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Complete Solutions in the Dilatation Theory of Elasticity with a Representation for Axisymmetry

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Abstract: In this paper, we present certain complete solutions in the dilatation theory of elasticity. This model can be derived as a special case of Eringen's linear theory of microstretched elastic solids when microrotations are absent. It is also a version of the theory of materials with voids. The dilatation theory can be considered the simplest theoretical model of microstructured materials and is suitable for investigating various phenomena that occur in engineering, geomechanics, and biomechanics. We establish three complete solutions to the displacement equations of equilibrium that are the counterpart of the Green–Lamé (GL), Boussinesq–Papkovich–Neuber (BPN), and Cauchy–Kowalevski–Somigliana (CKS) solutions of classical elasticity. The links between these BPN and CKS solutions are established. Then, we present a representation of the BPN solution in the case of axisymmetry. The results presented here are useful for solving axisymmetric problems in semi-infinite and infinite domains.

Keywords: dilatation elasticity; complete solutions; Boussinesq–Papkovich–Neuber solution; Green–Lamé solution; Cauchy–Kowalevski–Somigliana solution; axial symmetry

1. Introduction

The paper is concerned with the linear dilatation theory of elasticity. In this model, the basic independent kinematic variables are the components of the field $\mathbf{u}$ of the body point displacement and the scalar dilatation function φ. In contrast with the classical continuum theory, where all of the deformation characteristics depend on the displacement field $\mathbf{u}$, in this microstructural model, the dilatation of a small volume of the body does not depend on its centroidal motion, i.e., the relation $\varphi = \text{div}\mathbf{u}$ is rejected and φ is considered as an independent characteristic of deformation. This assumption leads to the theory of materials with voids proposed by Nunziato and Cowin [1,2]. This model can be also considered a special case of Eringen's microstretch theory [3], when the microrotations of the microparticles are absent. This theory has been investigated in detail regarding its fundamentals and applications (see, e.g., the works of Markov [4], Lakes [5], Iesan [6,7], De Cicco [8], and Birsan [9]). The increased interest in this topic is due to its suitability for investigating various phenomena occurring in engineering, geomechanics, and biomechanics. It is also useful for studying problems concerning the mechanical behaviour and fracturing of materials containing a large number of point defects or microvoids (see Ramizani and Jeany's work [10]). A review of theoretical advances and references to relevant contributions can be found in the book by Svanadze [11]. Recently, this field of theory has broadened its domain to include more complex models, including multi-porosity materials, chiral media, strain gradient elasticity and viscoelasticity, stress-driven elasticity, and non-local elasticity [12–18].

Studies determining the general analytic solutions of the governing equations in elastostatics and elastodynamics have been a core issue of the mathematical theory of elasticity. Outlines of the historical achievements in this topic can be found in the works of Gurtin [19], Sternberg [20], and Teodorescu [21]. The main theoretical efforts thus far

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have been directed at finding a unified method to derive the general solutions and a unified scheme to prove the related issue of completeness and uniqueness (see [22–24]). Further investigations have been devoted to extending the general elastic solutions to a variety of fields, such as thermoelasticity, piezoelasticity, anisotropic solids, fluent media, and so on (see, e.g., [25–27]). More recently, various studies have addressed the fields of generalized media, which are largely characterized by taking into account multifield coupling effects. In [28], Cowin established a complete solution to the displacement equations of equilibrium for linear Cosserat elasticity. A representation of this solution in terms of complex potentials in the context of microstretch elasticity was given by Iesan and Nappa [29]. Chandrasekaraiah and Cowin [30] presented a complete solution to Biot's poroelastic theory. A representation of a Galerkin-type solution in the theory of microfluids was established by Nappa in [31]. Xinsheng and Minzhong [32] derived general solutions in the case of axisymmetric Stokes flow. Other contributions are due to Ike [33], Fetecan and Vieru [34], and Markus and Mead [35]. For recent advances in the study of complete solutions for 3D problems, the reader is referred to the review article by M.Z. Wang et al. [36] and the references therein.

