .) : (0, +\infty) \to (0, 1]$ *is a CF,*
vi. $\Theta_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\sigma}) \geq 0$,
vii. $\Theta_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\sigma}) = \Theta_{\boldsymbol{\varpi}}(P\{z, \varkappa, \nu, \boldsymbol{\sigma}\}) = 0$ *if and only if* $z = \varkappa = \nu$, *where P is permutation,*
viii. $\Theta_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\zeta}(\varsigma + \boldsymbol{\varpi} + \boldsymbol{\sigma})) \leq \Theta_{\boldsymbol{\varpi}}(z, z, \gamma, \varsigma) \diamond \Theta_{\boldsymbol{\varpi}}(\varkappa, \varkappa, \gamma, \boldsymbol{\varpi}) \diamond \Theta_{\boldsymbol{\varpi}}(\nu, \nu, \gamma, \boldsymbol{\sigma})$,
ix. $\Theta_{\boldsymbol{\varpi}}(z, \varkappa, \nu, .) : (0, +\infty) \to (0, 1]$ *is a CF.*
x. $R_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\sigma}) \geq 0$,
xi. $R_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\sigma}) = R_{\boldsymbol{\varpi}}(P\{z, \varkappa, \nu, \boldsymbol{\sigma}\}) = 0$ *if and only if* $z = \varkappa = \nu$, *where P is permutation,*
xii. $R_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\zeta}(\varsigma + \boldsymbol{\varpi} + \boldsymbol{\sigma})) \leq R_{\boldsymbol{\varpi}}(z, z, \gamma, \varsigma) \diamond R_{\boldsymbol{\varpi}}(\varkappa, \varkappa, \gamma, \boldsymbol{\varpi}) \diamond R_{\boldsymbol{\varpi}}(\nu, \nu, \gamma, \boldsymbol{\sigma})$,
xiii. $R_{\boldsymbol{\varpi}}(z, \varkappa, \nu, .) : (0, +\infty) \to (0, 1]$ *is a CF.*

In this, membership, non-membership and neutral functions of $z, \varkappa$ *and* ν *concerning* σ *are* $E(z, \varkappa, \nu, \sigma)$, $\Theta(z, \varkappa, \nu, \sigma)$ *and* $R(z, \varkappa, \nu, \sigma)$.

Remark 2. *The NQSbMS definition is obtained by taking* $\mu = 1$ *in Definition 13.*

Example 3. *Let* $Z = \mathbb{R}^+ \cup \{0\}$. *Define* $E_{\boldsymbol{\varpi}}, \Theta_{\boldsymbol{\varpi}}$ *and* $R_{\boldsymbol{\varpi}}$ *by*

$$E_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\sigma}) = \begin{cases} 1, & \text{if } z = \varkappa = \nu, \\ \frac{\sigma}{\sigma + \left[\left|z - \frac{\nu}{2}\right| + \left|\varkappa - \frac{\nu}{2}\right|\right]^2}, & \text{otherwise,} \end{cases}$$

$$\Theta_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\sigma}) = \begin{cases} 0, & \text{if } z = \varkappa = \nu, \\ \frac{\left[\left|z - \frac{\nu}{2}\right| + \left|\varkappa - \frac{\nu}{2}\right|\right]^2}{\sigma + \left[\left|z - \frac{\nu}{2}\right| + \left|\varkappa - \frac{\nu}{2}\right|\right]^2}, & \text{otherwise.} \end{cases}$$

and

$$R_{\boldsymbol{\varpi}}(z, \varkappa, \nu, \boldsymbol{\sigma}) = \begin{cases} 0, & \text{if } z = \varkappa = \nu, \\ \frac{\left[\left|z - \frac{\nu}{2}\right| + \left|\varkappa - \frac{\nu}{2}\right|\right]^2}{\sigma}, & \text{otherwise.} \end{cases}$$

Then $(\mathbf{Z}, E_{\boldsymbol{\varpi}}, \Theta_{\boldsymbol{\varpi}}, R_{\boldsymbol{\varpi}}, *, \diamond, \boldsymbol{\zeta})$ *is said to be an NQSbMS with* $\boldsymbol{\zeta} = 2$. *Figures 7–9 show the graphical behavior of* $E_{\boldsymbol{\varpi}}, \Theta_{\boldsymbol{\varpi}}$ *and* $R_{\boldsymbol{\varpi}}$, *respectively.*

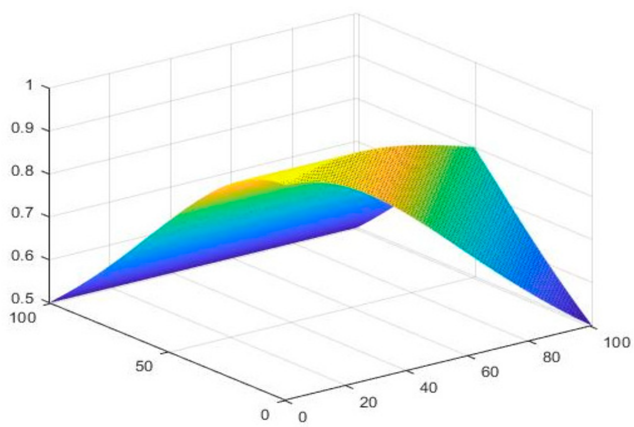

Figure 7. Shows the performance of $E_{\boldsymbol{\varpi}}$ for $\mathbf{Z} = [0, 1]$, $\boldsymbol{\sigma} = 1$.

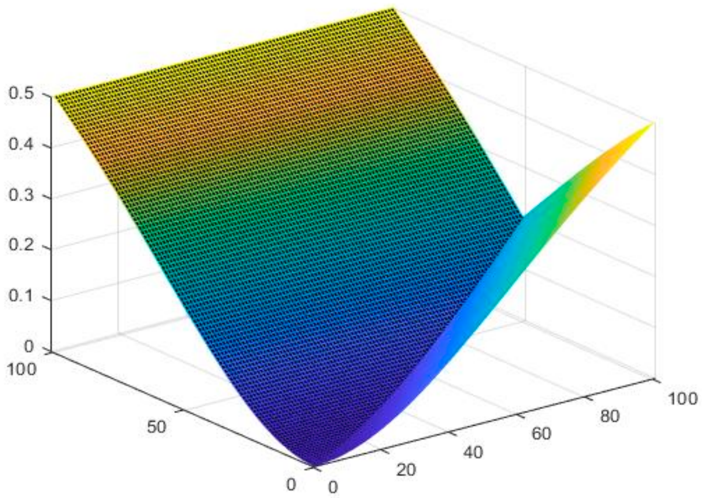

Figure 8. Shows the performance of Θ_{ϖ} for $\mathbf{Z} = [0,1],\ \sigma = 1$.

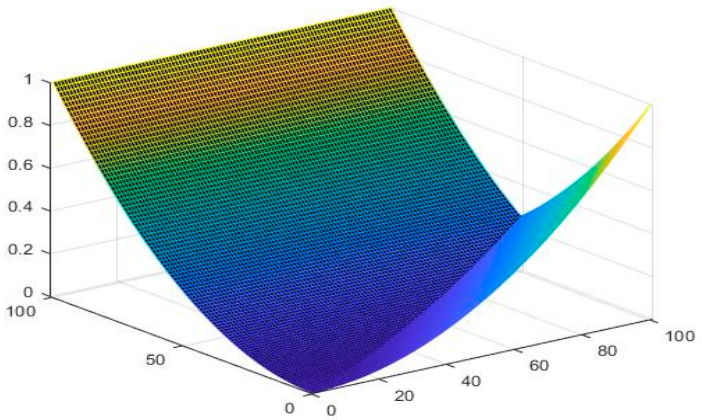

Figure 9. Shows the performance of R_{ϖ} for $\mathbf{Z} = [0,1],\ \sigma = 1$.

Remark 3. *If* $z \neq \varkappa \neq \nu$, *then by definition of* $E_{\varpi}, \Theta_{\varpi}$ *and* R_{ϖ} *in Example 3*

$$E_{\varpi}(z,\varkappa,\nu,\sigma) \neq E_{\varpi}(P\{z,\varkappa,\nu,\sigma\}),$$

$$\Theta_{\varpi}(z,\varkappa,\nu,\sigma) \neq \Theta_{\varpi}(P\{z,\varkappa,\nu,\sigma\}),$$

and

$$R_{\varpi}(z,\varkappa,\nu,\sigma) \neq R_{\varpi}(P\{z,\varkappa,\nu,\sigma\}).$$

Furthermore, $E_{\varpi}(z,z,\varkappa,\sigma) \neq E_{\varpi}(\varkappa,\varkappa,\nu,\sigma)$. *In general, NQSbMS is not symmetric.*

Definition 14. *A septuple* $(\mathbf{Z}, E_{\varpi\beta}, \Theta_{\varpi\beta}, R_{\varpi\beta}, *, \diamond, \boldsymbol{\zeta})$ *is an NPSbMS if* $\mathbf{Z} \neq \emptyset$ *is an arbitrary set,* $\boldsymbol{\zeta} \geq 1$ *is a real number,* $*$ *is a CtN,* $\diamond$ *is a CtCN,* $E_{\varpi\beta}$, $\Theta_{\varpi\beta}$ *and* $R_{\varpi\beta}$ *are NSs on* $\mathbf{Z}^3 \times [0,+\infty)$, *and fulfills the conditions below for all* $z,\varkappa,\nu,\gamma \in \mathbf{Z}$ *and* $\varsigma, \boldsymbol{\varpi}, \sigma > 0$:

(I) $E_{\varpi\beta}(z,\varkappa,\nu,\sigma) + \Theta_{\varpi\beta}(z,\varkappa,\nu,\sigma) + R_{\varpi\beta}(z,\varkappa,\nu,\sigma) \leq 3$,
(II) $E_{\varpi\beta}(z,\varkappa,\nu,\sigma) \geq 0$,
(III) $E_{\varpi\beta}(z,\varkappa,\nu,\sigma) = 1$ *if and only if* $z = \varkappa = \nu$,
(IV) $E_{\varpi\beta}(z,\varkappa,\nu,\boldsymbol{\zeta}(\varsigma + \boldsymbol{\varpi} + \sigma)) \geq E_{\varpi\beta}(z,z,\gamma,\varsigma) * E_{\varpi\beta}(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi}) * E_{\varpi\beta}(\nu,\nu,\gamma,\sigma)$,
(V) $E_{\varpi\beta}(z,\varkappa,\nu,.) : (0,+\infty) \to (0,1]$ *is a CF,*
(VI) $\Theta_{\varpi\beta}(z,\varkappa,\nu,\sigma) \geq 0$,
(VII) $\Theta_{\varpi\beta}(z,\varkappa,\nu,\sigma) = 0$ *if and only if* $z = \varkappa = \nu$,
(VIII) $\Theta_{\varpi\beta}(z,\varkappa,\nu,\boldsymbol{\zeta}(\varsigma + \boldsymbol{\varpi} + \sigma)) \leq \Theta_{\varpi\beta}(z,z,\gamma,\varsigma) \diamond \Theta_{\varpi\beta}(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi}) \diamond \Theta_{\varpi\beta}(\nu,\nu,\gamma,\sigma)$,

(IX) $\Theta_{\varpi\beta}(z,\varkappa,\nu,.):(0,+\infty)\to(0,1]$ *is a CF.*
(X) $R_{\varpi\beta}(z,\varkappa,\nu,\sigma)\geq 0,$
(XI) $R_{\varpi\beta}(z,\varkappa,\nu,\sigma)=0$ *if and only if* $z=\varkappa=\nu,$
(XII) $R_{\varpi\beta}(z,\varkappa,\nu,\zeta(\varsigma+\varpi+\sigma))\leq R_{\varpi\beta}(z,z,\gamma,\varsigma)\diamond R_{\varpi\beta}(\varkappa,\varkappa,\gamma,\varpi)\diamond R_{\varpi\beta}(\nu,\nu,\gamma,\sigma),$
(XIII) $R_{\varpi\beta}(z,\varkappa,\nu,.):(0,+\infty)\to(0,1]$ *is a CF.*

Remark 4. *The definition of NPSbMS is obtained if we can obtain* $\zeta=1$, *in the above definition.*

Example 4. *Suppose* $\mathbf{Z}=\mathbb{R}^{+}$. *Define* E_{ϖ} *and* Θ_{ϖ} *by*

$$E_{\varpi\beta}(z,\varkappa,\nu,\sigma)=\begin{cases}1, & \text{if } z=\varkappa=\nu,\\ \frac{\sigma}{\sigma+\left[|z^2-\nu^2|+|\varkappa^2-\nu^2|\right]^2}, & \text{otherwise,}\end{cases}$$

$$\Theta_{\varpi\beta}(z,\varkappa,\nu,\sigma)=\begin{cases}0, & \text{if } z=\varkappa=\nu,\\ \frac{\left[|z^2-\nu^2|+|\varkappa^2-\nu^2|\right]^2}{\sigma+\left[|z^2-\nu^2|+|z^2-\nu^2|\right]^2}, & \text{otherwise.}\end{cases}$$

and

$$R_{\varpi\beta}(z,\varkappa,\nu,\sigma)=\begin{cases}0, & \text{if } z=\varkappa=\nu,\\ \frac{\left[|z^2-\nu^2|+|\varkappa^2-\nu^2|\right]^2}{\sigma}, & \text{otherwise.}\end{cases}$$

Then $\left(\mathbf{Z},E_{\varpi\beta},\Theta_{\varpi\beta},\ R_{\varpi\beta},*,\diamond,\zeta\right)$ *is said to be an NPSbMS. Figures 10–12 show the graphical behavior of* $E_{\varpi\beta},\Theta_{\varpi\beta}$ *and* $R_{\varpi\beta}$, *respectively.*

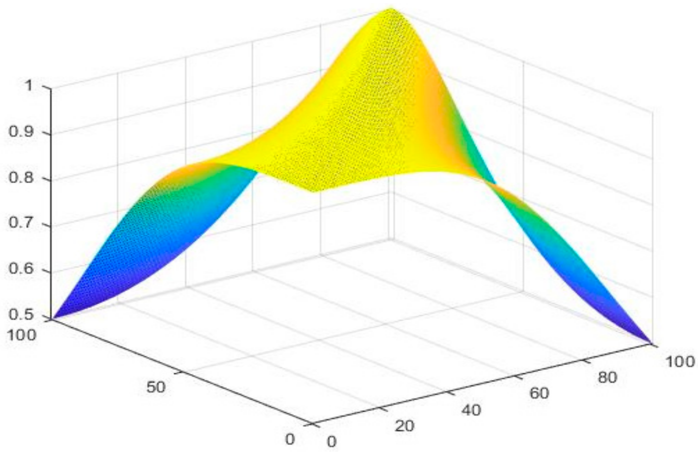

Figure 10. Shows the performance of $E_{\varpi\beta}$ for $\mathbf{Z}=[0,1],\ \sigma=1$.

