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The Variation of Constants Formula in Lebesgue Spaces with Variable Exponents

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Abstract: This study looks closely into the analysis of the variation of constants formula given by $\Phi(t) = S(t)\Phi(0) + \int_0^t S(t-\sigma)\mathcal{F}(\sigma, \Phi(\sigma))\,d\sigma$, for $t \in [0,T], T > 0$, within the context of modular function spaces L_ρ. Additionally, this research explores practical applications of the variation of constants formula in variable exponent Lebesgue spaces $L^{p(\cdot)}$. Specifically, the study examines these spaces under certain conditions applied to the exponent function $p(\cdot)$ and the functions $\mathcal{F}$ as well as the semigroup $S(t)$, utilizing the symmetry properties of the algebraic semigroup. This investigation sheds light on the intricate interplay between parameters and functions within these mathematical frameworks, offering valuable insights into their behavior and properties in $L^{p(\cdot)}$.

Keywords: integral equations; semigroup; modular function spaces; variable exponent spaces; fixed point

1. Introduction

The variation of constants formula plays a crucial role in various branches of mathematics, enabling the study of diverse phenomena through intricate mathematical techniques. This formula, expressed as

$$\Phi(t) = S(t)\Phi(0) + \int_0^t S(t-\sigma)\mathcal{F}(\sigma, \Phi(\sigma))\,d\sigma,$$

for $t \in [0,T]$, where $T > 0$, and $S(t)$ represents a one-parameter family of mappings from the modular function space L_ρ into itself. For more details about modular function spaces, we encourage the reader to refer to the following references [1–4]. In this paper, we explore the detailed complexities of this formula within the context of modular spaces L_ρ and variable exponent spaces $L^{p(\cdot)}$ by using the theory of semigroups [5–7], which has not yet been investigated for this context, especially in cases with non-standard growth; see [8,9] for partial differential equations. Such a combination of the theory of semigroups, the variation of constants formula, and variable-exponent spaces $L^{p(\cdot)}$ in the case of non-standard growth will be of interest and needs to be investigated.

The variation of constants formula is crucial for solving differential equations, particularly those with time-varying coefficients. This formula reveals a symmetry between solving differential equations and expressing them as integral equations, providing a systematic approach to finding solutions [6]. This formula systematically helps find solutions for equations with time-varying coefficients, making it a powerful tool in mathematical modeling. It is useful in fields like physics, engineering, economics, and biology, where systems often change over time. For instance, it is used in partial differential equations and delay differential equations, as discussed in [6,7]. By incorporating time-varying parameters, this formula helps us better understand complex real-world phenomena.

The importance of equations that can be solved using this formula is significant in both theoretical and applied mathematics. Introducing nonlinear operators through the variation of constants formula and combining it with linear semigroup theory [5–7] adds

Citation: Bachar, M. The Variation of Constants Formula in Lebesgue Spaces with Variable Exponents. *Symmetry* **2024**, *16*, 978. https://doi.org/10.3390/sym16080978

Academic Editor: Alexander Zaslavski

Received: 23 May 2024
Revised: 17 July 2024
Accepted: 21 July 2024
Published: 1 August 2024

complexity to the analysis. Solving these equations requires advanced mathematical tools like fixed-point theory, functional analysis, and integral operators. Additionally, applying the formula with nonlinear operators improves our understanding of dynamic system behavior. Its ability to solve equations not only interests mathematicians but also has practical implications, leading to the development of advanced techniques for thorough analysis. For more details, see [7].

We aim to explore the variation of constant formulas in modular function spaces L_ρ. Many authors have studied this for integral equations within the spaces of all ρ-continuous functions from the interval $[0,T]$ into L_ρ, denoted by $\mathcal{C}_\rho([0,T],L_\rho)$, where the Δ_2-type condition is satisfied. More details on such conditions can be found in [3,4,10–13].

The problem we are studying is in modular space, so we need to discuss modular space settings. Variable exponent $L^{p(\cdot)}$ spaces are important in analysis, and we focus on them in Section 4 where the solution of the differential equation is considered under non-standard growth conditions; see, for example [8,9,14]. The modular approach is advantageous because it avoids using the Luxemburg norm due to its complexity. Instead, we work with a convex regular modular function, which has the potential to be more accessible and effective for this purpose.

In the world of functional differential equations, semigroups are crucial for modeling how systems change over time. Pazy [5] and Engel and Nagel [6,7] explained in detail how semigroups offer a strong mathematical framework for studying the behavior of systems that evolve over time in a specific way. In simpler terms, semigroups here are like families of mathematical actions that follow certain rules when combined. Khamsi and Kozlowski [1] give the definition and characteristics of semigroups in the function space L_ρ. Understanding how semigroups behave over time is important for studying a formula that helps us track changes in L_ρ. This knowledge is valuable for the broader areas of functional analysis and functional differential equations.

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This article aims to investigate the existence of solutions for the variation of constants formula within the space $\mathcal{C}_\rho([0,T],L_\rho)$. Here, $\mathcal{C}\rho([0,T],L_\rho)$ represent the set of functions that are continuous with respect to ρ, mapping from the interval $[0,T]$ to L_ρ. To provide a comprehensive understanding, we begin in Section 2 by introducing the foundational concepts of modular function spaces and the theory of semigroups. Modular function spaces play a crucial role in functional analysis, offering a generalization of classical function spaces that can accommodate more complex structures and behaviors. Semigroups, on the other hand, provide a robust framework for analyzing the evolution of systems over time, particularly in the context of differential equations.