In this paper, complete solutions of the field equations of linear elastic dilatation theory are obtained. These solutions are analogous and include, as special cases, the BPN, GL, and CKS solutions of classical elastostatics. A connection between the BPN and CKS solutions is established. Then, we present the basic equations of the dilatation theory in the case of axisymmetry. The BPN solution of the displacement equilibrium system is particularized in the case of axisymmetry. This result is particularly useful for solving axisymmetric problems involving semi-infinite and infinite domains. The last section of this paper is devoted to discussions and conclusions.

2. Basic Equations

We consider a regular region B of a three-dimensional Euclidean space occupied by a linearly elastic material with voids. The region B is referred to as a system of Cartesian coordinates $O\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. We denote by ∂B the boundary of B and by $\mathbf{n}$ the outward unit normal vector of ∂B. We assume that the mass density ρ, in the deformed configuration, has the decomposition $\rho = \sigma\hat{\rho}$, where $\hat{\rho}$ is density of the matrix material and σ is the volume fraction field. In the non-deformed state, we have $\rho_0 = \sigma_0\hat{\rho_0}$, where ρ_0, $\hat{\textbf{æ}}_\mathbf{0}$, and σ_0 are the mass density, the density of the matrix material, and the volume fraction field in the reference configuration, respectively. We introduce the notation $\varphi = \sigma - \sigma_0$. The independent kinematic variables are the components of the displacement $u_i (i = 1, 2, 3)$ and the change in volume fraction φ. The governing equations of the linear theory of elastic materials with voids are as follows:

Geometrical equations:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

Equilibrium equations:

$$t_{ij,i} + b_j = 0, \quad h_{i,i} - p + q = 0 \tag{2}$$

where t_{ij} is the stress tensor, b_j is the body force, h_i is the equilibrated stress vector, p is the intrinsic equilibrated stress vector, and q is the intrinsic body force. The physical meaning and specific interpretations of h_i, p, and q have been discussed in [8].

Constitutive equations:

$$\begin{aligned} t_{ij} &= 2\mu e_{ij} + \lambda e_{rr} + \beta\varphi, \quad h_i = \alpha\varphi_{,i}, \\ p &= \beta u_{j,j} + \zeta\varphi, \end{aligned} \tag{3}$$

where μ, λ, β, α, and ζ are constitutive coefficients. We assume that the internal energy density is a positive definite. This assumption implies that

$$\mu > 0, \quad \alpha > 0, \quad \zeta > 0, \quad 2\mu + 3\lambda > 0, \quad (2\mu + 3\lambda)\zeta > 3\beta^2. \tag{4}$$

The boundary conditions at a regular point ∂B are expressed as

$$t_{ji}n_j = t_i, \quad h = h_i n_i \quad , \tag{5}$$

where t_i is the surface traction and h is the equilibrated surface force. With the help of Equations (1) and (3), the equilibrium Equation (2) can be rewritten in the following form:

$$\begin{aligned} &\mu\Delta\mathbf{u} + (\mu + \lambda)\nabla\text{div}\mathbf{u} + \beta\nabla\varphi + \mathbf{b} = \mathbf{0} \\ &\alpha\Delta\varphi - \zeta\varphi - \beta\text{div}\mathbf{u} + q = 0. \end{aligned} \tag{6}$$

In the following analysis, we study certain general solutions of the differential system of Equation (6).

3. Complete Solutions

We assume that all functions appearing in this discussion are continuous and differentiable up to the required order on B. We introduce the following notations:

$$c^2 = 2\mu + \lambda, \quad a^2 = \frac{\mu + \lambda}{2c^2}, \quad D = (\alpha\Delta - \zeta)c^2 + \beta^2 \tag{7}$$

Theorem 1. *Boussinesq –Papakovich–Neuber-type solution.*

Let

$$\begin{aligned} \mathbf{u} &= \mathbf{G} - a^2\nabla(\mathbf{x}\cdot\mathbf{G} + H), \\ \beta\varphi &= c^2a^2(\mathbf{x}\cdot\Delta\mathbf{G} + \Delta H), \end{aligned} \tag{8}$$

where $\mathbf{x}$ is the vector position of the generic point P, and $\mathbf{G}$ and H are arbitrary functions defined on B that satisfy

$$\begin{aligned} \Delta\mathbf{G} &= -\frac{1}{\mu}\mathbf{b} \\ D\Delta H &= \frac{\beta^2(1 - 2a^2)}{a^2}\text{div}\mathbf{G} - D(\mathbf{x}\cdot\Delta\mathbf{G}) - \frac{\beta q}{a^2}. \end{aligned} \tag{9}$$

Then, $\mathbf{u}$ and φ are a solution of the system of Equation (6).

