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Some Properties on Normalized Tails of Maclaurin Power Series Expansion of Exponential Function

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Abstract: In the paper, (1) in view of a general formula for any derivative of the quotient of two differentiable functions, (2) with the aid of a monotonicity rule for the quotient of two power series, (3) in light of the logarithmic convexity of an elementary function involving the exponential function, (4) with the help of an integral representation for the tail of the power series expansion of the exponential function, and (5) on account of Čebyšev's integral inequality, the authors (i) expand the logarithm of the normalized tail of the power series expansion of the exponential function into a power series whose coefficients are expressed in terms of specific Hessenberg determinants whose elements are quotients of combinatorial numbers, (ii) prove the logarithmic convexity of the normalized tail of the power series expansion of the exponential function, (iii) derive a new determinantal expression of the Bernoulli numbers, deduce a determinantal expression for Howard's numbers, (iv) confirm the increasing monotonicity of a function related to the logarithm of the normalized tail of the power series expansion of the exponential function, (v) present an inequality among three power series whose coefficients are reciprocals of combinatorial numbers, and (vi) generalize the logarithmic convexity of an extensively applied function involving the exponential function.

Keywords: Maclaurin power series expansion; normalized tail; exponential function; increasing property; logarithmic convexity; derivative formula; determinantal expression; monotonicity rule; integral representation; combinatorial number

MSC: Primary 41A58; Secondary 05A10; 11B65; 11B68; 11B83; 11C20; 15A15; 26A09; 26A48; 26A51; 26D15; 33B10

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1. Motivations

It is well-known that

$$e^u = \sum_{j=0}^{\infty} \frac{u^j}{j!} = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + \frac{u^6}{6!} + \cdots, \quad |u| \in \mathbb{R} \tag{1}$$

and

$$\begin{aligned} \frac{u}{e^u - 1} &= \sum_{j=0}^{\infty} B_j \frac{u^j}{j!} = 1 - \frac{u}{2} + \sum_{j=1}^{\infty} B_{2j} \frac{u^{2j}}{(2j)!} \\ &= 1 - \frac{u}{2} + \frac{u^2}{12} - \frac{u^4}{720} + \frac{u^6}{30240} - \frac{u^8}{1209600} + \cdots, \quad |u| < 2\pi. \end{aligned} \tag{2}$$

The generating function $\frac{u}{e^u-1}$ of the classical Bernoulli numbers B_j for $j \in \mathbb{N}_0$, its generalized expression $\frac{u}{\beta^u-\alpha^u}$ for $\beta > \alpha > 0$, and their reciprocals have been being systematically investigated and extensively applied by Qi and his coauthors from the late 1990s to the present. The first two papers about this topic are [1,2], while the first author of these two papers was a PhD student at the University of Science and Technology of China. The latest papers are [3,4].

In this paper, we start out from the logarithm of the reciprocal of the generating function $\frac{u}{\mathrm{e}^u-1}$ of the Bernoulli numbers B_j for $j \in \mathbb{N}_0$.

1.1. First Series Expansion

For $u \in \mathbb{R}$, let

$$F_1(u) = \begin{cases} \ln \dfrac{\mathrm{e}^u-1}{u}, & u \neq 0; \\ 0, & u = 0. \end{cases}$$

From ([5] Theorem 2.1) and Article 5 at the site http://rgmia.org/v11n1.php (accessed on 6 July 2024), we deduce that the function $F_1(u)$ is convex on $(-\infty,\infty)$ (see also Lemma 3 below), is 3-convex (that is, $F_1'''(u) \ge 0$) on $(-\infty,0)$, and is 3-concave (that is, $F_1'''(u) \le 0$) on $(0,\infty)$.

A simple differentiation yields

$$F_1'(u) = 1 - \frac{1}{u} + \frac{1}{\mathrm{e}^u-1} = \frac{1}{2} + \sum_{j=1}^{\infty} B_{2j}\frac{u^{2j-1}}{(2j)!}, \quad |u| < 2\pi,$$

where we used the Maclaurin power series expansion (2). Integrating on both sides yields

$$\begin{aligned} F_1(u) &= \frac{u}{2} + \sum_{j=1}^{\infty} \frac{B_{2j}}{2j}\frac{u^{2j}}{(2j)!} \\ &= \frac{u}{2} + \frac{u^2}{24} - \frac{u^4}{2880} + \frac{u^6}{181440} - \frac{u^8}{9676800} + \cdots, \quad |u| < 2\pi. \end{aligned} \tag{3}$$

The first Maclaurin power series expansion is achieved.

1.2. Second Series Expansion

Let

$$F_2(u) = \begin{cases} \ln \dfrac{2(\mathrm{e}^u-1-u)}{u^2}, & u \neq 0; \\ 0, & u = 0. \end{cases}$$

The reciprocal of the exponent of the function $F_2(u)$, that is, the function $\frac{u^2}{2}\frac{1}{\mathrm{e}^u-1-u}$, is a generating function of the Howard numbers A_j for $j \in \mathbb{N}_0$; see the paper ([6] p. 979, Equation (2.9)). In other words,

$$\begin{aligned} \frac{u^2}{2}\frac{1}{\mathrm{e}^u-1-u} &= \sum_{j=0}^{\infty} A_j \frac{u^j}{j!} \\ &= 1 - \frac{u}{3} + \frac{u^2}{36} + \frac{u^3}{540} - \frac{u^4}{6480} - \frac{u^5}{27216} - \frac{u^6}{4082400} + \cdots, \quad |u| < |u_0|, \end{aligned} \tag{4}$$

where $u_0 \neq 0$ is the zero, closest to the origin $u=0$, of the equation $\mathrm{e}^u-1-u=0$ on the complex plane $\mathbb{C}$. In ([7] Theorem 2.1), a closed-form expression for A_j was provided by

$$\begin{aligned} A_j = \frac{1}{(2j)!!} \sum_{k=0}^{j} \frac{\binom{j}{k}}{\binom{j+k}{k}} \sum_{n=0}^{j-k} n! \langle -2k \rangle_{j-k-n} \binom{j-k}{n}\binom{j+k}{n} \\ \times \sum_{m=0}^{k} (-1)^m \binom{k}{m} \sum_{p=0}^{j+k} 2^p \langle k-m \rangle_{j+k-p} \binom{j+k}{p} \frac{S(m+p,m)}{\binom{m+p}{m}}, \end{aligned} \tag{5}$$

where the shifting or falling factorial $\langle\rho\rangle_j$ is defined by

$$\langle\rho\rangle_j = \prod_{k=0}^{j-1}(\rho-k) = \begin{cases}\rho(\rho-1)\cdots(\rho-j+1), & j\in\mathbb{N}\\ 1, & j=0\end{cases}$$

for $\rho\in\mathbb{C}$ and the second kind of Stirling numbers $S(j,k)$ for $j\ge k\in\mathbb{N}_0$ can be analytically generated by

$$\left(\frac{\mathrm{e}^t-1}{t}\right)^\ell = \sum_{j=0}^{\infty}\frac{S(j+\ell,\ell)}{\binom{j+\ell}{\ell}}\frac{t^j}{j!}, \quad \ell\in\mathbb{N}_0.$$

It is clear that the closed-form formula (5) is not simple and beautiful. In Remark 2 below, we will derive a beautiful, symmetric, and determinantal expression for the Howard numbers A_j.