In Section 3, we introduce our main results concerning the existence of solutions for the variation of constants formula. This section presents a detailed examination of the conditions under which solutions exist, employing advanced mathematical tools and techniques. The variation of constants formula is pivotal in solving differential equations, especially those with dynamic coefficients, and understanding its solvability can significantly enhance our ability to model and predict the behavior of complex systems.

Finally, in Section 4, we apply our theoretical findings to the specific case of variable exponent spaces $L^{p(\cdot)}$. Variable exponent spaces generalize the classical Lebesgue spaces by allowing the exponent to vary as a function, thus providing a more flexible framework for dealing with non-standard growth conditions. This application sheds light on the intricate interplay between the parameters and functions within these spaces, revealing how changes in the exponent function can influence the behavior and properties of solutions. By exploring these interactions, we gain valuable insights into the underlying structures of $L^{p(\cdot)}$ spaces, which can inform further research and applications; see [7–9].

2. Modular Spaces

We denote by $\mathcal{M}_\infty$ the set of Lebesgue measurable functions defined on the interval $[0, T], T > 0$.

To avoid repeating things and build a solid foundation in the study of modular spaces, we suggest checking the detailed studies by Khamsi [1] and Diening [15]. These important works are like starting points for our discussions. In the next parts, we will look closely at two main ideas we need for our research: the modular function concept and the essential Δ_2-type condition, defined in [1,3,15]. With these basic concepts, we will carefully examine the variation of the constant formulas. We will present the definitions of a convex regular modular function based on the discussions in [1,3]. For a more in-depth understanding of the properties of the ρ-convex regular modular function, we direct the reader to Definition 2.2 and Theorem 2.3 in [3].

Definition 1 ([1,3])**.** *We define a convex regular modular function as a mapping $\rho : \mathcal{M}_\infty \to [0, \infty]$ that satisfies the following conditions:*

(1) $\rho(\phi) = 0$ *if and only if* $\phi = 0$.
(2) $\rho(a\phi) = \rho(\phi)$*, for* $|a| = 1$.
(3) $\rho(a\,\phi + (1-a)\,\psi) \leq a\,\rho(\phi) + (1-a)\,\rho(\psi)$*, for any* $a \in [0,1]$*, where* $\phi, \psi \in \mathcal{M}_\infty$.

We refer to [3,4], for the basic characteristics of modular function spaces given by

$$L_\rho = \{\phi \in X \mid \lim_{\lambda \to 0} \rho(\lambda\phi) = 0\}.$$

where

$$X = \{\phi \in \mathcal{M}_\infty \mid |\phi(\theta)| < \infty \ \rho\text{-a.e.}\}.$$

This definition ensures that the functions considered in L_ρ are bounded almost everywhere [3] with respect to the modular function ρ. The concept of ρ-almost everywhere (ρ-a.e.) is crucial for understanding the behavior and properties of functions within these spaces.

The Luxemburg norm in L_ρ is defined as follows:

$$\|\phi\|_\rho = \inf\left\{ t > 0 \mid \rho\left(\frac{1}{t}\phi\right) \leq 1 \right\}.$$

This norm provides a valuable tool for its topological properties within modular function spaces L_ρ; see, for example, [3,10,13].

As stated in [3], the Δ_2-type condition is crucial for analyzing L_ρ when it is satisfied, and we have

$$\omega_\rho(2) = \inf\{K : \rho(2\phi) \leq K\,\rho(\phi) \text{ where } \phi \in L_\rho\} < \infty.$$

In light of these conditions, it is readily observed that

$$\rho(\phi + \psi) \leq \frac{\omega_\rho(2)}{2}\Big(\rho(\phi) + \rho(\psi)\Big), \tag{1}$$

for any $\phi, \psi \in L_\rho$.

We give this definition from the book by Khamsi and Kozlowski [1] to facilitate the measurement of the modular function's growth ω_ρ.

Definition 2 ([1])**.** *Let ρ be a convex function modular. Define the growth function as follows:*

$$\omega_\rho(t) = \sup\left\{\frac{\rho(t\phi)}{\rho(\phi)} : 0 < \rho(f) < \infty\right\}$$

for all $t \geq 0$.

We note that the growth function $\omega_\rho(t)$ may become infinite for specific values of $t > 1$. However, if ρ satisfies the Δ_2-type condition, then the growth function $\omega_\rho(t)$ will be finite, as mentioned in Lemma 3.1 of [1]. Now, we present the definition of strongly continuous semigroups in the modular function space L_ρ. This definition is derived from the work of Khamsi and Kozlowski [1], specifically in Definitions 7.3 and 7.4, as mentioned in their book, concerning the symmetry properties of the algebraic semigroup, in the sense that it does not matter in which order the operations in t and s are applied; the result is the same.

Definition 3 ([1])**.** *A one-parameter family $S(t), t \geq 0$ of mappings from L_ρ into itself is said to be a strongly continuous semigroup on L_ρ if S satisfies the following conditions:*

(i) $S(0)\phi = \phi$ *for* $\phi \in L_\rho$, *(*$S(0) = I$ *is the identity operator on* L_ρ*)*
(ii) $S(t+s)\phi = S(t)S(s)\phi$ *for* $\phi \in L_\rho$ *and for every* $t, s \geq 0$,
(iii) *For each* $t \geq 0$, $S(t)$ *is strongly continuous for every* $\phi \in L_\rho$, *the following function*

$$\Lambda_\phi(t) = \rho(S(t)\phi - \phi)$$

is continuous at every $t \in [0, \infty)$.

