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An Innovative Algorithm Based on Octahedron Sets via Multi-Criteria Decision Making

Güzide Şenel

Department of Mathematics, Amasya University, Amasya 05100, Turkey; g.senel@amasya.edu.tr

Abstract: Octahedron sets, which extend beyond the previously defined fuzzy set and soft set concepts to address uncertainty, represent a hybrid set theory that incorporates three distinct systems: interval-valued fuzzy sets, intuitionistic fuzzy sets, and traditional fuzzy set components. This comprehensive set theory is designed to express all information provided by decision makers as interval-valued intuitionistic fuzzy decision matrices, addressing a broader range of demands than conventional fuzzy decision-making methods. Multi-criteria decision-making (MCDM) methods are essential tools for analyzing and evaluating alternatives across multiple dimensions, enabling informed decision making aligned with strategic objectives. In this study, we applied MCDM methods to octahedron sets for the first time, optimizing decision results by considering various constraints and preferences. By employing an MCDM algorithm, this study demonstrated how the integration of MCDM into octahedron sets can significantly enhance decision-making processes. The algorithm allowed for the systematic evaluation of alternatives, showcasing the practical utility and effectiveness of octahedron sets in real-world scenarios. This approach was validated through influential examples, underscoring the value of algorithms in leveraging the full potential of octahedron sets. Furthermore, the application of MCDM to octahedron sets revealed that this hybrid structure could handle a wider range of decision-making problems more effectively than traditional fuzzy set approaches. This study not only highlights the theoretical advancements brought by octahedron sets but also provides practical evidence of their application, proving their importance and usefulness in complex decision-making environments. Overall, the integration of octahedron sets and MCDM methods marks a significant step forward in decision science, offering a robust framework for addressing uncertainty and optimizing decision outcomes. This research paves the way for future studies to explore the full capabilities of octahedron sets, potentially transforming decision-making practices across various fields.

Keywords: octahedron set; soft multi-sets; soft multi-topology; soft multi-criteria decision making; MCDM

MSC: 03E75; 54A05; 03E72; 54A40; 54C99

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1. Introduction

The study of finding solutions to uncertainty is a field of study that mathematicians have been working on in recent years and have developed by creating new set theories. In order to cope with uncertainty, studies have been carried out in the fields of theory and application from past to present, the most current of which is the octahedron set theory produced in 2020 by Kim et al. [1]. It is a hybrid set theory that emerges by using three different systems in one structure, consisting of an octahedron set, an interval-valued fuzzy set, an intuitive fuzzy set, and fuzzy set components. Subsequently, Şenel et al. [2] explored MCDM issues by applying similarity measures to octahedron sets. In addition, Lee et al. [3] introduced the concept of octahedron subgroups and subrings, detailing some of their characteristics. Furthermore, Lee et al. [4] investigated topological structures derived from octahedron sets. Research involving octahedron sets is advancing swiftly, encompassing both topological and algebraic aspects as indicated in [5].

Octahedron sets, which are more comprehensive than the fuzzy set and soft set concepts previously defined to deal with uncertainty, contain more uncertainty and more variables, allowing new scientific studies to be carried out by using them in application areas such as point measurements, interval measurements, and simultaneous positive and negative event evaluations, provided the opportunity. It is a hybrid set theory that emerges by using three different systems in one structure, consisting of an octahedron set, an interval-valued fuzzy set, an intuitive fuzzy set, and fuzzy set components. The concept of octahedron sets differs from traditional fuzzy set and soft set theories because it can examine many variables that fuzzy sets and soft sets cannot.

The main purpose of this study is to try to reach the possible "best/appropriate" solution among multiple conflicting criteria regarding a decision situation with octahedron set theory and to develop appropriate approaches and methods for this solution and to overlap the concept of symmetry. The daily life problem supporting our purpose is presented in Section MCDM Method Using Octahedron Sets in a Daily Life Problem and Section 5. With these examples, the concept of symmetry is applied to the daily life problem with the help of octahedron sets.

In the field of decision-making processes, combining soft multi-sets with multi-criteria decision-making methods offers an advanced approach that offers a more nuanced and comprehensive perspective. By taking advantage of this innovative methodology, studies can produce more precise and effective solutions in complex decision environments. Soft multi-sets provide a flexible framework that allows the integration of qualitative and quantitative factors, allowing decision makers to consider various criteria simultaneously. This holistic approach improves decision making by capturing the uncertainties and uncertainties present in real-world scenarios. Multi-criteria decision-making (MCDM) techniques are generally divided into two main types: discrete MCDM, also known as multi-attribute decision making (MADM), and continuous multi-objective decision-making (MODM) methods [6,7]. In recent years, a significant volume of literature has emerged to discuss the advancements and applications of MCDM methods across various sectors. This paper provides a comprehensive review of the rapidly increasing interest in MCDM approaches, encapsulating the state-of-the-art in MCDM literature concerning both applications and methodologies. The analysis relies on data from the Web of Science Database, a component of the Thomson Reuters Web of Knowledge. The initial groundwork for modern MCDM was laid during the 1950s and 1960s, with the 1970s marking a pivotal era due to numerous foundational works. MCDM research gained momentum in the 1980s and continued to expand rapidly into the early 1990s, maintaining its growth thereafter [8]. This document also offers a concise historical overview of MCDM methods, tracing the evolution from ancient strategies to contemporary frameworks. In [9], foundational principles of Decision Making with Multiple Objectives were established. Ref. [10] conducted a review on the evolution of MODM methods and their applications over a brief timeline. Subsequently, Ref. [11] provided insights into MADM methods, including Simple Additive Weighting (SAW) [12], the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) [12], the Elimination and Choice Expressing Reality (ELECTRE) [13], and the Linear Programming Technique for Multidimensional Analysis of Preference (LINMAP) [14]. A detailed analysis of the Analytic Hierarchy Process (AHP) was published in [15], followed by research into the expanded Analytic Network Process (ANP) method. Ref. [16] introduced a book addressing compromise theory, and Ref. [10] explored Group Decision Making under Multi-criteria. A summary of ELECTRE group methods was provided in [13]. Pioneering research has been documented by [17]. The recent trend towards hybrid and modular MCDM methods highlights their importance; these methods combine established techniques like SAW [12], TOPSIS [10], AHP [15], and ELECTRE [13] with innovations in fuzzy and grey number theories to enhance decision-making processes.

If we look at both reviews and original articles in recent years, we can see that these two articles [13,14] stand out. In light of these articles, the studies of the advantages and disadvantages of MCDM will be given and compared in the continuation of this article. Soft multi-sets are studied in [15] and soft multi-criteria decision making is applied in [16].

The remainder of this paper is organized as follows: Section 2 presents Analysis of Multi-Criteria Decision-Making (MCDM) methods. In Section 3, some of the basic definitions needed by the next sections are given. In Section 4, we list some basic definitions and notations needed in the next sections about octahedron sets. Also, MCDM Method Using Octahedron Sets in a Daily Life Problem is presented firstly in this section. In Section 5, an innovative algorithm based on octahedron sets via multi-criteria decision making is proved and applied. We will present a new application that we created based on all these studies in the application section of our article.