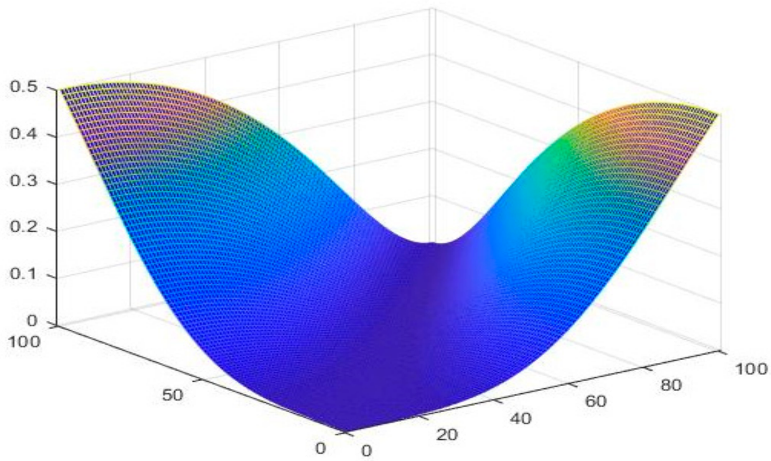

Figure 11. Shows the performance of $\Theta_{\varpi\beta}$ for $\mathbf{Z}=[0,1],\ \sigma=1$.

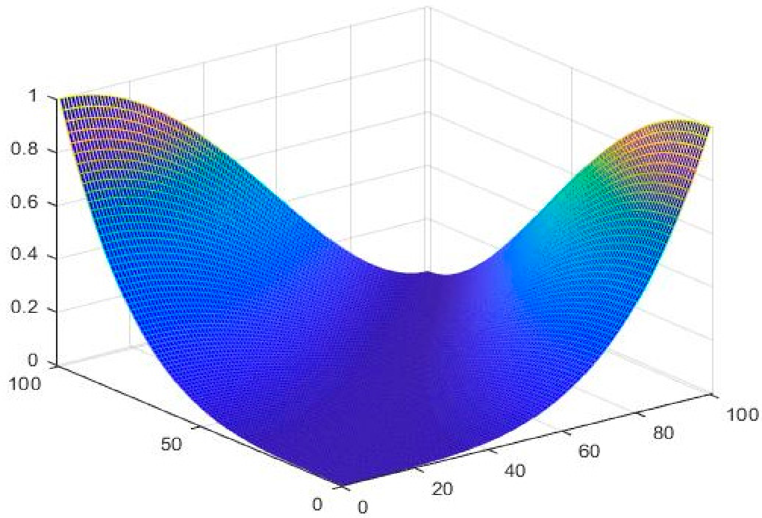

Figure 12. Shows the performance of $R_{\varpi\beta}$ for $\mathbf{Z} = [0,1]$, $\sigma = 1$.

Definition 15. *A sextuple* $(\mathbf{Z}, E_E, \Theta_E, R_E, *, \diamond)$ *is an NQNMS if* $\mathbf{Z} \neq \varnothing$ *is an arbitrary set,* $*$ *is a CtN,* $\diamond$ *is a CtCN,* E_E, Θ_E *and* R_E *are NSs on* $\mathbf{Z}^3 \times (0, +\infty)$*, and satisfies the below conditions for all* $z, \varkappa, \nu, \gamma \in \mathbf{Z}$ *and* $\varsigma, \boldsymbol{\varpi}, \boldsymbol{\sigma} > 0$:

(j) $E_E(z,\varkappa,\nu,\sigma) + \Theta_E(z,\varkappa,\nu,\boldsymbol{\sigma}) + R_E(z,\varkappa,\nu,\boldsymbol{\sigma}) \leq 3$,
(k) $E_E(z,\varkappa,\nu,\boldsymbol{\sigma}) > 0$,
(l) $E_E(z,\varkappa,\nu,\boldsymbol{\sigma}) = E_E(P\{z,\varkappa,\nu,\boldsymbol{\sigma}\}) = 1$ *if and only if* $z = \varkappa = \nu$, *where P is permutation,*
(m) $E_E(z,\varkappa,\nu,\varsigma + \boldsymbol{\varpi} + \boldsymbol{\sigma}) \geq E_E(z,z,\gamma,\varsigma) * E_E(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi}) * E_E(\nu,\nu,\gamma,\boldsymbol{\sigma})$,
(n) $E_E(z,\varkappa,\nu,.) : (0,+\infty) \to (0,1]$ *is a CF,*
(o) $\Theta_E(z,\varkappa,\nu,\boldsymbol{\sigma}) > 0$,
(p) $\Theta_E(z,\varkappa,\nu,\boldsymbol{\sigma}) = \Theta_E(P\{z,\varkappa,\nu,\boldsymbol{\sigma}\}) = 0$ *if and only if* $z = \varkappa = \nu$, *where P is permutation,*
(q) $\Theta_E(z,\varkappa,\nu,\varsigma + \varpi + \boldsymbol{\sigma}) \leq \Theta_E(z,z,\gamma,\varsigma) \diamond \Theta_E(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi}) \diamond \Theta_E(\nu,\nu,\gamma,\boldsymbol{\sigma})$,
(r) $\Theta_E(z,\varkappa,\nu,.) : (0,+\infty) \to (0,1]$ *is a CF.*
(s) $R_E(z,\varkappa,\nu,\boldsymbol{\sigma}) > 0$,
(t) $R_E(z,\varkappa,\nu,\boldsymbol{\sigma}) = R_E(P\{z,\varkappa,\nu,\boldsymbol{\sigma}\}) = 0$ *if and only if* $z = \varkappa = \nu$, *where P is permutation,*
(u) $R_E(z,\varkappa,\nu,\varsigma + \varpi + \boldsymbol{\sigma}) \leq R_E(z,z,\gamma,\varsigma) \diamond R_E(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi}) \diamond R_E(\nu,\nu,\gamma,\boldsymbol{\sigma})$,
(v) $R_E(z,\varkappa,\nu,.) : (0,+\infty) \to (0,1]$ *is a CF.*

Remark 5. *An NNMS satisfies the symmetric property, i.e.,* $E_E(z,z,\varkappa,\boldsymbol{\sigma}) = E_E(\varkappa,\varkappa,z,\boldsymbol{\sigma})$, $\Theta_E(z,z,\varkappa,\boldsymbol{\sigma}) = \Theta_E(\varkappa,\varkappa,z,\boldsymbol{\sigma})$ *and* $R_E(z,z,\varkappa,\boldsymbol{\sigma}) = R_E(\varkappa,\varkappa,z,\boldsymbol{\sigma})$*, but in NQNMS, the symmetric property is not fulfilled, i.e.,* $E_E(z,z,\varkappa,\boldsymbol{\sigma}) \neq E_E(\varkappa,\varkappa,z,\boldsymbol{\sigma})$, $\Theta_E(z,z,\varkappa,\boldsymbol{\sigma}) \neq \Theta_E(\varkappa,\varkappa,z,\boldsymbol{\sigma})$ *and* $R_E(z,z,\varkappa,\boldsymbol{\sigma}) \neq R_E(\varkappa,\varkappa,z,\boldsymbol{\sigma})$.

Definition 16. *A sextuple* $(\mathbf{Z}, E_q, \Theta_q, R_q, *, \diamond, \boldsymbol{\zeta})$ *is an NPNbMS if* $\mathbf{Z} \neq \varnothing$ *arbitrary set,* $*$ *is a CtN,* $\diamond$ *is a CtCN,* $\boldsymbol{\zeta} \geq 1$ *is a positive real number,* E_q, Θ_q *and* R_q *are NSs on* $\mathbf{Z}^3 \times (0, +\infty)$ *and fulfills the following conditions for all* $z, \varkappa, \nu, \gamma \in \mathbf{Z}$ *and* $\varsigma, \boldsymbol{\varpi}, \boldsymbol{\sigma} > 0$:

(x) $E_q(z,\varkappa,\nu,\boldsymbol{\sigma}) + \Theta_q(z,\varkappa,\nu,\boldsymbol{\sigma}) + R_q(z,\varkappa,\nu,\boldsymbol{\sigma}) \leq 3$,
(xi) $E_q(z,\varkappa,\nu,\boldsymbol{\sigma}) > 0$,
(xii) $E_q(z,\varkappa,\nu,\boldsymbol{\sigma}) = 1$ *if and only if* $z = \varkappa = \nu$,
(xiii) $E_q(z,\varkappa,\nu,\boldsymbol{\zeta}(\varsigma + \boldsymbol{\varpi} + \boldsymbol{\sigma})) \geq E_q(z,z,\gamma,\varsigma) * E_q(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi}) * E_q(\nu,\nu,\gamma,\boldsymbol{\sigma})$,
(xiv) $E_q(z,\varkappa,\nu,.) : (0,+\infty) \to (0,1]$ *is a CF,*
(xv) $\Theta_q(z,\varkappa,\nu,\boldsymbol{\sigma}) > 0$,
(xvi) $\Theta_q(z,\varkappa,\nu,\boldsymbol{\sigma}) = 0$ *if and only if* $z = \varkappa = \nu$,
(xvii) $\Theta_q(z,\varkappa,\nu,\boldsymbol{\zeta}(\varsigma + \boldsymbol{\varpi} + \boldsymbol{\sigma})) \leq \Theta_q(z,z,\gamma,\varsigma) \diamond \Theta_q(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi}) \diamond \Theta_q(\nu,\nu,\gamma,\boldsymbol{\sigma})$,
(xviii) $\Theta_q(z,\varkappa,\nu,.) : (0,+\infty) \to (0,1]$ *is a CF.*
(xix) $R_q(z,\varkappa,\nu,\boldsymbol{\sigma}) > 0$,

(xx) $R_q(z,\varkappa,\nu,\boldsymbol{\sigma}) = 0$ *if and only if* $z = \varkappa = \nu$,
(xxi) $R_q(z,\varkappa,\nu,\boldsymbol{\zeta}(\varsigma+\boldsymbol{\varpi}+\boldsymbol{\sigma})) \leq R_q(z,z,\gamma,\varsigma) \diamond R_q(\varkappa,\varkappa,\gamma,\boldsymbol{\varpi}) \diamond R_q(\nu,\nu,\gamma,\boldsymbol{\sigma})$,
(xxii) $R_q(z,\varkappa,\nu,.) : (0,+\infty) \to (0,1]$ *is a CF.*

Example 5. *Suppose that* $\mathbb{R}$ *equipped with a usual metric and* $\mathbf{Z} = \{\{z_n\} : \{z_n$ *is convergent in* $\mathbb{R}\}\}$. *Define CtN by* $\gamma * \beta * \pi = \gamma\beta\pi$ *and CtCN by* $\gamma \diamond \beta \diamond \pi = max\{\gamma,\beta,\pi\}$ *for all* $\gamma,\beta,\pi \in [0,1]$ *and*

$$E_q(z_n,\varkappa_n,\nu_n,\sigma) = \frac{\sigma}{\sigma + (|z_n - \nu_n| + |\varkappa_n - \nu_n|)^2}$$

$$\Theta_q(z_n,\varkappa_n,\nu_n,\sigma) = \frac{(|z_n - \nu_n| + |\varkappa_n - \nu_n|)^2}{\sigma + (|z_n - \nu_n| + |\varkappa_n - \nu_n|)^2}$$

and

$$R_q(z_n,\varkappa_n,\nu_n,\sigma) = \frac{(|z_n - \nu_n| + |\varkappa_n - \nu_n|)^2}{\sigma}$$

