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Editorial

Editorial for the Special Issue of "Fractional Differential and Fractional Integro-Differential Equations: Qualitative Theory, Numerical Simulations, and Symmetry Analysis"

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Citation: Tunç, C.; Yao, J.-C.; Benchohra, M.; El-Sayed, A.M.A. Editorial for the Special Issue of "Fractional Differential and Fractional Integro-Differential Equations: Qualitative Theory, Numerical Simulations, and Symmetry Analysis". *Symmetry* **2024**, *16*, 1193. https://doi.org/10.3390/sym16091193

Received: 6 August 2024
Accepted: 8 August 2024
Published: 11 September 2024

1. Introduction

The fractional calculus is a specific case of classical calculus, as is well known. This fact allows a significant advantage during the real world studies and their applications. Moreover, using classical calculus, countless real-world phenomena can also be represented theoretically as fractional-order differential equations. Thereby, fractional calculus and hence qualitative analysis of fractional-order differential and integro-differential equations has become a very hot and promising research topic in the pertinent literature. Indeed, in other words, ordinary differential equations, integral equations, integro-differential equations, fractional-order differential equations, fractional-order integro-differential equations, etc. can appear during various real world applications as proper mathematical models. Solving them explicitly for the nonlinear cases is a difficult task, sometimes it is impossible except numerically. Therefore qualitative analysis of that kind of mathematical models have a significant importance without solving them (see, for examples the books or the papers of Abbas et al. [1], Al-Ghafri et al. [2], Ali et al. [3], Awad and Alkhezi [4], Balachandran [5], Benchohra et al. [6], Burton [7], Diethelm [8], Khan et al. [9], Kilbas et al. [10], Miller and Ross [11], Podlubny [12], Tunç et al. ([13–15], Tunç and Tunç [16] and the references of these sources).

This Special Issue of *Symmetry* is devoted to qualitative studies of fractional differential equations, fractional integro-differential equations, etc., according to qualitative theory, numerical simulations, and symmetry analysis.

2. An Overview of Published Articles

Below, we have briefly summarized the contents of the published papers in this Special Issue and their contributions.

Using a hybrid approach, Sahraee and Arabameri (contribution 1) solved the time-fractional telegraph equation. In contribution 1, a combination of the finite difference and differential transform methods was used for the numerical simulations of the study. As a result, the equation of the study was first semi-discretized along the spatial ordinate. Later, the fractional differential transform method was used to solve the resulting system of ordinary differential equations. In particular cases, the present hybrid technique was applied for some known linear and nonlinear examples.

Sitho et al. (contribution 2) presented and examined a novel category of coupled and uncoupled systems. The considered systems include mixed-type ψ_1-Hilfer and ψ_2-Caputo fractional differential equations, which were supplemented with asymmetric and symmetric integro-differential nonlocal boundary conditions, respectively. In this paper, the Leray-Schauder alternatives were used to prove the existence result. Subsequently, the

Banach contraction mapping principle was used here to prove the uniqueness outcome of the paper.

Mukhtar (contribution 3) solved the second- and fourth-order fractional Boussinesq equations numerically using the variational iteration transform and the Adomian decomposition methods. Additionally, in the article, the ZZ transform and fractional Atangana-Baleanu operator were also utilized to provide equation solutions. In particular cases, two examples were provided to verify the methods and solutions. Here, it was conclude that the results of the article demonstrated that both the variational iteration and the Adomian decomposition methods were effective in obtaining accurate and reliable solutions for the time-fractional Boussinesq equation.

Azeem et al.'s work (contribution 4) discussed the consequences of infection in humans while taking into account a fractional-order cancer model using stem cells and chemotherapy. This article's mathematical model was examined using the highly efficient numerical method known in the pertinent literature as the Sumudu transform. This research additionally confirmed the positivity of solutions using the suggested technique's ABC operator. In the paper, the existence and uniqueness of the solutions for the fractional-order cancer system were also proved using fixed-point theory. The present analysis allows a better understanding of how to controlcancer disease in the human body.

Tunç et al. (contribution 5) dealt with certain nonlinear integro-differential equations of first order, which includes fading memory. Using the Lyapunov–Krasovski functional technique, the authors obtained new sufficient conditions under which several qualitative properties in connection with the considered mathematical model were guaranteed. Hence, several new results were proved on the mentioned properties of solutions, which are called asymptotic stability, boundedness, convergence, integrability, stability, and uniform stability. The novel outcomes and originality of this article were that the considered integro-differential equations are new mathematical models, and they also include some past mathematical models in the pertinent literature.