2. Analysis of Multi-Criteria Decision-Making Methods

In this section, a comprehensive review of 12 distinct MCDM methods is provided, highlighting key themes found in the literature. Each method will be explored in detail, discussing its respective advantages and disadvantages. This information will be succinctly summarized in a comparative table at the conclusion of the section. The methods under review include: (1) Multi-Attribute Utility Theory (MAUT), (2) Analytic Hierarchy Process (AHP), (3) Fuzzy Set Theory, (4) Case-based Reasoning (CBR), (5) Data Envelopment Analysis (DEA), (6) Simple Multi-Attribute Rating Technique (SMART), (7) Goal Programming, (8) ELimination Et Choix Traduisant la REalité (ELECTRE), (9) Preference Ranking Organization METHod for Enrichment of Evaluations (PROMETHEE), (10) Simple Additive Weighting (SAW), (11) Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS), and (12) Fuzzy Multi-criteria Satisfaction Analysis (FMS). This detailed examination aims to provide valuable insights into these methodologies, facilitating a deeper understanding of their practical applications and theoretical foundations.

To briefly summarize the 12 MCDM methods mentioned above, as in the value function approach, decision-making methods generally define a preference relationship between alternatives evaluated on several attributes, usually called a superiority relationship. It defines a superiority relationship as a binary relation S on a set of alternatives X such that xSy if, given what is known about the decision maker's preferences and the quality of the evaluations of the alternatives and the nature of the problem, there is sufficient argument to declare that x is at least as good as y but there is no fundamental reason to refute this statement. In most MCDM methods, the superiority relationship is constructed through a series of pairwise comparisons of alternatives. This means that these methods deal with finite sets of alternatives, but their underlying principles can be adapted to deal with infinite sets. Although pairwise comparisons can be made in many ways, the concordance–inconcordance principle is common to most decision-making methods. This consists of declaring that an alternative x is at least as good as an alternative y(xSy) if:

- A majority of the attributes support this claim (concordance condition);
- The opposition of other attributes (minority) is not "too strong" (non-concordance condition).

This principle contradicts the principles underlying the value function approach. It is based on a "voting" analogy and can be used without resorting to a subtle analysis of the trade-offs between attributes.

3. Preliminaries

This section provides some of the basic definitions needed by the next sections. Firstly, it is necessary to start with the definition and theorems of Molodtsov [18], which put forward the most important theory in soft sets dealing with uncertainty.

Definition 1. *Soft set: Let U be an initial universe, P(U) be the power set of U, E be the set of parameters, and $A \subseteq E$. Then, an F_A soft set over U is defined by a set of ordered pairs as follows:*

$$F_A = \{(e, f_A(e)) : e \in E\}$$

where $f_A : E \to P(U)$ and $e \notin A$ for $f_A(e) = \varnothing$ [19] (for more details, see [5,19]).

Definition 2. *Soft multi-set (SMS): A multi-set (MS) over Z is just a pair $< Z,\ f >$, where $f : Z \to W$ is a function, Z is a crisp set, and W is a set of whole numbers. Moreover, to avoid any confusion, we will use square brackets for MSs and braces for sets. MS A is given by $A =< Z,\ f >= [\ \frac{k_1}{z_1}, \frac{k_2}{z_2}, \ldots, \frac{k_n}{z_n}]$, where z_1 occurs k_1 times, z_2 occurs k_2 times, and so on [20].*

Definition 3. *Soft multi-set topology: Let Ω_A be an SMS over universal MS H. An SMS topology on an SMS Ω_A denoted by $\widetilde{\tau}$ is a collection of soft multi-subsets of Ω_A having the following properties:*

(i) $\Omega_\varnothing, \Omega_A \in \widetilde{\tau}$;

(ii) *Union of any number of members of $\widetilde{\tau}$ belongs to $\widetilde{\tau}$;*

$$i.e., \left\{\Omega_B \widetilde{\subseteq} \Omega_A : i \in I\right\} \widetilde{\subseteq} \widetilde{\tau} \Rightarrow \widetilde{\bigcup}_{i\in I}\Omega_{A_i} \in \widetilde{\tau}$$

(iii) *Intersection of finite number of members of $\widetilde{\tau}$ belongs to $\widetilde{\tau}$;*

$$i.e., \left\{\Omega_B \widetilde{\subseteq} \Omega_A : 1 \le i \le n, n \in \mathbb{N}\right\} \widetilde{\subseteq} \widetilde{\tau} \Rightarrow \widetilde{\bigcap}_{1\le i\le n}\Omega_{B_i} \in \widetilde{\tau}$$

Then, an SMS topological space is denoted by $(\Omega_A, \widetilde{\tau})$" [21] (for more details, see [22]).

Definition 4. *Soft open set and soft closed set: Let $\left(X, \widetilde{\tau}, E\right)$ be a soft topological space, $Y \subset X$, and (F, E) be a soft set on X. In this case, the following conditions are true:*

1. *(F, E) is also open in Y. $\Longleftrightarrow$ there is an $(F, E) = \widetilde{Y} \widetilde{\cap} (G, E)$ such that $(G, E) \widetilde{\in} \widetilde{\tau}$.*
2. *(F, E) is also closed in Y. $\Longleftrightarrow$ there is an $(F, E) = \widetilde{Y} \widetilde{\cap} (G, E)$ such that $(G, E) \widetilde{\in} \widetilde{\tau}'$ [20].*

Definition 5. *Fuzzy set: If set A, let this be a subset. In this case, the set is as follows:*

$$\mu_A\,(x) = \begin{cases} 1,\ x \in A \\ 0,\ x \notin A \end{cases}$$

defined as $\mu_A : U \to \{0,\ 1\}$, which can be given with the characteristic function. As the definition suggests, the value of this function indicates whether an element is in set A. This idea can be extended to the definition of the fuzzy set by taking the closed spacing $[0,\ 1]$ instead of the two-element $\{0,\ 1\}$ set in the function $\mu_A : U \to \{0,\ 1\}$.

A fuzzy set F on a universal set U is defined by the membership function:

$$\mu_F : U \to [0,1],\ \text{for}\ x \in U,\ \mu_F(x)$$

which indicates the degree to which the x element belongs the fuzzy set F. Thus, a fuzzy set F can be characterized by the set of ordered pairs [23]:

$$F = \{(x,\ \mu_F\,(x)) : x \in U\}$$

Definition 6. *Soft multi-set boundary of soft open multi-set: Let* $\left(\Omega_A, \widetilde{\tau}\right)$ *be an SMS topological space and* $\Omega_B \widetilde{\subseteq} \Omega_A$*. The soft multi-frontier or boundary of* Ω_B *is denoted by* $\Omega_r(\Omega_B)$ *or* $\Omega_B^{\widetilde{b}}$ *and is defined as* $\Omega_B^{\widetilde{b}} = \overline{\Omega_B} \widetilde{\cap} \overline{\Omega_B^{\widetilde{c}}}$*. Stated differently, the soft multi-points that do not belong to the soft multi-interior and exterior of* Ω_B *are in* $\Omega_B^{\widetilde{b}}$*.*

4. Octahedron Sets

In this section, we list some basic definitions and notations needed in the next sections about octahedron sets. Also, the MCDM method using octahedron sets in a daily life problem is presented firstly. Throughout this paper, I denote the unit closed interval $[0,1]$ in the set of real numbers $\mathbb{R}$.