We note that the growth function $\omega_\rho(t)$ may become infinite for specific values of $t > 1$. However, if ρ satisfies the Δ_2-type condition, then the growth function $\omega_\rho(t)$ will be finite, as was mentioned in Lemma 3.1 of [1]. Now, we present the definition of strongly continuous semigroups in the modular function space L_ρ. This definition is derived from the work of Khamsi and Kozlowski [1], specifically in Definitions 7.3 and 7.4, as mentioned in their book.

Definition 4 ([1])**.** *A one-parameter family $S(t), t \geq 0$ of mappings from L_ρ into itself is said to be a strongly continuous semigroup on L_ρ if S satisfies the following conditions:*

(i) $S(0)\phi = \phi$ *for* $\phi \in L_\rho$, *(*$S(0) = I$ *is the identity operator on* L_ρ*)*
(ii) $S(t+s)\phi = S(t)S(s)\phi$ *for* $\phi \in L_\rho$ *and for every* $t, s \geq 0$,
(iii) *For each* $t \geq 0$, $S(t)$ *is strongly continuous for every* $\phi \in L_\rho$, *the following function*

$$\Lambda_\phi(t) = \rho(S(t)\phi - \phi)$$

is continuous at every $t \in [0, \infty)$.

The linear operator A is defined by

$$A\phi = \lim_{t\to 0}\frac{S(t)\phi - \phi}{t}, \text{ for } \phi \in D(A),$$

where

$$D(A) = \left\{\phi \in L_\rho : \lim_{t\to 0}\frac{S(t)\phi - \phi}{t} \text{ exists}\right\},$$

is the infinitesimal generator of the semigroup $S(t)$, $D(A)$ is the domain of A. The linear operator $A : L_\rho \to L_\rho$ is said to be ρ-bounded on L_ρ if there exists a constant $M > 0$ such that,

$$\rho(A\phi) \leq M\rho(\phi), \text{ for every } \phi \in L_\rho. \tag{2}$$

Using Proposition 3.7 from [1], we establish the following lemma to describe the relationship between modular and norm convergence in modular function spaces, where the linear operator A is ρ-bounded on L_ρ.

Lemma 1. *Let A be a ρ-bounded linear operator on L_ρ, i.e., there exists a constant $M > 0$ such that $\rho(A\phi) \leq M\rho(\phi)$ for every $\phi \in L_\rho$. Then, for any $\phi \in L_\rho$, there exists a constant $\widetilde{M} = \max(M, 1) > 1$ such that $\|A\phi\|_\rho \leq \widetilde{M}\|\phi\||_\rho$.*

Proof. Using the fact that A is ρ-bounded, then there exists a constant $M > 0$ such that,

$$\rho(A\phi) \leq M\rho(\phi), \text{ for every } \phi \in L_\rho.$$

Let $\phi \in L_\rho$ such that $\|\phi\|_\rho \leq 1$, then by using part (b) of Proposition 3.7 in [1], we have

$$\rho(A\phi) \leq M.$$

If $M > 1$, then

$$\begin{aligned} \rho(\tfrac{1}{M}A\phi) &\leq 1 \\ \|\tfrac{1}{M}A\phi\|_\rho &\leq 1 \end{aligned}$$

hence

$$\|A\phi\|_\rho \leq M.$$

If $M \leq 1$, then $\rho(A\phi) \leq 1$, using Proposition 3.7 again, we have

$$\|A\phi\|_\rho \leq 1. \tag{3}$$

Thus,

$$\|A\phi\|_\rho \leq \widetilde{M}\|\phi\|_\rho, \text{ where } \widetilde{M} = max(M, 1). \tag{4}$$

□

In the next part, we will explore how to solve the variation of constants formula. We will mainly look at finding solutions within the larger context of the general space $\mathcal{C}\rho([0, T], L_\rho)$, as discussed in [3].

3. The Variation of Constants Formula on $\mathcal{C}_\rho([0, T], L_\rho)$

Within this section, we adopt the notation $\Omega = [0, T] \times L_\rho$, where $T > 0$. The primary objective of this section is to explore the feasibility of solving the variation of constants formula, with a particular emphasis on its applicability within the framework of modular function spaces L_ρ

$$\Phi(t) = S(t)\Phi(0) + \int_0^t S(t-\sigma)\mathcal{F}(\sigma, \Phi(\sigma))\, d\sigma, t \in [0, T], \tag{5}$$

where $\Phi \in L_\rho$, and $\mathcal{F} : \Omega \to \mathbb{R}$, and $S(t)$, represents a one-parameter family of mappings from L_ρ into itself.

Under the Δ_2-type condition, we suggest considering the references [3,10,13] for the concept of the ρ-continuous space $\mathcal{C}_\rho([0, T], L_\rho)$, and we define $\rho_{\mathcal{C}_\rho} : \mathcal{C}_\rho([0, T], L_\rho) \to [0, \infty]$ by

$$\rho_{\mathcal{C}_\rho}(\Phi) = \sup_{\vartheta \in [0,T]} \rho(\Phi(\vartheta)),$$

where $\Phi \in \mathcal{C}_\rho([0, T], L_\rho)$.