Proof.

$$\begin{aligned} &\mu\Delta\mathbf{u} + (\mu + \lambda)\nabla\text{div}\mathbf{u} + \beta\nabla\varphi + \mathbf{b} = \mu\Delta\mathbf{G} - a^2c^2\nabla\Delta(\mathbf{x}\cdot\mathbf{G}) + a^2c^2\nabla(\mathbf{x}\cdot\Delta\mathbf{G}) \\ &+ 2a^2c^2\nabla\text{div}\mathbf{G} + \mathbf{b} = \mu\Delta\mathbf{G} + \mathbf{b} = \mathbf{0}, \end{aligned}$$

when we have used the relation

$$\Delta(\mathbf{x}\cdot\mathbf{G}) = \mathbf{x}\cdot\Delta\mathbf{G} + 2\text{div}\mathbf{G}. \tag{10}$$

Moreover, we have

$$\begin{aligned} &\alpha\Delta\varphi - \zeta\varphi - \beta\text{div}\mathbf{u} + q = \frac{a^2}{\beta}[(\alpha\Delta + \zeta)c^2 + \beta^2]\mathbf{x}\cdot\Delta\mathbf{G} + 2a^2\beta\text{div}\mathbf{G} - \\ &\beta\text{div}\mathbf{G} + \frac{a^2}{\beta}[(\alpha\Delta - \zeta)c^2 + \beta^2]\Delta H + q = \\ &\frac{a^2}{\beta}[D(\mathbf{x}\cdot\Delta\mathbf{G}) - \frac{(1 - 2a^2)\beta^2}{a^2}\text{div}\mathbf{G} + D\Delta H + q\frac{\beta}{a^2}] = 0. \end{aligned}$$

□

In most applications of elastostatics, the Boussinesq–Papkovich–Neuber solution is preferred over the Boussinesq–Somigliana–Galerkin solution. The reasons for this are discussed by Gurtin (see [18], p. 142). As a corollary of Theorem 1, we obtain the following result:

(i) If $\mathbf{b} = 0$ and $q = 0$, the relations in Equation (8) reduce to

$$\begin{aligned} \mathbf{u} &= \mathbf{G} - a^2\nabla(\mathbf{x}\cdot\mathbf{G} + H) \\ \beta\varphi &= c^2a^2\Delta H, \end{aligned} \tag{11}$$

where $\mathbf{G}$ and H satisfy

$$\Delta\mathbf{G} = 0, \quad D\Delta H = \frac{\beta^2(1-2a^2)}{a^2}\text{div}\mathbf{G}. \tag{12}$$

(ii) Let $\mathbf{e}$ be a given unit vector. If we take

$$\mathbf{G} = G\mathbf{e}, \tag{13}$$

then Equation (11) becomes

$$\begin{aligned} \mathbf{u} &= G\mathbf{e} - a^2\nabla(H + zG) \\ \beta\varphi &= c^2a^2\Delta H, \end{aligned} \tag{14}$$

where $z = \mathbf{x}\cdot\mathbf{e}$ is the coordinate in direction of $\mathbf{e}$. The conditions in Equation (12) take the following form:

$$\Delta G = 0, \quad D\Delta H = \frac{\beta^2(1-2a^2)}{a^2}\text{div}(G\mathbf{e}). \tag{15}$$

