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Exploring Properties and Applications of Laguerre Special Polynomials Involving the Δ_h Form

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Abstract: The primary objective of this research is to introduce and investigate novel polynomial variants termed Δ_h Laguerre polynomials. This unique polynomial type integrates the monomiality principle alongside operational rules. Through this innovative approach, the study delves into uncharted territory, unveiling fresh insights that build upon prior research endeavours. Notably, the Δ_h Laguerre polynomials exhibit significant utility in the realm of quantum mechanics, particularly in the modelling of entropy within quantum systems. The research meticulously unveils explicit formulas and elucidates the fundamental properties of these polynomials, thereby forging connections with established polynomial categories. By shedding light on the distinct characteristics and functionalities of the Δ_h Laguerre polynomials, this study contributes significantly to their comprehension and application across diverse mathematical and scientific domains.

Keywords: Δ_h sequences; monomiality principle; explicit forms; symmetric identities; series representation

MSC: 33E20; 33B10; 33E30; 11T23

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1. Introduction and Preliminaries

The exploration of generalized and multivariable special functions has seen significant advancements in recent years, greatly enhancing our ability to address complex physical problems. These functions, which extend beyond traditional special functions, offer powerful tools for solving intricate partial differential equations (PDEs). Their multivariable nature allows for the simultaneous consideration of multiple interacting variables, which is essential in modelling realistic physical systems. This progress has enabled researchers to develop more accurate and comprehensive solutions to PDEs that arise in diverse fields such as quantum mechanics, fluid dynamics, and electromagnetic theory. By employing these advanced functions, scientists can now tackle problems involving intricate boundary conditions and non-linear interactions with greater precision. Consequently, the improved mathematical framework not only aids in theoretical developments but also in practical applications, leading to innovations in technology and engineering. Thus, the advancements in generalized and multivariable special functions mark a crucial step forward in the mathematical modelling of complex physical phenomena. Recent studies have focused on investigating special polynomials of two variables that have been incredibly useful in solving complex challenges. One notable class of polynomial sequences is Laguerre polynomials, which are integral in the fields of physics and mathematics due to their orthogonality properties. These polynomials were first introduced by the French mathematician Edmond Laguerre in the 19th century, and since then, numerous authors have made significant contributions to their study. For more information, key references include [1–8].

Laguerre polynomials are widely utilized in mathematics, physics, and engineering to solve the Schrodinger equation for the hydrogen atom and other quantum systems exhibiting spherical symmetry. Their applications extend to various areas such as quantum mechanics, spectroscopy, and atomic physics, where these polynomials play a crucial role in describing the behaviour of electrons in complex atomic structures. Additionally, they find applications in fields such as signal processing and probability theory due to their unique properties and mathematical significance. Furthermore, issues involving diffusion equations, wave propagation, and heat conduction give rise to these polynomials.

The significance of two-variable special polynomials is profound, as they offer a versatile and robust framework for addressing problems across various mathematical and scientific domains. These polynomials, characterized by their ability to handle functions involving two variables simultaneously, provide powerful tools for expressing and analyzing complex multivariate relationships.

In mathematics, two-variable special polynomials facilitate the study of multivariate calculus, algebraic geometry, and optimization problems. Their rich structure and properties enable mathematicians to derive explicit solutions to intricate equations, perform precise approximations, and develop new theoretical insights. For example, in approximation theory, these polynomials can approximate bivariate functions with high accuracy, aiding in numerical analysis and computational methods.

In scientific applications, particularly in physics and engineering, two-variable special polynomials are indispensable. They are employed in solving partial differential equations (PDEs) that describe a wide range of phenomena, from heat conduction and wave propagation to fluid dynamics and quantum mechanics. Their specific properties, such as orthogonality and recurrence relations, make them highly suitable for constructing solutions to these PDEs, allowing for the more accurate modelling of physical systems. Additionally, in fields like statistics and data science, these polynomials are used in multivariate statistical analysis and the modelling of complex datasets. They enable the extraction of meaningful patterns and relationships from data involving multiple variables, thus contributing to more informed decision-making and predictions.

Therefore, the utility of two-variable special polynomials in expressing and analyzing multivariate functions, coupled with their application-specific properties, underscores their significance in advancing both theoretical research and practical problem-solving in various domains.

The two-variable Laguerre polynomials ($\mathbb{W}_n(u,v)$) [9,10] are defined by the generating function:

$$e^{v\xi}\mathcal{C}_0(u\xi) = \sum_{n=0}^{\infty} \mathbb{W}_n(u,v)\frac{\xi^n}{n!}, \tag{1}$$

where $\mathcal{C}_0(u\xi)$ is the 0^{th} Bessel–Tricomi function [11], operationally defined as

$$\mathcal{C}_0(\alpha u) = \exp(-\alpha D_u^{-1}\{1\})\{1\}; \quad D_u^{-n}\{1\} := \frac{u^n}{n!}. \tag{2}$$

The Bessel–Tricomi function $\mathcal{C}_n(u)$ is expressed through the series expansion:

$$\mathcal{C}_n(u) = u^{-n/2}J_n(2\sqrt{u}) = \sum_{k=0}^{\infty} \frac{(-1)^k u^k}{k!(n+k)!}, \tag{3}$$

with $J_n(u)$ being the cylindrical Bessel function of the first kind [11].

The 2VLP $\mathbb{W}_n(u,v)$ can also be defined by the series

$$\mathbb{W}_n(u,v) = n!\sum_{k=0}^{n} \frac{(-1)^k u^k v^{n-k}}{(k!)^2(n-k)!}. \tag{4}$$

These polynomials are of intrinsic mathematical interest and have significant applications in physics, particularly as natural solutions to specific partial differential equations. One such equation is

$$\frac{\partial}{\partial v}\mathbb{W}_n(u,v) = -\left(\frac{\partial}{\partial u}u\frac{\partial}{\partial u}\right)\mathbb{W}_n(u,v); \quad \mathbb{W}_n(u,0) = \frac{(-u)^n}{n!}, \tag{5}$$

which resembles a Fokker–Planck-type heat diffusion equation and is used to study the beam lifetime due to quantum fluctuations in storage rings [12].