Observe that $(\mathbf{Z}, E_q, \Theta_q, R_q, *, \diamond, \boldsymbol{\zeta})$ *is an NPNbMS but it is not an NNbMS. For this, take* $\{z_n\} = \frac{2}{n}$, $\{\varkappa_n\} = \frac{3}{n}$ *and* $\{\nu_n\} = \frac{5}{n}$. *Then,* $\{z_n\} \neq \{\varkappa_n\} \neq \{\nu_n\}$ *for* $\{z_n\}$, $\{\varkappa_n\}$ *and* $\{\nu_n\}$ *in* $\mathbf{Z}$ *but*

$$E_q(z_n,\varkappa_n,\nu_n,\boldsymbol{\sigma}) = 1,$$
$$\Theta_q(z_n,\varkappa_n,\nu_n,\boldsymbol{\sigma}) = 0,$$
$$R_q(z_n,\varkappa_n,\nu_n,\boldsymbol{\sigma}) = 0.$$

Remark 6. *Every NNbMS is an NPNbMS but the converse need not be true.*

Definition 17. *Suppose* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ *is a symmetric NNbMS. A sequence* $\{z_n\}$ *in* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ *is called convergent, if* $E(z_n,z_n,z,\boldsymbol{\sigma}) \to 1$, $\Theta(z_n,z_n,z,\boldsymbol{\sigma}) \to 0$ *and* $R(z_n,z_n,z,\boldsymbol{\sigma}) \to 0$ *or* $E(z,z,z_n,\boldsymbol{\sigma}) \to 1$, $\Theta(z,z,z_n,\boldsymbol{\sigma}) \to 0$ *and* $R(z,z,z_n,\boldsymbol{\sigma}) \to 0$ *as* $n \to +\infty$ *for every* $\boldsymbol{\sigma} > 0$. *That is, for* $\varsigma > 0$ *and* $\boldsymbol{\sigma} > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$, $E(z_n,z_n,z,\boldsymbol{\sigma}) > 1 - \varsigma$, $\Theta(z_n,z_n,z,\boldsymbol{\sigma}) < \varsigma$ *and* $R(z_n,z_n,z,\boldsymbol{\sigma}) < \varsigma$ *or* $E(z,z,z_n,\boldsymbol{\sigma}) > 1 - \varsigma$, $\Theta(z,z,z_n,\boldsymbol{\sigma}) < \varsigma$ *and* $R(z,z,z_n,\boldsymbol{\sigma}) < \varsigma$.

Lemma 1. *Suppose* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ *is a symmetric NNbMS with CtN* $\gamma * \beta * \pi = \gamma\beta\pi$ *and CtCN* $\gamma \diamond \beta \diamond \pi = max\{\gamma,\beta,\pi\}$. *Suppose* $\mathbf{Z}$ *has a convergent sequence which is* $\{z_n\}$. *If sequence* $\{z_n\}$ *converges to* z *and* $\varkappa$ *then* $z = \varkappa$. *That is, the limit of* $\{z_n\}$ *is unique if it exists.*

Proof. Let $\{z_n\}$ be a convergent sequence in $\mathbf{Z}$ and converges to z and $\varkappa$. Then, $E(z,z,z_n,\boldsymbol{\varpi}) \to 1$, $\Theta(z,z,z_n,\boldsymbol{\varpi}) \to 0$ and $R(z,z,z_n,\boldsymbol{\varpi}) \to 0$ as $n \to +\infty$ for every $\boldsymbol{\varpi} > 0$ and $E(\varkappa,\varkappa,z_n,\boldsymbol{\sigma} - 2\boldsymbol{\varpi}) \to 1$, $\Theta(\varkappa,\varkappa,z_n,\boldsymbol{\sigma} - 2\boldsymbol{\varpi}) \to 0$ and $R(\varkappa,\varkappa,z_n,\boldsymbol{\sigma} - 2\boldsymbol{\varpi}) \to 0$ as $n \to +\infty$ for every $\boldsymbol{\sigma} - 2\boldsymbol{\varpi} > 0$.

$$E(z,z,\varkappa,\boldsymbol{\sigma}) \geq E(z,z,z_n,\boldsymbol{\varpi}) * E(z,z,z_n,\boldsymbol{\varpi}) * E\left(\varkappa,\varkappa,z_n,\frac{\boldsymbol{\sigma}}{\boldsymbol{\zeta}} - 2\boldsymbol{\varpi}\right) \to 1 * 1 * 1 = 1,$$

$$\Theta(z,z,\varkappa,\boldsymbol{\sigma}) \leq \Theta(z,z,z_n,\boldsymbol{\varpi}) \diamond \Theta(z,z,z_n,\boldsymbol{\varpi}) \diamond \Theta\left(\varkappa,\varkappa,z_n,\frac{\boldsymbol{\sigma}}{\boldsymbol{\zeta}} - 2\boldsymbol{\varpi}\right) \to 0 \diamond 0 \diamond 0 = 0,$$

and

$$R(z,z,\varkappa,\boldsymbol{\sigma}) \leq R(z,z,z_n,\boldsymbol{\varpi}) \diamond R(z,z,z_n,\boldsymbol{\varpi}) \diamond R\left(\varkappa,\varkappa,z_n,\frac{\boldsymbol{\sigma}}{\boldsymbol{\zeta}} - 2\boldsymbol{\varpi}\right) \to 0 \diamond 0 \diamond 0 = 0,$$

as $n \to +\infty$. □

Definition 18. *Let* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ *be a symmetric NNbMS. A sequence* $\{z_n\}$ *is a Cauchy sequence (CS), if for all* $\varsigma > 0$ *and* $\boldsymbol{\sigma} > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that*

$$E(z_n, z_n, z_m, \boldsymbol{\sigma}) > 1 - \varsigma,$$
$$\Theta(z_n, z_n, z_m, \boldsymbol{\sigma}) < \varsigma,$$
$$R(z_n, z_n, z_m, \boldsymbol{\sigma}) < \varsigma.$$

or

$$E(z_m, z_m, z_n, \boldsymbol{\sigma}) > 1 - \varsigma,$$
$$\Theta(z_m, z_m, z_n, \boldsymbol{\sigma}) < \varsigma,$$
$$R(z_m, z_m, z_n, \boldsymbol{\sigma}) < \varsigma,$$

for each $n, m \geq n_0$.

Definition 19. *Let* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ *be a symmetric NNbMS.* $\mathbf{Z}$ *is said to be a complete symmetric NNbMS if every Cauchy in* $\mathbf{Z}$ *is convergent in* $\mathbf{Z}$.

Definition 20. *Let* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ *be a symmetric NNbMS. A subset* A *of* $\mathbf{Z}$ *is called* F *-bounded if there exist* $\boldsymbol{\sigma} > 0$ *and* $0 < \varsigma < 1$ *such that*

$$E(z, z, \varkappa, \boldsymbol{\sigma}) > 1 - \varsigma,$$
$$\Theta(z, z, \varkappa, \boldsymbol{\sigma}) < \varsigma,$$
$$R(z, z, \varkappa, \boldsymbol{\sigma}) < \varsigma,$$

for each $z, \varkappa \in A$.

Definition 21. *Let* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ *be an NNbMS. A mapping* $\Omega : \mathbf{Z} \to \mathbf{Z}$ *is said to be a neutrosophic* β*-contraction if for each* $z, \varkappa, \nu \in \mathbf{Z}$ *and for some* $q \in (0, 1)$*, we obtain*

$$E(\Omega(z), \Omega(z), \Omega(\varkappa), q\boldsymbol{\sigma}) \geq E(z, z, \varkappa, \boldsymbol{\sigma}),$$
$$\Theta(\Omega(z), \Omega(z), \Omega(\varkappa), q\boldsymbol{\sigma}) \leq \Theta(z, z, \varkappa, \boldsymbol{\sigma}),$$
$$R(\Omega(z), \Omega(z), \Omega(\varkappa), q\boldsymbol{\sigma}) \leq R(z, z, \varkappa, \boldsymbol{\sigma}).$$

Lemma 2. *Let* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ *be an NNbMS with CtN* $\gamma * \beta * \pi = \gamma\beta\pi$ *and CtCN* $\gamma \diamond \beta \diamond \pi = max\{\gamma, \beta, \pi\}$*. Assume that a sequence* $\{z_n\}$ *converges to* z *in* $\mathbf{Z}$*. Then, sequence* $\{z_n\}$ *is called a CS in* $\mathbf{Z}$.

Proof. There is $p \in \mathbb{N}$ for every $\boldsymbol{\varpi}, \boldsymbol{\sigma} > 0$, we have

$$E(z_n, z_n, z, \boldsymbol{\varpi}) \to 1,$$
$$\Theta(z_n, z_n, z, \boldsymbol{\varpi}) \to 0,$$
$$R(z_n, z_n, z, \boldsymbol{\varpi}) \to 0,$$

as $n \to +\infty$, and

$$E\left(z_{n+p}, z_{n+p}, z, \frac{\sigma}{\zeta} - 2\boldsymbol{\varpi}\right) \to 1,$$
$$\Theta\left(z_{n+p}, z_{n+p}, z, \frac{\sigma}{\zeta} - 2\boldsymbol{\varpi}\right) \to 0,$$
$$R\left(z_{n+p}, z_{n+p}, z, \frac{\sigma}{\zeta} - 2\boldsymbol{\varpi}\right) \to 0$$

as $n \to +\infty$, for all $\frac{\sigma}{\zeta} - 2\boldsymbol{\varpi} > 0$.

$$(z_n, z_n, z_{n+p}, \boldsymbol{\sigma}) \geq E(z_n, z_n, z, \boldsymbol{\varpi}) * E(z_n, z_n, z, \boldsymbol{\varpi}) * E\left(z_{n+p}, z_{n+p}, z, \frac{\sigma}{\zeta} - 2\boldsymbol{\varpi}\right)$$
$$\to 1 * 1 * 1 = 1 \text{ as } n \to +\infty,$$

$$E\Theta(z_n, z_n, z_{n+p}, \boldsymbol{\sigma}) \leq \Theta(z_n, z_n, z, \boldsymbol{\varpi}) \diamond \Theta(z_n, z_n, z, \boldsymbol{\varpi}) \diamond \Theta\left(z_{n+p}, z_{n+p}, z, \frac{\sigma}{\zeta} - 2\boldsymbol{\varpi}\right)$$
$$\to 0 \diamond 0 \diamond 0 = 0 \text{ as } n \to +\infty.$$

and

$$R\left(z_n, z_n, z_{n+p}, \sigma\right) \leq R(z_n, z_n, z, \varpi) \diamond R(z_n, z_n, z, \varpi) \diamond R\left(z_{n+p}, z_{n+p}, z, \frac{\sigma}{\zeta} - 2\varpi\right)$$
$$\rightarrow 0 \diamond 0 \diamond 0 = 0 \text{ as } n \rightarrow +\infty.$$

Hence, $\{z_n\}$ is a CS. □

Definition 22. *Assume that the symmetric NNbMSs are $(\boldsymbol{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ and $(\boldsymbol{Z}', E', \Theta', R', *', \diamond', \boldsymbol{\zeta}')$. Then a function $\Omega : \boldsymbol{Z} \rightarrow \boldsymbol{Z}'$ is said to be continuous at a point $z \in \boldsymbol{Z}$ if it is sequentially continuous at z, that is, whenever $\{z_n\}$ is convergent to z, we have $\{\Omega z_n\}$ converging to $\Omega(z)$.*

Proposition 1. *Let $(\boldsymbol{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ be symmetric NNbMSs and Ω be a fuzzy q-contraction. If any FP z of Ω satisfies*

$$E(z, z, z, \sigma) > 0,$$
$$\Theta(z, z, z, \sigma) < 1,$$
$$R(z, z, z, \sigma) < 1,$$

then

$$E(z, z, z, \sigma) = 1,$$
$$\Theta(z, z, z, \sigma) = 0,$$
$$R(z, z, z, \sigma) = 0.$$

Proof. Given that Ω is a fuzzy q-contraction, suppose a point $z \in \boldsymbol{Z}$ is an FP of Ω, then we obtain

$$E(z, z, z, \sigma) = E(\Omega(z), \Omega(z), \Omega(z), \sigma)$$
$$\geq E\left(z, z, z, \frac{\sigma}{q}\right) \geq E\left(z, z, z, \frac{\sigma}{q^2}\right)$$
$$\geq \cdots \geq E\left(z, z, z, \frac{\sigma}{q^n}\right) \rightarrow 1,$$

as $n \rightarrow +\infty$ and so

$$E(z, z, z, \sigma) = 1.$$

and

$$\Theta(z, z, z, \sigma) = \Theta(\Omega(z), \Omega(z), \Omega(z), \sigma)$$
$$\leq \Theta\left(z, z, z, \frac{\sigma}{q}\right) \leq \Theta\left(z, z, z, \frac{\sigma}{q^2}\right)$$
$$\leq \cdots \leq \Theta\left(z, z, z, \frac{\sigma}{q^n}\right) \rightarrow 0,$$

as $n \rightarrow +\infty$ and so

$$\Theta(z, z, z, \sigma) \rightarrow 0.$$

Similarly,

$$R(z, z, z, \sigma) = R(\Omega(z), \Omega(z), \Omega(z), \sigma)$$
$$\leq R\left(z, z, z, \frac{\sigma}{q}\right) \leq R\left(z, z, z, \frac{\sigma}{q^2}\right)$$
$$\leq \cdots \leq R\left(z, z, z, \frac{\sigma}{q^n}\right) \rightarrow 0$$

as $n \rightarrow +\infty$ and so

$$R(z, z, z, \sigma) \rightarrow 0.$$

□

Lemma 3. *Let $(\boldsymbol{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ be symmetric NNbMSs. Let $\{z_n\}$ and $\{\varkappa_n\}$ be two sequences in $\boldsymbol{Z}$ and suppose $z_n \rightarrow z$, $\varkappa_n \rightarrow \varkappa$, as $n \rightarrow +\infty$, $E(z, z, \varkappa, \sigma_n) \rightarrow E(z, z, \varkappa, \sigma)$, $\Theta(z, z, \varkappa, \sigma_n) \rightarrow \Theta(z, z, \varkappa, \sigma)$ and $R(z, z, \varkappa, \sigma_n) \rightarrow R(z, z, \varkappa, \sigma)$ as $n \rightarrow +\infty$. Then $E(z_n, z_n, \varkappa_n, \sigma_n) \rightarrow E(z, z, z, \sigma)$, $\Theta(z_n, z_n, \varkappa_n, \sigma_n) \rightarrow \Theta(z, z, z, \sigma)$ and $R(z_n, z_n, \varkappa_n, \sigma_n) \rightarrow R(z, z, z, \sigma)$ as $n \rightarrow +\infty$.*