Direct differentiating results in

$$F_2'(u) = 1-\frac{2}{u}+\frac{u}{\mathrm{e}^u-1-u} = \frac{1}{3}+2\sum_{j=1}^{\infty}\frac{A_{j+1}}{(j+1)!}u^j.$$

Accordingly, we arrive at

$$\begin{aligned}F_2(u) &= \frac{u}{3}+2\sum_{j=1}^{\infty}\frac{A_{j+1}}{j+1}\frac{u^{j+1}}{(j+1)!}\\ &= \frac{u}{3}+\frac{u^2}{36}+\frac{u^3}{810}-\frac{u^4}{12960}-\frac{u^5}{68040}-\frac{u^6}{12247200}+\frac{u^7}{6123600}+\cdots.\end{aligned} \tag{6}$$

The second Maclaurin power series expansion is attained.

1.3. Motivations and Problems

It is known that, for $n\in\mathbb{N}$ and $u\in\mathbb{R}$, the quantity

$$R_n(u) = \mathrm{e}^u-\sum_{k=0}^{n-1}\frac{u^k}{k!}$$

is called the nth tail of the Maclaurin power series expansion (1). In what follows, we consider the function

$$f_n(u) = \begin{cases}\dfrac{n!}{u^n}\left(\mathrm{e}^u-\displaystyle\sum_{j=0}^{n-1}\frac{u^j}{j!}\right), & u\ne0\\ 1, & u=0\end{cases} \tag{7}$$

for $n\in\mathbb{N}$. We call this quantity the nth normalized tail of the Maclaurin power series expansion (1).

Motivated by the new Maclaurin power series expansions (3) and (6), we now propose the following problems.

1. What is the Maclaurin power series expansion of the logarithm of the nth normalized tail

$$F_n(u) = \begin{cases}\ln\left[\dfrac{n!}{u^n}\left(\mathrm{e}^u-\displaystyle\sum_{k=0}^{n-1}\frac{u^k}{k!}\right)\right], & u\ne0\\ 0, & u=0\end{cases} \tag{8}$$

   around $u=0$ for $n\in\mathbb{N}$? What about the monotonicity and convexity of $F_n(u)$ on $(-\infty,\infty)$?

2. For $n \in \mathbb{N}$, is the function

$$R_{n,0}(u) = \begin{cases} \dfrac{F_n(u)}{u}, & u \neq 0 \\ \dfrac{1}{n+1}, & u = 0 \end{cases} \tag{9}$$

increasing on $(-\infty, \infty)$?

3. For $n > m \in \mathbb{N}$, does the function

$$R_{n,m}(u) = \begin{cases} \dfrac{F_n(u)}{F_m(u)}, & u \neq 0 \\ \dfrac{m+1}{n+1}, & u = 0 \end{cases}$$

have a unique minimum on $(-\infty, \infty)$?

In this paper, we will provide solutions regarding the first two problems, but we leave the third problem as an open problem.

2. Preliminaries

For solving the first two problems mentioned above, we now prepare the following six lemmas.

Lemma 1 ([8])**.** *For a real variable $z \in I \subseteq \mathbb{R}$ and a fixed integer $j \in \mathbb{N}_0$, let $\phi(z)$ and $\varphi(z) \neq 0$ be two j-time differentiable functions, where I denotes an interval on $\mathbb{R}$. Then, the jth derivative of the quotient $\frac{\phi(z)}{\varphi(z)}$ is*

$$\frac{\mathrm{d}^j}{\mathrm{d}z^j}\left[\frac{\phi(z)}{\varphi(z)}\right] = (-1)^j \frac{|W_{(j+1)\times(j+1)}(z)|}{\varphi^{j+1}(z)}, \quad j \in \mathbb{N}_0, \tag{10}$$

where the $(j+1)\times(j+1)$ order matrix $W_{(j+1)\times(j+1)}(z)$ is defined by

$$W_{(j+1)\times(j+1)}(z) = \begin{pmatrix} U_{(j+1)\times 1}(z) & V_{(j+1)\times j}(z) \end{pmatrix}_{(j+1)\times(j+1)},$$

the $(j+1)\times 1$ order matrix $U_{(j+1)\times 1}(z)$ is of elements $u_{k,1}(z) = \phi^{(k-1)}(z)$ for $1 \le k \le j+1$, the $(j+1)\times j$ order matrix $V_{(j+1)\times j}(z)$ is of elements $v_{\ell,m}(z) = \binom{\ell-1}{m-1}\varphi^{(\ell-m)}(z)$ for $1 \le \ell \le j+1$ and $1 \le m \le j$, and the quantity $|W_{(j+1)\times(j+1)}(z)|$ is the determinant of the $(j+1)\times(j+1)$ order matrix $W_{(j+1)\times(j+1)}(z)$.

The Formula (10) is a higher-order derivative formula for the ratio of two differentiable functions in terms of the determinant of a specific Hessenberg matrix. Sergei M. Sitnik (Voronezh Institute of the Ministry of Internal Affairs of Russia) provided the Formula (10) and related references to Qi via e-mails on 25 September 2014 and thereafter. Qi first applied the Formula (10) in the paper [9]. Hereafter, Qi and his coauthors have been employing the Formula (10) for extensively studying many mathematical problems. The latest two papers applying the Formula (10) by Qi are [3,10].

Lemma 2 ([11])**.** *Let $A_j, B_j \in \mathbb{R}$ for $j \in \mathbb{N} \cup \{0\}$ be two real sequences and let the Maclaurin power series*

$$U(z) = \sum_{j=0}^{\infty} A_j z^j \quad \text{and} \quad V(z) = \sum_{j=0}^{\infty} B_j z^j$$

be convergent on $(-R, R)$ for some positive number $R > 0$. If $B_j > 0$ and the quotient $\frac{A_j}{B_j}$ is increasing for $j \in \mathbb{N}_0$, then the quotient $\frac{U(z)}{V(z)}$ is also increasing on $(0, R)$.

Lemma 2 is called the monotonicity rule for the quotient of two Maclaurin power series. There exists a nice article [12] for reviewing, surveying, retrospecting, explaining, correcting, and generalizing several monotonicity rules.

Lemma 3 ([5] Theorem 2.1)**.** *For two fixed numbers α and β such that $\beta > \alpha > 0$, define the function*

$$h_{\alpha,\beta}(u) = \begin{cases} \dfrac{\beta^u - \alpha^u}{u}, & u \neq 0 \\ \ln \dfrac{\beta}{\alpha}, & u = 0 \end{cases}$$

for $u \in \mathbb{R}$. Then, the function $h_{\alpha,\beta}(u)$ is logarithmically convex on $(-\infty, \infty)$.

Lemma 4 ([13] p. 502)**.** *For $u \in \mathbb{R}$ and $m \in \{0\} \cup \mathbb{N}$, we have*

$$R_{m+1}(u) = \mathrm{e}^u - \sum_{j=0}^{m} \frac{u^j}{j!} = \frac{u^{m+1}}{(m+1)!} \left[1 + u \int_0^1 v^{m+1} \mathrm{e}^{u(1-v)} \, \mathrm{d}v\right].$$

Lemma 5 (Čebyšev's integral inequality [14] p. 239, Chapter IX)**.** *Let $f, h : [\alpha, \beta] \to (-\infty, \infty)$ be two integrable functions, either both increasing or both decreasing. Moreover, let $q : [\alpha, \beta] \to [0, \infty)$ be a non-negative and integral function. Then,*

$$\int_\alpha^\beta q(v) \, \mathrm{d}v \int_\alpha^\beta q(v) f(v) h(v) \, \mathrm{d}v \geq \int_\alpha^\beta q(v) f(v) \, \mathrm{d}v \int_\alpha^\beta q(v) h(v) \, \mathrm{d}v. \tag{11}$$

If one of the functions f and h is non-increasing and the other non-decreasing, then the inequality in (11) is reversed. The equality in (11) validates if and only if one of the functions f and h reduces to a scalar.