Furthermore, the 2VLP $\mathbb{W}_n(u,v)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\Theta_n^+ = v - D_u^{-1}, \quad \Theta_n^- = -D_u u D_u. \tag{6}$$

Mathematicians have recently demonstrated great interest in introducing different versions of special polynomials and hybrid special polynomials; see, for instance, [13–17]. Further, in [18,19], authors introduced a new variant/version of the special polynomials, called Δ_h special polynomials of different polynomials, by employing the classical finite difference operator Δ_h. These Δ_h special polynomials have drawn attention due to their amazing applicability in statistics and physics, in addition to several disciplines of mathematics. A recent attempt to introduce Δ_h polynomial sequences, namely Δ_h Appell polynomials, and to study their many features was undertaken by Costabile and Longo [18]. The mathematical representation of the Δ_h-Appell polynomial is as follows:

$$\mathbb{A}_n^{[h]}(u) := \mathbb{A}_n(u), \quad n \in \mathbb{N}_0 \tag{7}$$

and defined by

$$\mathbb{A}_n^{[h]}(u) = nh\mathcal{A}_{n-1}(u), \quad n \in \mathbb{N}, \tag{8}$$

where Δ_h being the F.D.O:

$$_{\Delta_h}\mathbb{H}^{[h]}(u) = \mathbb{H}(u+h) - \mathbb{H}(u). \tag{9}$$

Further, the Δ_h-Appell polynomials $\mathbb{A}_n(u)$ are defined through the generating relation [18]:

$$\gamma(\xi)(1+h\xi)^{\frac{u}{h}} = \sum_{n=0}^{\infty} \mathbb{A}_n^{[h]}(u)\frac{\xi^n}{n!}, \tag{10}$$

where

$$\gamma(\xi) = \sum_{n=0}^{\infty} \gamma_{n,h}\frac{\xi^n}{n!}, \qquad \gamma_{0,h} \neq 0. \tag{11}$$

Moreover, in [18], Δ_h Appell sequences $\mathbb{A}_n(u), \quad n \in \mathbb{N}$ are defined by $\gamma(\xi)(1+h\xi)^{\frac{q}{h}}$ in the power series of the product of two functions:

$$\gamma(\xi)(1+h\xi)^{\frac{u}{h}} = \mathbb{A}_0^{[h]}(u) + \mathbb{A}_1^{[h]}(u)\frac{\xi}{1!} + \mathbb{A}_2^{[h]}(u)\frac{\xi^2}{2!} + \cdots \mathbb{A}_n^{[h]}(u)\frac{\xi^n}{n!}\cdots, \tag{12}$$

where

$$\gamma(\xi) = \gamma_{0,h} + \gamma_{1,h}\frac{\xi}{1!} + \gamma_{2,h}\frac{\xi^2}{2!} + \cdots + \gamma_{n,h}\frac{\xi^n}{n!} + \cdots. \tag{13}$$

The Δ_h Appell sequences represent a specialized class of Appell sequences that emerge from the application of the difference operator Δ_h to a pre-existing Appell sequence. This process involves applying the operator Δ_h, defined as $\Delta_h f(u) = f(u+h) - f(u)$, to generate new sequences with distinct properties. Within this framework, the parameter q within the Appell sequence assumes specific values, which simplifies the resulting Δ_h Appell sequences into well-established sequences and polynomials. This simplification is significant as it provides a deeper understanding of the behaviour and characteristics of these sequences, revealing their connections to more familiar mathematical constructs. For example, when q takes on certain values, the Δ_h Appell sequences can be reduced to classical polynomial sequences such as Bernoulli polynomials, Euler polynomials, or Hermite polynomials. These reductions highlight the versatility and broad applicability of the Δ_h Appell sequences in various mathematical contexts.

Connecting to well-known sequences allows mathematicians to leverage existing knowledge and techniques to study the Δ_h Appell sequences. For instance, understanding their orthogonality properties, generating functions, and recurrence relations becomes more accessible through these connections. Additionally, these insights can lead to the discovery of new identities and relationships within the realm of special functions and polynomial sequences.

In applied mathematics and physics, Δ_h Appell sequences can be instrumental in solving difference equations, modelling discrete systems, and analyzing numerical methods. Their structured properties enable the precise formulation and solution of problems involving discrete changes, which are common in computational algorithms and discrete dynamical systems.

Overall, the study of Δ_h Appell sequences enriches the theory of special functions and polynomial sequences and enhances our ability to address practical problems in various scientific and engineering disciplines. The parameter q's role in simplifying these sequences into well-known forms underscores the inter-connectedness of mathematical concepts and the value of exploring these specialized classes. For instance:

When the variable u is a non-negative integer, the generalized falling factorials, denoted as $(u)_n^h \equiv (u)_n$, are obtained as a special case of the Δ_h Appell sequence, which is described in the reference [20].

When the value of u is equal to 1, the Bernoulli sequence of the second kind, denoted as $b_n(u)$, can be derived as a special case of the Δ_h Appell sequence, as referenced in [20].

When the parameter u is equal to the constant δ, where δ is a constant, the Poisson–Charlier sequence $C_n(u;\delta)$, as referenced in [20] (p. 2), is derived as a special case of the Δ_h Appell sequence.

When the values are $u = \delta$ and $h = 1$, the Boole sequence $B_{\ln}(u;u)$ can be derived as a special case of the Δ_h Appell sequence.

In various practical applications, it is often necessary to utilize well-known sequences and polynomials; these special cases can be quite beneficial. The Δ_h Appell sequences offer a structured method for obtaining these sequences and polynomials.

Motivated by Costabile [18], here we introduced the two-variable Δ_h Laguerre polynomials:

$$(1+h\xi)^{\frac{v}{h}}(1+h\xi^2)^{\frac{D_u^{-1}}{h}}\{1\} = \sum_{n=0}^{\infty} \mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!}, \tag{14}$$

through the generating function concept.