In order to establish the existence of solutions to the variation of constants formula, we need to impose certain conditions on the function $\mathcal{F}$ and the semigroup $S(t)$. These conditions ensure that the integral equation behaves well within the modular function space L_ρ and that the solutions remain within the desired function space throughout the interval $[0,T]$.

Let $\mathcal{F}:\Omega\to\mathbb{R}$, such that $\mathcal{F}(\cdot,\Phi(\cdot))\in L_\rho$, and $S(\cdot)\Phi\in L_\rho$, where $\Phi\in L_\rho$.

Hypothesis 1. *For each $\Phi\in\mathcal{C}_\rho([0,T],B)$, where B is any nonempty ρ-bounded subset of L_ρ, we assume $\sigma\to\mathcal{F}(\sigma,\Phi(\sigma))\in\mathcal{C}_\rho([0,T],L_\rho)$;*

Hypothesis 2. *S is a strongly continuous semigroup on $\mathcal{C}\rho([0,1],L_\rho)$ into itself, associated to the infinitesimal generator linear operator ρ-bounded A, with domain $D(A)$ such that*

$$A\Phi=\lim_{t\to0^+}\frac{S(t)\Phi-\Phi}{t},\ \textit{for}\ \Phi\in D(A),$$

where

$$D(A)=\{\Phi\in\mathcal{C}\rho([0,T],L_\rho):\lim_{t\to0}\frac{S(t)\Phi-\Phi}{t}\ \textit{exists}\}.$$

The following theorem provides a characterization of a ρ-bounded linear operator A on L_ρ that serves as the infinitesimal generator of a strongly continuous semigroup.

Theorem 1. *Consider ρ a convex, regular modular function that satisfies the Δ_2-type condition. Let A be a linear operator that is ρ-bounded on L_ρ. If A fulfills this criterion, it acts as the infinitesimal generator for a strongly continuous semigroup $S(t)$ for all $\phi\in L_\rho$, where $t\in[0,T]$ and $T>0$. In other words, for each $\phi\in L_\rho$, the function*

$$\Lambda_\phi(t)=\rho(S(t)\phi-\phi)$$

is continuous for every t within the interval $[0,T]$.

Proof. Let us introduce the operator

$$S(t)\phi=exp(tA)\phi=\sum_{n=0}^{\infty}\frac{t^n}{n!}A^n\phi,\tag{6}$$

where A is a ρ-bounded linear operator on L_ρ. Let us first show that the operator

$$A:L_\rho\to L_\rho$$

is continuous with respect to the Luxemburg norm:

$$\|\phi\|_\rho=\inf\left\{t>0;\rho\left(\frac{1}{t}\phi\right)\le1\right\}.$$

Using the fact that A is a ρ-bounded, there exists a constant $M>0$ such that,

$$\rho(A\phi)\le M\rho(\phi),\ \text{for every}\ \phi\in L_\rho.\tag{7}$$

Then, using again Lemma 1, there is a constant $\widetilde{M}=max(M,1)>1$ such that

$$\|A\phi\|_\rho\le\widetilde{M}\|\phi\|_\rho.\tag{8}$$

This shows that the expression (6) is well defined. Utilizing the convexity property of ρ and the Δ_2-type condition, which guarantees that $\omega_\rho(exp(T))$ is bounded as mentioned in Lemma 3.1 in [1], the operator $S(t)$ is a ρ-bounded operator on L_ρ for every $t\in[0,T]$, satisfying $S(0)\phi=\phi$ and the algebraic semigroup property (ii) in Definition 4. In fact, let $\phi\in L_\rho$. Using the convexity property of ρ in Definition 1, A is a ρ-bounded linear operators on L_ρ and the fact that $\omega_\rho(exp(T)) < \infty$, and $\frac{1}{exp(T)}\sum_{n=0}^{\infty}\frac{t^n}{n!} = \frac{exp(t)}{exp(T)} < 1$, we obtain for every $t \in [0, T]$ and $\phi \in L_\rho$,

$$\begin{aligned}
\rho(S(t)\phi) &= \rho(exp(tA)\phi) \\
&= \rho\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}A^n\phi\right) \\
&\leq \sum_{n=0}^{\infty}\frac{1}{exp(T)}\frac{t^n}{n!}\rho(exp(T)A^n\phi) \\
&\leq \frac{\omega_\rho(exp(T))}{exp(T)}\sum_{n=0}^{\infty}\frac{t^n}{n!}\rho(A^n\phi), \\
&\leq \frac{\omega_\rho(exp(T))}{exp(T)}\sum_{n=0}^{\infty}\frac{t^n}{n!}M^n\rho(\phi), \\
&\leq \omega_\rho(exp(T))\frac{exp(TM)}{exp(T)}\rho(\phi).
\end{aligned}$$

It is clear that $S(0)\phi = \phi$, and

$$\begin{aligned}
S(t)S(s)\phi &= \sum_{n=0}^{\infty}\frac{t^n}{n!}A^n\sum_{k=0}^{\infty}\frac{s^k}{k!}A^k\phi \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^{(n-k)}A^{n-k}}{(n-k)!}\frac{s^kA^k}{k!}\phi \\
&= \sum_{n=0}^{\infty}\frac{(t+s)^nA^n}{n!}\phi \\
&= S(t+s)\phi.
\end{aligned}$$