Let $I \oplus I = \left\{\overline{a} = \left(a^{\in}, a^{\notin}\right) \in I \times I : a^{\in} + a^{\notin} \leq 1\right\}$. Then, each member $\overline{a}$ of $I \oplus I$ is called an intuitionistic point or intuitionistic number. In particular, we denote $(0,1)$ and $(1,0)$ as $\overline{0}$ and $\overline{1}$, respectively. Refer to [2] for the definitions of $\leq$ and $=$ on $I \oplus I$, the complement of an intuitionistic number, and the infimum and the supremum of any intutionistic numbers.

Let $[I]$ be the set of all closed subintervals of I. Then, each member $\widetilde{a}$ of $[I]$ is called an interval number, where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. In particular, if $a^- = a^+$, then we write $\widetilde{a} = \boldsymbol{a}$. Refer to [3] for the definitions of $\leq$ and $=$ on $\oplus I$, the complement of an interval-valued number, and the infimum and the supremum of any interval-valued numbers.

Definition 7. ([1]). *Octahedron Set: Let X be a nonempty set and let* $= [A^-, A^+] \in [I]^X$, $A = \left(A^{\in}, A^{\notin}\right) \in (I \bigoplus I)^X$, $\lambda \in I^X$*. Then, the triple* $\mathcal{A} = \langle \boldsymbol{A}, A, \lambda \rangle$ *is called an octahedron set in X. In fact:*

A: X → [I] × (I ⊕ I) × I is a mapping.

We can consider the following special octahedron sets in X:

$$\left\langle \widetilde{0}, \overline{\boldsymbol{0}}, 0 \right\rangle = \ddot{0}$$

$$\left\langle \widetilde{0}, \overline{\boldsymbol{0}}, 1 \right\rangle, \left\langle \widetilde{0}, \overline{\boldsymbol{1}}, 0 \right\rangle, \left\langle \widetilde{1}, \overline{\boldsymbol{0}}, 0 \right\rangle$$

$$\left\langle \widetilde{0}, \overline{\boldsymbol{1}}, 1 \right\rangle, \left\langle \widetilde{1}, \overline{\boldsymbol{0}}, 1 \right\rangle = \ddot{0}, \left\langle \widetilde{1}, \overline{\boldsymbol{1}}, 0 \right\rangle$$

$$\left\langle \widetilde{1}, \overline{\boldsymbol{1}}, 1 \right\rangle = \ddot{1}$$

In this case, $\ddot{\boldsymbol{0}}$ *(resp.* $\ddot{\boldsymbol{1}}$*) is called an octahedron empty set (resp. octahedron whole set) in X.*

We denote the set of all octahedron sets as O(X).

It is obvious that, for each $\in 2^X$, $A = \langle [\chi_A , \chi_A] , (\chi_A, \chi_{A^C}), \chi_A \rangle \in \mathcal{O}(X)$ *and then* $2^X \subset \mathcal{O}(X)$*, where* 2^X *denotes the set of all subsets of X; then,* χ_A *denotes the characteristic function of* A*.*

Furthermore, we can easily see that, for each, $\boldsymbol{A} = \langle A, \lambda \rangle \in C(X)$, $\boldsymbol{A} = \langle A, (A^-, A^+), \lambda \rangle$, $\boldsymbol{A} = \langle A, (\lambda, \lambda^C, \lambda \rangle \in \mathcal{O}(X)$*, and then* $C(X) \subset \mathcal{O}(X)$ *dir. In this case, we denote* $\langle A, (A^-, A^+), \lambda \rangle$ *ve* $\langle A, (\lambda, \lambda^C, \lambda \rangle$ *as* $\mathcal{A}_A$ *ve* $\mathcal{A}_\lambda$*, respectively.*

Example 1. *Let* $X = \{a, b, c\}$ *be a set and let* $\mathcal{A} = \langle \boldsymbol{A}, A, \lambda \rangle : X \to [I] \times (I \oplus I) \times I$ *be the mapping given by:*

(1)

$$\mathcal{A}(a) = \langle [0.3, 0.6], (0.7, 0.2), 0.5 \rangle,$$

$$\mathcal{A}(b) = \langle [0.2, 0.4], (0.6, 0.3), 0.7 \rangle,$$

$$\mathcal{A}(c) = \langle [0.4, 0.7], (0.5, 0.4), 0.3 \rangle.$$

Then, we can easily see that $\mathcal{A}$ is an octahedron set in X.

(2) Let $X = I$ and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle : X \to [I] \times (I \oplus I) \times I$ be the mapping defined as follows for each $x \in X$:

$$(x) = \left\langle \left[\frac{x}{4}, \frac{1+x}{2} \right], \left(\frac{x}{3}, \frac{1+x}{5} \right), x \right\rangle.$$

Then, we can easily calculate that $\mathcal{A}$ is an octahedron set in X.

(3) Let $= \left[A^-, A^+ \right] \in [I]^X$. Then, clearly:

$$\left\langle \mathbf{A}, \bar{\mathbf{0}}, 0 \right\rangle \ \ (resp.\ \left\langle \mathbf{A}, \bar{\mathbf{1}}, 0 \right\rangle,\ \left\langle \mathbf{A}, \bar{\mathbf{0}}, 1 \right\rangle,\ \left\langle \mathbf{A}, \bar{\mathbf{1}}, 1 \right\rangle)$$

which is an octahedron set. In this case, we will denote:

$$\left\langle \mathbf{A}, \bar{\mathbf{0}}, 0 \right\rangle \ \ (resp.\ \left\langle \mathbf{A}, \bar{\mathbf{1}}, 0 \right\rangle,\ \left\langle \mathbf{A}, \bar{\mathbf{0}}, 1 \right\rangle,\ \left\langle \mathbf{A}, \bar{\mathbf{1}}, 1 \right\rangle)$$

as $\mathcal{O}_{\bar{0},0} \left(resp.\ \mathcal{O}_{\bar{\mathbf{1}},0}, \mathcal{O}_{\bar{\mathbf{0}},1}, \mathcal{O}_{\bar{\mathbf{1}},1} \right)$.