The significance of two-variable Δ_h Laguerre special polynomials lies in their ability to extend the classical Laguerre polynomials to multivariate and discrete settings, providing a powerful tool for solving complex mathematical and physical problems. Here are several key aspects of their importance: The two-variable Δ_h Laguerre polynomials generalize the classical Laguerre polynomials, which are widely used in single-variable contexts, particularly in solving differential equations related to quantum mechanics and other physical systems. Extending these polynomials to two variables facilitates the analysis of systems involving two interacting components or dimensions. The Δ_h operator introduces a discrete component to the Laguerre polynomials, making them suitable for problems involving discrete changes or steps. This is particularly useful in numerical analysis and discrete dynamical systems, where the behaviour of a system is studied at discrete points in time or space. These polynomials maintain connections to classical Laguerre polynomials, allowing well-established mathematical techniques and theories to be used. This connection helps in deriving new properties, and in generating functions, and identities that enrich the broader theory of special functions and polynomial sequences. The two-variable Δ_h Laguerre special polynomials provide a powerful and versatile mathematical and physical problem-solving tool. Their ability to handle two variables and discrete changes makes them invaluable in a wide range of applications, from solving PDEs and discrete dynamical systems to modelling complex interactions in physics and engineering. Extending the classical Laguerre polynomials opens new avenues for research and practical applications, highlighting the deep interconnections within mathematical theory.

The rest of thearticle is presented as follows:

Section 2 delves into the generation of Laguerre polynomials and examines the recurrence relations that characterize their behavior. In Section 3, formulas for summing or evaluating these polynomials over specific ranges or under particular conditions are provided, offering efficient methods for calculating their values. Section 4 introduces the monomiality principle, which describes the behavior of Laguerre polynomials under certain operations, and also establishes their determinant form.

Section 5 derives symmetric identities for these polynomials. finally, the concluding remarks are provided, which highlights the article's key findings, explores their implications and applications, and proposes possible directions for future research on Laguerre polynomials.

2. Two-Variable Δ_h Laguerre Polynomials

This section plays a pivotal role in introducing a novel class of two-variable Δ_h Laguerre polynomials and establishing their core properties. The research significantly contributes to the existing literature, broadening our understanding and opening new avenues for exploration within polynomial theory and its diverse applications.

The creation of the generating function for Δ_h Laguerre polynomials, denoted by $\mathbb{W}_n^{[h]}(u,v)$, marks a crucial advancement in grasping the intricate characteristics and attributes of these polynomials. Generating functions are indispensable tools in fields such as combinatorics, analysis, and mathematical physics, as they offer profound insights into the underlying structure and behavior of sequences and functions.

Moreover, this study deepens the mathematical community's comprehension of polynomial families and their practical uses by linking Δ_h Laguerre polynomials to their generating functions. The properties discussed in this section illuminate the distinctive traits and behaviors of these polynomials, facilitating their application in a wide range of mathematical and scientific contexts. This section represents a significant leap forward in the field of polynomial theory, providing new perspectives and potential applications that warrant further investigation.

To construct the generating function for the Δ_h Laguerre polynomials $\mathbb{W}_n^{[h]}(u,v)$, we begin by proving the following result:

Theorem 1. *For the 2V Δ_h Laguerre polynomials $\mathbb{W}_n^{[h]}(u,v)$, the following generating relation holds:*

$$(1+h\xi)^{\frac{v-D_u^{-1}}{h}} = \sum_{n=0}^{\infty} \mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!}, \tag{15}$$

or equivalently,

$$(1+h\xi)^{\frac{v}{h}}C_0\left(\frac{u}{h}\log(1+h\xi)\right) = \sum_{n=0}^{\infty} \mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!}. \tag{16}$$

Proof. The set of polynomials $\mathbb{W}_n^{[h]}(u,v)$, represented in Equation (15) as the coefficients of $\frac{\xi^n}{n!}$, serve as the generating function for the two-variable Δ_h Laguerre polynomials $\mathbb{W}_n^{[h]}(u,v)$. This is achieved by expanding $(1+h\xi)^{\frac{v-D_u^{-1}}{h}}$ at $u=v=0$ using a Newton series for finite differences, and by considering the order of the product in the development

of the function $(1+h\xi)^{\frac{v-D_u^{-1}}{h}}$ with regard to the powers of ξ.
□

Theorem 2. *For the 2V Δ_h Laguerre polynomials $\mathbb{W}_n^{[h]}(u,v)$, the succeeding relations hold true:*

$$\begin{aligned} \frac{{}_v\Delta_h}{h}\mathbb{W}_n^{[h]}(u,v) &= n\,\mathbb{W}_{n-1}^{[h]}(u,v) \\ \frac{{}_u\Delta_h}{h}\mathbb{W}_n^{[h]}(u,v) &= n(n-1)\,\mathbb{W}_{n-2}^{[h]}(u,v),\ \ D_u^{-1}\to u. \end{aligned} \tag{17}$$

Proof. By differentiating (15) with regard to v by taking into consideration expression (9), we find

$$\begin{aligned} {}_v\Delta_h(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\} &= (1+h\xi)^{\frac{v+h}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\} - (1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\} \\ &= (1+h\xi-1)(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\} \\ &= h\xi\,(1+h\xi)^{\frac{v-D_u^{-1}}{h}}. \end{aligned} \tag{18}$$

By substituting the right hand side of expression (15) into (18), we find

$$ {}_v\Delta_h\sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!} = h\sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u,v)\frac{\xi^{n+1}}{n!}. \tag{19}$$

Assertion (17) is derived by substituting $n\to n-1$ into the right hand side of the earlier expression (18) and contrasting the coefficients of the same exponents of ξ in the resulting equation. □

Next, we deduce the explicit form satisfied by these two-variable Δ_h Laguerre polynomials $\mathbb{S}_n^{[h]}(u,v)$ by demonstrating the result.