The representations in Equations (14) and (15) are known as Boussinesq's solutions. Both solutions (i) and (ii) are extremely useful in axisymmetric problems. The next step is to establish the Green–Lamé-type representation of the solution. As in classical elasticity, we assume that $\mathbf{b}$ admits the Helmotz decomposition

$$\mathbf{b} = -(\nabla\eta + \text{curl}\gamma), \tag{16}$$

where η and γ are arbitrary functions defined on B,

Theorem 2. *Green–Lamé-type solution.*

Let

$$\begin{aligned} \mathbf{u} &= \nabla\Psi + \text{curl}\Phi \\ \beta\varphi &= \eta - c^2\Delta\Psi, \end{aligned} \tag{17}$$

where Ψ and Φ are functions that satisfy

$$\begin{aligned} \mu\Delta\Phi &= \gamma \\ D\Delta\Psi &= (\alpha\Delta - \zeta)\eta + \beta q. \end{aligned} \tag{18}$$

Then, $\mathbf{u}$ and φ are a solution of the system in Equation (6).

Proof.

$$\begin{aligned} &\mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\text{div}\mathbf{u} + \beta\nabla\varphi + \mathbf{b} = (2\mu+\lambda)\nabla\Delta\Psi + \\ &\mu\text{curl}\Delta\Phi + \nabla\eta - c^2\nabla\Delta\Psi - \nabla\eta - \text{curl}\gamma = \text{curl}(\mu\Delta\Phi - \gamma) = \mathbf{0} \end{aligned}$$

Substituting Equation (17) in the second equation of Equation (6), we have

$$\begin{aligned}&\alpha\Delta\varphi-\zeta\varphi-\beta\mathrm{div}\mathbf{u}+q=\\&\frac{1}{\beta}(\alpha\Delta-\zeta)\eta-\frac{1}{\beta}(\alpha\Delta-\zeta)c^2\Delta\Psi-\beta\Delta\Psi+q=\\&\frac{1}{\beta}\{(\alpha\Delta-\zeta)\eta+\beta q-[(\alpha\Delta-\zeta)c^2+\beta^2]\Delta\psi\}=0.\end{aligned}$$

□

The proof of Theorem 2 is thus complete. In the next theorem, we obtain a representation of $\mathbf{u}$ and φ that is called Cauchy–Kowalevski–Somigliana solution. We introduce the following notation:

$$D^*=2(\alpha\Delta-\zeta)c^2+\frac{\beta^2}{a^2}. \tag{19}$$

Theorem 3. *Cauchy–Kowalevski–Somigliana-type solution.*

Let

$$\begin{aligned}\mathbf{u}=&\frac{1}{\alpha}[D\Delta\mathbf{P}-a^2\nabla(D^*\mathrm{div}\mathbf{P}+\beta Q)]\\\varphi=&\frac{1}{\alpha}\Delta(a^2c^2Q+\mu\beta\mathrm{div}\mathbf{P})\end{aligned} \tag{20}$$

where $\mathbf{P}$ and Q are functions defined on B that satisfy

$$\begin{aligned}D\Delta^2\mathbf{P}=&-\frac{\alpha\mathbf{b}}{\mu}\\D\Delta Q=&-\frac{\alpha Q}{a^2}\end{aligned} \tag{21}$$

Then, $\mathbf{u}$ and φ are solution of the system of Equation (6).

Proof.

$$\begin{aligned}&\Delta\mathbf{u}+(\mu+\lambda)\nabla\mathrm{div}\mathbf{u}+\beta\nabla\varphi+\mathbf{b}=\\&\frac{1}{\alpha}\{\mu D\Delta^2\mathbf{P}-[(2\mu+\lambda)a^2D^*-(\mu+\lambda)D-\mu\beta^2]\Delta\nabla\mathrm{div}\mathbf{P}-[\beta a^2(2\mu+\lambda)-a^2c^2\beta]\Delta\nabla Q+\alpha\mathbf{b}\}=\\&\frac{1}{\alpha}\{\mu D\Delta^2\mathbf{P}-[(\mu+\lambda)(\alpha\Delta-\zeta)c^2+\frac{(\mu+\lambda)\beta^2}{2a^2}-(\mu+\lambda)(\alpha\Delta-\zeta)c^2-\\&(\mu+\lambda)(\alpha\Delta-\zeta)c^2-(\mu+\lambda)\beta^2-\mu\beta^2]\Delta\nabla\mathrm{div}\mathbf{P}+\alpha\mathbf{b}\}=\\&\frac{1}{\alpha}(\mu D\Delta^2\mathbf{P}+\alpha\mathbf{b})=\mathbf{0}\end{aligned}$$