Lemma 6 ([15,16])**.** *Let the functions $U(y)$, $V(y) > 0$, and $W(y,t) > 0$ be integrable in $y \in (\alpha, \beta)$.*

1. *If the quotients $\frac{\partial W(y,t)/\partial t}{W(y,t)}$ and $\frac{U(y)}{V(y)}$ are both increasing or both decreasing in $y \in (\alpha, \beta)$, then the quotient*
$$R(t) = \frac{\int_\alpha^\beta W(y,t) U(y) \, \mathrm{d}y}{\int_\alpha^\beta W(y,t) V(y) \, \mathrm{d}y}$$
is increasing in t.
2. *If one of the quotients $\frac{\partial W(y,t)/\partial t}{W(y,t)}$ or $\frac{U(y)}{V(y)}$ is increasing and another one of them is decreasing in $y \in (\alpha, \beta)$, then the quotient $R(t)$ is decreasing in t.*

Lemma 6 is a new monotonicity rule, not included in the nice article [12], which was established and applied in recent years, and has been generalized in the paper [17].

There have been several independent developments of the monotonicity rules for the ratios between two differentiable functions, two Maclaurin power series, two Laplace transforms, two integrals, and the like. For more details, please refer to the newly published papers [18–24], ([25] Lemma 4), the arXiv preprints [26,27], and closely related references therein.

In July 2023, a Chinese mathematician Zhen-Hang Yang drafted a review and survey work about the monotonicity rules for many various ratios and reported it at Guangdong University of Education.

3. A New General Maclaurin Power Series Expansion

After preparing necessary knowledge, in what follows in this section, we will establish a new general Maclaurin power series expansion of the logarithmic expression $F_n(t)$ defined by (8).

Theorem 1. *For $n \in \mathbb{N}$, the function $F_n(t)$ defined in (8) can be expanded into the Maclaurin power series*

$$F_n(t) = \sum_{\ell=1}^{\infty} (-1)^{\ell-1} D_\ell(n) \frac{t^\ell}{\ell!}, \tag{12}$$

where the determinant $D_\ell(n)$ is defined by

$$\begin{aligned} D_{\ell+1}(n) &= \begin{vmatrix} \frac{1}{\binom{n+1}{1}} & \frac{\binom{0}{0}}{\binom{n}{0}} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{\binom{n+2}{2}} & \frac{\binom{1}{0}}{\binom{n+1}{1}} & \frac{\binom{1}{1}}{\binom{n}{0}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{\binom{n+3}{3}} & \frac{\binom{2}{0}}{\binom{n+2}{2}} & \frac{\binom{2}{1}}{\binom{n+1}{1}} & \frac{\binom{2}{2}}{\binom{n}{0}} & 0 & \cdots & 0 & 0 \\ \frac{1}{\binom{n+4}{4}} & \frac{\binom{3}{0}}{\binom{n+3}{3}} & \frac{\binom{3}{1}}{\binom{n+2}{2}} & \frac{\binom{3}{2}}{\binom{n+1}{1}} & \frac{\binom{3}{3}}{\binom{n}{0}} & \cdots & 0 & 0 \\ \frac{1}{\binom{n+5}{5}} & \frac{\binom{4}{0}}{\binom{n+4}{4}} & \frac{\binom{4}{1}}{\binom{n+3}{3}} & \frac{\binom{4}{2}}{\binom{n+2}{2}} & \frac{\binom{4}{3}}{\binom{n+1}{1}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\binom{n+\ell-1}{\ell-1}} & \frac{\binom{\ell-2}{0}}{\binom{n+\ell-2}{\ell-2}} & \frac{\binom{\ell-2}{1}}{\binom{n+\ell-3}{\ell-3}} & \frac{\binom{\ell-2}{2}}{\binom{n+\ell-4}{\ell-4}} & \frac{\binom{\ell-2}{3}}{\binom{n+\ell-5}{\ell-5}} & \cdots & \frac{\binom{\ell-2}{\ell-2}}{\binom{n}{0}} & 0 \\ \frac{1}{\binom{n+\ell}{\ell}} & \frac{\binom{\ell-1}{0}}{\binom{n+\ell-1}{\ell-1}} & \frac{\binom{\ell-1}{1}}{\binom{n+\ell-2}{\ell-2}} & \frac{\binom{\ell-1}{2}}{\binom{n+\ell-3}{\ell-3}} & \frac{\binom{\ell-1}{3}}{\binom{n+\ell-4}{\ell-4}} & \cdots & \frac{\binom{\ell-1}{\ell-2}}{\binom{n+1}{1}} & \frac{\binom{\ell-1}{\ell-1}}{\binom{n}{0}} \\ \frac{1}{\binom{n+\ell+1}{\ell+1}} & \frac{\binom{\ell}{0}}{\binom{n+\ell}{\ell}} & \frac{\binom{\ell}{1}}{\binom{n+\ell-1}{\ell-1}} & \frac{\binom{\ell}{2}}{\binom{n+\ell-2}{\ell-2}} & \frac{\binom{\ell}{3}}{\binom{n+\ell-3}{\ell-3}} & \cdots & \frac{\binom{\ell}{\ell-2}}{\binom{n+2}{2}} & \frac{\binom{\ell}{\ell-1}}{\binom{n+1}{1}} \end{vmatrix} \\ &= \begin{vmatrix} A_{\ell+1,1}(n) & B_{\ell+1,\ell}(n) \end{vmatrix}_{(\ell+1)\times(\ell+1)} \end{aligned}$$

for $n \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$, in which the matrices $A_{\ell+1,1}(n)$ and $B_{\ell+1,\ell}(n)$ are defined by

$$A_{\ell+1,1}(n) = \left(\alpha_{i,j}(n)\right)_{1\le i\le \ell+1, j=1}, \quad \alpha_{i,1}(n) = \frac{1}{\binom{n+i}{i}}$$

and

$$B_{\ell+1,\ell}(n) = \left(\beta_{i,j}(n)\right)_{1\le i\le \ell+1, 1\le j\le \ell}, \quad \beta_{i,j} = \begin{cases} 0, & \ell \ge j > i \in \mathbb{N}; \\ \frac{\binom{i-1}{j-1}}{\binom{n+i-j}{i-j}}, & 1 \le j \le i \le \ell+1. \end{cases}$$