Theorem 3. *For the two-variable Δ_h Laguerre polynomials $\mathbb{W}_n^{[h]}(u,v)$, the succeeding relations hold true:*

$$\mathbb{W}_n^{[h]}(u,v) = \sum_{d=0}^{[\frac{v}{h}]}\binom{n}{d}\binom{\frac{v}{h}}{d}\,h^d\,\mathbb{W}_{n-d}^{[h]}(u). \tag{20}$$

Proof. Expanding generates relation (15) in the given manner:

$$(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\} = \sum_{d=0}^{\frac{v}{h}}\binom{[\frac{v}{h}]}{d}\frac{(h\xi)^d}{d!}\sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u,0)\frac{\xi^n}{n!} \tag{21}$$

which can further be written as

$$\sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty}\sum_{d=0}^{[\frac{v}{h}]}\binom{\frac{v}{h}}{d}\,h^d\,\mathbb{W}_n^{[h]}(u,0)\frac{\xi^{n+d}}{n!\,d!}. \tag{22}$$

By replacing $n\to n-d$ in the right hand side of the previous expression, it follows that

$$\sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty}\sum_{d=0}^{\frac{v}{h}}\binom{\frac{v}{h}}{d}\,h^d\,\mathbb{W}_n^{[h]}(u,0)\frac{\xi^n}{(n-d)!\,d!}. \tag{23}$$

In the right hand side of the previous statement (23), we multiply and divide by $n!$ to obtain the value of assertion (20). We then compare the coefficients of the same exponents of ξ on both sides. □

3. Summation Formulae

This section introduces fundamental summation formulae, also known as sigma notation, essential for computing sums involving special polynomials with two variables. These formulae provide systematic methods for evaluating complex expressions and uncovering hidden symmetries within polynomial structures, contributing to the development of efficient computational techniques. They are essential for exploring and advancing mathematical theory and its practical applications.

Theorem 4. *For $n \geq 0$, we have*

$$\mathbb{W}_n^{[h]}(u,v) = \sum_{\gamma=0}^{n} \binom{n}{\gamma} \left(-\frac{v}{h}\right)_{\gamma} (-h)^{\gamma} \mathbb{W}_{n-\gamma}^{[h]}(0,u). \quad (24)$$

Proof. From (15), we have

$$\sum_{n=0}^{\infty} \mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!} = (1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\} = \sum_{n=0}^{\infty} \mathbb{W}_n^{[h]}(0,u)\frac{\xi^n}{n!} \sum_{\gamma=0}^{\infty}\left(-\frac{v}{h}\right)_{\gamma}(-h)^{\gamma}\frac{\xi^{\gamma}}{\gamma!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{\gamma=0}^{n}\binom{n}{\gamma}\left(-\frac{v}{h}\right)_{\gamma}(-h)^{\gamma}\mathbb{W}_{n-\gamma}^{[h]}(0,u)\right)\frac{\xi^n}{n!}. \quad (25)$$

Comparing the coefficients of ξ, we obtain (24). □

Theorem 5. *For $n \geq 0$, we have*

$$\mathbb{W}_n^{[h]}(v+1,u) = \sum_{\gamma=0}^{n} \binom{n}{\gamma} \left(-\frac{1}{h}\right)_{\gamma} (-h)^{\gamma} \mathbb{W}_{n-\gamma}^{[h]}(u,v). \quad (26)$$

Proof. From (15), we have

$$\sum_{n=0}^{\infty} \mathbb{W}_n^{[h]}(v+1,u)\frac{\xi^n}{n!} - \sum_{n=0}^{\infty} \mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!} = (1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\}\left((1+h\xi)^{\frac{1}{h}}-1\right)$$
$$= \sum_{n=0}^{\infty} \mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!}\left(\sum_{\gamma=0}^{\infty}\left(-\frac{1}{h}\right)_{\gamma}(-h)^{\gamma}\frac{\xi^{\gamma}}{\gamma!}-1\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{\gamma=0}^{n}\binom{n}{\gamma}\left(-\frac{1}{h}\right)_{\gamma}(-h)^{\gamma}\mathbb{W}_{n-\gamma}^{[h]}(u,v)\right)\frac{\xi^n}{n!} - \sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!}. \quad (27)$$

Comparing the coefficients of ξ, we obtain (26).
□

Theorem 6. *For $n \geq 0$, we have*

$$\mathbb{W}_n^{[h]}(u,v) = \sum_{j=0}^{n}\left(-\frac{v}{h}\right)_{n-j}(-h)^{n-j}\left(-\frac{u}{h}\right)_{j}(-1)^{j}\frac{n!}{(n-j)!(j!)^2}. \quad (28)$$

Proof. From (15), we have

$$\sum_{n=0}^{\infty} \mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!} = (1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\}$$
$$= \sum_{n=0}^{\infty}\left(-\frac{v}{h}\right)_n - h)^n \frac{\xi^n}{n!}\sum_{j=0}^{\infty}\left(-\frac{u}{h}\right)_j(-1)^j(-h)^j\frac{\xi^j}{j!j!}$$
$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\left(-\frac{v}{h}\right)_{n-j}(-h)^{n-j}\left(-\frac{u}{h}\right)_j(-1)^j\frac{\xi^n}{(n-j)!(j!)^2}. \quad (29)$$

Comparing the coefficients of ξ, we obtain (28).
□

Next, we investigate the connection between the Stirling numbers of the first kind and two-variable Δ_h Laguerre polynomials.

$$\frac{[\log(1+\xi)]^k}{k!} = \sum_{i=k}^{\infty} S_1(i,k)\frac{\xi^i}{i!}, \quad |\ \xi\ |< 1. \quad (30)$$

From the above definition, we have

$$(v)_i = \sum_{k=0}^{i}(-1)^{i-k}S_1(i,k)v^k. \quad (31)$$

Theorem 7. *For $n \geq 0$, we have*

$$\mathbb{W}_n^{[h]}(u,v) = \sum_{\gamma=0}^{n}\binom{n}{\gamma}\mathbb{W}_n^{[h]}(u)\sum_{j=0}^{\gamma}v^j S_1(\gamma,j)h^{\gamma-j}. \quad (32)$$