Proof. Since $\lim_{n\to+\infty} z_n$, $\lim_{n\to+\infty} z_n = z$, $\lim_{n\to+\infty} \varkappa_n = \varkappa$, $\lim_{n\to+\infty} E(z,\, z, \varkappa, \boldsymbol{\sigma}_n) = E(z,\, z, \varkappa, \boldsymbol{\sigma})$, $\lim_{n\to+\infty} \Theta(z,\, z, \varkappa, \boldsymbol{\sigma}_n) = \Theta(z,\, z, \varkappa, \boldsymbol{\sigma})$ and $\lim_{n\to+\infty} R(z,\, z, \varkappa, \boldsymbol{\sigma}_n) = R(z,\, z, \varkappa, \boldsymbol{\sigma})$, there is $n_0 \in \mathbb{N}$ such that $|\boldsymbol{\sigma} - \boldsymbol{\sigma}_n| < \delta$ for $n \geq n_0$ and $\delta < \frac{\sigma}{2}$, so we have

$$
\begin{gathered}
E(z_n, z_n, \varkappa_n, \boldsymbol{\sigma}_n) \geq E(z_n, z_n, \varkappa_n, \boldsymbol{\sigma} - \delta) \\
\geq E\left(z_n, z_n, z, \frac{\delta}{3\zeta}\right) * E\left(z_n, z_n, z, \frac{\delta}{3\zeta}\right) * E\left(\varkappa_n, \varkappa_n, z, \frac{\sigma}{\zeta} - \frac{5\delta}{3\zeta}\right) \\
\geq E\left(z_n, z_n, z, \frac{\delta}{3\zeta}\right) * E\left(z_n, z_n, z, \frac{\delta}{3\zeta}\right) * E\left(\varkappa_n, \varkappa_n, \varkappa, \frac{\delta}{6\zeta^2}\right) \\
* E\left(\varkappa_n, \varkappa_n, \varkappa, \frac{\delta}{6\zeta^2}\right) * E\left(\varkappa, \varkappa, z, \frac{\sigma}{\zeta^2} \frac{\delta}{6\zeta^2}\right),
\end{gathered}
$$

and

$$
\begin{gathered}
E(z, z, \varkappa, \boldsymbol{\sigma} + 2\delta) \geq E(z, z, \varkappa, \boldsymbol{\sigma}_n + 2\delta) \\
\geq E\left(z, z, z_n, \frac{\delta}{3\zeta}\right) * E\left(z, z, z_n, \frac{\delta}{3\zeta}\right) * E\left(\varkappa, \varkappa, z_n, \frac{\sigma_n}{\zeta} + \frac{\delta}{3\zeta}\right) \\
\geq E\left(z, z, z_n, \frac{\delta}{3\zeta}\right) * E\left(z, z, z_n, \frac{\delta}{3\zeta}\right) * E\left(\varkappa, \varkappa, z_n, \frac{\delta}{6\zeta^2}\right) \\
* E\left(\varkappa, \varkappa, \varkappa_n, \frac{\delta}{6\zeta^2}\right) * E\left(z_n, z_n, \nu_n, \frac{\sigma_n}{\zeta^2}\right). \\
\Theta(z_n, z_n, \varkappa_n, \boldsymbol{\sigma}_n) \leq \Theta(z_n, z_n, \varkappa_n, \boldsymbol{\sigma} - \delta) \\
\leq \Theta\left(z_n, z_n, z, \frac{\delta}{3\zeta}\right) \diamond \Theta\left(z_n, z_n, z, \frac{\delta}{3\zeta}\right) \diamond \Theta\left(\varkappa_n, \varkappa_n, z, \frac{\sigma}{\zeta} - \frac{5\delta}{3\zeta}\right) \\
\leq \Theta\left(z_n, z_n, z, \frac{\delta}{3\zeta}\right) \diamond \Theta\left(z_n, z_n, z, \frac{\delta}{3\zeta}\right) \diamond \Theta\left(\varkappa_n, \varkappa_n, \varkappa, \frac{\delta}{6\zeta^2}\right) \\
\diamond \Theta\left(\varkappa_n, \varkappa_n, \varkappa, \frac{\delta}{6\zeta^2}\right) \diamond \Theta\left(\varkappa, \varkappa, z, \frac{\sigma}{\zeta^2} \frac{\delta}{6\zeta^2}\right),
\end{gathered}
$$

and we have

$$
\begin{gathered}
\Theta(z, z, \varkappa, \boldsymbol{\sigma} + 2\delta) \leq \Theta(z, z, \varkappa, \boldsymbol{\sigma}_n + 2\delta) \\
\leq \Theta\left(z, z, z_n, \frac{\delta}{3\zeta}\right) \diamond \Theta\left(z, z, z_n, \frac{\delta}{3\zeta}\right) \diamond \Theta\left(\varkappa, \varkappa, z_n, \frac{\sigma_n}{\zeta} + \frac{\delta}{3\zeta}\right) \\
\leq \Theta\left(z, z, z_n, \frac{\delta}{3\zeta}\right) \diamond \Theta\left(z, z, z_n, \frac{\delta}{3\zeta}\right) \diamond \Theta\left(\varkappa, \varkappa, z_n, \frac{\delta}{6\zeta^2}\right) \\
\diamond \Theta\left(\varkappa, \varkappa, \varkappa_n, \frac{\delta}{6\zeta^2}\right) \diamond \Theta\left(z_n, z_n, \nu_n, \frac{\sigma_n}{\zeta^2}\right).
\end{gathered}
$$

Similarly,

$$
\begin{gathered}
R(z_n, z_n, \varkappa_n, \boldsymbol{\sigma}_n) \leq R(z_n, z_n, \varkappa_n, \boldsymbol{\sigma} - \delta) \\
\leq R\left(z_n, z_n, z, \frac{\delta}{3\zeta}\right) \diamond R\left(z_n, z_n, z, \frac{\delta}{3\zeta}\right) \diamond R\left(\varkappa_n, \varkappa_n, z, \frac{\sigma}{\zeta} - \frac{5\delta}{3\zeta}\right) \\
\leq R\left(z_n, z_n, z, \frac{\delta}{3\zeta}\right) \diamond R\left(z_n, z_n, z, \frac{\delta}{3\zeta}\right) \diamond R\left(\varkappa_n, \varkappa_n, \varkappa, \frac{\delta}{6\zeta^2}\right) \\
\diamond R\left(\varkappa_n, \varkappa_n, \varkappa, \frac{\delta}{6\zeta^2}\right) \diamond R\left(\varkappa, \varkappa, z, \frac{\sigma}{\zeta^2} \frac{\delta}{6\zeta^2}\right),
\end{gathered}
$$

and

$$
\begin{gathered}
R(z, z, \varkappa, \boldsymbol{\sigma} + 2\delta) \leq R(z, z, \varkappa, \boldsymbol{\sigma}_n + 2\delta) \\
\leq R\left(z, z, z_n, \frac{\delta}{3\zeta}\right) \diamond R\left(z, z, z_n, \frac{\delta}{3\zeta}\right) \diamond R\left(\varkappa, \varkappa, z_n, \frac{\sigma_n}{\zeta} + \frac{\delta}{3\zeta}\right) \\
\leq R\left(z, z, z_n, \frac{\delta}{3\zeta}\right) \diamond R\left(z, z, z_n, \frac{\delta}{3\zeta}\right) \diamond R\left(\varkappa, \varkappa, z_n, \frac{\delta}{6\zeta^2}\right) \\
\diamond R\left(\varkappa, \varkappa, \varkappa_n, \frac{\delta}{6\zeta^2}\right) \diamond R\left(z_n, z_n, \nu_n, \frac{\sigma_n}{\zeta^2}\right).
\end{gathered}
$$

By Definition of 17 and combining the arbitrary nature of δ and the continuity for $E(z,\, z, \varkappa,\, .)$, $\Theta(z,\, z, \varkappa,\, .)$ and $R(z,\, z, \varkappa,\, .)$ concerning $\boldsymbol{\sigma}$. for large enough n, by using Definition 12, we obtain

$$
\begin{gathered}
E(z,\, z,\, \varkappa, \boldsymbol{\sigma}) \geq E(z_n,\, z_n,\, \varkappa_n, \boldsymbol{\sigma}_n) \geq E(\varkappa,\, \varkappa,\, z, \boldsymbol{\sigma}) \\
E(z,\, z,\, \varkappa, \boldsymbol{\sigma}) \geq E(z_n,\, z_n,\, \nu_n, \boldsymbol{\sigma}_n), \\
\geq E(z,\, z,\, \varkappa, \boldsymbol{\sigma}),
\end{gathered}
$$

and consequently, by using Definition 12, we have

$$\lim_{n\to+\infty} E(z_n,\, z_n,\, \varkappa_n, \sigma_n) = E(z,\, z,\, \varkappa, \sigma).$$
$$\Theta(z,\, z,\, \varkappa, \sigma) \le \Theta(z_n,\, z_n,\, \varkappa_n, \sigma_n) \le \Theta(\varkappa,\, \varkappa,\, z, \sigma)$$
$$\Theta(z,\, z,\, \varkappa, \sigma) \le \Theta(z_n,\, z_n,\, \nu_n, \sigma_n),$$
$$\le \Theta(z,\, z,\, \varkappa, \sigma).$$

We obtain

$$\lim_{n\to+\infty} \Theta(z_n,\, z_n,\, \varkappa_n, \sigma_n) = \Theta(z,\, z,\, \varkappa, \sigma).$$

and

$$R(z,\, z,\, \varkappa, \sigma) \le R(z_n,\, z_n,\, \varkappa_n, \sigma_n) \le R(\varkappa,\, \varkappa,\, z, \sigma)$$
$$R(z,\, z,\, \varkappa, \sigma) \le R(z_n,\, z_n,\, \nu_n, \sigma_n),$$
$$\le R(z,\, z,\, \varkappa, \sigma).$$

Consequently,

$$\lim_{n\to+\infty} R(z_n,\, z_n,\, \varkappa_n, \sigma_n) = R(z,\, z,\, \varkappa, \sigma).$$

□

Lemma 4. *Let* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ *be symmetric NNbMSs. If there exists* $q \in (0,1)$ *such that* $E(z,z,\varkappa,\sigma) \ge E\left(z,z,\, \varkappa, \frac{\sigma}{q}\right)$, $\Theta(z,z,\varkappa,\sigma) \le \Theta\left(z,z,\, \varkappa, \frac{\sigma}{q}\right)$ *and* $R(z,z,\varkappa,\sigma) \le R\left(z,z,\, \varkappa, \frac{\sigma}{q}\right)$ *for all* $z,\ \varkappa \in \mathbf{Z},\ \sigma > 0$ *and*

$$\lim_{\sigma\to+\infty} E(z,\varkappa,\nu,\sigma) = 1,$$
$$\lim_{\sigma\to+\infty} \Theta(z,\varkappa,\nu,\sigma) = 0,$$
$$\lim_{\sigma\to+\infty} R(z,\varkappa,\nu,\sigma) = 0.$$

Then $z = \varkappa$.

Proof. Assume that there exists $q \in (0,1)$ such that $E(z,z,\varkappa,\sigma) \ge E\left(z,z,\varkappa, \frac{\sigma}{q}\right)$, $\Theta(z,z,\varkappa,\sigma)$ $\le \Theta\left(z,z,\varkappa, \frac{\sigma}{q}\right)$ and $R(z,z,\varkappa,\sigma) \le R\left(z,z,\varkappa, \frac{\sigma}{q}\right)$, for all $z,\ \varkappa \in \mathbf{Z}$ and $\sigma > 0$. Then,

$$E(z,z,\varkappa,\sigma) \ge E\left(z,z,\varkappa, \frac{\sigma}{q}\right) \ge E\left(z,z,\varkappa, \frac{\sigma}{q^2}\right),$$

and so

$$E(z,z,\varkappa,\sigma) \ge E\left(z,z,\varkappa, \frac{\sigma}{q^n}\right),$$

for positive integer n. Taking the limit as $n \to +\infty$, $E(z,z,\varkappa,\sigma) \ge 1$,

$$\Theta(z,z,\varkappa,\sigma) \le \Theta\left(z,z,\varkappa, \frac{\sigma}{q}\right) \le \Theta\left(z,z,\varkappa, \frac{\sigma}{q^2}\right),$$

and so

$$\Theta(z,z,\varkappa,\sigma) \le \Theta\left(z,z,\varkappa, \frac{\sigma}{q^n}\right),$$

For positive integer n. Taking the limit as $\to +\infty$, $\Theta(z,z,\varkappa,\sigma) = 0$,

$$R(z,z,\varkappa,\sigma) \le R\left(z,z,\varkappa, \frac{\sigma}{q}\right) \le R\left(z,z,\varkappa, \frac{\sigma}{q^2}\right),$$

and so

$$R(z,z,\varkappa,\sigma) \le R\left(z,z,\varkappa, \frac{\sigma}{q^n}\right),$$

For positive integer n. Taking the limit as $n \to +\infty$, $R(z, z, \varkappa, \sigma) = 0$ and hence $z = \varkappa$. □

5. Application in FP Theory

In this, we describe the application of a Banach contraction principle via neutrosophic $\bar{q}$-contraction in symmetric NNbMSs

Theorem 1. *Suppose $(\boldsymbol{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ is a symmetric complete NNbMSs with*

$$\lim_{\sigma \to +\infty} E(z, \varkappa, \nu, \sigma) = 1, \; \lim_{\sigma \to +\infty} \Theta(z, \varkappa, \nu, \sigma) = 0, \; \lim_{\sigma \to +\infty} R(z, \varkappa, \nu, \sigma) = 0 \tag{4}$$

and Ω be a neutrosophic $q-$ contraction. Then, Ω has a unique FP.