To prove the strong continuity of the semigroup, we again utilize the power series estimation provided in Equation (6). We obtain:

$$\begin{aligned}
\Lambda_\phi(t) &= \rho(S(t)\phi - S(0)\phi) \\
&= \rho\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}A^n\phi - \phi\right) \\
&= \rho\left(\sum_{n=0}^{\infty}\frac{t^{n+1}}{(n+1)!}A^{n+1}\phi\right) \\
&\leq \frac{1}{exp(T)-1}\sum_{n=0}^{\infty}\frac{t^{n+1}}{(n+1)!}\rho\Big((exp(T)-1)A^{n+1}\phi\Big) \\
&\leq \frac{\omega_\rho(exp(T)-1)}{exp(T)-1}\sum_{n=0}^{\infty}\frac{t^{n+1}}{(n+1)!}\rho\Big(A^{n+1}\phi\Big), \\
&\leq \frac{\omega_\rho(exp(T)-1)}{exp(T)-1}\sum_{n=0}^{\infty}\frac{t^{n+1}}{(n+1)!}M^{n+1}\rho(\phi), \\
&\leq \frac{\omega_\rho(exp(T)-1)}{exp(T)-1}Mt\sum_{n=0}^{\infty}\frac{t^n}{n!}M^n\rho(\phi), \\
&\leq \frac{\omega_\rho(exp(T)-1)}{exp(T)-1}Mtexp(Mt)\rho(\phi),
\end{aligned}$$

Then, $\Lambda_\phi(t)$ is continuous for ρ-bounded linear operators A on L_ρ for $t \in [0, T]$, and we have, for sufficiently small $t > 0$

$$\begin{aligned}
\rho\left(\frac{S(t)\phi - S(0)\phi}{t} - A\phi\right) &= \rho\left(\frac{At\sum_{n=0}^{\infty}\frac{t^n}{(n+1)!}A^n}{t}\phi - A\phi\right) \\
&= \rho\left(A\sum_{n=0}^{\infty}\frac{t^n}{(n+1)!}A^n\phi - A\phi\right) \\
&\leq M\rho\left(\sum_{n=2}^{\infty}\frac{t^{n-1}}{n!}A^{n-1}\phi\right) \\
&\leq M\sum_{n=2}^{\infty}\frac{1}{exp(1)-2}\frac{1}{n!}\rho\Big((exp(1)-2)t^{n-1}A^{n-1}\phi\Big) \\
&\leq M\sum_{n=2}^{\infty}\frac{1}{n!}t^{n-1}\rho\Big(A^{n-1}\phi\Big) \\
&\leq \sum_{n=2}^{\infty}\frac{1}{n!}t^{n-1}M^n\rho(\phi) \\
&= \Big(\frac{exp(tM)-1}{t} - M\Big)\rho(\phi)
\end{aligned}$$

which implies that A is its infinitesimal generator. □

By using the Definition 4, then for every $t, s \in [0, T]$ and $\phi \in L_\rho$:

$$\begin{aligned}
\lim_{s\to t}\rho(S(s)\phi - S(t)\phi) &= \lim_{s\to t}\rho\Big(S(t)\Big(S(s-t)\phi - \phi\Big)\Big) \\
&\leq M\lim_{s\to t}\rho\Big(S(s-t)\phi - \phi\Big) = 0,
\end{aligned}$$

therefore

$$\lim_{s\to t}\rho(S(s)\phi - S(t)\phi) = 0 \tag{9}$$

We will now explore the Poincaré operator $\mathcal{J} : \mathcal{C}\rho([0, T], L_\rho) \to \mathcal{C}\rho([0, T], L_\rho)$ associated with the equations (5) within the modular function space L_ρ, as follows:

$$\mathcal{J}(\Phi)(\bullet) = S(\bullet)\Phi(0) + \int_0^{\bullet} S(\bullet - \sigma)\mathcal{F}(\sigma, \Phi(\sigma))\, d\sigma. \tag{10}$$

where $S \in \mathcal{C}\rho([0, T], L_\rho)$ and $\mathcal{F} : \Omega \to \mathbb{R}$ and satisfy $(H1)$, $(H2)$, and there exist nonnegative constants M and $M_{\mathcal{F}}$ such that

$$\begin{cases} \rho\Big(\mathcal{F}(\cdot, \Phi(\cdot))\Big) & \leq & M_{\mathcal{F}}\, \rho_{\mathcal{C}_\rho}(\Phi), \\ \rho(A\Phi) & \leq & M\rho(\Phi), \end{cases}$$

for any $\Phi \in \mathcal{C}_\rho([0, T], L_\rho)$ and $t \in [0, T]$.

We are prepared to present the primary outcome of our study, focusing on the existence of a solution to the variation of constants formula, as specified by (5) within the space $\mathcal{C}\rho([0, T], L_\rho)$.

Theorem 2. *Let ρ be a convex, regular modular function that satisfies the Δ_2-type condition. Assume that*

$$T\omega_\rho(e^T)\frac{e^{TM}}{e^T}M_{\mathcal{F}} < \frac{2}{\omega_\rho(2)}.$$

Then operator $\mathcal{J} : \mathcal{C}\rho([0,T], L_\rho) \to \mathcal{C}\rho([0,T], L_\rho)$ *given by (10) has a solution in* $\mathcal{C}_\rho([0,T], L_\rho)$.