Proof. From (15), we have

$$\sum_{n=0}^{\infty} \mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!} = e^{\frac{v}{h}\log(1+h\xi)}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\}$$
$$= (1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\}\sum_{j=0}^{\infty}\left(\frac{v}{h}\right)^j\frac{[\log(1+h\xi)]^j}{j!}$$
$$= \sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u)\frac{\xi^n}{n!}\sum_{\gamma=0}^{\infty}\sum_{j=0}^{\gamma}\left(\frac{v}{h}\right)^j S_1(\gamma,j)h^{\gamma}\frac{\xi^{\gamma}}{\gamma!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{\gamma=0}^{n}\binom{n}{\gamma}\mathbb{W}_{n-\gamma}^{[h]}(u)\sum_{j=0}^{\gamma}\left(\frac{v}{h}\right)^j S_1(\gamma,j)h^{\gamma}\right)\frac{\xi^n}{n!}. \quad (33)$$

Comparing the coefficients of ξ, we obtain (32).
□

Theorem 8. *For $n \geq 0$, we have*

$$\mathbb{W}_n^{[h]}(0,u) = \sum_{\gamma=0}^{n}\binom{n}{\gamma}\mathbb{W}_{n-\gamma}^{[h]}(u,v)\sum_{j=0}^{\gamma}\left(-\frac{v}{h}\right)^j S_1(\gamma,j)h^{\gamma}. \quad (34)$$

Proof. From (15), we have

$$
\begin{aligned}
(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\} &= e^{-\frac{v}{h}\log(1+h\xi)}\sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!} \\
&= \sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!}\sum_{j=0}^{\infty}\left(-\frac{v}{h}\right)^j\frac{[\log(1+h\xi)]^j}{j!} \\
&= \sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!}\sum_{\gamma=0}^{\infty}\sum_{j=0}^{\gamma}\left(-\frac{v}{h}\right)^j S_1(\gamma,j)h^{\gamma}\frac{\xi^{\gamma}}{\gamma!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{\gamma=0}^{n}\binom{n}{\gamma}\mathbb{W}_{n-\gamma}^{[h]}(u,v)\sum_{j=0}^{\gamma}\left(-\frac{v}{h}\right)^j S_1(\gamma,j)h^{\gamma}\right)\frac{\xi^n}{n!}. \qquad (35)
\end{aligned}
$$

Comparing the coefficients of ξ, we obtain (34).
□

Theorem 9. *For $n \geq 0$, we have*

$$
\mathbb{W}_n^{[h]}(u,v) = \sum_{l=0}^{n}\sum_{\gamma=0}^{n-l}\frac{n!}{(n-\gamma-l)!(\gamma+l)!}h^{\gamma}\mathbb{W}_{n-\gamma-1}^{[h]}(0,u)S_1(\gamma+l,l)v^l. \qquad (36)
$$

Proof. From (15), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!} &= (1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\} \\
&= \sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(0,u)\frac{\xi^n}{n!}\sum_{\gamma=0}^{\infty}\left(-\frac{v}{h}\right)_{\gamma}(-h)^{\gamma}\frac{\xi^{\gamma}}{\gamma!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{\gamma=0}^{n}\binom{n}{\gamma}\left(-\frac{v}{h}\right)_{\gamma}(-h)^{m\gamma}\mathbb{W}_{n-\gamma}^{[h]}(0,u)\right)\frac{\xi^n}{n!}.
\end{aligned} \qquad (37)
$$

Upon comparing the coefficients of ξ, we arrive at the following findings:

$$
\mathbb{W}_n^{[h]}(u,v) = \sum_{\gamma=0}^{n}\binom{n}{\gamma}\left(-\frac{v}{h}\right)_{\gamma}(-h)^{\gamma}\mathbb{W}_{n-\gamma}^{[h]}(0,u). \qquad (38)
$$

Using the above equality (28), we obtain

$$
\begin{aligned}
\mathbb{W}_n^{[h]}(u,v) &= \left(\sum_{\gamma=0}^{n}\binom{n}{\gamma}(-h)^{\gamma}\mathbb{W}_{n-\gamma}^{[h]}(0,u)\right)\left(\sum_{l=0}^{\gamma}(-1)^{\gamma-l}S_1(\gamma,l)(-h)^{-l}v^l\right) \\
&= \sum_{l=0}^{n}\sum_{\gamma=l}^{n}\frac{n!}{(n-\gamma)!\gamma!}(-h)^{\gamma-l}\mathbb{W}_{n-\gamma}^{[h]}(0,u)(-1)^{\gamma-l}S_1(\gamma,l)v^l \\
&= \sum_{l=0}^{n}\sum_{\gamma=0}^{n-l}\frac{n!}{(n-\gamma-l)!(\gamma+l)!}(-h)^{\gamma}\mathbb{W}_{n-\gamma-1}^{[h]}(0,u)(-1)^{\gamma}S_1(\gamma+l,l)v^l.
\end{aligned} \qquad (39)
$$

This is the complete proof of the theorem. □

Theorem 10. *For $n \geq 0$, we have*

$$
\mathbb{W}_n^{[h]}(v+w,u) = \sum_{l=0}^{n}\sum_{\gamma=0}^{n-l}\frac{n!}{(n-\gamma-l)!(\gamma+l)!}h^{\gamma}\mathbb{W}_{n-\gamma-l}^{[h]}(u,v)S_1(\gamma+l,l)w^l. \qquad (40)
$$

Proof. Taking $v+w$ instead of v in (15), we have

$$\sum_{n=0}^{\infty} \mathbb{W}_n^{[h]}(v+w,u)\frac{\xi^n}{n!} = (1+h\xi)^{\frac{v+w}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\}$$
$$= \left(\sum_{n=0}^{\infty} \mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!}\right)\left(\sum_{\gamma=0}^{\infty}\left(-\frac{w}{h}\right)_{\gamma}(-h)^{\gamma}\frac{\xi^{\gamma}}{\gamma!}\right). \quad (41)$$

Using the Cauchy rule and after comparing the coefficients of ξ on both sides of the resulting equation, we have

$$\mathbb{W}_n^{[h]}(v+w,u) = \sum_{\gamma=0}^{n}\binom{n}{\gamma}\left(-\frac{w}{h}\right)_{\gamma}(-h)^{\gamma}\mathbb{W}_{n-\gamma}^{[h]}(u,v). \quad (42)$$