Proof. Let $z_0 \in \boldsymbol{Z}$ and by using the iterative process, we create a sequence $\{z_n\}$ which satisfies $z_n = \Omega^n(z_0)$, $n \in \mathbb{N}$. Since $n, \sigma > 0$, we obtain

$$\begin{aligned}
E(z_n, \, z_n, z_{n+1}, \, q\sigma) &= E(\Omega z_{n-1}, \Omega z_{n-1}, \, \Omega z_n, \, q\sigma) \\
&\geq E(z_{n-1}, z_{n-1}, \, z_n, \, \sigma) \\
&\geq E\left(z_{n-2}, z_{n-2}, \, z_n, \frac{\sigma}{q}\right) \\
&\;\;\vdots \\
&\geq E\left(z_0, z_0, \, z_1, \frac{\sigma}{q^{n-1}}\right), \\
\Theta(z_n, \, z_n, z_{n+1}, \, q\sigma) &= \Theta(\Omega z_{n-1}, \Omega z_{n-1}, \, \Omega z_n, \, q\sigma) \\
&\leq \Theta(z_{n-1}, z_{n-1}, \, z_n, \, \sigma) \\
&\leq \Theta\left(z_{n-2}, z_{n-2}, \, z_n, \frac{\sigma}{q}\right) \\
&\;\;\vdots \\
&\leq \Theta\left(z_0, z_0, \, z_1, \frac{\sigma}{q^{n-1}}\right),
\end{aligned}$$

and

$$\begin{aligned}
R(z_n, \, z_n, z_{n+1}, \, q\sigma) &= R(\Omega z_{n-1}, \Omega z_{n-1}, \, \Omega z_n, \, q\sigma) \\
&\leq R(z_{n-1}, z_{n-1}, \, z_n, \, \sigma) \\
&\leq R\left(z_{n-2}, z_{n-2}, \, z_n, \frac{\sigma}{q}\right) \\
&\;\;\vdots \\
&\leq R\left(z_0, z_0, \, z_1, \frac{\sigma}{q^{n-1}}\right).
\end{aligned}$$

Hence,

$$E(z_n, \, z_n, z_{n+1}, \, q\sigma) \geq \; E\left(z_0, z_0, \, z_1, \frac{\sigma}{q^{n-1}}\right),$$

by using (iv) of Definition 11, we obtain

$$\begin{aligned}
E(z_n, \, z_n, z_{n+p}, \, \sigma) \geq \; & E\left(z_n, z_n, \, z_{n+1}, \frac{\sigma}{3\zeta}\right) * E\left(z_n, z_n, \, z_{n+1}, \frac{\sigma}{3\zeta}\right) * E\left(z_{n+p}, z_{n+p}, \, z_{n+1}, \frac{\sigma}{3\zeta}\right). \\
= \; & E\left(z_n, z_n, \, z_{n+1}, \frac{\sigma}{3\zeta}\right) * E\left(z_n, z_n, \, z_{n+1}, \frac{\sigma}{3\zeta}\right) * E\left(z_{n+1}, z_{n+1}, \, z_{n+p}, \frac{\sigma}{3\zeta}\right) \text{ [by using Definition 12]} \\
\geq \; & E\left(z_n, z_n, \, z_{n+1}, \frac{\sigma}{3\zeta}\right) * E\left(z_n, z_n, \, z_{n+1}, \frac{\sigma}{3\zeta}\right) * E\left(z_{n+1}, z_{n+1}, \, z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right) \\
& * E\left(z_{n+1}, z_{n+1}, \, z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right) * E\left(z_{n+p}, z_{n+p}, \, z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right), \\
= \; & E\left(z_n, z_n, \, z_{n+1}, \frac{\sigma}{3\zeta}\right) * E\left(z_n, z_n, \, z_{n+1}, \frac{\sigma}{3\zeta}\right) * E\left(z_{n+1}, z_{n+1}, \, z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right) \\
& * E\left(z_{n+1}, z_{n+1}, \, z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right) * E\left(z_{n+2}, z_{n+2}, \, z_{n+p}, \frac{\sigma}{(3\zeta)^2}\right),
\end{aligned}$$

$$\geq E\left(z_0, z_0,\ z_1, \frac{\sigma}{q^n(3\zeta)}\right) * E\left(z_0, z_0,\ z_1, \frac{\sigma}{q^n(3\zeta)}\right) * E\left(z_0, z_0,\ z_1, \frac{\sigma}{q^{n+1}(3\zeta)^2}\right)$$
$$*E\left(z_0, z_0,\ z_1, \frac{\sigma}{q^{n+1}(3\zeta)^2}\right).$$

By (4) neutrosophic q⁻contraction (since, $q < 1$) so

$$\lim_{n\to+\infty} E(z_n,\ z_n, z_{n+1},\ \sigma) = 1 * 1 * \ \cdots * 1 = 1,$$

as $n \to +\infty$, and

$$\Theta(z_n,\ z_n, z_{n+1},\ q\sigma) \leq \Theta\left(z_0, z_0,\ z_1, \frac{\sigma}{q^{n-1}}\right),$$

by using (viii) of Definition 11, we obtain

$$\Theta(z_n,\ z_n, z_{n+p},\ \sigma) \leq \Theta\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond \Theta\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond \Theta\left(z_{n+p}, z_{n+p},\ z_{n+1}, \frac{\sigma}{3\zeta}\right).$$
$$= \Theta\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond \Theta\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond \Theta\left(z_{n+1}, z_{n+1},\ z_{n+p}, \frac{\sigma}{3\zeta}\right), \text{ [by Definition 12]}$$
$$\leq \Theta\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond \Theta\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond \Theta\left(z_{n+1}, z_{n+1},\ z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right)$$
$$\diamond\Theta\left(z_{n+1}, z_{n+1},\ z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right) \diamond \Theta\left(z_{n+p}, z_{n+p},\ z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right),$$
$$= \Theta\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond \Theta\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond \Theta\left(z_{n+1}, z_{n+1},\ z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right)$$
$$\diamond\Theta\left(z_{n+1}, z_{n+1},\ z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right) \diamond \Theta\left(z_{n+2}, z_{n+2},\ z_{n+p}, \frac{\sigma}{(3\zeta)^2}\right)$$
$$\leq \Theta\left(z_0, z_0,\ z_1, \frac{\sigma}{q^n(3\zeta)}\right) \diamond \Theta\left(z_0, z_0,\ z_1, \frac{\sigma}{q^n(3\zeta)}\right) \diamond \Theta\left(z_0, z_0,\ z_1, \frac{\sigma}{q^{n+1}(3\zeta)^2}\right)$$
$$\diamond\Theta\left(z_0, z_0,\ z_1, \frac{\sigma}{q^{n+1}(3\zeta)^2}\right).$$

By (4), neutrosophic q⁻contraction (i.e., $q < 1$) and taking $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} \Theta(z_n,\ z_n, z_{n+1},\ \sigma) = 0 \diamond 0 \diamond \ \cdots \diamond 0 = 0.$$

Similarly,

$$R(z_n,\ z_n, z_{n+1},\ q\sigma) \leq R\left(z_0, z_0,\ z_1, \frac{\sigma}{q^{n-1}}\right),$$

by using (xii) of Definition 11, we obtain

$$R(z_n,\ z_n, z_{n+p},\ \sigma) \leq R\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond R\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond R\left(z_{n+p}, z_{n+p},\ z_{n+1}, \frac{\sigma}{3\zeta}\right).$$
$$= R\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond R\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond R\left(z_{n+1}, z_{n+1},\ z_{n+p}, \frac{\sigma}{3\zeta}\right), \text{ [by Definition 12]}$$
$$\leq R\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond R\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond R\left(z_{n+1}, z_{n+1},\ z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right)$$
$$\diamond R\left(z_{n+1}, z_{n+1},\ z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right) \diamond R\left(z_{n+p}, z_{n+p},\ z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right),$$
$$= R\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond R\left(z_n, z_n,\ z_{n+1}, \frac{\sigma}{3\zeta}\right) \diamond R\left(z_{n+1}, z_{n+1},\ z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right)$$
$$\diamond R\left(z_{n+1}, z_{n+1},\ z_{n+2}, \frac{\sigma}{(3\zeta)^2}\right) \diamond R\left(z_{n+2}, z_{n+2},\ z_{n+p}, \frac{\sigma}{(3\zeta)^2}\right)$$
$$\leq R\left(z_0, z_0,\ z_1, \frac{\sigma}{q^n(3\zeta)}\right) \diamond R\left(z_0, z_0,\ z_1, \frac{\sigma}{q^n(3\zeta)}\right) \diamond R\left(z_0, z_0,\ z_1, \frac{\sigma}{q^{n+1}(3\zeta)^2}\right)$$
$$\diamond R\left(z_0, z_0,\ z_1, \frac{\sigma}{q^{n+1}(3\zeta)^2}\right).$$

By (4), neutrosophic q-contraction (i.e., $q < 1$) and taking $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} R(z_n,\ z_n, z_{n+1},\ \boldsymbol{\sigma}) = 0 \diamond 0 \diamond\ \cdots \diamond 0 = 0.$$

Hence, $\{z_n\}$ is CS. Therefore, $(\boldsymbol{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ is a symmetric complete NNbMSs, there exists $z \in \boldsymbol{Z}$, we have

$$\lim_{n\to+\infty} z_n = z.$$

The point z is an FP of Ω, as we will demonstrate below:

$$\begin{aligned} E(\Omega(z),\ \Omega(z),\ z,\ \boldsymbol{\sigma}) &\geq E\left(\Omega(z), \Omega(z),\ z_n, \tfrac{\sigma}{3\zeta}\right) * E\left(\Omega(z), \Omega(z),\ z_n, \tfrac{\sigma}{3\zeta}\right) \\ &\quad * E\left(z, z,\ \Omega(z_n), \tfrac{\sigma}{3\zeta}\right) \\ &\geq E\left(z, z,\ z_n, \tfrac{\sigma}{3\zeta q}\right) * E\left(z, z,\ z_n, \tfrac{\sigma}{3\zeta q}\right) * E\left(z, z,\ z_{n+1}, \tfrac{\sigma}{3\zeta}\right). \end{aligned}$$

subsequently Ω is neutrosophic q-contraction and $\Omega(z_n) = z_{n+1}$ as $n \to +\infty$

$$\to 1 * 1 * 1 = 1$$

and

$$\begin{aligned} \Theta(\Omega(z),\ \Omega(z),\ z,\ \boldsymbol{\sigma}) &\leq \Theta\left(\Omega(z), \Omega(z),\ z_n, \tfrac{\sigma}{3\zeta}\right) \diamond \Theta\left(\Omega(z), \Omega(z),\ z_n, \tfrac{\sigma}{3\zeta}\right) \\ &\quad \diamond \Theta\left(z, z,\ \Omega(z_n), \tfrac{\sigma}{3\zeta}\right) \\ &\leq \Theta\left(z, z,\ z_n, \tfrac{\sigma}{3\zeta q}\right) \diamond \Theta\left(z, z,\ z_n, \tfrac{\sigma}{3\zeta q}\right) \diamond \Theta\left(z, z,\ z_{n+1}, \tfrac{\sigma}{3\zeta}\right), \end{aligned}$$

Since Ω is neutrosophic q-contraction and $\Omega(z_n) = z_{n+1}$ as $n \to +\infty$

$$\to 0 \diamond 0 \diamond 0 = 0.$$

similarly, we have

$$\begin{aligned} R(\Omega(z),\ \Omega(z),\ z,\ \boldsymbol{\sigma}) &\leq R\left(\Omega(z), \Omega(z),\ z_n, \tfrac{\sigma}{3\zeta}\right) \diamond R\left(\Omega(z), \Omega(z),\ z_n, \tfrac{\sigma}{3\zeta}\right) \\ &\quad \diamond R\left(z, z,\ \Omega(z_n), \tfrac{\sigma}{3\zeta}\right) \\ &\leq R\left(z, z,\ z_n, \tfrac{\sigma}{3\zeta q}\right) \diamond R\left(z, z,\ z_n, \tfrac{\sigma}{3\zeta q}\right) \diamond R\left(z, z,\ z_{n+1}, \tfrac{\sigma}{3\zeta}\right), \end{aligned}$$