and

$$\begin{aligned}&\alpha\Delta\varphi-\zeta\varphi-\beta\mathrm{div}\mathbf{u}+q=\\&\frac{1}{\alpha}\{[(\alpha\Delta-\zeta)c^2+\beta^2]a^2\Delta Q+\beta[\mu(\alpha\Delta-\zeta)-D+a^2D^*]\Delta\mathrm{div}\mathbf{P}+q\alpha\}=\\&\frac{1}{\alpha}\{a^2D\Delta Q+\beta[(\alpha\Delta-\zeta)(\mu-c^2+2a^2c^2)-\beta^2+a^2\frac{\beta^2}{a^2}]\Delta\mathrm{div}\mathbf{P}+q\alpha\}=\\&\frac{1}{\alpha}(a^2D\Delta Q+q\alpha)=0.\end{aligned}$$

□

The complete solutions established in this section are the counterpart of the analogous complete solutions of the classical elasticity theory. The completeness of these solutions can be proven following the method of classical elasticity.

4. Links between the BPN and CKS Solutions

We introduce the following notations:

$$s^2 = \frac{1}{\alpha}(\zeta - \frac{\beta^2}{2\mu+\lambda}), \quad d^2 = \frac{1}{\alpha}(\zeta - \frac{\beta^2}{\mu+\lambda})$$
$$l^2 = \frac{2\mu\beta^2}{\alpha(\mu+\lambda)(2\mu+\lambda)}, \quad a_1 = \frac{2\beta}{\alpha(\mu+\lambda)},$$
$$a_2 = \frac{1}{\mu(2\mu+\lambda)}, \quad a_3 = \frac{2}{\mu+\lambda}. \tag{22}$$

Equation (9) related to the BPN-type solution can be rewritten in the following form:

$$\Delta \mathbf{G} = -\frac{1}{\mu}\mathbf{b}$$
$$(\Delta - s^2)\Delta H = -(\Delta - s^2)\Delta(\mathbf{x}\cdot\mathbf{G}) - 2d^2 \text{div}\mathbf{G} - a_1 q. \tag{23}$$

If we let $\mathbf{b} = 0$, $q = 0$, then Equation (23) takes the following form:

$$\Delta \mathbf{G} = 0, \quad (\Delta - s^2)\Delta H = l^2 \text{div}\mathbf{G}. \tag{24}$$

Similarly, the version of Equation (21) related to the CKS-type solution becomes

$$(\Delta - s^2)\Delta\Delta\mathbf{P} = -a_2\mathbf{b},$$
$$(\Delta - s^2)\Delta Q = -a_3 q. \tag{25}$$

When $\mathbf{b} = 0$ and $q = 0$, Equation (25) reduces to

$$(\Delta - s^2)\Delta\Delta\mathbf{P} = 0, \quad (\Delta - s^2)\Delta Q = 0. \tag{26}$$

Straightforward calculations lead to the following relations between the two solutions:

$$\mathbf{G} = c^2(\Delta - s^2)\Delta\mathbf{P},$$
$$H = \frac{1}{\alpha}(\beta Q - D^* \text{div}\mathbf{P}) - \mathbf{x}\cdot\mathbf{G}. \tag{27}$$

In the same way, it is not difficult to determine links between the GL solution and the BPN and CKS solutions.