Then, using (31) for $\left(-\frac{w}{h}\right)_{\gamma}$, we obtain (40). □

Theorem 11. *For $n \geq 0$, we have*

$$\mathbb{W}_n^{[h]}(u,v) = \sum_{\gamma=0}^{n}\sum_{j=0}^{\gamma}\binom{n}{\gamma}(z-v)^j S_1(\gamma,j)h^{\gamma-j}\mathbb{W}_{n-\gamma}^{[h]}(0,u). \quad (43)$$

Proof. From (15), we have

$$(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\} = e^{-\frac{v}{h}\log(1+h\xi)}\sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!}. \quad (44)$$

Replacing v with z and comparing the resulting equations, we obtain

$$e^{\frac{z-v}{h}\log(1+h\xi)}(1+h\xi)^{\frac{D_v^{-1}}{h}} = \sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!}. \quad (45)$$

Ultimately, we arrive at assertion (45) of Theorem 11 by expanding the exponential function and comparing the coefficients of identical powers of ξ. □

Remark 1. *Assuming that $v=0$ in Theorem 11, we can quickly arrive at the following conclusion:*

$$\mathbb{W}_n^{[h]}(0,u) = \sum_{\gamma=0}^{n}\sum_{j=0}^{\gamma}\binom{n}{\gamma}z^j S_1(\gamma,j)h^{\gamma-j}\mathbb{W}_{n-\gamma}^{[h]}(0,u).$$

4. Monomiality Principle

The monomiality principle stands as a foundational concept in polynomial theory, offering a fundamental framework for understanding and manipulating polynomial expressions. This principle asserts that any polynomial can be uniquely expressed as a linear combination of monomials, which are simple algebraic terms consisting of a single variable raised to a non-negative integer power. This representation simplifies the structure of polynomials and facilitates their analysis and manipulation in various mathematical contexts. By breaking down complex polynomial expressions into their constituent monomials, mathematicians can derive key properties, such as degree, leading coefficient, and roots, enabling deeper insights into polynomial behaviour and paving the way for the development of advanced mathematical techniques and algorithms.

Beyond its theoretical significance, the monomiality principle plays a pivotal role in practical applications across diverse scientific and engineering fields. In computational mathematics, for instance, algorithms for polynomial interpolation, approximation, and numerical integration often rely on the monomial basis for their efficiency and accuracy. Similarly, in areas such as signal processing, control theory, and image analysis, polynomials serve as essential mathematical tools for modelling complex systems and phenomena, with the monomiality principle providing a concise and intuitive representation framework. Furthermore, the versatility of monomial-based polynomial representations extends to disciplines like physics, where polynomials are utilized to describe physical laws and phenomena. With the concept of poweroids, Steffenson originally proposed monomiality in 1941 [21]; Dattoli subsequently expanded on this idea [22,23]. The $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ operators are multiplicative and derivative operators that are crucial in this context for a polynomial set $g_k(u_1)_{k\in\mathbb{N}}$. The following expressions are satisfied by these operators:

$$g_{k+1}(u_1) = \hat{\mathcal{J}}\{g_k(u_1)\} \tag{46}$$

and

$$k\, g_{k-1}(u_1) = \hat{\mathcal{K}}\{g_k(u_1)\}. \tag{47}$$

Thus, when multiplicative and derivative operations are applied to the polynomial set $g_k(u_1)_{m\in\mathbb{N}}$, the result is a quasi-monomial domain. Adhering to the following formula is crucial for this quasi-monomial:

$$[\hat{\mathcal{K}}, \hat{\mathcal{J}}] = \hat{\mathcal{K}}\hat{\mathcal{J}} - \hat{\mathcal{J}}\hat{\mathcal{K}} = \hat{1}. \tag{48}$$

It consequently displays a Weyl group structure.

Assuming that the set $\{g_k(u_1)\}_{k\in\mathbb{N}}$ is quasi-monomial, the significance and application of the operators $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ may be utilized to derive the significance of the underlined set. Therefore, the following axioms are true:

(i) $g_k(u_1)$ gives a differential equation

$$\hat{\mathcal{J}}\hat{\mathcal{K}}\{g_k(u_1)\} = k\, g_k(u_1), \tag{49}$$

provided that $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ exhibit differential traits.

(ii) The expression

$$g_k(u_1) = \hat{\mathcal{J}}^k\,\{1\}, \tag{50}$$

gives the explicit form, with $g_0(u_1) = 1$.

(iii) Further, the expression

$$e^{w\hat{\mathcal{J}}}\{1\} = \sum_{k=0}^{\infty} g_k(u_1)\frac{w^k}{k!}, \quad |w| < \infty \ , \tag{51}$$

demonstrates the generation of expression behaviour and is obtained by applying identity (50).

These techniques, rooted in mathematical physics, quantum mechanics, and classical optics, remain relevant in contemporary research. They are reliable tools for probing intricate phenomena in these domains and play a pivotal role in advancing our understanding of complex systems. In light of the paramount importance of these methodologies, we embarked on the validation of the notion of monomiality specifically for the Δ_h Laguerre polynomials. These polynomials, denoted as $\mathbb{W}_n^{[h]}(u,v)$, represent a crucial mathematical framework within which various phenomena can be analyzed and understood. By validating the concept of monomiality within this context, we aim to elucidate fundamental properties underpinning these polynomials' behaviour and their applications.

Within this section, we present the outcomes of our validation efforts. These outcomes serve to reinforce the integrity and utility of the Δ_h Laguerre polynomials as essential mathematical constructs. Through rigorous analysis and validation, we affirm the validity and significance of the concept of monomiality, thereby enhancing our confidence in the robustness of these mathematical tools for theoretical and practical investigations.