First proof. A direct differentiation provides

$$\begin{aligned} F_n'(t) &= \frac{\frac{1}{t}\left[(t-n)\,\mathrm{e}^t + n + \sum_{r=1}^{n-1}(n-r)\frac{t^r}{r!}\right]}{\mathrm{e}^t - \sum_{r=0}^{n-1}\frac{t^r}{r!}} \\ &= \frac{\frac{1}{t^{n+1}}\left[(t-n)\,\mathrm{e}^t + n + \sum_{r=1}^{n-1}(n-r)\frac{t^r}{r!}\right]}{\frac{1}{t^n}\left(\mathrm{e}^t - \sum_{r=0}^{n-1}\frac{t^r}{r!}\right)} \\ &= \frac{\sum_{r=0}^{\infty}\frac{(r+1)!}{(n+r+1)!}\frac{t^r}{r!}}{\sum_{r=0}^{\infty}\frac{r!}{(n+r)!}\frac{t^r}{r!}} \\ &= \frac{\sum_{r=0}^{\infty} C_{n,r+1}\frac{t^r}{r!}}{\sum_{r=0}^{\infty} C_{n,r}\frac{t^r}{r!}} \\ &= \frac{\phi_n(t)}{\varphi_n(t)}, \end{aligned}$$

where we denote

$$\phi_n(t) = \sum_{r=0}^{\infty} \frac{(r+1)!}{(n+r+1)!} \frac{t^r}{r!} = \sum_{r=0}^{\infty} C_{n,r+1} \frac{t^r}{r!},$$
$$\varphi_n(t) = \sum_{r=0}^{\infty} \frac{r!}{(n+r)!} \frac{t^r}{r!} = \sum_{r=0}^{\infty} C_{n,r} \frac{t^r}{r!},$$

and

$$C_{n,r} = \frac{r!}{(n+r)!}, \quad r \in \mathbb{N}_0, \quad n \in \mathbb{N}. \tag{13}$$

Therefore, it is easy to see that

$$F_n'(0) = \frac{1}{n+1}, \quad \phi_n^{(r)}(0) = C_{n,r+1}, \quad \varphi_n^{(r)}(0) = C_{n,r}$$

for $n \in \mathbb{N}$ and $r \in \mathbb{N}_0$.

Employing the derivative formula (10) and simplifying leads to

$$\begin{aligned}
F_n^{(\ell+1)}(0) &= \lim_{t\to 0}\left[\frac{\phi_n(t)}{\varphi_n(t)}\right]^{(\ell)} \\
&= \frac{(-1)^\ell}{\varphi_n^{\ell+1}(0)} \begin{vmatrix}
\phi_n(0) & \varphi_n(0) & 0 & 0 & \cdots & 0 \\
\phi_n'(0) & \varphi_n'(0) & \binom{1}{1}\varphi_n(0) & 0 & \cdots & 0 \\
\phi_n''(0) & \varphi_n''(0) & \binom{2}{1}\varphi_n'(0) & \binom{2}{2}\varphi_n(0) & \cdots & 0 \\
\phi_n^{(3)}(0) & \varphi_n^{(3)}(0) & \binom{3}{1}\varphi_n''(0) & \binom{3}{2}\varphi_n'(0) & \cdots & 0 \\
\phi_n^{(4)}(0) & \varphi_n^{(4)}(0) & \binom{4}{1}\varphi_n^{(3)}(0) & \binom{4}{2}\varphi_n''(0) & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\phi_n^{(\ell)}(0) & \varphi_n^{(\ell)}(0) & \binom{\ell}{1}\varphi_n^{(\ell-1)}(0) & \binom{\ell}{2}\varphi_n^{(\ell-2)}(0) & \cdots & \binom{\ell}{\ell-1}\varphi_n'(0)
\end{vmatrix} \\
&= \frac{(-1)^\ell}{C_{n,0}^{\ell+1}} \begin{vmatrix}
C_{n,1} & C_{n,0} & 0 & 0 & \cdots & 0 \\
C_{n,2} & C_{n,1} & \binom{1}{1}C_{n,0} & 0 & \cdots & 0 \\
C_{n,3} & C_{n,2} & \binom{2}{1}C_{n,1} & \binom{2}{2}C_{n,0} & \cdots & 0 \\
C_{n,4} & C_{n,3} & \binom{3}{1}C_{n,2} & \binom{3}{2}C_{n,1} & \cdots & 0 \\
C_{n,5} & C_{n,4} & \binom{4}{1}C_{n,3} & \binom{4}{2}C_{n,2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
C_{n,\ell+1} & C_{n,\ell} & \binom{\ell}{1}C_{n,\ell-1} & \binom{\ell}{2}C_{n,\ell-2} & \cdots & \binom{\ell}{\ell-1}C_{n,1}
\end{vmatrix} \\
&= (-1)^\ell \begin{vmatrix}
\frac{C_{n,1}}{C_{n,0}} & \frac{C_{n,0}}{C_{n,0}} & 0 & 0 & 0 & \cdots & 0 \\
\frac{C_{n,2}}{C_{n,0}} & \frac{C_{n,1}}{C_{n,0}} & \binom{1}{1}\frac{C_{n,0}}{C_{n,0}} & 0 & 0 & \cdots & 0 \\
\frac{C_{n,3}}{C_{n,0}} & \frac{C_{n,2}}{C_{n,0}} & \binom{2}{1}\frac{C_{n,1}}{C_{n,0}} & \binom{2}{2}\frac{C_{n,0}}{C_{n,0}} & 0 & \cdots & 0 \\
\frac{C_{n,4}}{C_{n,0}} & \frac{C_{n,3}}{C_{n,0}} & \binom{3}{1}\frac{C_{n,2}}{C_{n,0}} & \binom{3}{2}\frac{C_{n,1}}{C_{n,0}} & \binom{3}{3}\frac{C_{n,0}}{C_{n,0}} & \cdots & 0 \\
\frac{C_{n,5}}{C_{n,0}} & \frac{C_{n,4}}{C_{n,0}} & \binom{4}{1}\frac{C_{n,3}}{C_{n,0}} & \binom{4}{2}\frac{C_{n,2}}{C_{n,0}} & \binom{4}{3}\frac{C_{n,1}}{C_{n,0}} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{C_{n,\ell+1}}{C_{n,0}} & \frac{C_{n,\ell}}{C_{n,0}} & \binom{\ell}{1}\frac{C_{n,\ell-1}}{C_{n,0}} & \binom{\ell}{2}\frac{C_{n,\ell-2}}{C_{n,0}} & \binom{\ell}{3}\frac{C_{n,\ell-3}}{C_{n,0}} & \cdots & \binom{\ell}{\ell-1}\frac{C_{n,1}}{C_{n,0}}
\end{vmatrix}
\end{aligned}$$