Since Ω is neutrosophic q-contraction and $\Omega(z_n) = z_{n+1}$ as $n \to +\infty$

$$\to 0 \diamond 0 \diamond 0 = 0.$$

That is $\Omega(z) = z$, hence, z is an FP of Ω. Now, we evaluate the uniqueness; let $\Omega(\varkappa) = \varkappa$ for some $\varkappa \in \mathbf{Z}$, then

$$\begin{aligned} E(\varkappa,\ \varkappa, z, \boldsymbol{\sigma}) &= E(\Omega(\varkappa),\ \Omega(\varkappa),\ \Omega(z), \boldsymbol{\sigma}) \\ &\geq E\left(\varkappa,\ \varkappa, z, \tfrac{\sigma}{q}\right), \\ = E\left(\Omega(\varkappa),\ \Omega(\varkappa),\ \Omega(z), \tfrac{\sigma}{q}\right) &\geq E\left(\varkappa,\ \varkappa, z, \tfrac{\sigma}{q^2}\right) \geq \cdots \geq E\left(\varkappa,\ \varkappa, z, \tfrac{\sigma}{q^n}\right) \to 1, \end{aligned}$$

as $n \to +\infty$.

$$\begin{aligned} \Theta(\varkappa,\ \varkappa, z, \boldsymbol{\sigma}) &= \Theta(\Omega(\varkappa),\ \Omega(\varkappa),\ \Omega(z), \boldsymbol{\sigma}) \\ &\leq \Theta\left(\varkappa,\ \varkappa, z, \tfrac{\sigma}{q}\right), \\ = \Theta\left(\Omega(\varkappa),\ \Omega(\varkappa),\ \Omega(z), \tfrac{\sigma}{q}\right) &\leq \Theta\left(\varkappa,\ \varkappa, z, \tfrac{\sigma}{q^2}\right) \leq \cdots \leq \Theta\left(\varkappa,\ \varkappa, z, \tfrac{\sigma}{q^n}\right) \to 0, \end{aligned}$$

as $n \to +\infty$ and

$$\begin{aligned} R(\varkappa, \varkappa, z, \sigma) &= R(\Omega(\varkappa), \Omega(\varkappa), \Omega(z), \sigma) \\ &\leq R\left(\varkappa, \varkappa, z, \frac{\sigma}{q}\right), \\ = R\left(\Omega(\varkappa), \Omega(\varkappa), \Omega(z), \frac{\sigma}{q}\right) &\leq R\left(\varkappa, \varkappa, z, \frac{\sigma}{q^2}\right) \leq \cdots \leq R\left(\varkappa, \varkappa, z, \frac{\sigma}{q^n}\right) \to 0, \end{aligned}$$

as $n \to +\infty$. That is $z = \varkappa$. □

Example 6. *Suppose* $\mathbf{Z} = [0,1]$ *and* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ *is a symmetric complete NNbMSs where* E, Θ *and* R *are defined by*

$$\begin{aligned} E(z, \varkappa, \nu, \sigma) &= \frac{\sigma}{\sigma + [|z-\nu| + |\varkappa - \nu|]^2}, \\ \Theta(z, \varkappa, \nu, \sigma) &= \frac{[|z-\nu| + |\varkappa-\nu|]^2}{\sigma + [|z-\nu| + |\varkappa-\nu|]^2}, \\ R(z, \varkappa, \nu, \sigma) &= \frac{[|z-\nu| + |\varkappa-\nu|]^2}{\sigma}, \text{for all } z, \varkappa, \nu \in \mathbf{Z}, \ \sigma > 0. \end{aligned}$$

Let $\Omega(z) = \lambda z, \lambda < \frac{\sqrt{2}}{2}, z \in \mathbf{Z}, \sigma > 0$. *Then, for* $\frac{1}{2} > q$

$$\begin{aligned} E(\Omega(z), \Omega(z), \Omega(\varkappa), \sigma) &= \frac{\sigma}{\sigma + [|\Omega(z) - \Omega(\varkappa)| + |\Omega(z) - \Omega(\varkappa)|]^2}, \\ \frac{\sigma}{\sigma + [2|\Omega(z) - \Omega(\varkappa)|]^2} &= \frac{\sigma}{\sigma + [2|\lambda z - \lambda\varkappa|]^2} \\ = \frac{\sigma}{\sigma + 4\lambda^2 |z-\varkappa|^2} &= \frac{\frac{\sigma}{\lambda^2}}{\frac{\sigma}{\lambda^2} + [|z-\varkappa| + |z-\varkappa|]^2} \\ = \frac{\frac{\sigma}{q}}{\frac{\sigma}{q} + [|z-\varkappa| + |z-\varkappa|]^2} &= E\left(z, z, \varkappa, \frac{\sigma}{q}\right), \end{aligned}$$

where $\lambda^2 = q$, *and*

$$\begin{aligned} \Theta(\Omega(z), \Omega(z), \Omega(\varkappa), \sigma) &= \frac{[|\Omega(z) - \Omega(\varkappa)| + |\Omega(z) - \Omega(\varkappa)|]^2}{\sigma + [|\Omega(z) - \Omega(\varkappa)| + |\Omega(z) - \Omega(\varkappa)|]^2}, \\ \frac{[2|\Omega(z) - \Omega(\varkappa)|]^2}{\sigma + [2|\Omega(z) - \Omega(\varkappa)|]^2} &= \frac{[2|\lambda z - \lambda\varkappa|]^2}{\sigma + [2|\lambda z - \lambda\varkappa|]^2} \\ = \frac{4\lambda^2|z-\varkappa|^2}{\sigma + 4\lambda^2|z-\varkappa|^2} &= \frac{[|z-\varkappa| + |z-\varkappa|]^2}{\frac{\sigma}{\lambda^2} + [|z-\varkappa| + |z-\varkappa|]^2} \\ = \frac{[|z-\varkappa| + |z-\varkappa|]^2}{\frac{\sigma}{q} + [|z-\varkappa| + |z-\varkappa|]^2} &= \Theta\left(z, z, \varkappa, \frac{\sigma}{q}\right), \end{aligned}$$

where $\lambda^2 = q$. *Similarly,*

$$\begin{aligned} R(\Omega(z), \Omega(z), \Omega(\varkappa), \sigma) &= \frac{[|\Omega(z) - \Omega(\varkappa)| + |\Omega(z) - \Omega(\varkappa)|]^2}{\sigma}, \\ \frac{[2|\Omega(z) - \Omega(\varkappa)|]^2}{\sigma} &= \frac{[2|\lambda z - \lambda\varkappa|]^2}{\sigma} \\ = \frac{4\lambda^2|z-\varkappa|^2}{\sigma} &= \frac{[|z-\varkappa| + |z-\varkappa|]^2}{\frac{\sigma}{\lambda^2}} \\ = \frac{[|z-\varkappa| + |z-\varkappa|]^2}{\frac{\sigma}{q}} &= R\left(z, z, \varkappa, \frac{\sigma}{q}\right), \end{aligned}$$

where $\lambda^2 = q$. *Therefore, all the conditions of Theorem 1 are satisfied and 0 is a unique FP of* Ω *in* $\mathbf{Z}$. *Let* $\Theta : (0, +\infty) \to (0, +\infty)$ *as*

$$\Theta(\sigma) = \int_0^\sigma \phi(\sigma) d\sigma, \ \text{ for all } \sigma > 0,$$

be a non-decreasing and CF. Moreover, for every $\varsigma > 0$, $\phi(\varsigma) > 0$. *This implies* $\phi(\sigma) = 0 \ if \ \sigma = 0$.

Theorem 2. *Suppose* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ *is a complete symmetric NNbMSs and* $\Omega : \mathbf{Z} \to \mathbf{Z}$ *is a mapping satisfying*

$$\int_0^{E(\Omega(z),\ \Omega(z),\ \Omega(\varkappa),\ q\sigma)} \phi(\sigma)d\sigma \geq \int_0^{E(z,\ z,\varkappa,\sigma)} \phi(\sigma)d\sigma,$$
$$\int_0^{\Theta(\Omega(z),\ \Omega(z),\ \Omega(\varkappa),\ q\sigma)} \phi(\sigma)d\sigma \leq \int_0^{\Theta(z,\ z,\varkappa,\sigma)} \phi(\sigma)d\sigma,$$
$$\int_0^{R(\Omega(z),\ \Omega(z),\ \Omega(\varkappa),\ q\sigma)} \phi(\sigma)d\sigma \leq \int_0^{R(z,\ z,\varkappa,\sigma)} \phi(\sigma)d\sigma,$$

for all $z,\ \varkappa \in \mathbf{Z},\ \phi \in \Theta$ *and* $q \in (0,1)$. *Then there exists a unique FP of* Ω.

Proof. It is immediate and by using Theorem 1 letting $\phi(1) = 1$. □

6. Application to Integral Equations

Solving equations in any form is one of the most important and interesting aspects of mathematics. There are several approaches to solving various types of equations. Identifying the solution of a problem whether it is singular or multiple. One of the main methods that has made significant progress in the study of IEs is FP theory, which is an iterative procedure with a variety of applications. To determine if a differential or integral problem has a solution, FP theory is essential.

In this section, we prove that Theorem 1 is valid for a specific nonlinear integral problem. The following theorem provides an answer to the question whether "The solution for a specific nonlinear IE exists or not". Assume the set of real-valued CFs on a bounded interval $[0, I]$ is denoted by $\mathbf{Z} = C[0,\ I]$.

Then $(\mathbf{Z}, E, \Theta, *, \diamond, \boldsymbol{\zeta})$ are complete symmetric NNbMSs defined by $E,\ \Theta : \mathbf{Z}^3 \times (0, +\infty) \to [0,1]$ by

$$E(z,\ \varkappa, \nu, \sigma) = \frac{\sigma}{\sigma + \sup\limits_{\boldsymbol{\varpi} \in [0,I]} [|z(\boldsymbol{\varpi}) - \nu(\boldsymbol{\varpi})| + |\varkappa(\boldsymbol{\varpi}) + \nu(\boldsymbol{\varpi})|]^2} \tag{5}$$

$$\Theta(z,\ \varkappa, \nu, \sigma) = \frac{\sup\limits_{\boldsymbol{\varpi} \in [0,I]} [|z(\boldsymbol{\varpi}) - \nu(\boldsymbol{\varpi})| + |\varkappa(\boldsymbol{\varpi}) + \nu(\boldsymbol{\varpi})|]^2}{\sigma + \sup\limits_{\boldsymbol{\varpi} \in [0,I]} [|z(\boldsymbol{\varpi}) - \nu(\boldsymbol{\varpi})| + |\varkappa(\boldsymbol{\varpi}) + \nu(\boldsymbol{\varpi})|]^2} \tag{6}$$

$$R(z,\ \varkappa, \nu, \sigma) = \frac{\sup\limits_{\boldsymbol{\varpi} \in [0,I]} [|z(\boldsymbol{\varpi}) - \nu(\boldsymbol{\varpi})| + |\varkappa(\boldsymbol{\varpi}) + \nu(\boldsymbol{\varpi})|]^2}{\sigma} \tag{7}$$

for $\boldsymbol{\sigma} > 0$ and for all $z, \varkappa, \nu \in \mathbf{Z}$ and let

$$z(\sigma) = g(\boldsymbol{\sigma}) + \int_0^I A(\boldsymbol{\sigma}, \boldsymbol{\varpi}) H(\boldsymbol{\sigma}, \boldsymbol{\varpi}, z(\boldsymbol{\varpi})) d\boldsymbol{\varpi}, \tag{8}$$

where $I > 0$ and $g : [0, I] \to R$ and $H : [0,1]^2 \times R \to R$ are CFs.