Wang et al. (contribution 6) focused on the analysis of image segmentation algorithms, which is based on an evolutionary technique. The study included further adaptations of these evolutionary algorithms to quantum algorithms, as well as a discussion of the image segmentation algorithms that use the clustering technique and the one that converts the color model from RGB to HSI. In the study, the genetic algorithm was utilized to determine the best parameters to select and improve the segmentation impact.

Tusset et al. (contribution 7) developed a mathematical model to describe the dynamic behavior of a bioreactor, where a fermentation process took place. The temperature of the bioreactor, which is regulated by the flow of refrigerant fluid through the reactor jacket, was considered in the analysis. During the process, the reactor temperature was maintained by an ideal LQR control that was acting on the water flow through a jacket. In the study, a reduced-order model of the system was also taken into consideration for the control design. Additionally, in this study, a model incorporating the fractional order heat exchange between the reactor and the jacket using the Riemann-Liouville differential operators was proposed in light of the heat transfer symmetry, which was observed in reactors. As a result, the numerical simulation showed that the suggested control was reliable in the face of disturbances in the inlet temperature reactor and effective in maintaining the temperature at the appropriate levels.

Salem and Cichoń (contribution 8) investigated the tempered fractional integral operators moving on subspaces of $L\_1[a,b]$, which are called Orlicz or Hölder spaces. The authors proved that Orlicz spaces can be converted into (generalized) Hölder spaces. In particular case, the authors mapped Hölder spaces into the same class of spaces. The obtained results were extensions of classical results for the Riemann–Liouville fractional operator and constituted the basis for the use of generalized operators in the study of differential and integral equations.

Ahmad et al. (contribution 9) dealt with formulation of a new couple system of multi-term fractional differential equations including time delay, as well as supplementation with nonlocal boundary conditions. First the model formulation was constructed, later the existence and the uniqueness conditions were constructed, utilizing fixed point theory and functional analysis. Moreover, some results related to Ulam's type stability were also obtained, and an example was provided to illustrate the outcomes of the present study.

Li et al. (contribution 10) studied a kind of non-linear fractional hybrid differential equations involving three mixed fractional orders with boundary conditions. Under weaker conditions, a formula of solutions was constructed, and the existence results regarding solutions of the considered problem were studied. The results of this article can be used to solve more general fractional hybrid equations, for example, fractional hybrid Langevin equations. Moreover, the form of the solution for this kind of equation can provide a theoretical basis for the further study of the positive solution and its symmetry. In the current article, in a particular case, a numerical example was presented to show the application of the main result of the article.

El-Sayed et al. (contribution 11) discussed the existence of solutions for two initial value problems, which involve a quadratic functional integro-differential equation of arbitrary (fractional) orders and its corresponding integer order equation. In the present article, the existence of the maximal and minimal solutions were also proven. The sufficient conditions for the uniqueness of the solutions were also obtained, and the continuous dependence of the unique solution on some data was studied. Lastly, an investigation was conducted on the continuation of the arbitrary (fractional) orders problem to the integer order problem.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

List of Contributions

1. Sahraee, Z.; Arabameri, M. A Semi-Discretization Method Based on Finite Difference and Differential Transform Methods to Solve the Time-Fractional Telegraph Equation. *Symmetry* **2023**, *15*, 1759. https://doi.org/10.3390/sym15091759.
2. Sitho, S.; Ntouyas, S.K.; Sudprasert, C.; Tariboon, J. Systems of Sequential ψ1-Hilfer and ψ2-Caputo Fractional Differential Equations with Fractional Integro-Differential Nonlocal Boundary Conditions. *Symmetry* **2023**, *15*, 680. https://doi.org/10.3390/sym15030680.
3. Mukhtar, S. Numerical Analysis of the Time-Fractional Boussinesq Equation in Gradient Unconfined Aquifers with the Mittag-Leffler Derivative. *Symmetry* **2023**, *15*, 608. https://doi.org/10.3390/sym15030608.
4. Azeem, M.; Farman, M.; Akgül, A.; De la Sen, M. Fractional Order Operator for Symmetric Analysis of Cancer Model on Stem Cells with Chemotherapy. *Symmetry* **2023**, *15*, 533. https://doi.org/10.3390/sym15020533.
5. Tunç, C.; Tunç, O.; Yao, J.-C. On the Enhanced New Qualitative Results of Nonlinear Integro-Differential Equations. *Symmetry* **2023**, *15*, 109. https://doi.org/10.3390/sym15010109.
6. Wang, W.; Yousaf, M.; Liu, D.; Sohail, A. A Comparative Study of the Genetic Deep Learning Image Segmentation Algorithms. *Symmetry* **2022**, *14*, 1977. https://doi.org/10.3390/sym14101977.
7. Tusset, A.M.; Inacio, D.; Fuziki, M.E.K.; Costa, P.M.L.Z.; Lenzi, G.G. Dynamic Analysis and Control for a Bioreactor in Fractional Order. *Symmetry* **2022**, *14*, 1609. https://doi.org/10.3390/sym14081609.
8. Salem, H.A.H.; Cichoń, M. Analysis of Tempered Fractional Calculus in Hölder and Orlicz Spaces. *Symmetry* **2022**, *14*, 1581. https://doi.org/10.3390/sym14081581.
9. Ahmad, D.; Agarwal, R.P.; ur Rahman, G.U. Formulation, Solution's Existence, and Stability Analysis for Multi-Term System of Fractional-Order Differential Equations. *Symmetry* **2022**, *14*, 1342. https://doi.org/10.3390/sym14071342.
10. Li, F.; Zhang, L.; Wang, H. On Fractional Hybrid Non-Linear Differential Equations Involving Three Mixed Fractional Orders with Boundary Conditions. *Symmetry* **2022**, *14*, 1189. https://doi.org/10.3390/sym14061189.
11. El-Sayed, A.M.A.; Hamdallah, E.M.A.; Ba-Ali, M.M.S. Qualitative Study for a Delay Quadratic Functional Integro-Differential Equation of Arbitrary (Fractional) Orders. *Symmetry* **2022**, *14*, 784. https://doi.org/10.3390/sym14040784.