Proof. Following a similar argument to Theorem 5.2 in [3], we consider the mesh points:

$$\zeta_i^n = \frac{it}{n}, \quad i = 0, \dots, n,$$

where $n = 1, 2, \cdots$ and $t \leq T$. Utilizing the fact that the operator A is ρ-bounded on L_ρ, and the Fatou property (see [1]), we obtain

$$\begin{aligned}
\rho((\mathcal{J}\Phi)(t) - (\mathcal{J}\widetilde{\Phi})(t)) &= \rho\Big(\int_0^t S(t-\sigma)\mathcal{F}(\sigma, \Phi(\sigma))\,d\sigma - \int_0^t S(t-\sigma)\mathcal{F}(\sigma, \widetilde{\Phi}(\sigma))\,d\sigma\Big) \\
&\leq \omega_\rho(\exp(T))\frac{e^{TM}}{e^T} \liminf_{n\to\infty} \sum_{i=0}^{n-1} \frac{t}{n}\rho(\mathcal{F}(\zeta_i^n, \Phi(\zeta_i^n)) - \mathcal{F}(\zeta_i^n, \widetilde{\Phi}(\zeta_i^n))) \\
&\leq \omega_\rho(\exp(T))\frac{e^{TM}}{e^T} \liminf_{n\to\infty} \sum_{i=0}^{n-1} \frac{t}{n} \sup_{\zeta\in[0,T]} \rho\Big(\mathcal{F}(\zeta, \Phi(\zeta)) - \mathcal{F}(\zeta, \widetilde{\Phi}(\zeta))\Big) \\
&\leq \omega_\rho(\exp(T))\tfrac{e^{TM}}{e^T} t \sup_{\zeta\in[0,T]} \rho\Big(\mathcal{F}(\zeta, \Phi(\zeta)) - \mathcal{F}(\zeta, \widetilde{\Phi}(\zeta))\Big).
\end{aligned}$$

For each $\Phi \in \mathcal{C}_\rho([0,T], L_\rho)$, we set

$$\widetilde{\mathcal{F}}(\Phi)(\zeta) = \mathcal{F}(\zeta, \widetilde{\Phi}(\zeta)),$$

which implies, for any $t \in [0,T]$,

$$\rho_{\mathcal{C}_\rho}(\mathcal{J}(\Phi) - \mathcal{J}(\widetilde{\Phi})) \leq \omega_\rho(e^T)\frac{e^{TM}}{e^T} T \rho_{\mathcal{C}_\rho}(\widetilde{\mathcal{F}}(\Phi) - \widetilde{\mathcal{F}}(\widetilde{\Phi})),$$

for any $\Phi, \widetilde{\Phi} \in \mathcal{C}_\rho([0,T], L_\rho)$. Let us now consider a ρ-bounded nonempty subset $B \subset L_\rho$. Let us prove that $\mathcal{J}\Big(\mathcal{C}_\rho([0,T], B)\Big)$ is $\rho_{\mathcal{C}_\rho}$-relatively compact. Using the property (1), we have

$$\begin{aligned}
\rho\Big((\mathcal{J}\Phi)(t) - (\mathcal{J}\Phi)(\widetilde{t})\Big) &\leq \frac{\omega_\rho(2)}{2}\rho\Big(S(t)\Phi(0) - S(\widetilde{t})\Phi(0)\Big) \\
&\quad + \frac{\omega_\rho(2)}{2}\rho\Big(\int_{\widetilde{t}}^t S(t-\zeta)\mathcal{F}(\zeta, \Phi(\zeta))\,d\zeta\Big) \\
&\leq \frac{\omega_\rho(2)}{2}\rho\Big(S(t)\Phi(0) - S(\widetilde{t})\Phi(0)\Big) \\
&\quad + \omega_\rho(e^T)\frac{e^{TM}}{e^T}\frac{\omega_\rho(2)}{2} \sup_{\zeta\in[0,T]} \rho(\mathcal{F}(\zeta, \Phi(\zeta))) \liminf_{n\to\infty} \sum_{i=0}^{n-1} \frac{t-\widetilde{t}}{n} \\
&= \frac{\omega_\rho(2)}{2}\rho\Big(S(t)\Phi(0) - S(\widetilde{t})\Phi(0)\Big) \\
&\quad + (t-\widetilde{t})\omega_\rho(e^T)\frac{e^{TM}}{e^T}\frac{\omega_\rho(2)}{2} \sup_{\zeta\in[0,T]} \rho\Big(\mathcal{F}(\zeta, \Phi(\zeta))\Big),
\end{aligned}$$

for any $t, \widetilde{t}$ in $[0,T]$ and $\Phi \in \mathcal{C}_\rho([0,T], B)$. Again, by using the fact that ρ satisfies the Δ_2-type condition, we establish that the family $\mathcal{J}\Big(\mathcal{C}_\rho([0,T], B)\Big)$ is equicontinuous with respect to the Luxemburg norm $\|\cdot\|_{\mathcal{C}_\rho}$. We deduce the complete continuity of the operator $\mathcal{J}$ with respect to $\rho_{\mathcal{C}_\rho}$ by using the Arzelà-Ascoli theorem.