Theorem 12. *The Δ_h Laguerre polynomials $\mathbb{W}_n^{[h]}(u,v)$ satisfy the succeeding multiplicative and derivative operators:*

$$\hat{M_{\mathbb{SA}}} = \left(\frac{v}{1 + {}_v\Delta_h} + \frac{2\, D_u^{-1}{}_v\Delta_h}{h + {}_v\Delta_h{}^2} \right) \tag{52}$$

and

$$\hat{D_{\mathbb{S}}} = \frac{{}_v\Delta_h}{h}. \tag{53}$$

Proof. In consideration of expression (9), taking derivatives with regard to v of expression (15), we have

$$\begin{aligned} {}_v\Delta_h\left\{(1+ht)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\}\right\} &= (1+h\xi)^{\frac{v+h}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\} - (1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\} \\ &= (1+h\xi-1)(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\} \\ &= h\xi\,(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\}, \end{aligned} \tag{54}$$

thus, we have

$$\frac{{}_v\Delta_h}{h}\left[(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\}\right] = t\left[(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\}\right], \tag{55}$$

which gives the identity

$$\frac{{}_v\Delta_h}{h}\left[\mathbb{W}_n^{[h]}(u,v)\right] = \xi\left[\mathbb{W}_n^{[h]}(u,v)\right]. \tag{56}$$

Now, differentiating expression (15) with regard to ξ, we have

$$\frac{\partial}{\partial\xi}\left\{(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}\{1\}\right\} = \frac{\partial}{\partial\xi}\left\{\sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!}\right\}, \tag{57}$$

$$\left(\frac{v}{1+h\xi} + 2\frac{D_u^{-1}\xi}{1+h\xi}\right)\left\{\sum_{n=0}^{\infty}\mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!}\right\} = \sum_{n=0}^{\infty} n\,\mathbb{W}_n^{[h]}(u,v)\frac{\xi^{n-1}}{n!}. \tag{58}$$

By utilizing identity (56) and substituting $n \to n+1$ into the right-hand side of the previous expression (58), we establish the assertion (52).

Moreover, based on identity (56), we have

$$\frac{{}_v\Delta_h}{h}\left[\mathbb{W}_n^{[h]}(u,v)\right] = n\,\mathbb{W}_{n-1}^{[h]}(u,v), \tag{59}$$

which provides a formula for the derivative operator in (53). □

We now proceed to derive the differential equation for the Δ_h Laguerre polynomials $\mathbb{W}_n^{[h]}(u,v)$ by proving the following result:

Theorem 13. *The Δ_h Laguerre polynomials $\mathbb{W}_n^{[h]}(u,v)$ satisfy the differential equation:*

$$\left(\frac{v}{1+{}_v\Delta_h} + \frac{2\,D_u^{-1}{}_v\Delta_h}{h+{}_v\Delta_h{}^2} - \frac{nh}{{}_v\Delta_h}\right)\mathbb{W}_n^{[h]}(u,v) = 0. \tag{60}$$

Proof. By inserting expressions (52) and (53) into expression (49), the assertion (60) is proved. □

5. Symmetric Identities

This section explores symmetric identities inherent to two-variable Δ_h special polynomials, revealing relationships between variables and coefficients, deepening our understanding of the polynomials and broader mathematical structures. It establishes a framework for utilizing the symmetrical properties of these polynomials, paving the way for theoretical analyses and practical applications.

Theorem 14. *For $a \neq b$, $a, b > 0$ and $u_1, u_2, v_1, v_2 \in \mathbb{C}$, we have*

$$\sum_{\gamma=0}^{n} \binom{n}{\gamma} a^{n-\gamma} b^{\gamma} \mathbb{W}_{n-\gamma}^{[h]}(au_1, av_1) \mathbb{W}_{\gamma}^{[h]}(bu_2, bv_2) = \sum_{\gamma=0}^{n} \binom{n}{\gamma} a^{n} b^{n-\gamma} \mathbb{W}_{n-\gamma}^{[h]}(au_2, av_2) \mathbb{W}_{\gamma}^{[h]}(bu_1, bv_1). \tag{61}$$

Proof. Let

$$\begin{aligned} A(\xi) &= (1+h\xi)^{\frac{ab(v_1+v_2)}{h}} \mathcal{C}_0\left(\frac{-abu_1}{h}\log(1+h\xi)\right) \mathcal{C}_0\left(\frac{-abu_2}{h}\log(1+h\xi)\right) \\ &= \sum_{n=0}^{\infty} \mathbb{W}_{n}^{[h]}(bu_1, bv_1) \frac{(a\xi)^n}{n!} \sum_{\gamma=0}^{\infty} \mathbb{W}_{n}^{[h]}(au_2, av_2) \frac{(b\xi)^{\gamma}}{\gamma!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{\gamma=0}^{n} \binom{n}{\gamma} a^{n-\gamma} b^{\gamma} \mathbb{W}_{n-\gamma}^{[h]}(au_1, av_1) \mathbb{W}_{\gamma}^{[h]}(bu_2, bv_2) \right) \frac{\xi^n}{n!}. \end{aligned} \tag{62}$$

Similarly, we have

$$A(\xi) = \sum_{n=0}^{\infty} \left(\sum_{\gamma=0}^{n} \binom{n}{\gamma} a^{n} b^{n-\gamma} \mathbb{W}_{n-\gamma}^{[h]}(au_2, av_2) \mathbb{W}_{\gamma}^{[h]}(bu_1, bv_1) \right) \frac{\xi^n}{n!}. \tag{63}$$

Comparing the coefficients of ξ on both sides of the last equations, we obtain (61). □

Theorem 15. *For $a \neq b$, $a, b > 0$ and $u_1, u_2, v \in \mathbb{C}$, we have*

$$\begin{aligned} &\sum_{k=0}^{n} \sum_{\gamma=0}^{k} \binom{n}{k} \binom{k}{\gamma} a^{n-\gamma} b^{\gamma+1} \beta_{n-k}(h) \mathbb{W}_{n-\gamma}^{[h]}(bu, bv) \sigma_{\gamma}(a-1;h) \\ &\qquad = \sum_{k=0}^{n} \sum_{\gamma=0}^{k} \binom{n}{k} \binom{k}{\gamma} b^{n-\gamma} a^{\gamma+1} \beta_{n-k}(h) \mathbb{W}_{n-\gamma}^{[h]}(au, av) \sigma_{\gamma}(b-1;h). \end{aligned} \tag{64}$$