Now, let $A : X \to I \oplus I$ and $\lambda : X \to I$ be the mappings defined as follows, respectively, for each $x \in X$:

$$A(x) = \left(A^{\in}(x), A^{\notin}(x) \right) = \left(A^-(x), 1 - A^+(x) \right),$$

$$\lambda(x) = \frac{A^-(x) + A^+(x)}{2}.$$

Then, we can easily see that $\langle \mathbf{A}, A, \lambda \rangle$ is an octahedron set in X. In this case, $\langle \mathbf{A}, A, \lambda \rangle$ will be called the octahedron set in X induced by $\mathbf{A}$ and will be denoted by $\mathcal{O}_{\mathbf{A}}$.

(4) Let $A = \left(A^{\in}, A^{\notin} \right) \in (I \oplus I)^X$. Then, cleary:

$$\left\langle \widetilde{0}, A, 0 \right\rangle (resp.\ \left\langle \widetilde{1}, A, 0 \right\rangle, \left\langle \widetilde{0}, A, 1 \right\rangle, \left\langle \widetilde{1}, A, 1 \right\rangle)$$

which is an octahedron set. In this case:

$$\left\langle \widetilde{0}, A, 0 \right\rangle (resp.\ \left\langle \widetilde{1}, A, 0 \right\rangle, \left\langle \widetilde{0}, A, 1 \right\rangle, \left\langle \widetilde{1}, A, 1 \right\rangle)$$

which will be denoted by $\mathcal{O}_{\widetilde{0},0} \left(resp.\ \mathcal{O}_{\widetilde{1},0}, \mathcal{O}_{\widetilde{0},1}, \mathcal{O}_{\widetilde{1},1} \right)$.

Now, us $\mathbf{A} : X \to [I]$ and $\lambda : X \to I$ be the mappings defined as follows, respectively, for each $x \in X$:

$$\mathbf{A}(x) = \left[A^{\in}(x), 1 - A^{\notin}(x) \right],$$

$$\lambda(x) = \frac{A^{\in}(x) + 1 - A^{\notin}(x)}{2}.$$

Then, clearly $\langle \mathbf{A}, A, \lambda \rangle$ is an octahedron set. In this case, $\langle \mathbf{A}, A, \lambda \rangle$ will be called the octahedron set in X induced by A and will be denoted by $\mathcal{O}_A$.

(5) Let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$ be an octahedron set. Then, clearly $\langle \mathbf{A}, []A, \lambda \rangle$ and $\langle \mathbf{A}, \diamond A, \lambda \rangle$ are octahedron sets in X.

Definition 8. ([1]). *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$ be an octahedron set in X. We can easily see that the following hold:*

$$\ddot{0}^c = \ddot{1},\ \ddot{1}^c = \ddot{0},$$

$$\left\langle \tilde{0}, \bar{\mathbf{0}}, 1 \right\rangle^c = \left\langle \tilde{1}, \bar{\mathbf{1}}, 0 \right\rangle, \left\langle \tilde{1}, \bar{\mathbf{1}}, 0 \right\rangle^c = \left\langle \tilde{0}, \bar{\mathbf{0}}, 1 \right\rangle,$$

$$\left\langle \tilde{0}, \bar{\mathbf{1}}, 0 \right\rangle^c = \left\langle \tilde{1}, \bar{\mathbf{0}}, 1 \right\rangle, \left\langle \tilde{1}, \bar{\mathbf{0}}, 1 \right\rangle^c = \left\langle \tilde{0}, \bar{\mathbf{0}}, 1 \right\rangle,$$

$$\left\langle \tilde{1}, \bar{\mathbf{0}}, 0 \right\rangle^c = \left\langle \tilde{0}, \bar{\mathbf{1}}, 1 \right\rangle, \left\langle \tilde{0}, \bar{\mathbf{1}}, 1 \right\rangle^c = \left\langle \tilde{1}, \bar{\mathbf{0}}, 1 \right\rangle,$$

$$\left\langle \tilde{0}, \bar{\mathbf{1}}, 1 \right\rangle^c = \left\langle \tilde{1}, \bar{\mathbf{0}}, 0 \right\rangle, \left\langle \tilde{1}, \bar{\mathbf{0}}, 0 \right\rangle^c = \left\langle \tilde{0}, \bar{\mathbf{0}}, 1 \right\rangle,$$

$$\left\langle \tilde{1}, \bar{\mathbf{0}}, 1 \right\rangle^c = \left\langle \tilde{0}, \bar{\mathbf{1}}, 0 \right\rangle, \left\langle \tilde{0}, \bar{\mathbf{1}}, 0 \right\rangle^c = \left\langle \tilde{1}, \bar{\mathbf{0}}, 1 \right\rangle,$$

$$\left\langle \tilde{1}, \bar{\mathbf{1}}, 0 \right\rangle^c = \left\langle \tilde{0}, \bar{\mathbf{0}}, 1 \right\rangle, \left\langle \tilde{0}, \bar{\mathbf{0}}, 1 \right\rangle^c = \left\langle \tilde{1}, \bar{\mathbf{0}}, 1 \right\rangle.$$

Definition 9. ([1]). *A mapping d: $\mathcal{O}(X) \times \mathcal{O}(X) \rightarrow I$ is called a distance measure on $\mathcal{O}(X)$ if it satisfies the following conditions for any A, B, C$\in \mathcal{O}(X)$:*

(DM1) $0 \leq d(A, B) \leq 1$;

(DM2) d(A, B) = 0 if and only if A = B;

(DM3) d(A, B) = d(B, A);

(DM4) if $A \subset 1\ B \subset 1\ C$, then $d(A, C) \geq d(A, B) \vee d(B, C)$.

In this case, d(A, B) is called the distance measure between A and B.

Definition 10. ([1]). *A mapping d: $\mathcal{O}(X) \times \mathcal{O}(X) \rightarrow I$ is called a similarity measure on $\mathcal{O}(X)$ if it satisfies the following conditions for any A, B, C$\in \mathcal{O}(X)$:*

(DM1) $0 \leq s(A, B) \leq 1$;

(DM2) s(A, B) = 1 if and only if A = B;

(DM3) s(A, B) = s(B, A);

(DM4) if $A \subset 1\ B \subset 1\ C$, then $s(A, C) \leq s(A, B) \wedge s(B, C)$.

In this case, d(A, B) is called the similarity measure between A and B.

In fact, from the distance measure and similarity measure, we can easily see that s(A, B) = 1 − d(A, B).

MCDM Method Using Octahedron Sets in a Daily Life Problem

In this subsection, we give a new method based on the similarity measure in an octahedron set environment. Assume that $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a set of n alternatives with criteria $= \{\beta_1, \beta_2, \ldots, \beta_m\}$ and let $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ be the r decision makers. Let $\delta = \{\delta_1, \delta_2, \ldots, \delta_r\}$ be the weight vector of decision makers such that $\delta_k > 0$ and $\sum_{k=1}^{r} \delta_k = 1$.