$$
= (-1)^{\ell}\begin{vmatrix}
\frac{1}{\binom{n+1}{1}} & \frac{\binom{0}{0}}{\binom{n}{0}} & 0 & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{\binom{n+2}{2}} & \frac{\binom{1}{0}}{\binom{n+1}{1}} & \frac{\binom{1}{1}}{\binom{n}{0}} & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{\binom{n+3}{3}} & \frac{\binom{2}{0}}{\binom{n+2}{2}} & \frac{\binom{2}{1}}{\binom{n+1}{1}} & \frac{\binom{2}{2}}{\binom{n}{0}} & 0 & \cdots & 0 & 0 \\
\frac{1}{\binom{n+4}{4}} & \frac{\binom{3}{0}}{\binom{n+3}{3}} & \frac{\binom{3}{1}}{\binom{n+2}{2}} & \frac{\binom{3}{2}}{\binom{n+1}{1}} & \frac{\binom{3}{3}}{\binom{n}{0}} & \cdots & 0 & 0 \\
\frac{1}{\binom{n+5}{5}} & \frac{\binom{4}{0}}{\binom{n+4}{4}} & \frac{\binom{4}{1}}{\binom{n+3}{3}} & \frac{\binom{4}{2}}{\binom{n+2}{2}} & \frac{\binom{4}{3}}{\binom{n+1}{1}} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{\binom{n+\ell-1}{\ell-1}} & \frac{\binom{\ell-2}{0}}{\binom{n+\ell-2}{\ell-2}} & \frac{\binom{\ell-2}{1}}{\binom{n+\ell-3}{\ell-3}} & \frac{\binom{\ell-2}{2}}{\binom{n+\ell-4}{\ell-4}} & \frac{\binom{\ell-2}{3}}{\binom{n+\ell-5}{\ell-5}} & \cdots & \frac{\binom{\ell-2}{\ell-2}}{\binom{n}{0}} & 0 \\
\frac{1}{\binom{n+\ell}{\ell}} & \frac{\binom{\ell-1}{0}}{\binom{n+\ell-1}{\ell-1}} & \frac{\binom{\ell-1}{1}}{\binom{n+\ell-2}{\ell-2}} & \frac{\binom{\ell-1}{2}}{\binom{n+\ell-3}{\ell-3}} & \frac{\binom{\ell-1}{3}}{\binom{n+\ell-4}{\ell-4}} & \cdots & \frac{\binom{\ell-1}{\ell-2}}{\binom{n+1}{1}} & \frac{\binom{\ell-1}{\ell-1}}{\binom{n}{0}} \\
\frac{1}{\binom{n+\ell+1}{\ell+1}} & \frac{\binom{\ell}{0}}{\binom{n+\ell}{\ell}} & \frac{\binom{\ell}{1}}{\binom{n+\ell-1}{\ell-1}} & \frac{\binom{\ell}{2}}{\binom{n+\ell-2}{\ell-2}} & \frac{\binom{\ell}{3}}{\binom{n+\ell-3}{\ell-3}} & \cdots & \frac{\binom{\ell}{\ell-2}}{\binom{n+2}{2}} & \frac{\binom{\ell}{\ell-1}}{\binom{n+1}{1}}
\end{vmatrix}
$$
$$
= (-1)^{\ell} D_{\ell+1}(n)
$$

for $\ell \in \mathbb{N}_0$. Consequently, from the fact that $F_n(0) = 0$, we arrive at

$$
F_n(t) = \sum_{\ell=1}^{\infty} F_n^{(\ell)}(0)\frac{t^{\ell}}{\ell!} = \sum_{\ell=1}^{\infty} (-1)^{\ell-1} D_{\ell}(n)\frac{t^{\ell}}{\ell!}.
$$

The first proof of Theorem 1 is thus complete. □

Second proof. The function $f_n(u)$ defined in (7) can be formulated as

$$
f_n(u) = n!\sum_{j=0}^{\infty} \frac{u^j}{(j+n)!}, \quad n \in \mathbb{N}.
$$

Then

$$
F_n'(u) = [\ln f_n(u)]' = \frac{\sum_{j=1}^{\infty} \frac{j u^{j-1}}{(j+n)!}}{\sum_{j=0}^{\infty} \frac{u^j}{(j+n)!}} = \frac{\sum_{j=0}^{\infty} \frac{(j+1)u^j}{(j+n+1)!}}{\sum_{j=0}^{\infty} \frac{u^j}{(j+n)!}}. \tag{14}
$$

Let

$$
\phi_n(u) = \sum_{j=0}^{\infty} \frac{(j+1)u^j}{(j+n+1)!} \quad \text{and} \quad \varphi_n(u) = \sum_{j=0}^{\infty} \frac{u^j}{(j+n)!}.
$$

Then, for $m \in \mathbb{N}_0$,

$$
\phi_n^{(m)}(0) = \frac{(m+1)!}{(n+m+1)!} \quad \text{and} \quad \varphi_n^{(m)}(0) = \frac{m!}{(n+m)!}.
$$

Accordingly, utilizing the derivative formula (10), we obtain

$$
F_n^{(\ell+1)}(0) = \lim_{u\to 0}\left[\frac{\phi_n(u)}{\varphi_n(u)}\right]^{(\ell)}
$$
$$
= \frac{(-1)^{\ell}}{\varphi_n^{\ell+1}(0)}\begin{vmatrix}
\phi_n(0) & \varphi_n(0) & 0 & 0 & \cdots & 0 & 0 \\
\phi_n'(0) & \varphi_n'(0) & \binom{1}{1}\varphi_n(0) & 0 & \cdots & 0 & 0 \\
\phi_n''(0) & \varphi_n''(0) & \binom{2}{1}\varphi_n'(0) & \binom{2}{2}\varphi_n(0) & \cdots & 0 & 0 \\
\phi_n^{(3)}(0) & \varphi_n^{(3)}(0) & \binom{3}{1}\varphi_n''(0) & \binom{3}{2}\varphi_n'(0) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_n^{(\ell-1)}(0) & \varphi_n^{(\ell-1)}(0) & \binom{\ell-1}{1}\varphi_n^{(\ell-2)}(0) & \binom{\ell-1}{2}\varphi_n^{(\ell-3)}(0) & \cdots & \binom{\ell-1}{\ell-2}\varphi_n'(0) & \binom{\ell-1}{\ell-1}\varphi_n(0) \\
\phi_n^{(\ell)}(0) & \varphi_n^{(\ell)}(0) & \binom{\ell}{1}\varphi_n^{\ell-1}(0) & \binom{\ell}{2}\varphi_n^{(\ell-2)}(0) & \cdots & \binom{\ell}{\ell-2}\varphi_n''(0) & \binom{\ell}{\ell-1}\varphi_n'(0)
\end{vmatrix}
$$

$$
\begin{aligned}
&= (-1)^{\ell}(n!)^{\ell+1}\begin{vmatrix}
\frac{1!}{(n+1)!} & \frac{0!}{n!} & 0 & 0 & \cdots & 0 & 0\\
\frac{2!}{(n+2)!} & \frac{1!}{(n+1)!} & \binom{1}{1}\frac{0!}{n!} & 0 & \cdots & 0 & 0\\
\frac{3!}{(n+3)!} & \frac{2!}{(n+2)!} & \binom{2}{1}\frac{1!}{(n+1)!} & \binom{2}{2}\frac{0!}{n!} & \cdots & 0 & 0\\
\frac{4!}{(n+4)!} & \frac{3!}{(n+3)!} & \binom{3}{1}\frac{2!}{(n+2)!} & \binom{3}{2}\frac{1!}{(n+1)!} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{\ell!}{(n+\ell)!} & \frac{(\ell-1)!}{(n+\ell-1)!} & \binom{\ell-1}{1}\frac{(\ell-2)!}{(n+\ell-2)!} & \binom{\ell-1}{2}\frac{(\ell-3)!}{n+\ell-3} & \cdots & \binom{\ell-1}{\ell-2}\frac{1!}{(n+1)!} & \binom{\ell-1}{\ell-1}\frac{0!}{n!}\\
\frac{(\ell+1)!}{(n+\ell+1)!} & \frac{\ell!}{(n+\ell)!} & \binom{\ell}{1}\frac{(\ell-1)!}{(n+\ell-1)!} & \binom{\ell}{2}\frac{(\ell-2)!}{(n+\ell-2)!} & \cdots & \binom{\ell}{\ell-2}\frac{2!}{(n+2)!} & \binom{\ell}{\ell-1}\frac{1!}{(n+1)!}
\end{vmatrix}\\
&= (-1)^{\ell}D_{\ell+1}(n)
\end{aligned}
$$