Consider the set

$$\Omega = \{\Phi \in \mathcal{C}_\rho([0,T], L_\rho) : \Phi = \nu\mathcal{J}(\Phi), \text{ for some } \nu \text{ in } [0,1]\}.$$

We claim that Ω is $\rho_{\mathcal{C}_\rho}$-bounded. Indeed, the set Ω is nonempty, since it contains the zero function. Next, let $\Phi \in \Omega$. We have $\Phi = \nu \mathcal{J}(\Phi)$, for some $\nu \in [0,1]$, which implies

$$\begin{aligned}
\rho(\Phi(t)) &= \rho\Big(\nu S(t)\Phi(0) + \nu \int_0^t S(t-\zeta)\mathcal{F}(\zeta,\Phi(\zeta))\, d\zeta\Big) \\
&\leq \nu\rho\Big(S(t)\Phi(0) + \int_0^t S(t-\zeta)\mathcal{F}(\zeta,\Phi(\zeta))\, d\zeta\Big) \\
&\leq \nu \frac{\omega_\rho(2)}{2}\Big(\rho(S(t)\Phi(0)) + \rho\Big(\int_0^t S(t-\zeta)\mathcal{F}(\zeta,\Phi(\zeta))\, d\zeta\Big)\Big),
\end{aligned}$$

for any $t \in [0,T]$. By virtue of the Fatou property, one has

$$\begin{aligned}
\rho(\Phi(t)) &\leq \omega_\rho(e^T)\frac{e^{TM}}{e^T}\frac{\omega_\rho(2)}{2}\Big[\rho(\Phi(0)) + \liminf_{n\to\infty}\sum_{i=0}^{n-1}(\zeta_{i+1}^n - \zeta_i^n)\rho(\mathcal{F}(\vartheta,\zeta_i^n,\Phi(\zeta_i^n)))\Big], \\
&\leq \omega_\rho(e^T)\frac{e^{TM}}{e^T}\frac{\omega_\rho(2)}{2}\Big[\rho(\Phi(0)) + \sup_{\zeta\in[0,T]}\rho\Big(\mathcal{F}(\zeta,\Phi(\zeta))\Big)\liminf_{n\to\infty}\sum_{i=0}^{n-1}\Big(\frac{t}{n}\Big)\Big], \\
&= \omega_\rho(e^T)\frac{e^{TM}}{e^T}\frac{\omega_\rho(2)}{2}\Big(\rho(\Phi(0)) + T\sup_{\zeta\in[0,T]}\rho\Big(\mathcal{F}(\zeta,\Phi(\zeta))\Big)\Big), \\
&\leq \omega_\rho(e^T)\frac{e^{TM}}{e^T}\frac{\omega_\rho(2)}{2}\Big(\rho(\Phi(0)) + TM_{\mathcal{F}}\rho_{\mathcal{C}_\rho}(\Phi)\Big),
\end{aligned}$$

for any $t \in [0,T]$, which implies

$$\rho_{\mathcal{C}_\rho}(\Phi)\Big(1 - T\frac{\omega_\rho(2)}{2}(\omega_\rho(e^T)\frac{e^{TM}}{e^T}M_{\mathcal{F}})\Big) \leq \omega_\rho(e^T)\frac{e^{TM}}{e^T}\frac{\omega_\rho(2)}{2}\rho(\Phi(0)),$$

Since $T\omega_\rho(e^T)\frac{e^{TM}}{e^T}M_{\mathcal{F}} < \frac{2}{\omega_\rho(2)}$, we obtain

$$\rho_{\mathcal{C}_\rho}(\Phi) \leq \frac{\omega_\rho(2)}{2}\frac{\omega_\rho(e^T)\frac{e^{TM}}{e^T}\rho(\Phi(0))}{1 - \frac{\omega_\rho(2)}{2}(T\omega_\rho(e^T)\frac{e^{TM}}{e^T}M_{\mathcal{F}})}.$$

Therefore, S is bounded with respect to $\rho_{\mathcal{C}_\rho}$. Revisiting Schaeffer's theorem ensures the existence of a fixed point for $\mathcal{J}$, leading to the solution of the equation (FIE) in $\mathcal{C}_\rho([0,T],L_\rho)$. This follows from utilizing the Δ_2-type condition and the fact that, in this scenario, Ω is bounded with respect to the associated Luxemburg norm. □

In the following section, we employ our established existence result within the framework of variable-exponent Lebesgue spaces. This application enables the identification of minimal conditions that must be satisfied by the exponent function $p(\cdot)$, the functions $\mathcal{F}$, and the semigroup $S(t)$.

4. Application to Variable-Exponent Lebesgue Spaces

In this section, we explore the implications of Theorem 2 for variable exponent Lebesgue spaces $L^{p(\cdot)}$; see [15,16]. We start by presenting the basic definitions and key properties of this space. Let us consider

$$p : [0,1] \to [1,\infty], p^- := \operatorname{ess\,inf}_{x\in[0,1]} p(x) \;\; \textit{and} \quad p^+ := \operatorname{ess\,sup}_{x\in[0,1]} p(x).$$

The space $L^{p(\cdot)}([0,1])$ is defined by

$$L^{p(\cdot)} := L^{p(\cdot)}([0,1]) = \{\phi \in L^0([0,1]) : \varrho_{L^{p(\cdot)}}(\lambda\phi) < \infty \quad \text{for some } \lambda > 0\},$$

where $L^0([0,1])$ is the space of Lebesgue measurable functions on $[0,1]$ with $\mathbb{R}$-valued, and

$$\varrho_{L^{p(\cdot)}}(\phi) = \int_0^1 \mid \phi(x) \mid^{p(x)} dx.$$

The Luxemburg norm is defined by:

$$\|\phi\|_{L^{p(\cdot)}} = \inf\left\{ t > 0 \mid \varrho_{L^{p(\cdot)}}\left(\frac{1}{t}\phi\right) \leq 1 \right\}.$$