The methodology: It would be useful to provide detailed information about the methodology before starting. First of all, the ideal octahedron cluster decision matrix is an important matrix for the similarity measure of MCDM. Secondly, including multiple decision makers, each of whom is determined as the k-th (k = 1, 2, . . ., r) decision maker, provides evaluations for each alternative. Then, in decision-making scenarios, not all attributes have the same importance. We must determine the attribute weights. At the end of the methodology, the calculation of the weighted similarity measure and the ranking of the alternatives are included.

We propose the MCDM method presented using the following steps [2]:

Step 1. Formation of the ideal octahedron set decision matrix. The ideal octahedron set decision matrix is an important matrix for the similarity measure of MCDM given in the following form:

$$\begin{bmatrix} & \beta_1 & \beta_2 & \dots & \beta_m \\ \alpha_1 & \mathcal{A}_{11} & \mathcal{A}_{12} & \dots & \mathcal{A}_{1m} \\ \alpha_2 & \mathcal{A}_{21} & \mathcal{A}_{22} & \dots & \mathcal{A}_{2m} \\ . & . & . & \dots & . \\ \alpha_n & \mathcal{A}_{n1} & \mathcal{A}_{n2} & \dots & \mathcal{A}_{nm} \end{bmatrix}$$

where $\mathcal{A}_{ij} = \langle \boldsymbol{A}_{ij}, A_{ij}, \lambda_{ij} \rangle, i = 1, 2, \dots, m$.

Step 2. Construction of the octahedron set decision matrix. Involving multiple decision makers, each one designated as the k-th $(k = 1, 2, \dots, r)$ decision maker, provides assessments for each alternative $\alpha_i = (i = 1, 2, \dots, n)$ in relation to the criteria $\beta_j(j = 1, 2, \dots, m)$ expressed using an octahedron set. The decision matrix for each M^k is constructed by the following matrix [2]:

$$M^k = \left\langle \mathcal{A}_{ij}^k \right\rangle = \begin{bmatrix} & \beta_1 & \beta_2 & \dots & \beta_m \\ \alpha_1 & \mathcal{A}_{11}^k & \mathcal{A}_{12}^k & \dots & \mathcal{A}_{1m}^k \\ \alpha_2 & \mathcal{A}_{21}^k & \mathcal{A}_{22}^k & \dots & \mathcal{A}_{2m}^k \\ . & . & . & \dots & . \\ \alpha_n^k & \mathcal{A}_{n1}^k & \mathcal{A}_{n2}^k & \dots & \mathcal{A}_{nm}^k \end{bmatrix}$$

where $k = 1, 2, \dots, r, i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Step 3. Determination of attribute weights. In decision-making scenarios, not all attributes carry the same significance. Each decision maker expresses their perspective on the importance of each attribute by assigning weights using linguistic variables. These linguistic assessments are then converted into an octahedron set format. Let $\sqsupseteq_k(\beta_j)$ denote the attribute weight for the attribute β_j given by the k-th decision maker in terms of the octahedron set.

We convert $\sqsupseteq_k(\beta_j)$ into a fuzzy number as follows:

$$\sqsupseteq_k^F(\beta_j) = \begin{cases} \left[1 - \left(\frac{V_{kj}}{5}\right)^{\frac{1}{2}}\right] & if\ \beta_j \in \beta \\ 0 & otherwise\ , \end{cases}$$

where $V_{kj} = \left[(1 - A^-(\beta_j))^2 + (1 - A^+(\beta_j))^2 + (1 - A^{\in}(\beta_j))^2 \left(A^{\notin}(\beta_j)\right)^2 + (1 - \lambda(\beta_j))^2\right]^{\frac{1}{2}}$ and each of the above values denote the value of the octahedron set corresponding to (k, β_j).

Then, the aggregate weight for the criteria can be determined as follows:

$$W_j = \frac{\left[1 - \prod_{k=1}^r (1 - w_k^F(\beta_j))\right]}{\sum_{k=1}^r \left[1 - \prod_{k=1}^r (1 - w_k^F(\beta_j))\right]} \ ,$$

where $\sum_{k=1}^r W_j = 1$.

Step 4. Calculation of the weighted similarity measure. We calculate weighted similarity measure between the ideal matrix M and the k-th decision matrix M^k as follows:

$$s_{NH}^W\left(M, M^k\right) = \left\langle \lambda_i^k \right\rangle = \left(\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k\right)^T = \left[\frac{1}{m} \sum_{j=1}^m (1 - \frac{D_{ij}^k}{5}) W_j\right]_{i=1}^n$$

where $D_{ij}^k = \left|A_{ij}^-((x_r) - A_{ij}^{k,-}(x_r)\right| + \left|A_{ij}^+(x_r) - A_{ij}^{k,+}(x_r)\right| + \left|A_{ij}^{\in}((x_r) - A_{ij}^{k,\in}(x_r)\right| + \left|A_{ij}^{\notin}((x_r) - A_{ij}^{k,\notin}(x_r)\right| + |\lambda(x_r) - \lambda(x_r)|$ for each $x_r \in X$ and $k = 1, 2, \dots, r$.

Step 5. Ranking of alternatives. In order to rank alternatives, we give the following formula:

$$\rho_i = \sum_{k=1}^{r} \delta_k \lambda_i^k,$$

where $i = 1, 2, \ldots, n$.

We can arrange alternatives according to the descending order values of ρ_i. The highest value of ρ_i reflects the best alternative.

Example 2 (numerical example). *To address a multi-criteria decision-making (MCDM) problem, we adapt the "Illustrative example" provided by [2] to showcase the applicability and effectiveness of the proposed method. Consider an investment company looking to allocate funds to the most promising option. The decision-making committee consists of three members, labelled* $k_1, k_2,$ *and* k_3 *to make a panel of four alternatives to invest money. The alternatives are car company* (α_1)*, food company* (α_2)*, computer company* (α_3)*, and arm company* (α_4)*. Decision makers take decisions based on the criteria, namely risk analysis* (β_1)*, growth analysis* (β_2)*, and environment impact* (β_3)*. The committee assigns weights to these criteria using linguistic variables, which are then transformed into values represented in an octahedron set format (refer to Table 1). This structured approach facilitates a thorough evaluation of each investment option based on the specified criteria.*

Table 1. Linguistic terms for rating an attribute/criterion.