for $\ell \in \mathbb{N}_0$. The second proof of Theorem 1 is complete. □

Remark 1. *If taking $n=1$ in Theorem 12, we derive*

$$
D_1(1) = \left|\frac{1}{\binom{2}{1}}\right| = \frac{1}{2}, \qquad D_2(1) = \begin{vmatrix} \frac{1}{\binom{2}{1}} & \frac{\binom{0}{0}}{\binom{1}{0}} \\ \frac{1}{\binom{3}{2}} & \frac{\binom{1}{0}}{\binom{2}{1}} \end{vmatrix} = -\frac{1}{12},
$$

$$
D_3(1) = \begin{vmatrix} \frac{1}{\binom{2}{1}} & \frac{\binom{0}{0}}{\binom{1}{0}} & 0 \\ \frac{1}{\binom{3}{2}} & \frac{\binom{1}{0}}{\binom{2}{1}} & \frac{\binom{1}{1}}{\binom{1}{0}} \\ \frac{1}{\binom{4}{3}} & \frac{\binom{2}{0}}{\binom{3}{2}} & \frac{\binom{2}{1}}{\binom{2}{1}} \end{vmatrix} = 0, \qquad D_4(1) = \begin{vmatrix} \frac{1}{\binom{2}{1}} & \frac{\binom{0}{0}}{\binom{1}{0}} & 0 & 0 \\ \frac{1}{\binom{3}{2}} & \frac{\binom{1}{0}}{\binom{2}{1}} & \frac{\binom{1}{1}}{\binom{1}{0}} & 0 \\ \frac{1}{\binom{4}{3}} & \frac{\binom{2}{0}}{\binom{3}{2}} & \frac{\binom{2}{1}}{\binom{2}{1}} & \frac{\binom{2}{2}}{\binom{1}{0}} \\ \frac{1}{\binom{5}{4}} & \frac{\binom{3}{0}}{\binom{4}{3}} & \frac{\binom{3}{1}}{\binom{3}{2}} & \frac{\binom{3}{2}}{\binom{2}{1}} \end{vmatrix} = \frac{1}{120},
$$

and

$$
D_5(1) = \begin{vmatrix} \frac{1}{\binom{2}{1}} & \frac{\binom{0}{0}}{\binom{1}{0}} & 0 & 0 & 0 \\ \frac{1}{\binom{3}{2}} & \frac{\binom{1}{0}}{\binom{2}{1}} & \frac{\binom{1}{1}}{\binom{1}{0}} & 0 & 0 \\ \frac{1}{\binom{4}{3}} & \frac{\binom{2}{0}}{\binom{3}{2}} & \frac{\binom{2}{1}}{\binom{2}{1}} & \frac{\binom{2}{2}}{\binom{1}{0}} & 0 \\ \frac{1}{\binom{5}{4}} & \frac{\binom{3}{0}}{\binom{4}{3}} & \frac{\binom{3}{1}}{\binom{3}{2}} & \frac{\binom{3}{2}}{\binom{2}{1}} & \frac{\binom{3}{3}}{\binom{1}{0}} \\ \frac{1}{\binom{6}{5}} & \frac{\binom{4}{0}}{\binom{5}{4}} & \frac{\binom{4}{1}}{\binom{4}{3}} & \frac{\binom{4}{2}}{\binom{3}{2}} & \frac{\binom{4}{3}}{\binom{2}{1}} \end{vmatrix} = 0.
$$

Then, we have

$$
\begin{aligned}
F_1(t) &= D_1(1)t - D_2(1)\frac{t^2}{2} + D_3(1)\frac{t^3}{6} - D_4(1)\frac{t^4}{24} + D_5(1)\frac{t^5}{120} + \cdots \\
&= \frac{t}{2} + \frac{t^2}{24} - \frac{t^4}{2880} + \cdots
\end{aligned}
$$

which coincides with the first three terms of the Maclaurin power series expansion (3).

Comparing the Maclaurin power series expansion (3) with the Maclaurin power series expansion (12) for $n=1$ reveals two equalities

$$
D_{2k+1}(1) = 0 \quad \text{and} \quad B_{2k} = -2kD_{2k}(1)
$$

for $k \in \mathbb{N}$. The last equality presents a new determinantal expression of the Bernoulli number B_{2k}, or the last equality provides a computation of the determinant D_k for $k \in \mathbb{N}_0$.

Regarding the Bernoulli numbers B_{2k}, Qi and his coauthors have investigated many years and obtained a number of significant results such as explicit and closed-form expressions, recursive relations, determinantal expressions, a two-side inequality for the quotient of $\frac{B_{2k+2}}{B_{2k}}$, identities, logarithmic convexity and increasing monotonicity of the Bernoulli numbers B_{2k} and their quotients

$\frac{B_{2k+2}}{B_{2k}}$, *signs of several Toeplitz–Hessenberg determinants of elements involving the Bernoulli numbers* B_k, *generalizations, and the like. In the paper [3], there was a concise review and survey on these results.*

Remark 2. *If setting* $n = 2$ *in Theorem 12, we acquire*

$$D_1(2) = \begin{vmatrix} \frac{1}{\binom{3}{1}} \end{vmatrix} = \frac{1}{3}, \qquad D_2(2) = \begin{vmatrix} \frac{1}{\binom{3}{1}} & \frac{\binom{0}{0}}{\binom{2}{0}} \\ \frac{1}{\binom{4}{2}} & \frac{\binom{1}{0}}{\binom{3}{1}} \end{vmatrix} = -\frac{1}{18},$$

$$D_3(2) = \begin{vmatrix} \frac{1}{\binom{3}{1}} & \frac{\binom{0}{0}}{\binom{2}{0}} & 0 \\ \frac{1}{\binom{4}{2}} & \frac{\binom{1}{0}}{\binom{3}{1}} & \frac{\binom{1}{1}}{\binom{2}{0}} \\ \frac{1}{\binom{5}{3}} & \frac{\binom{2}{0}}{\binom{4}{2}} & \frac{\binom{2}{1}}{\binom{3}{1}} \end{vmatrix} = \frac{1}{135}, \qquad D_4(2) = \begin{vmatrix} \frac{1}{\binom{3}{1}} & \frac{\binom{0}{0}}{\binom{2}{0}} & 0 & 0 \\ \frac{1}{\binom{4}{2}} & \frac{\binom{1}{0}}{\binom{3}{1}} & \frac{\binom{1}{1}}{\binom{2}{0}} & 0 \\ \frac{1}{\binom{5}{3}} & \frac{\binom{2}{0}}{\binom{4}{2}} & \frac{\binom{2}{1}}{\binom{3}{1}} & \frac{\binom{2}{2}}{\binom{2}{0}} \\ \frac{1}{\binom{6}{4}} & \frac{\binom{3}{0}}{\binom{5}{3}} & \frac{\binom{3}{1}}{\binom{4}{2}} & \frac{\binom{3}{2}}{\binom{3}{1}} \end{vmatrix} = \frac{1}{540},$$

and

$$D_5(2) = \begin{vmatrix} \frac{1}{\binom{3}{1}} & \frac{\binom{0}{0}}{\binom{2}{0}} & 0 & 0 & 0 \\ \frac{1}{\binom{4}{2}} & \frac{\binom{1}{0}}{\binom{3}{1}} & \frac{\binom{1}{1}}{\binom{2}{0}} & 0 & 0 \\ \frac{1}{\binom{5}{3}} & \frac{\binom{2}{0}}{\binom{4}{2}} & \frac{\binom{2}{1}}{\binom{3}{1}} & \frac{\binom{2}{2}}{\binom{2}{0}} & 0 \\ \frac{1}{\binom{6}{4}} & \frac{\binom{3}{0}}{\binom{5}{3}} & \frac{\binom{3}{1}}{\binom{4}{2}} & \frac{\binom{3}{2}}{\binom{3}{1}} & \frac{\binom{3}{3}}{\binom{2}{0}} \\ \frac{1}{\binom{7}{5}} & \frac{\binom{4}{0}}{\binom{6}{4}} & \frac{\binom{4}{1}}{\binom{5}{3}} & \frac{\binom{4}{2}}{\binom{4}{2}} & \frac{\binom{4}{3}}{\binom{3}{1}} \end{vmatrix} = -\frac{1}{567}.$$