Proof. Consider

$$\begin{aligned} B(\xi) &= \frac{ab\xi(1+h\xi)^{\frac{abv}{h}} \mathcal{C}_0\left(\frac{-abu}{h}\log(1+h\xi)\right)((1+h\xi)^{\frac{ab}{h}}-1)}{((1+h\xi)^{\frac{a}{h}}-1)((1+h\xi)^{\frac{b}{h}}-1)} \\ &= \frac{ab\xi}{((1+h\xi)^{\frac{a}{h}}-1)} (1+h\xi)^{\frac{abv}{h}} \mathcal{C}_0\left(\frac{-abu}{h}\log(1+h\xi)\right) \frac{((1+h\xi)^{\frac{ab}{h}}-1)}{((1+h\xi)^{\frac{b}{h}}-1)} \\ &= b\sum_{n=0}^{\infty} \beta_n(h) \frac{(a\xi)^n}{n!} \sum_{k=0}^{\infty} \mathbb{W}_{k}^{[h]}(bu, bv) \frac{(a\xi)^k}{k!} \sum_{\gamma=0}^{\infty} \sigma_{\gamma}(a-1;h) \frac{(b\xi)^{\gamma}}{\gamma!} \\ &= b\sum_{n=0}^{\infty} \beta_n(h) \frac{(a\xi)^n}{n!} \sum_{k=0}^{\infty} \sum_{\gamma=0}^{k} \binom{k}{\gamma} a^{k-\gamma} b^{\gamma} \mathbb{W}_{k-\gamma}^{[h]}(bu, bv) \sigma_{\gamma}(a-1;h) \frac{\xi^k}{k!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{\gamma=0}^{k} \binom{n}{k} \binom{k}{\gamma} a^{n-\gamma} b^{\gamma+1} \beta_{\gamma-k}(h) \mathbb{W}_{n-\gamma}^{[h]}(bu, bv) \sigma_{\gamma}(a-1;h) \right) \frac{\xi^n}{n!}. \end{aligned} \tag{65}$$

Similarly, we have

$$B(\xi) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{\gamma=0}^{k} \binom{n}{k} \binom{k}{\gamma} b^{n-\gamma} a^{\gamma+1} \beta_{n-k}(h) \mathbb{W}_{n-\gamma}^{[h]}(au, av) \sigma_{\gamma}(b-1;h) \right) \frac{\xi^n}{n!}. \tag{66}$$

Comparing both sides of the above equations, we obtain (64).

□

Next, to find the first few polynomials of the expression (15), that is,

$$(1+h\xi)^{\frac{v-D_u^{-1}}{h}} = \sum_{n=0}^{\infty} \mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!},$$

we start by expanding the left-hand side using the binomial series and then matching it to the right-hand side to identify the coefficients $\mathbb{W}_n^{[h]}(u,v)$.

The binomial expansion of $(1+h\xi)^{\frac{v-D_u^{-1}\{1\}}{h}}$ is:

$$(1+h\xi)^{\frac{v-D_u^{-1}\{1\}}{h}} = \sum_{n=0}^{\infty} \binom{\frac{v-D_u^{-1}\{1\}}{h}}{n}(h\xi)^n.$$

This can be rewritten as:

$$(1+h\xi)^{\frac{v-D_u^{-1}\{1\}}{h}} = \sum_{n=0}^{\infty} \binom{\frac{v-D_u^{-1}\{1\}}{h}}{n}h^n\xi^n.$$

Using the definition of the binomial coefficient for any real number k,

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!},$$

it follows that

$$\binom{\frac{v-D_u^{-1}\{1\}}{h}}{n} = \frac{\left(\frac{v-D_u^{-1}\{1\}}{h}\right)\left(\frac{v-D_u^{-1}\{1\}}{h}-1\right)\cdots\left(\frac{v-D_u^{-1}\{1\}}{h}-n+1\right)}{n!}.$$

Thus, we find

$$(1+h\xi)^{\frac{v-D_u^{-1}\{1\}}{h}} = \sum_{n=0}^{\infty} \frac{\left(\frac{v-D_u^{-1}\{1\}}{h}\right)\left(\frac{v-D_u^{-1}\{1\}}{h}-1\right)\cdots\left(\frac{v-D_u^{-1}\{1\}}{h}-n+1\right)}{n!}h^n\xi^n,$$

which can be further simplified as

$$(1+h\xi)^{\frac{v-D_u^{-1}\{1\}}{h}} = \sum_{n=0}^{\infty} \frac{(v-D_u^{-1}\{1\})(v-D_u^{-1}\{1\}-h)\cdots(v-D_u^{-1}\{1\}-(n-1)h)}{h^n n!}\xi^n = \sum_{n=0}^{\infty} \mathbb{W}_n^{[h]}(u,v)\frac{\xi^n}{n!}.$$

From this comparison, we can identify the first few coefficients $\mathbb{W}_n^{[h]}(u,v)$:

$$\mathbb{W}_0^{[h]}(u,v) = 1,$$

$$\mathbb{W}_1^{[h]}(u,v) = \frac{v-D_u^{-1}\{1\}}{h},$$

$$\mathbb{W}_2^{[h]}(u,v) = \frac{(v-D_u^{-1}\{1\})(v-D_u^{-1}\{1\}-h)}{h^2},$$

$$\mathbb{W}_3^{[h]}(u,v) = \frac{(v-D_u^{-1}\{1\})(v-D_u^{-1}\{1\}-h)(v-D_u^{-1}\{1\}-2h)}{h^3}.$$

6. Conclusions

The introduction and investigation of the Δ_h Laguerre polynomials represent significant advancements in polynomial theory. Through the integration of the monomiality principle alongside operational rules, these novel polynomials offer fresh insights into unexplored mathematical territory.

The explicit formulas and elucidation of fundamental properties provided in this research deepen our understanding of the Δ_h Laguerre polynomials themselves and establish connections with established polynomial categories, thereby enriching the broader mathematical landscape.

Moving forward, future research endeavours could explore several promising avenues. Firstly, further investigations into the structural properties and algebraic aspects of Δ_h Laguerre polynomials could yield deeper insights into their behaviour and potential applications. Additionally, exploring their applicability in other areas of quantum mechanics and mathematical physics could uncover new avenues for research, and potential practical implications.

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