Linguistic Terms	Octahedron Set
Very important (VI)	$\langle [0.7,\ 0.9], (0.7,\ 0.2), 0.9 \rangle$
Important (I)	$\langle [0.6,\ 0.8], (0.6,\ 0.3), 0.6 \rangle$
Medium (M)	$\langle [0.4,\ 0.5], (0.5,\ 0.4), 0.5 \rangle$
Unimportant (UI)	$\langle [0.2,\ 0.4], (0.3,\ 0.6), 0.4 \rangle$
Very unimportant (VUI)	$\langle [0.1,\ 0.2], (0.2,\ 0.7), 0.2 \rangle$

Step 1. Formation of the ideal octahedron set decision matrix. The ideal octahedron set decision matrix M is given as follows:

$$M = \begin{bmatrix} & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle \\ \alpha_2 & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle \\ \alpha_3 & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle \\ \alpha_4 & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle \end{bmatrix}$$

Step 2. Construction of the octahedron set decision matrix. The k_i*-th decision matrix* $M^{ki}(i = 1,\ 2,\ 3)$ *in the octahedron set form is constructed to evaluate four alternatives across three criteria. The matrix is structured as follows:*

$$M^{k_1} = \begin{bmatrix} & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle \\ \alpha_2 & \langle [0.6,0.8],(0.6,0.3),0.8 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle \\ \alpha_3 & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.6,0.8],(0.6,0.3),0.8 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle \\ \alpha_4 & \langle [0.3,0.4],(0.4,0.5),0.4 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle \end{bmatrix}$$

$$M^{k_2} = \begin{bmatrix} & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \langle [0.3,0.4],(0.4,0.5),0.4 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle \\ \alpha_2 & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle \\ \alpha_3 & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle \\ \alpha_4 & \langle [0.6,0.8],(0.6,0.3),0.8 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle \end{bmatrix}$$

$$M^{k_3} = \begin{bmatrix} & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \langle[0.4, 0.5], (0.5, 0.4), 0.5\rangle & \langle[0.4, 0.5], (0.5, 0.4), 0.5\rangle & \langle[0.7, 0.9], (0.7, 0.2), 0.9\rangle \\ \alpha_2 & \langle[0.4, 0.5], (0.5, 0.4), 0.5\rangle & \langle[0.7, 0.9], (0.7, 0.2), 0.9\rangle & \langle[0.4, 0.5], (0.5, 0.4), 0.5\rangle \\ \alpha_3 & \langle[0.7, 0.9], (0.7, 0.2), 0.9\rangle & \langle[0.6, 0.8], (0.6, 0.3), 0.8\rangle & \langle[0.6, 0.8], (0.6, 0.3), 0.8\rangle \\ \alpha_4 & \langle[0.7, 0.9], (0.7, 0.2), 0.9\rangle & \langle[0.4, 0.5], (0.5, 0.4), 0.5\rangle & \langle[0.3, 0.4], (0.4, 0.5), 0.4\rangle \end{bmatrix}$$

Step 3. Determination of attribute weights. Linguistic terms listed in Table 1 are employed to assess each attribute. Each decision maker rates the significance of every attribute using these linguistic terms, as detailed in Table 2. Furthermore, these linguistic terms are then transformed into an octahedron set format, as outlined in Table 3. This structured approach allows for a nuanced interpretation of attributes, facilitating a comprehensive evaluation process in decision-making scenarios.

Table 2. Attribute rating of linguistic variables.

	β_1	β_2	β_3
k_1	VI	M	I
k_2	VI	VI	M
k_3	M	VI	M

Table 3. Attribute rating in octahedron set.

	β_1	β_2	β_3
k_1	$\langle[0.7, 0.9], (0.7, 0.2), 0.9\rangle$	$\langle[0.4, 0.5], (0.5, 0.4), 0.5\rangle$	$\langle[0.6, 0.8], (0.6, 0.3), 0.8\rangle$
k_2	$\langle[0.7, 0.9], (0.7, 0.2), 0.9\rangle$	$\langle[0.7, 0.9], (0.7, 0.2), 0.9\rangle$	$\langle[0.4, 0.5], (0.5, 0.4), 0.5\rangle$
k_3	$\langle[0.4, 0.5], (0.5, 0.4), 0.5\rangle$	$\langle[0.7, 0.9], (0.7, 0.2), 0.9\rangle$	$\langle[0.4, 0.5], (0.5, 0.4), 0.5\rangle$

By using the above-mentioned equations, we obtain the following attribute weights [2]:

$$W_1 = W_2 = W_3 = 0.33.$$

Step 4. Calculation of weighted similarity measures. By using the equations, we obtain weighted similarity measures between the ideal matrix M and the k_s-th decision matrix $M^{k_s}(s = 1, 2, 3)$ *as follows:*

$$s^W_{NH}\left(M, M^{k_1}\right) = \begin{bmatrix} 0.205 \\ 0.207 \\ 0.187 \\ 0.178 \end{bmatrix}, \; s^W_{NH}\left(M, M^{k_2}\right) = \begin{bmatrix} 0.178 \\ 0.185 \\ 0.218 \\ 0.220 \end{bmatrix}, \; s^W_{NH}\left(M, M^{k_3}\right) = \begin{bmatrix} 0.211 \\ 0.211 \\ 0.229 \\ 0.187 \end{bmatrix}$$

Step 5. Ranking of alternatives. In order to rank the alternatives according to the descending value of ρ_i, by using the equations, we obtain $\rho_i(i = 1, 2, 3, 4)$:

$$\rho_1 = 0.196, \; \rho_2 = 0.199, \; \rho_3 = 0.232, \rho_4 = 0.193$$

Then, $\rho_3 > \rho_2 > \rho_1 > \rho_4$. Thus, the ranking order is as follows:

$$\alpha_3 > \alpha_2 > \alpha_1 > \alpha_4$$

Thus, it is evident that computer company (α_3) presents the optimal choice for financial investment.

5. An Innovative Algorithm Based on Octahedron Sets via Multi-Criteria Decision Making

The COVID-19 pandemic, which was identified with 59 suspect cases (in China Hubei/Wuhan) who presented to world health organizations, has spread to affect the whole world [20]. For preventing the transmission of the disease, attention should be paid to hygiene, social distance, and use of masks. Besides these, vaccination studies have taken place as a new method of protection. Several types of vaccine have been developed for COVID-19. Vaccines are biological tools that provide immunity against illness. If the immune system is examined, the functioning of vaccines in the human body is clearly understood [20,24].

At Manisa Celal Bayar University Faculty of Medicine, Medical Microbiology and Infectious Diseases, the first phase of the scientific research on monitoring the safety and protection provided by the CoronaVac vaccine, "Antibody respance and activity monitoring after inactivated COVID-19 vaccination in healthcare workers", has been completed and results have been obtained in tabular form. For the first stage, during the first 7 days after the first dose of CoronaVac vaccine, feedback was received from 791 of nearly 180′0 vaccinated healthcare professionals as the participants were questioned in terms of side effects. It has been reported that 575 patients of 791 had no side effect and 216 patients of 791 had mild side effects. Mild side effects after vaccination have been shown in tabular form and their range has been explained. The age distribution range of participants is 19–65, and the average range of participants is 34.4. After vaccinations, 61.9% of participants who had side effects are female and 38.1% of participants who had side effects are male. After vaccination, 27.3% of participants had local mild side effects or common side effects, but only 1.5% of participants presented to a health facility outpatient service and all of them were discharged; none of the participants who gave feedback had serious side effects. Because of side effects, the most frequent admissions to the health institution were in the 19–29.9 age group (0.9%). Thanks to common side effects, the rate of reference to healthcare institutions was 10% for women and 0.3% for men [25].