5. Axially Symmetric Problems

We assume that the region B refers to the half-space $x_3 > 0$. We are interested in axially symmetric problems with the displacement field $\mathbf{u}$ and the volume fraction φ being specified in cylindrical coordinates as follows:

$$u_r = u(r,z), \quad u_\theta = 0, \quad u_z = w(r,z)$$
$$\varphi = \varphi(r,z). \tag{28}$$

The geometrical equations become

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{u}{r}, \quad e_{zz} = \frac{\partial w}{\partial z}$$
$$e_{r\theta} = 0, \quad e_{\theta z} = 0, \quad e_{zz} = \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}). \tag{29}$$

The constitutive equations are given by

$$\begin{aligned}
&\tau_{rr} = 2\mu e_{rr} + \lambda e + \beta\varphi, \\
&\tau_{\theta\theta} = 2\mu\frac{u}{r} + \lambda e + \beta\varphi, \\
&\tau_{zz} = 2\mu e_{zz} + \lambda e + \beta\varphi, \\
&\tau_{rz} = 2\mu e_{rz}, \quad \tau_{r\theta} = \tau_{\theta z} = 0, \\
&h_r = \alpha\frac{\partial\varphi}{\partial r}, \quad h_\theta = 0, \quad h_z = \alpha\frac{\partial\varphi}{\partial z}, \\
&p = -\beta e - \zeta\varphi,
\end{aligned} \tag{30}$$

where

$$e = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z}. \tag{31}$$

The equilibrium equations take the form

$$\begin{aligned}
&\frac{\partial\tau_{rr}}{\partial r} + \frac{\partial\tau_{rz}}{\partial z} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) + b_r = 0, \\
&\frac{\partial\tau_{rz}}{\partial r} + \frac{\partial\tau_{zz}}{\partial z} + \frac{1}{r}\tau_{rz} + b_z = 0, \\
&\frac{1}{r}\frac{\partial}{\partial r}(rh_r) + \frac{\partial h_z}{\partial z} - p + q = 0.
\end{aligned} \tag{32}$$

Equation (32) can be expressed in terms of u, w, and φ, as follows:

$$\begin{aligned}
&\mu(\Delta - \frac{1}{r^2})u + (\lambda + \mu)\frac{\partial e}{\partial r} + \beta\frac{\partial\varphi}{\partial r} + b_r = 0, \\
&\mu\Delta w + (\lambda + \mu)\frac{\partial e}{\partial z} + \beta\frac{\partial\varphi}{\partial z} + b_z = 0, \\
&\alpha\Delta\varphi - \beta e - \zeta\varphi + q = 0,
\end{aligned} \tag{33}$$

where $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{\partial^2}{\partial z^2}$.

The Boussinesq–Papkovich–Neuber-Type Representation for Axisymmetry

In the system of Equation (8), we express the functions G and H in cylindrical coordinates. We assume that

$$G_r = \xi(r,z), \quad G_\theta = 0, \quad G_z = \psi(r,z) \quad H = \chi(r,z). \tag{34}$$

From Equations (8), (28) and (34), we have

$$u = \xi - a^2\frac{\partial\Gamma}{\partial r}, \quad w = \psi - a^2\frac{\partial\Gamma}{\partial z}, \quad \beta\varphi = a^2c^2(\Delta\Gamma - 2\kappa), \tag{35}$$

where

$$\Gamma = r\xi + z\psi + \chi. \tag{36}$$

We introduce the following notation:

$$\kappa = \frac{1}{r}\frac{\partial}{\partial r}(r\xi) + \frac{\partial\psi}{\partial z}. \tag{37}$$

From Equations (31) and (36), we obtain

$$e = \kappa - a^2 \Delta\Gamma. \tag{38}$$

If ξ, ψ, and χ satisfy the following equations

$$(\Delta - \frac{1}{r^2})\xi = -\frac{b_r}{\mu}, \quad \Delta\psi = -\frac{b_z}{\mu},$$
$$D\Delta\Gamma - D^*\kappa = -q\frac{\beta}{a^2} \tag{39}$$

then u, w, and φ are a solution of the system in Equation (33).

In what follows, we will use the following relation:

$$\Delta\Gamma = r(\Delta - \frac{1}{r^2})\xi + z\Delta\psi + \Delta\chi + 2\kappa. \tag{40}$$

Now, we consider the case in which $b_r = b_z = 0$, $q = 0$.