Assuming the condition $1 < p^- < p^+ < \infty$, we ensure the Δ_2-type condition for $L^{p(\cdot)}$ holds iff $p^+ < \infty$. Additionally, under the same condition $1 < p^- < p^+ < \infty$, the Luxemburg norm $\| \cdot \|_{L^{p(\cdot)}}$ is uniformly convex. For more details regarding these conditions and their significant properties, we refer to [4,15,17] and the references therein. Next, we consider the integral equation

$$\Phi(t) = S(t)\Phi(0) + \int_0^t S(t-\zeta)\mathcal{F}(\zeta, \Phi(\zeta))\, d\zeta, t \in [0,1]. \tag{11}$$

where $\Phi \in L^{p(\cdot)}$, $S(\cdot)\Phi \in L^{p(\cdot)}$, and $\mathcal{F} : \Omega \to \mathbb{R}$. The existence of a solution Φ of the Equation (11) will follow along the same lines as those of the investigation developed in the previous section. First, we assume there exists a constant $C_{\mathcal{F}} \geq 0$ and all $\phi \in L^{p(\cdot)}$, we have

$$|\mathcal{F}(\zeta, \phi(\zeta))| \leq C_{\mathcal{F}}\, |\phi(\zeta)|.$$

And S is a strongly continuous semigroup on $\mathcal{C}\rho([0,1], L^{p(\cdot)})$ into itself, associated to the infinitesimal generator linear operator ρ-bounded A by M, with domain $D(A)$ such that

$$A\Phi = \lim_{t \to 0} \frac{S(t)\Phi - \Phi}{t} \text{ for } \Phi \in D(A),$$

where

$$D(A) = \{\Phi \in \mathcal{C}\rho([0,1], L^{p(\cdot)}) : \lim_{t \to 0} \frac{S(t)\Phi - \Phi}{t} \text{ exists}\}.$$

Let $\Phi \in \mathcal{C}_{\varrho_{L^{p(\cdot)}}}([0,1], L^{p(\cdot)})$. Set

$$\phi := \Phi(\cdot) \in L^{p(\cdot)}.$$

Moreover,

$$\begin{aligned}
\varrho_{L^{p(\cdot)}}(\mathcal{F}(\cdot, \Phi(\cdot))) &= \int_0^1 |\mathcal{F}(\zeta, \phi(\zeta))|^{p(\zeta)} d\zeta \\
&\leq \int_0^1 (C_{\mathcal{F}} |\phi(\zeta)|)^{p(\zeta)}\ d\zeta \\
&\leq \int_0^1 (C_{\mathcal{F}} |\phi(\zeta)|)^{p(\zeta)}\ d\zeta \\
&\leq \int_0^1 (\overline{C_{\mathcal{F}}}\ (|\phi(\zeta)|)^{p(\zeta)}\ d\zeta \\
&= \overline{M}_{\mathcal{F}}\ \rho_{\mathcal{C}_{\varrho_{L^{p(\cdot)}}}}(\Phi),
\end{aligned}$$

where $\overline{C_{\mathcal{F}}} = \max\{C_{\mathcal{F}}^{p^-}, C_{\mathcal{F}}^{p^+}\}$, $\varrho_{\mathcal{C}_{\varrho_{L^{p(\cdot)}}}}(\Phi) = \sup_{\vartheta \in [0,1]} \varrho_{L^{p(\cdot)}}(\Phi(\vartheta))$, and $\overline{M}_{\mathcal{F}} = \overline{C_{\mathcal{F}}}$. So, if $\overline{C_{\mathcal{F}}} < e^{(1-p^+)\ln 2 - M + 1}$, it means that when $C_{\mathcal{F}} \leq 1$, we get

$$p^- \ln(C_{\mathcal{F}}) + p^+ \ln 2 < \ln 2 - M + 1.$$

And if $C_{\mathcal{F}} \geq 1$, we get
$$p^+ \Big(\ln(C_{\mathcal{F}}) + \ln 2 \Big) < \ln 2 - M + 1.$$

In both cases, Theorem 2 will imply that (11) has a solution in $\mathcal{C}_{\varrho_{L^{p(\cdot)}}}([0,1], L^{p(\cdot)})$ and provide information on how M should be to ensure the solution exists, where $1 < p^- < p^+ < \infty$.

This investigation seeks to understand the behavior of solutions in variable-exponent Lebesgue spaces, providing a broader framework for analyzing differential equations in non-standard growth conditions. The variation of the constant formula (11) plays a crucial role in this analysis as it encapsulates the state of the system over time. By studying (11), we can gain insights into the stability, long-term behavior, and boundedness of the system, which are essential for both theoretical understanding and practical applications. The motivation for exploring these properties lies in extending classical results to more complex and realistic scenarios with non-standard growth conditions, thereby enhancing our ability to model and control various dynamic systems. For example, future investigations could focus on applying these results to specific types of differential equations or exploring the impact of different growth conditions on the stability and boundedness of solutions.

Funding: This research is funded by Researchers Supporting Project Number (RSPD2024R963) King Saud University, Riyadh Saudi Arabia.

Data Availability Statement: Data sharing is not applicable to this article, as no datasets were generated or analyzed during the current study.

Acknowledgments: The author would like to extend sincere appreciation to the Researchers Supporting Program for funding this work under Researchers Supporting Project number RSPD2024R963, King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The author declares no conflicts of interest.

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