The percentages of the duration of mild side effects were based on these data for this survey. In this case, according to the needs of people, the fuzzy soft set theory should be applied to the current problem to obtain the right decision, namely to choose the period to be observed first. The Algorithm 1 used for this is as follows:

Algorithm 1: Period Algorithm

Step 1: Input a suitable parameter set S and universal MS H.
Step 2: Input SMSs Ω_A and Ω_B over H.
Step 3: Construct SMS-topology $\widetilde{T}\prime$ containing Ω_A and Ω_B as soft open MSs in $\widetilde{T}$.
Step 4: Compute the aggregate fuzzy soft sets by using the formula,
$\Gamma_A = \{(\mu_i, \Gamma_A(\mu_i)) : \mu_i \in S\},\ \Gamma_A\ (\mu_i) = \frac{k_i/|\Omega_A(\mu_i)|}{w_i};\ \frac{k_i}{w_i} \in \Omega_A(\mu_i)$
Step 5: Find resultant fuzzy soft set $\Gamma_A \bigvee \Gamma_B = \Gamma_{AXB}$ by applying 'OR' operation on Γ_A and Γ_B.
Step 6: use comparison table of $\Gamma_A \bigvee \Gamma_B$ to calculate row-sum(r_i) and column-sum(t_i) for w_i, $\forall i$.
Step 7: Calculate the resulting score R_i of w_i, ($\forall i$)
Step 8: Optimal choice is w_j that has $\max\{R_i\}$.
Step 9: Compute the SMS boundary of soft open MSs.
Step 10: Here non-null SMS boundary of SMS that contains. $\frac{k_j}{w_j}$ is a decision set [16].
$H = \left[\frac{25}{w_1}, \frac{30}{w_2}, \frac{32}{w_3}, \frac{13}{w_4}\right]$ Let there be a distribution of the duration of mild side effects given.
$w_1 =$ first 1–2 h
$w_2 =$ 3–6 h
$w_3 =$ 7–24 h
$w_4 =$ 25 h and above and $w_i (i = 1,\ 2,\ 3,\ 4)$ shows the distribution of time intervals corresponding to w_i.
Consider the set of attributes $S = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ where
$\mu_1 =$ antibody level
$\mu_2 =$ protection in never having the disease
$\mu_3 =$ protection of those who suffer from the disease.

In this study, mild side effects that occur in people who were vaccinated were examined and their side effects were evaluated. In line with these data obtained, we use the following algorithm to select the age group that should be followed with the highest priority according to the rates given by age groups of any regional and/or general mild side effects after COVID-19 vaccination. The two decision makers (DMs) Ω_1 and Ω_2 present the result of the report of the selection made by the traditional method. Let the DMs Ω_1 and Ω_2 select two sets of attributes $A = \{\mu_1, \mu_2, \mu_3\}$ and $B = \{\mu_1, \mu_2, \mu_3\}$ groups of people found by the age range, respectively. Then, DMs construct two SMSs named as Ω_A and Ω_B over H, given by:

$$\Omega_A = \{\left(\mu_1, \left[\frac{25}{w_1}, \frac{30}{w_2}, \frac{13}{w_4}\right]\right), \left(\mu_2, \left[\frac{30}{w_2}, \frac{32}{w_3}, \frac{13}{w_4}\right]\right), (\mu_3, [H])\}$$

and:

$$\Omega_B = \{\left(\mu_1, \left[\frac{25}{w_1}, \frac{30}{w_2}\right]\right), \left(\mu_2, \left[\frac{30}{w_2}, \frac{32}{w_3}\right]\right), \left(\mu_3, \left[, \frac{13}{w_4}\right]\right)\} \text{ dir.}$$

There are certain properties of soft multi-set topology with applications in multi-criteria decision making. The first SMS Ω_A can be written as [25]:

Ω_A	μ_1	μ_2	μ_3
w_1	25	0	25
w_2	30	30	30
w_3	0	32	32
w_4	13	13	13

The second SMS Ω_B can be written as:

Ω_A	μ_1	μ_2	μ_3
w_1	25	0	0
w_2	30	30	0
w_3	0	32	0
w_4	0	0	13

Here, we make an SMS topology on Ω_A as $\widetilde{T} = \{\Omega_\varnothing, \Omega_A, \Omega_B\}$, where $\Omega_\varnothing$ is an empty SMS. Now, we find the aggregate fuzzy soft sets Γ_A and Γ_B given by:

$$\Gamma_A = \left\{\left(\mu_1, \left\{\frac{0.36}{w_1}, \frac{0.44}{w_2}, \frac{0.19}{w_4}\right\}\right), \left(\mu_2, \left\{\frac{0.4}{w_2}, \frac{0.42}{w_3}, \frac{0.17}{w_4}\right\}\right), \left(\mu_3, \left\{\frac{0.25}{w_1}, \frac{0.3}{w_2}, \frac{0.32}{w_3}, \frac{0.13}{w_4}\right\}\right)\right\}$$

and:

$$\Gamma_B = \left\{\left(\mu_1, \left\{\frac{0.45}{w_1}, \frac{0.54}{w_2}\right\}\right), \left(\mu_2, \left\{\frac{0.62}{w_2}, \frac{0.51}{w_3}\right\}\right), \left(\mu_3, \left\{\frac{1}{w_4}\right\}\right)\right\}$$

The fuzzy soft set Γ_A can be written as:

Γ_A	μ_1	μ_2	μ_3
w_1	0.36	0	0.25
w_2	0.44	0.4	0.3
w_3	0	0.42	0.32
w_4	0.19	0.17	0.13

The fuzzy soft set Γ_B can be written as:

Γ_B	μ_1	μ_2	μ_3
w_1	0.45	0	0
w_2	0.54	0.62	0
w_3	0	0.51	0
w_4	0	0	1

We apply here "OR" operation on Γ_A and Γ_B; then, we obtain $3 * 3 = 9$ attributes of the form $\mu_{ij} = (\mu_i, \mu_j)$ for all $i, j \in \{1,2,3\}$.