Accordingly, we obtain

$$\begin{aligned} F_2(u) &= D_1(2)\frac{u}{1!} - D_2(2)\frac{u^2}{2!} + D_3(2)\frac{u^3}{3!} - D_4(2)\frac{u^4}{4!} + D_5(2)\frac{u^5}{5!} + \cdots \\ &= \frac{u}{3} + \frac{u^2}{36} + \frac{u^3}{810} - \frac{u^4}{12960} - \frac{u^5}{68040} - \cdots \end{aligned}$$

which are coincident with the first five terms of the power series expansion (6).

Comparing the Maclaurin power series expansion (6) and the Maclaurin power series expansion (12) for $n = 2$ *yields*

$$A_n = (-1)^{n-1}\frac{n}{2}D_n(2), \quad n \ge 2. \tag{15}$$

This surprisingly establishes a connection between the sequence A_n *and the determinant* D_n*, and presents a determinantal formula of the quantities* A_n *studied in the papers [6,7]. It is clear that the determinantal expression (15) for* A_n *is more beautiful and symmetric than the one expressed in Equation (5).*

4. Increasing Monotonicity and Logarithmic Convexity

In this section, we prove the increasing property of the functions $R_{n,0}(u)$ in (9) on $(-\infty, \infty)$ and derive logarithmic convexity of the function $F_n(u)$.

Theorem 2. *For* $n \in \mathbb{N}$*, the function* $R_{n,0}(u)$ *in (9) is increasing on* $(-\infty, \infty)$.

Proof. A straightforward calculation results in

$$R'_{n,0}(u) = \frac{uF'_n(u) - F_n(u)}{u^2}$$

and

$$[uF_n'(u) - F_n(u)]' = uF_n''(u).$$

Hence, in order to prove the increasing property of $R_{n,0}(u)$ on $(-\infty, \infty)$, it is sufficient to show that the function $uF_n'(u) - F_n(u)$ is positive on $\mathbb{R}$. Since the limit

$$\lim_{u\to 0}[uF_n'(u) - F_n(u)] = 0$$

is valid, it is sufficient to show that the second derivative $F_n''(u)$ is positive on $(-\infty, \infty)$. Therefore, it suffices to prove that the first derivative

$$F_n'(u) = \frac{\sum_{j=0}^{\infty} C_{n,j+1} \frac{u^j}{j!}}{\sum_{j=0}^{\infty} C_{n,j} \frac{u^j}{j!}} \tag{16}$$

is increasing on $(-\infty, \infty)$, where $C_{n,j}$ is defined by (13). It is apparent that the sequence $\frac{C_{n,j+1}}{C_{n,j}} = \frac{j+1}{n+j+1}$ is increasing in $j \in \mathbb{N}_0$. This can also be verified from (14). Making use of Lemma 2 results in the increasing property of $F_n'(u)$, that is, $F_n''(u) > 0$, on the interval $[0, \infty)$. Consequently, the function $R_{n,0}(u)$ with $n \in \mathbb{N}$ is thus increasing on the interval $[0, \infty)$.

The function $R_{n,0}(u)$ can be reformulated as

$$R_{n,0}(u) = \frac{1}{u}\int_0^u F_n'(t)\,\mathrm{d}t = \int_0^1 F_n'(uv)\,\mathrm{d}v.$$

In order to prove the increasing property of $R_{n,0}(u)$ on $(-\infty, \infty)$, it is enough to show that the function $F_n(u)$ is convex on $(-\infty, \infty)$. Lemma 3 for $\alpha = 1$ and $\beta = e$ means that the function $F_1(u)$ is convex on $(-\infty, \infty)$. Hence, the function $R_{1,0}(u)$ is increasing on $(-\infty, \infty)$.

From the first derivative (16), we find

$$F_n''(u) = \left[\frac{\sum_{j=0}^{\infty} C_{n,j+1} \frac{u^j}{j!}}{\sum_{j=0}^{\infty} C_{n,j} \frac{u^j}{j!}}\right]' = \frac{\left(\sum_{j=0}^{\infty} C_{n,j+2} \frac{u^j}{j!}\right)\left(\sum_{j=0}^{\infty} C_{n,j} \frac{u^j}{j!}\right) - \left(\sum_{j=0}^{\infty} C_{n,j+1} \frac{u^j}{j!}\right)^2}{\left(\sum_{j=0}^{\infty} C_{n,j} \frac{u^j}{j!}\right)^2}.$$

In order to prove $F_n''(u) > 0$ on $(-\infty, \infty)$, it suffices to prove that its numerator is positive, that is, the inequality

$$\left[\sum_{j=0}^{\infty} \frac{1}{\binom{n+j+2}{n}} \frac{u^j}{j!}\right]\left[\sum_{j=0}^{\infty} \frac{1}{\binom{n+j}{n}} \frac{u^j}{j!}\right] \ge \left[\sum_{j=0}^{\infty} \frac{1}{\binom{n+j+1}{n}} \frac{u^j}{j!}\right]^2, \quad n \in \mathbb{N} \tag{17}$$

is valid on $(-\infty, \infty)$, which is equivalent to

$$f_n(u) f_n''(u) \ge [f_n'(u)]^2, \quad n \in \mathbb{N} \tag{18}$$

on $(-\infty, \infty)$, where $f_n(u)$ defined by (7) satisfies the recursive relation

$$\frac{f_{n+1}(u)}{n+1} = \frac{f_n(u) - 1}{u} = \int_0^1 f_n'(ut)\,\mathrm{d}t.$$

From Lemma 4 and by integration by parts, we derive

$$\begin{aligned} f_n(u) &= 1 + u\int_0^1 v^n \,\mathrm{e}^{u(1-v)}\,\mathrm{d}v \\ &= 1 - \int_0^1 v^n \frac{\mathrm{d}\mathrm{e}^{u(1-v)}}{\mathrm{d}v}\,\mathrm{d}v \end{aligned}$$

$$
\begin{aligned}
&= n \int_0^1 v^{n-1}\,\mathrm{e}^{u(1-v)}\,\mathrm{d}v,\\
f_n'(u) &= n \int_0^1 v^{n-1}(1-v)\,\mathrm{e}^{u(1-v)}\,\mathrm{d}v,\\
f_n''(u) &= n \int_0^1 v^{n-1}(1-v)^2\,\mathrm{e}^{u(1-v)}\,\mathrm{d}v
\end{aligned}
$$

for $n \in \mathbb{N}$. Then, the inequality (18) becomes

$$
\int_0^1 v^{n-1}\,\mathrm{e}^{u(1-v)}\,\mathrm{d}v \int_0^1 (1-v)^2 v^{n-1}\,\mathrm{e}^{u(1-v)}\,\mathrm{d}v \ge \left[\int_0^1 (1-v)v^{n-1}\,\mathrm{e}^{u(1-v)}\,\mathrm{d}v\right]^2
$$

for $n \in \mathbb{N}$ and $u \in (-\infty, \infty)$. This integral inequality follows from an immediate application of Lemma 5 with $q(v) = v^{n-1}\,\mathrm{e}^{u(1-v)} \in \mathbb{N}_0$ and $f(v) = h(v) = 1 - v$ on the interval $[\alpha, \beta] = [0, 1]$. Consequently, the second derivative $F_n''(u)$ is positive, and then the function $R_{n,0}(u)$ is increasing, on $(-\infty, \infty)$. The proof of Theorem 2 is thus complete. □