Theorem 3. *Let* $(\mathbf{Z}, E, \Theta, R, *, \diamond, \boldsymbol{\zeta})$ *be a symmetric complete NNbMSs provided in (5), (6) and (7). Define the integral operator* $\Omega : \mathbf{Z} \to \mathbf{Z}$ *by*

$$\Omega(z(\sigma)) = g(\sigma) + \int_0^I A(\sigma, \varpi) H(\sigma, \varpi, z(\varpi)) d\varpi, \tag{9}$$

for all $z \in \mathbf{Z}$ *and* $\boldsymbol{\sigma}, \boldsymbol{\varpi} \in [0, I]$. *Assume that the following axioms are fulfilled;*

(a) *For all* $\sigma, \tau \in [0, I]$ *and* $z, \varkappa \in \mathbf{Z}$

$$|H(\sigma, \varpi, z(\varpi)) - H(\sigma, \varpi, \varkappa(\varpi))| \leq |z(\varpi) - \varkappa(\varpi)|. \tag{10}$$

(b) *For all* $\sigma, \varpi \in [0, I]$,

$$\underset{\boldsymbol{\varpi}\in[0,I]}{sup}\left|\int_0^I \left(A(\boldsymbol{\sigma},\boldsymbol{\varpi})^2\right)d\boldsymbol{\varpi}\right| \leq q < 1. \tag{11}$$

Then $z^* \in \mathbf{Z}$ *is a unique solution for (8).*

Proof. For each $z, \varkappa \in \mathbf{Z}$, we obtain

$$\begin{aligned}
E(\Omega(z), \Omega(z),\ \Omega(\varkappa), q\sigma) &= \frac{q\sigma}{q\sigma + \sup_{\varpi\in[0,I]} [|\Omega(z(\sigma)) - \Omega(\varkappa(\sigma))| + |\Omega(z(\sigma)) - \Omega(\varkappa(\sigma))|]^2}, \\
&= \frac{q\sigma}{q\sigma + \sup_{\varpi\in[0,I]} [2|\Omega(z(\sigma)) - \Omega(\varkappa(\sigma))|]^2} \\
&= \frac{q\sigma}{q\sigma + \sup_{\varpi\in[0,I]} 4\left|\int_0^I (A(\sigma,\varpi)H(\sigma,\varpi,z(\varpi)) - A(\sigma,\varpi)H(\sigma,\varpi,z(\varpi)))d\varpi\right|^2} \\
&\geq \frac{q\sigma}{q\sigma + \sup_{\varpi\in[0,I]} 4\left|\int_0^I (A(\varpi,\sigma))^2 d\varpi\right| \int_0^I |[H(\sigma,\varpi,z(\varpi)) - H(\sigma,\varpi,\varkappa(\varpi))]d\varpi|^2} \\
&\geq \frac{q\sigma}{q\sigma + 4q\int_0^I |(z(\varpi) - \varkappa(\varpi))d\varpi|^2} \\
&\geq \frac{\sigma}{\sigma + \sup_{\varpi\in[0,I]} 4|z(\varpi) - \varkappa(\varpi)|^2} \\
&\geq \frac{\sigma}{\sigma + \sup_{\varpi\in[0,I]} [|z(\varpi) - \varkappa(\varpi)| + |z(\varpi) - \varkappa(\varpi)|]^2} \\
&= E(z, z, \varkappa, \sigma).
\end{aligned}$$

and,

$$\begin{aligned}
\Theta(\Omega(z), \Omega(z),\ \Omega(\varkappa), q\sigma) &= \frac{\sup_{\varpi\in[0,I]} [|\Omega(z(\sigma)) - \Omega(\varkappa(\sigma))| + |\Omega(z(\sigma)) - \Omega(\varkappa(\sigma))|]^2}{q\sigma + \sup_{\varpi\in[0,I]} [|\Omega(z(\sigma)) - \Omega(\varkappa(\sigma))| + |\Omega(z(\sigma)) - \Omega(\varkappa(\sigma))|]^2}, \\
&= \frac{\sup_{\varpi\in[0,I]} [2|\Omega(z(\sigma)) - \Omega(\varkappa(\sigma))|]^2}{q\sigma + \sup_{\varpi\in[0,I]} [2|\Omega(z(\sigma)) - \Omega(\varkappa(\sigma))|]^2} \\
&= \frac{\sup_{\varpi\in[0,I]} 4\left|\int_0^I (A(\sigma,\varpi)H(\sigma,\varpi,z(\varpi)) - A(\sigma,\varpi)H(\sigma,\varpi,z(\varpi)))d\varpi\right|^2}{q\sigma + \sup_{\varpi\in[0,I]} 4\left|\int_0^I (A(\sigma,\varpi)H(\sigma,\varpi,z(\varpi)) - A(\sigma,\varpi)H(\sigma,\varpi,z(\varpi)))d\varpi\right|^2} \\
&\leq \frac{\sup_{\varpi\in[0,I]} 4\left|\int_0^I (A(\varpi,\sigma))^2 d\varpi\right| \int_0^I |[H(\sigma,\varpi,z(\varpi)) - H(\sigma,\varpi,\varkappa(\varpi))]d\varpi|^2}{q\sigma + \sup_{\varpi\in[0,I]} 4\left|\int_0^I (A(\varpi,\sigma))^2 d\varpi\right| \int_0^I |[H(\sigma,\varpi,z(\varpi)) - H(\sigma,\varpi,\varkappa(\varpi))]d\varpi|^2} \\
&\leq \frac{4q\int_0^I |(z(\varpi) - \varkappa(\varpi))d\varpi|^2}{q\sigma + 4q\int_0^I |(z(\varpi) - \varkappa(\varpi))d\varpi|^2} \\
&\leq \frac{\sup_{\varpi\in[0,I]} 4|z(\varpi) - \varkappa(\varpi)|^2}{\sigma + \sup_{\varpi\in[0,I]} 4|z(\varpi) - \varkappa(\varpi)|^2} \\
&\leq \frac{\sup_{\varpi\in[0,I]} [|z(\varpi) - \varkappa(\varpi)| + |z(\varpi) - \varkappa(\varpi)|]^2}{\sigma + \sup_{\varpi\in[0,I]} [|z(\varpi) - \varkappa(\varpi)| + |z(\varpi) - \varkappa(\varpi)|]^2} \\
&= \Theta(z, z, \varkappa, \sigma).
\end{aligned}$$

Similarly,

$$\begin{aligned}
R(\Omega(z),\Omega(z),\ \Omega(\varkappa),q\sigma) &= \frac{\sup_{\varpi\in[0,I]}[|\Omega(z(\sigma))-\Omega(\varkappa(\sigma))|+|\Omega(z(\sigma))-\Omega(\varkappa(\sigma))|]^2}{q\sigma},\\
&= \frac{\sup_{\varpi\in[0,I]}[2|\Omega(z(\sigma))-\Omega(\varkappa(\sigma))|]^2}{q\sigma}\\
&= \frac{\sup_{\varpi\in[0,I]}4\left|\int_0^I(A(\sigma,\varpi)H(\sigma,\varpi,z(\varpi))-A(\sigma,\varpi)H(\sigma,\varpi,z(\varpi)))d\varpi\right|^2}{q\sigma}\\
&\leq \frac{\sup_{\varpi\in[0,I]}4\left|\int_0^I(A(\varpi,\sigma))^2d\varpi\right|\int_0^I|[H(\sigma,\varpi,z(\varpi))-H(\sigma,\varpi,\varkappa(\varpi))]d\varpi|^2}{q\sigma}\\
&\leq \frac{4q\int_0^I|(z(\varpi)-\varkappa(\varpi))d\varpi|^2}{q\sigma}\\
&\leq \frac{\sup_{\varpi\in[0,I]}4|z(\varpi)-\varkappa(\varpi)|^2}{\sigma}\\
&\leq \frac{\sup_{\varpi\in[0,I]}[|z(\varpi)-\varkappa(\varpi)|+|z(\varpi)-\varkappa(\varpi)|]^2}{\sigma}\\
&= R(z,z,\varkappa,\sigma).
\end{aligned}$$

Since all conditions of Theorem 1 are fulfilled, then the IE (8) has a unique solution. □

7. Application to Linear Equations

Assume $\mathbf{Z}=\mathbb{R}^n$ and define a complete symmetric NNbMS on $\mathbf{Z}^3\times(0,+\infty)$ by

$$E(z,\varkappa,\nu,\sigma) = \frac{\sigma}{\sigma+[\sum_{i=1}^n|z_i-\varkappa_i|+\sum_{i=1}^n|\varkappa_i-\nu_i|]^2}, \tag{12}$$

$$\Theta(z,\varkappa,\nu,\sigma) = \frac{[\sum_{i=1}^n|z_i-\varkappa_i|+\sum_{i=1}^n|\varkappa_i-\nu_i|]^2}{\sigma+[\sum_{i=1}^n|z_i-\varkappa_i|+\sum_{i=1}^n|\varkappa_i-\nu_i|]^2}, \tag{13}$$

$$\Theta(z,\varkappa,\nu,\sigma) = \frac{[\sum_{i=1}^n|z_i-\varkappa_i|+\sum_{i=1}^n|\varkappa_i-\nu_i|]^2}{\sigma}, \tag{14}$$

for all $z,\varkappa,\nu\in\mathbb{R}^n$ and $\boldsymbol{\zeta}=2$, if

$$\left[\max_{1\leq j\leq n}\sum_{i=1}^n|\pi_{ij}|\right]^2 \leq q < 1. \tag{15}$$

The following linear equations have only one solution.

$$\begin{cases}\pi_{11}z_1+\pi_{12}z_2+\cdots+\pi_{1n}z_n = d_1,\\ \pi_{21}z_1+\pi_{22}z_2+\cdots+\pi_{2n}z_n = d_2,\\ \qquad\vdots\\ \pi_{n1}z_1+\pi_{n2}z_2+\cdots+\pi_{nn}z_n = d_n.\end{cases} \tag{16}$$

Proof. Assume $\Omega:\ \mathbf{Z}\to\mathbf{Z}$ be described by $\Omega(z)=\pi z+d$ where $z,d\in\mathbb{R}^n$ and

$$\pi = \begin{bmatrix}\pi_{11} & \pi_{12} & \cdots & \pi_{1n}\\ \pi_{21} & \pi_{22} & \cdots & \pi_{2n}\\ \vdots & \vdots & & \\ \pi_{n1} & \pi_{n2} & \cdots & \pi_{nn}\end{bmatrix}.$$

For $z, \varkappa \in \mathbb{R}^n$, we obtain

$$E(\Omega(z),\ \Omega(z),\ \Omega(\varkappa),\ q\sigma) = \frac{q\sigma}{q\sigma+4\left[\sum_{i=1}^{n}\left|\sum_{j=1}^{n}\pi_{ij}\left(z_j-\varkappa_j\right)\right|\right]^2}$$
$$\geq \frac{q\sigma}{q\sigma+4\left[\sum_{i=1}^{n}\sum_{j=1}^{n}|\pi_{ij}|\left|\left(z_j-\varkappa_j\right)\right|\right]^2} = \frac{q\sigma}{q\sigma+\left[\sum_{j=1}^{n}2\left|z_j-\varkappa_j\right|\sum_{i=1}^{n}|\pi_{ij}|\right]^2}$$
$$\geq \frac{q\sigma}{q\sigma+\left[\max\limits_{1\leq j\leq n}\sum_{i=1}^{n}|\pi_{ij}|\right]^2\left[\sum_{j=1}^{n}2\left|z_j-\varkappa_j\right|\right]^2}.$$

Using (15), we have

$$\geq \frac{q\sigma}{q\sigma+q\left[\sum_{j=1}^{n}2\left|z_j-\varkappa_j\right|\right]^2}$$
$$= \frac{\sigma}{\sigma+\left[\sum_{j=1}^{n}\left|z_j-\varkappa_j\right|+\left|z_j-\varkappa_j\right|\right]^2} = E(z,\ z,\ \varkappa,\sigma).$$

and

$$\Theta(\Omega(z),\ \Omega(z),\ \Omega(\varkappa),\ q\sigma) = \frac{4\left[\sum_{i=1}^{n}\left|\sum_{j=1}^{n}\pi_{ij}\left(z_j-\varkappa_j\right)\right|\right]^2}{q\sigma+4\left[\sum_{i=1}^{n}\left|\sum_{j=1}^{n}\pi_{ij}\left(z_j-\varkappa_j\right)\right|\right]^2}$$
$$\leq \frac{4\left[\sum_{i=1}^{n}\sum_{j=1}^{n}|\pi_{ij}|\left|\left(z_j-\varkappa_j\right)\right|\right]^2}{q\sigma+4\left[\sum_{i=1}^{n}\sum_{j=1}^{n}|\pi_{ij}|\left|\left(z_j-\varkappa_j\right)\right|\right]^2} = \frac{\left[\sum_{j=1}^{n}2\left|z_j-\varkappa_j\right|\sum_{i=1}^{n}|\pi_{ij}|\right]^2}{q\sigma+\left[\sum_{j=1}^{n}2\left|z_j-\varkappa_j\right|\sum_{i=1}^{n}|\pi_{ij}|\right]^2}$$
$$\leq \frac{\left[\max\limits_{1\leq j\leq n}\sum_{i=1}^{n}|\pi_{ij}|\right]^2\left[\sum_{j=1}^{n}2\left|z_j-\varkappa_j\right|\right]^2}{q\sigma+\left[\max\limits_{1\leq j\leq n}\sum_{i=1}^{n}|\pi_{ij}|\right]^2\left[\sum_{j=1}^{n}2\left|z_j-\varkappa_j\right|\right]^2}.$$

Using (15)

$$\leq \frac{q\left[\sum_{j=1}^{n}2\left|z_j-\varkappa_j\right|\right]^2}{q\sigma+q\left[\sum_{j=1}^{n}2\left|z_j-\varkappa_j\right|\right]^2} = \frac{\left[\sum_{j=1}^{n}\left|z_j-\varkappa_j\right|+\left|z_j-\varkappa_j\right|\right]^2}{\sigma+\left[\sum_{j=1}^{n}\left|z_j-\varkappa_j\right|+\left|z_j-\varkappa_j\right|\right]^2} = \Theta(z,\ z,\ \varkappa,\sigma).$$

and

$$R(\Omega(z),\ \Omega(z),\ \Omega(\varkappa),\ q\sigma) = \frac{4\left[\sum_{i=1}^{n}\left|\sum_{j=1}^{n}\pi_{ij}\left(z_j-\varkappa_j\right)\right|\right]^2}{q\sigma}$$
$$\leq \frac{4\left[\sum_{i=1}^{n}\sum_{j=1}^{n}|\pi_{ij}|\left|\left(z_j-\varkappa_j\right)\right|\right]^2}{q\sigma} = \frac{\left[\sum_{j=1}^{n}2\left|z_j-\varkappa_j\right|\sum_{i=1}^{n}|\pi_{ij}|\right]^2}{q\sigma}$$
$$\leq \frac{\left[\max\limits_{1\leq j\leq n}\sum_{i=1}^{n}|\pi_{ij}|\right]^2\left[\sum_{j=1}^{n}2\left|z_j-\varkappa_j\right|\right]^2}{q\sigma}.$$

Using (15)

$$\leq \frac{q\left[\sum_{j=1}^{n}2\left|z_j-\varkappa_j\right|\right]^2}{q\sigma} = \frac{\left[\sum_{j=1}^{n}\left|z_j-\varkappa_j\right|+\left|z_j-\varkappa_j\right|\right]^2}{\sigma} = R(z,\ z,\ \varkappa,\sigma).$$

Therefore, all the conditions of Theorem 1 are satisfied, and Ω is a neutrosophic q^{-} contraction. There is a unique solution of the SLEs (16) in $\boldsymbol{Z}$. □

8. Application to Nonlinear Fractional Differential Equation

In this part, we apply Theorem 1 to determine the existence and uniqueness of a solution to nonlinear FDE given by

$$D_{\pi}^{\alpha}z(\varrho) = \psi(\varrho, z(\varrho))\ (\varrho \in (0,1),\ \alpha \in (1,2]),$$

with boundary conditions

$$z(0) = 0,\ z'(0) = Iz(\varrho)\ \varrho \in (0,1),$$

where D^{α}_{π} means caputo fractional derivative of order α, defined by

$$D^{\alpha}_{\pi}\psi(\varrho) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\varrho} (\varrho - \varpi)^{n-\alpha-1}\psi^n(\varpi)d\varpi\ (n-1 < \alpha < n,\ n = [\alpha]+1),$$

and $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ is a CF. We suppose that $Z = C([0,1],\mathbb{R})$, from $[0,1]$ into $\mathbb{R}$ with supremum $|z| = \sup_{\theta \in [0,1]} |z(\varrho)|$.