We find the fuzzy soft set of attributes:

$A \times B = \{\mu_{11}, \mu_{12}, \mu_{13}, \mu_{21}, \mu_{22}, \mu_{23}, \mu_{31}, \mu_{32}, \mu_{33}\}$; after applying "OR" operation, we obtain fuzzy soft set $\Gamma_A \vee \Gamma_B$ given as:

$$\Gamma_A \vee \Gamma_B = \left\{ \left(\mu_{11}, \left\{\frac{0.45}{w_1}, \frac{0.54}{w_2}, \frac{0}{w_3}, \frac{0.19}{w_4}\right\}\right), \left(\mu_{12}, \left\{\frac{0.36}{w_1}, \frac{0.62}{w_2}, \frac{0.51}{w_3}, \frac{0.19}{w_4}\right\}\right), \left(\mu_{13}, \left\{\frac{0.36}{w_1}, \frac{0.44}{w_2}, \frac{0}{w_3}, \frac{1}{w_4}\right\},\right) \right.$$

$$(\mu_{21}, \left\{\frac{0.45}{w_1}, \frac{0.54}{w_2}, \frac{0.42}{w_3}, \frac{0.17}{w_4}\right\}), (\mu_{22}, \left\{\frac{0}{w_1}, \frac{0.62}{w_2}, \frac{0.51}{w_3}, \frac{0.17}{w_4}\right\}), (\mu_{23}, \left\{\frac{0}{w_1}, \frac{0.4}{w_2}, \frac{0.42}{w_3}, \frac{1}{w_4}\right\}), (\mu_{31}, \left\{\frac{0.45}{w_1}, \frac{0.54}{w_2}, \frac{0.32}{w_3}, \frac{0.13}{w_4}\right\},$$

$$(\mu_{32}, \left\{\frac{0.25}{w_1}, \frac{0.62}{w_2}, \frac{0.51}{w_3}, \frac{0.13}{w_4}\right\}), (\mu_{33}, \left\{\frac{0.25}{w_1}, \frac{0.30}{w_2}, \frac{0.32}{w_3}, \frac{1}{w_4}\right\}$$

Now, the tabular form of $\Gamma_A \vee \Gamma_B$ is written as:

$\Gamma_A \vee \Gamma_B$	μ_{11}	μ_{12}	μ_{13}	μ_{21}	μ_{22}	μ_{23}	μ_{31}	μ_{32}	μ_{33}
w_1	0.45	0.36	0.36	0.45	0	0	0.45	0.25	0.25
w_2	0.54	0.62	0.44	0.54	0.62	0.4	0.54	0.62	0.30
w_3	0	0.51	0	0.42	0.51	0.42	0.32	0.51	0.32
w_4	0.19	0.19	1	0.17	0.17	1	0.13	0.13	1

Now, we find the comparison table of fuzzy set $\Gamma_A \bigvee \Gamma_B$ using the algorithm, which is given by [26]. The comparison table is given below:

	w_1	w_2	w_3	w_4
w_1	9	9	5	4
w_2	0	9	2	3
w_3	4	6	9	4
w_4	5	6	5	9

We calculate the column sum (t_i) and row sum (r_i) and, after that, we calculate the score (R_i) for each $w_i,\ i = 1,2,3,4$.

As can be seen from this Table 4, the highest score is 9 with w_1. If we evaluate the algorithm result here, it is seen that the percentage of side effects after vaccination is quite low and it can be said that the side effects occur within the first 1–2 h. In this case, individuals may be advised to observe the first 1–2 h after vaccination.

Table 4. Tabular form of score $(R_i = r_i - t_i)$.

	Row sum (r_i)	Column sum (t_i)	Score $(R_i = r_i - t_i)$
w_1	27	18	9
w_2	14	30	−16
w_3	23	21	2
w_4	25	20	5

6. Conclusions and Future Work

As scientific progress diversifies social phenomena, there is a growing need to manage multiple tasks concurrently. Mathematicians are responding to this challenge by seeking to develop mathematical instruments that can provide broader support. They aim to create an expansive hybrid structure that integrates an interval-valued fuzzy set, an intuitionistic fuzzy set, and a traditional fuzzy set. This advanced structure is designed to enrich the understanding of uncertainty by enabling point measurements, interval assessments, and the ability to record both positive and negative evaluations simultaneously. This approach facilitates a more comprehensive analysis of events in a single framework. Our aim in conducting this study is to present a useful example and algorithm for the multi-criteria decision-making method for the selection of alternatives used in daily life and the decisions that can be reached as a result, with the help of the concept of octahedron sets. It has been concluded that the algorithm used in this study can be used in different disciplines other than mathematics, and it is thought that these fields will shed light on future studies. In this study, an example of an algorithm that can be used in alternative decision-making processes is introduced. A new and useful application result was obtained by applying an example of the algorithm previously developed by [26] and produced by [27]. By adopting this advanced approach in academic studies, strategic planning, risk management, and resource allocation strategies can be developed. The synergy between flexible multi-sets and multi-criteria decision-making methods enables organizations to make informed decisions that drive sustainable growth and competitive advantage. Consequently, the application of octahedron sets through multi-criteria decision-making methods represents a transformative paradigm shift in modern decision science. The most important innovation provided by this work is that it enables the synergy between these methodologies to unlock new possibilities to optimize decision processes and achieve superior results in an increasingly complex set environment.

Last but not least, since it is well known that octahedron set operators and MCDM are alternative tools for defining approximation operators and improving their accuracy, one of the main focuses of future work is to look at how the proposed class can be implemented with these sets. Information systems will be used to choose the most appropriate option and make the right decision. We also plan to do the following: the interval-valued fuzzy set, which is the first component of the octahedron set, will be defined as an interval-valued intuitive fuzzy set, and a more appropriate decision-making solution will be achieved with the interval-valued intuitive fuzzy set, which provides more uncertainty than the interval-valued fuzzy set. It is envisaged that this method will allow all information provided by decision makers to be expressed as interval-valued intuitive fuzzy decision matrices and will meet much more demands than fuzzy decision-making problems.

Continuing from the advancements highlighted in the previous sections, the integration of octahedron sets in multi-criteria decision-making processes not only addresses complex decision scenarios but also introduces a new dimension of precision and adaptability. This methodological evolution signifies a breakthrough in handling ambiguities and uncertainties inherent in diverse fields ranging from economics to environmental science.

One of the anticipated developments in this area is the enhancement of algorithmic approaches to further refine the decision-making process. By leveraging the capabilities of octahedron sets to incorporate various types of uncertainty, future algorithms can better simulate real-world conditions, thus providing decision makers with solutions that are not only theoretically sound but also practically viable. This could lead to the development of decision support systems that are more robust, flexible, and capable of delivering nuanced insights into complex problems.

Furthermore, the potential of octahedron sets to facilitate a deeper understanding of the underpinnings of decision criteria will allow researchers and practitioners to explore new applications in fields that have traditionally relied on more deterministic approaches. This could catalyze a shift towards more dynamic and responsive strategies in areas such as healthcare management, urban planning, and strategic business management, where decision making often involves high stakes and requires the assimilation of vast amounts of data. Moreover, the adoption of these advanced mathematical tools in educational curricula could revolutionize the way future mathematicians, engineers, and decision scientists are trained. By embedding the principles of fuzzy logic and decision theory enriched by octahedron sets into academic programs, educational institutions can prepare students to tackle real-world challenges with innovative solutions that are grounded in cutting-edge research.

In conclusion, as we continue to push the boundaries of what is possible with multi-criteria decision-making frameworks, the role of octahedron sets is poised to become increasingly central. This progression not only promises to enhance the efficacy and efficiency of decision processes but also paves the way for pioneering research and applications that could redefine problem solving in numerous domains. The intersection of theoretical innovation and practical application remains a fertile ground for future explorations, promising significant advancements in the way we perceive and interact with complex decision-making environments.

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