Corollary 1. *The function $f_n(u)$ in (7) is increasing and logarithmically convex on $(-\infty, \infty)$. Equivalently, the function $F_n(u)$ in (8) is increasing and convex on $(-\infty, \infty)$.*

First proof. It is general knowledge that

$$
\mathrm{e}^u = \sum_{j=0}^{n-1} \frac{u^j}{j!} + \frac{1}{(n-1)!}\int_0^u (u-v)^{n-1}\,\mathrm{e}^v\,\mathrm{d}v = \sum_{j=0}^{n-1} \frac{u^j}{j!} + \frac{u^n}{(n-1)!}\int_0^1 (1-v)^{n-1}\,\mathrm{e}^{uv}\,\mathrm{d}v.
$$

As a result, we arrive at

$$
F_n(u) = \ln\left[n\int_0^1 (1-v)^{n-1}\,\mathrm{e}^{uv}\,\mathrm{d}v\right], \quad n \in \mathbb{N},
$$

which is increasing in $u \in (-\infty, \infty)$. Then, we obtain the integral representation

$$
F_n'(u) = \frac{\int_0^1 (1-v)^{n-1} v\,\mathrm{e}^{uv}\,\mathrm{d}v}{\int_0^1 (1-v)^{n-1}\,\mathrm{e}^{uv}\,\mathrm{d}v}, \quad n \in \mathbb{N}.
$$

Applying Lemma 6 to $W(v,u) = \mathrm{e}^{uv}$, $U(v) = (1-v)^{n-1}v$, and $V(v) = (1-v)^{n-1}$ such that both $\frac{\partial W(v,u)/\partial u}{W(v,u)} = v$ and $\frac{U(v)}{V(v)} = v$ are increasing on $(0,1)$, we derive that the first derivative $F_n'(u)$ for $n \in \mathbb{N}$ is increasing in $u \in (-\infty, \infty)$. As a result, the function $F_n(u)$ for $n \in \mathbb{N}$ is convex in $u \in (-\infty, \infty)$. □

Second proof. This comes from reorganizing a part of the proof of Theorem 2. □

Corollary 2. *The inequality (17) is valid on $(-\infty, \infty)$.*

Proof. This follows from reorganizing a part of the proof of Theorem 2. □

Remark 3. *Corollary 1 generalizes Lemma 3 with $\alpha = 1$ and $\beta = e$.*

5. Conclusions

In this paper, we obtained the following interesting and significant results.

1. For $n \in \mathbb{N}$, the new general Maclaurin power series expansion (12) of the function $F_n(t)$ defined by (8) was established in Theorem 1, from which two special Maclaurin power series expansions (3) and (6) can be derived immediately.
2. A new determinantal expression $B_{2k} = -2kD_{2k}(1)$ of the Bernoulli numbers B_{2k} for $k \in \mathbb{N}$ was deduced in Remark 1.

3. A determinantal expression (15) for Howard's numbers A_n for $n \geq 2$, which are generated by (4), was deduced in Remark 2.
4. For $n \in \mathbb{N}$, the function $R_{n,0}(u)$ defined in (9) was proved in Theorem 2 to be increasing on $(-\infty, \infty)$.
5. For $n \in \mathbb{N}$, the function $f_n(u)$ defined in (7), the tail of the Maclaurin power series expansion of the exponential function e^u, was proved in the proof of Theorem 2 to be increasing and logarithmically convex on $(-\infty, \infty)$.
6. For $n \in \mathbb{N}$, the function $F_n(u)$ defined in (8) was proved in the proof of Theorem 2 to be increasing and convex on $(-\infty, \infty)$.
7. The inequality (17) is valid on $(-\infty, \infty)$.
8. Lemma 3 with $\alpha = 1$ and $\beta = e$ was generalized in the proof of Theorem 2.

By the way, we point out that the ideas of constructing the function $F_n(u)$ and studying its properties can be further concluded in the following ways. Suppose that a real function $f(u)$ has a formal Maclaurin power series expansion

$$f(u) = \sum_{j=0}^{\infty} f^{(j)}(0) \frac{u^j}{j!}. \tag{19}$$

If $f^{(n+1)}(0) \neq 0$ for some integer $n \in \mathbb{N}_0$, we can consider the function

$$\frac{(n+1)!}{f^{(n+1)}(0)} \frac{1}{u^{n+1}} \left[f(u) - \sum_{j=0}^{n} f^{(j)}(0) \frac{u^j}{j!} \right],$$

call it the nth normalized tail of the formal Maclaurin power series expansion (19), study its monotonicity and its (logarithmic) convexity or concavity, expand its logarithm into a Maclaurin power series around $u = 0$, and investigate the monotonicity and (logarithmic) convexity or concavity of the quotient of two functions with consecutively different values of n. Concretely speaking, we can take $f(u)$ as any one of the elementary functions such as $\ln u$, $\sin u$, $\cos u$, $\tan u$, $\cot u$, $\arcsin u$, $\arccos u$, and their integer powers $(\ln u)^m$, $(\sin u)^m$, $(\cos u)^m$, $(\tan u)^m$, $(\cot u)^m$, $(\arcsin u)^m$, and $(\arccos u)^m$ for $m \in \mathbb{N}$. More significantly, we can take $f(u)$ as any one of the generating functions

$$\frac{u}{\mathrm{e}^u - 1}, \quad \frac{2\,\mathrm{e}^u}{\mathrm{e}^{2u} + 1}, \quad [\ln(1+u)]^m, \quad \left(\frac{\mathrm{e}^u - 1}{u} \right)^m, \quad \frac{2}{1 + \sqrt{1 - 4u}}, \quad \frac{1}{\sqrt{1 - 6u + u^2}}$$

of the Bernoulli numbers B_j, the Euler numbers E_j, the Stirling numbers of the first kind $s(j,k)$, the Stirling numbers of the second kind $S(j,k)$, the Catalan numbers C_j, and the central Delannoy numbers $D(j)$ for $j \in \mathbb{N}_0$.

So far, Qi and his coauthors have initially investigated several of the simple functions mentioned above and published the papers [28–34] and ([35] Remark 7), for example. In particular, by the study of the normalized tails associated with the generating function $\frac{u}{\mathrm{e}^u - 1}$ of the Bernoulli numbers B_j in [33], Qi and his coauthors derived an interesting problem on the monotonic properties of the ratios of any two Bernoulli polynomials $B_j(x)$ and arrived at many significant and novel results in ([33] Proposition 1) and the arXiv preprint at the site https://doi.org/10.48550/arxiv.2405.05280; see also ([31] Remarks 5 and 6). These events demonstrated that the normalized tails, also known as the normalized remainders, associated with the Maclaurin or formal power series expansions of analytic or generating functions in analysis and combinatorial number theory, firstly and creatively designed by Qi, deserve to be extensively and deeply investigated by mathematicians.

We believe that the ideas and techniques used in this paper will attract more and more mathematicians to conduct increasingly better research in mathematics.

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