The Riemann–Liouville fractional integral of order α is given by

$$I^{\alpha}\psi(\varrho) = \frac{1}{\Gamma(\alpha)}\int_0^{\varrho} (\varrho - \varpi)^{\alpha-1}\psi(\varpi)d\varpi\ (\alpha > 0).$$

We first provide a nonlinear FDE in an appropriate form and then investigate the existence of a solution. Now, we suppose the following FDE

$$D^{\alpha}_{\pi}z(\varrho) = \psi(\varrho, z(\varrho))\ (\varrho \in (0,1),\ \alpha \in (1,2]), \tag{17}$$

with the boundary conditions

$$z(0) = 0, z'(0) = Iz(\varrho)(\varrho \in (0,1)),$$

where

i. $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ is a CF,

ii. $z(\varrho) : [0,1] \to \mathbb{R}$ is continuous,

and fulfill the axioms below:

$$|\psi(\varrho, z) - \psi(\varrho, \varkappa) - \psi(\varrho, \nu)| \leq ЛL|z - \varkappa - \nu|,$$

for all $\varrho \in [0,1]$ and L is a constant with $LЛ < 1$, where

$$Л = \frac{1}{\Gamma(\alpha+1)} + \frac{2\varkappa^{\alpha+1}\Gamma(\alpha)}{(2-\varkappa^2)\Gamma(\alpha+1)}.$$

Then there exists a unique solution to Equation (17).

Proof. Assume that

$$E(z, \varkappa, \nu, \sigma) = \frac{\sigma}{\sigma + |z - \varkappa - \nu|^p}$$

$$\Theta(z, \varkappa, \nu, \sigma) = \frac{|z - \varkappa - \nu|^p}{\sigma + |z - \varkappa - \nu|^p},$$

$$R(z, \varkappa, \nu, \sigma) = \frac{|z - \varkappa - \nu|^p}{\sigma},\ \text{for all } z, \varkappa \in Z \text{ and } \sigma > 0,$$

defined by $\gamma * \beta * \pi = \gamma\beta\pi$, and $\gamma \diamond \beta \diamond \pi = \max\{\gamma, \beta, \pi\}$. Let $|z - \varkappa - \nu| = \sup_{\varrho \in [0,1]} |z - \varkappa - \nu|^p$, for all $z, \varkappa \in Z$. Then $(Z, E, \Theta, R, *, \diamond, \zeta)$ is a complete NNbMS. We describe a mapping $\Omega : Z \to Z$ by

$$\Omega z(\varrho) = \frac{1}{\Gamma(\alpha)}\int_0^{\varrho} (\varrho - \varpi)^{\alpha-1}\psi(\varpi,\ z(\varpi))d\varpi + \frac{2\varrho}{(2-\varkappa^2)\Gamma(\alpha)}\int_0^{\varkappa}\left(\int_0^{\varpi} (\varpi - m)^{\alpha-1}\psi(m,\ z(m))dm\right)d\varpi \tag{18}$$

for all $\varrho \in [0,1]$. An Equation (17) has a solution $z \in Z$ if $z(\varrho) = \Omega z(\varrho)$ for all $\varrho \in [0,1]$. Now

$$E(z(\varrho), \varkappa(\varrho), \nu(\varrho), \sigma) = \frac{\sigma}{\sigma + |z(\varrho) - \varkappa(\varrho) - \nu(\varrho)|^p},$$

$$\Theta(z(\varrho), \varkappa(\varrho), \nu(\varrho), \sigma) = \frac{|z(\varrho) - \varkappa(\varrho) - \nu(\varrho)|^p}{\sigma + |z(\varrho) - \varkappa(\varrho) - \nu(\varrho)|^p}, \tag{19}$$

$$R(z(\varrho), \varkappa(\varrho), \nu(\varrho), \sigma) = \frac{|z(\varrho) - \varkappa(\varrho) - \nu(\varrho)|^p}{\sigma}. \tag{20}$$

$$\begin{aligned}
&|\Omega z(\varrho) - \Omega\varkappa(\varrho) - \Omega\nu(\varrho)| \\
&= \left|\frac{1}{\Gamma(\alpha)}\int_0^{\varrho} (\varrho - \varpi)^{\alpha-1}\psi(\varpi, z(\varpi))d\varpi + \frac{2\varrho}{(2-\varkappa^2)\Gamma(\alpha)}\int_0^{\varkappa}\left(\int_0^{\varpi}(\varpi - m)^{\alpha-1}\psi(m, z(m))dm\right)d\boldsymbol{\varpi}\right| \\
&- \left|\frac{1}{\Gamma(\alpha)}\int_0^{\varrho} (\varrho - \varpi)^{\alpha-1}\psi(\varpi, \varkappa(\varpi))d\varpi + \frac{2\varrho}{(2-\varkappa^2)\Gamma(\alpha)}\int_0^{\varkappa}\left(\int_0^{\varpi}(\varpi - m)^{\alpha-1}\psi(m, \varkappa(m))dm\right)d\varpi\right| \\
&- \left|\frac{1}{\Gamma(\alpha)}\int_0^{\varrho} (\varrho - \varpi)^{\alpha-1}\psi(\varpi, \nu(\varpi))d\varpi + \frac{2\varrho}{(2-\varkappa^2)\Gamma(\alpha)}\int_0^{\varkappa}\left(\int_0^{\varpi}(\varpi - m)^{\alpha-1}\psi(m, \nu(m))dm\right)d\varpi\right|.
\end{aligned}$$

That is,

$$\begin{aligned}
&|\Omega z(\varrho) - \Omega\varkappa(\varrho) - \Omega\nu(\varrho)| \\
&= \Big| \frac{1}{\Gamma(\alpha)}\int_0^{\varrho} (\varrho - \varpi)^{\alpha-1}\psi(\varpi, z(\varpi))d\varpi + \frac{2\varrho}{(2-\varkappa^2)\Gamma(\alpha)}\int_0^{\varkappa}\left(\int_0^{\varpi}(\varpi - m)^{\alpha-1}\psi(m, z(m))dm\right)d\varpi \\
&- \frac{1}{\Gamma(\alpha)}\int_0^{\varrho} (\varrho - \varpi)^{\alpha-1}\psi(\varpi, \varkappa(\varpi))d\varpi - \frac{2\varrho}{(2-\varkappa^2)\Gamma(\alpha)}\int_0^{\varkappa}\left(\int_0^{\varpi}(\varpi - m)^{\alpha-1}\psi(m, \varkappa(m))dm\right)d\varpi \\
&- \frac{1}{\Gamma(\alpha)}\int_0^{\varrho} (\varrho - \varpi)^{\alpha-1}\psi(\varpi, \nu(\varpi))d\varpi - \frac{2\varrho}{(2-\varkappa^2)\Gamma(\alpha)}\int_0^{\varkappa}\left(\int_0^{\varpi}(\varpi - m)^{\alpha-1}\psi(m, \nu(m))dm\right)d\varpi \Big| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{\varrho} (\varrho - \varpi)^{\alpha-1}|\psi(\varpi, z(\varpi)) - \psi(\varpi, \varkappa(\varpi)) - \psi(\varpi, \nu(\varpi))|d\varpi \\
&+ \frac{2\varrho}{(2-\varkappa^2)\Gamma(\alpha)}\int_0^{\varkappa}\left(\int_0^{\varpi}(\varpi - m)^{\alpha-1}|\psi(m, z(m)) - \psi(m, \varkappa(m)) - \psi(m, \nu(m))|dm\right)d\varpi \\
&\le \frac{L|z-\varkappa-\nu|}{\Gamma(\alpha)}\int_0^{\varrho}(\varrho - \varpi)^{\alpha-1}d\varpi + \frac{2L|z-\varkappa|}{\Gamma(\alpha)}\int_0^{\varkappa}\left(\int_0^{\varpi}(\varpi - m)^{\alpha-1}dm\right)d\varpi \\
&\le \frac{L|z-\varkappa-\nu|}{\Gamma(\alpha+1)} + \frac{2\varkappa^{\alpha+1}L|z-\varkappa|\Gamma(\alpha)}{(2-\varkappa^2)\Gamma(\alpha+2)} \\
&\le L|z-\varkappa-\nu|\left(\frac{1}{\Gamma(\alpha+1)} + \frac{2\varkappa^{\alpha+1}\Gamma(\alpha)}{(2-\varkappa^2)\Gamma(\alpha+2)}\right) = L\text{Л}|z-\varkappa-\nu|.
\end{aligned}$$

Utilizing the fact $L\text{Л} < 1$ and (19), we have

$$\begin{aligned}
E(\Omega z(\varrho), \Omega\varkappa(\varrho), \Omega\nu(\varrho), \eta\sigma) &= \frac{\eta\sigma}{\eta\sigma + |\Omega z(\varrho) - \Omega\varkappa(\varrho) - \Omega\nu(\varrho)|^p} \ge \frac{\eta\sigma}{\eta\sigma + L\text{Л}|z(\varrho) - \varkappa(\varrho) - \nu(\varrho)|^p} \\
&\ge \frac{\sigma}{\sigma + |z(\varrho) - \varkappa(\varrho) - \nu(\varrho)|^p} = E(z(\varrho), \varkappa(\varrho), \nu(\varrho), \sigma), \\
\Theta(\Omega z(\varrho), \Omega\varkappa(\varrho), \Omega\nu(\varrho), \eta\sigma) &= \frac{|\Omega z(\varrho) - \Omega\varkappa(\varrho) - \Omega\nu(\varrho)|^p}{\eta\sigma + |\Omega z(\varrho) - \Omega\varkappa(\varrho) - \Omega\nu(\varrho)|^p} \le \frac{L\text{Л}|z(\varrho) - \varkappa(\varrho) - \nu(\varrho)|^p}{\eta\sigma + L\text{Л}|z(\varrho) - \varkappa(\varrho) - \nu(\varrho)|^p} \\
&\le \frac{|z(\varrho) - \varkappa(\varrho) - \nu(\varrho)|^p}{\sigma + |z(\varrho) - \varkappa(\varrho) - \nu(\varrho)|^p} = \Theta(z(\varrho), \varkappa(\varrho), \nu(\varrho), \sigma),
\end{aligned}$$

and

$$\begin{aligned}
R(\Omega z(\varrho), \Omega\varkappa(\varrho), \Omega\nu(\varrho), \eta\sigma) &= \frac{|\Omega z(\varrho) - \Omega\varkappa(\varrho) - \Omega\nu(\varrho)|^p}{\eta\sigma} \le \frac{L\text{Л}|z(\varrho) - \varkappa(\varrho) - \nu(\varrho)|^p}{\eta\sigma} \\
&\le \frac{|z(\varrho) - \varkappa(\varrho) - \nu(\varrho)|^p}{\sigma} = R(z(\varrho), \varkappa(\varrho), \nu(\varrho), \sigma).
\end{aligned}$$

All axioms of Theorem 1 are satisfied. This shows that Ω has unique solution. □

9. Conclusions

In this study, we presented several new concepts including NNbMS, NQSbMS, NPSbMS, NQNMS and NPNbMS. Further, we established several FP results in the framework of NNbMS and proved a well-known decomposition theorem. Furthermore, we presented several non-trivial examples with their graphs for better understanding by the readers and to show the superiority of the introduced definitions and results. At the end, we presented the existence and uniqueness of a solution of an integral equation, SLEs and nonlinear FDE by applying the main results. This work is extendable in the context of neutrosophic N_β controlled metric spaces, neutrosophic quasi N_β controlled metric spaces, neutrosophic pseudo N_β partial metric spaces and many other structures.

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