References

1. Abbas, S.; Benchohra, M.; Graef, J.R.; Henderson, J. *Implicit Fractional Differential and Integral Equations: Existence and Stability*; De Gruyter: Berlin, Germany, 2018; Volume 26.
2. Al-Ghafri, K.S.; Alabdala, A.T.; Redhwan, S.S.; Bazighifan, O.; Ali, A.H.; Iambor, L.F. Symmetrical Solutions for Non-Local Fractional Integro-Differential Equations via Caputo–Katugampola Derivatives. *Symmetry* **2023**, *15*, 662. [CrossRef]
3. Ali, I.; Yaseen, M.; Khan, S. Addressing Volterra Partial Integro-Differential Equations through an Innovative Extended Cubic B-Spline Collocation Technique. *Symmetry* **2023**, *15*, 1851. [CrossRef]
4. Awad, Y.; Alkhezi, Y. Solutions of Second-Order Nonlinear Implicit ψ-Conformable Fractional Integro-Differential Equations with Nonlocal Fractional Integral Boundary Conditions in Banach Algebra. *Symmetry* **2024**, *16*, 1097. [CrossRef]
5. Balachandran, K. *An Introduction to Fractional Differential Equations*; Springer: Singapore, 2023.
6. Benchohra, M.; Karapınar, E.; Lazreg, J.E.; Salim, A. *Advanced Topics in Fractional Differential Equations—A Fixed Point Approach*; Springer: Cham, Germany, 2023.
7. Burton, T.A. An existence theorem for a fractional differential equation using progressive contractions. *J. Fract. Calc. Appl.* **2017**, *8*, 168–172.
8. Diethelm, K. *The Analysis of Fractional Differential Equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type*; Springer: Berlin/Heidelberg, Germany, 2010.
9. Khan, H.; Tunç, C.; Khan, A. Stability results and existence theorems for nonlinear delay-fractional differential equations with ϕ∗p-operator. *J. Appl. Anal. Comput.* **2020**, *10*, 584–597.
10. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204, pp. 1–523.
11. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; John Wiley & Sons, Inc.: New York, NY, USA, 1993.
12. Podlubny, I. *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*; Academic Press, Inc.: San Diego, CA, USA, 1998; p. 198.
13. Tunç, C.; Alshammari, F.S.; Akyildiz, F.T. Existence and Uniqueness of Solutions of Hammerstein-Type Functional Integral Equations. *Symmetry* **2023**, *15*, 2205. [CrossRef]
14. Tunç, O.; Atan, Ö.; Tunç, C.; Yao, J.-C. Qualitative Analyses of Integro-Fractional Differential Equations with Caputo Derivatives and Retardations via the Lyapunov–Razumikhin Method. *Axioms* **2021**, *10*, 58. [CrossRef]
15. Tunç, C.; Tunç, O.; Yao, J.-C. On the Enhanced New Qualitative Results of Nonlinear Integro-Differential Equations. *Symmetry* **2023**, *15*, 109. [CrossRef]
16. Tunç, O.; Tunç, C. On Ulam Stabilities of Delay Hammerstein Integral Equation. *Symmetry* **2023**, *15*, 1736. [CrossRef]