The relation in Equation (40) reduces to

$$\Delta\Gamma = \Delta\chi + 2\kappa, \tag{41}$$

and from Equation (35), we have

$$\beta\varphi = a^2 c^2 \Delta\chi. \tag{42}$$

Then, Equation (39) becomes

$$(\Delta - \frac{1}{r^2})\xi = 0, \quad \Delta\psi = 0,$$
$$D\Delta\chi = \frac{\beta^2(1 - 2a^2)}{a^2}\kappa. \tag{43}$$

6. Discussion and Conclusions

1. The dilatation theory has been interpreted as a theory for an elastic solid containing a large number of microvoids. The theoretical advantage of this theory is that it is able to account for the non-local interactions between voids solely by means of the first gradient of the void ratio. Moreover, the constitutive equations contain only five material constants, which can be calculated based on certain requirements of equivalence for two single solutions for microporous and dilatation bodies, respectively. In recent decades, this theory has been widely discussed and criticized. The main objection concerns the balance equation of equilibrated forces that appears obscure from a physical point of view (see R. de Boer's work [37]). Several interpretations of these equilibrated forces have been given and this theory is now accepted by the scientific community.
2. In the BPN solution, the displacement field $\mathbf{u}$ and the dilatation field φ are represented by means of two auxiliary functions $\mathbf{G}$ and H. The displacement $\mathbf{u}$ is expressed as a linear combination of the first derivative of $\mathbf{G}$ and H, while φ is a linear combination of the second derivative of $\mathbf{G}$ and H. The coefficients in both of these linear combinations depend on the spatial variable $\mathbf{x}$. If we put $\beta = 0$ into Equations (8) and (9), we have $\varphi = 0$ and

$$\mathbf{u} = \mathbf{G} - a^2\nabla(\mathbf{x} \cdot \mathbf{G} + H), \tag{44}$$

$$\Delta\mathbf{G} = -\frac{1}{\mu}\mathbf{b}, \quad \Delta H = \frac{1}{\mu}\mathbf{x} \cdot \mathbf{b}. \tag{45}$$

Equations (44) and (45) are the BPN solution in classical elasticity.

3. In a similar way, by equalizing β to 0 in the GL solution in Equations (17) and (18), the function φ disappears and

$$\mathbf{u} = \nabla\Psi + \mathrm{curl}\boldsymbol{\Phi}, \tag{46}$$

$$\mu\Delta\boldsymbol{\Phi} = \gamma, \quad c^2\Delta\Psi = \eta. \tag{47}$$

Equations (46) and (47) show the representation of the GL solution in classical elasticity.

4. In the theory under discussion, the CKS representation of the solution is expressed in terms of two auxiliary functions P and Q, in contrast with the classical elasticity solution, where the displacement $\mathbf{u}$ depends on one auxiliary function.
If we let $\beta = 0$ and $\zeta = 0$, Equations (20) and (21) reduce to

$$\mathbf{u} = c^2\Delta\mathbf{p} - 2a^2c^2\nabla\mathrm{div}\mathbf{p}, \tag{48}$$

$$\Delta\mathbf{p} = -\frac{\mathbf{b}}{\mu c^2}, \tag{49}$$

where we have set $\Delta\mathbf{P} = \mathbf{p}$. Equations (48) and (49) give the well-known CKS solution in classical elastostatics.

5. In Section 3, we have established a connection between the BPN and CKS solutions. By introducing appropriate notations, we have obtained analytical formulas that are more suitable for dealing with topics like fundamental solutions and steady vibrations.
6. Some theoretical aspects are still open and have not been addressed in this study. It might be useful to present a heuristic method of obtaining a general solution via the Boussinesq–Somigliana–Galerkin solution. It is also interesting to establish a constructive scheme for the study of general elastic solutions. Problems in transversally isotropic elasticity, planar problems, and axisymmetric problems can be investigated under the framework of this scheme. The results presented in the paper are useful for the determination of the stress and displacement fields in three fundamental three-dimensional axisymmetric elasticity problems: the Kelvin problem of a concentrated force in the interior of an infinite space, the Boussinesq problem of a concentrated force orthogonal to the boundary of a half-space, and the Mindlin problem of a concentrated force in the interior of a half-space.

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