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Conformable Double Laplace Transform Method (CDLTM) and Homotopy Perturbation Method (HPM) for Solving Conformable Fractional Partial Differential Equations

Musa Rahamh GadAllah and Hassan Eltayeb Gadain *

Mathematics Department, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; magal@ksu.edu.sa
* Correspondence: hgadain@ksu.edu.sa

Abstract: In the present article, the method which was obtained from a combination of the conformable fractional double Laplace transform method (CFDLTM) and the homotopy perturbation method (HPM) was successfully applied to solve linear and nonlinear conformable fractional partial differential equations (CFPDEs). We included three examples to help our presented technique. Moreover, the results show that the proposed method is efficient, dependable, and easy to use for certain problems in PDEs compared with existing methods. The solution graphs show close contact between the exact and CFDLTM solutions. The outcome obtained by the conformable fractional double Laplace transform method is symmetrical to the gain using the double Laplace transform.

Keywords: conformable partial derivative; conformable double Laplace decomposition method; homotopy perturbation method

MSC: 35A44; 65M44; 35A22

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1. Introduction

Several applications in advanced science are modeled by linear and nonlinear fractional partial differential equations (FPDEs), for instance, in mathematics, fluids mechanics, physics, biology, chemistry, economics, electromagnetic theory, signal processing, etc. The authors in [1] utilized the homotopy perturbation method (HPM) to handle nonlinear ordinary differential equations and partial differential equations. Through this, solving FPDEs lured the concern of many authors (see [2–6]). Various researchers assisted with fractional derivatives of FPDEs, for example, Caputo [7], Liouville [8], and Ross [9]. The homotopy perturbation method has been extended and used to acquire approximate and exact solutions of fractional-order linear and nonlinear PDEs [10–12]. Recently, in [13], the authors extend the familiar limit definition of the derivative of the function and offer the so-called conformable fractional derivative (CFD). Many scientists have been applying the CFD in numerous employments (see [14,15]). The Laplace transform method (LTM) in [16,17] is one of the famous methods for gaining the approximate and exact solutions of FDEs. The researcher in [18] constructed the single conformable Laplace transform method (SCLTM) to solve FPDEs in the conformable fractional derivative sense. The authors in [19] solved time fractional diffusion-wave (TFDW) equations by employing the meshless generalized finite difference method. Series of time-independent integer-order boundary value problems were obtained by transforming the time-fractional diffusion-wave initial-boundary value problem, and the linear algebraic systems were built by the application of the GFDM [20]. The researchers in [21] found the numerical solution of fractional-time-space differential equations with the spectral fractional Laplacian by using the generalized exponential time-differentiating method. In [22], the authors presented a new method of the double Laplace transform called the conformable double Laplace transform (CDLT) and implemented it to solve the conformable fractional partial differential equation. Many strong methods have been improved and developed to gain numerical and exact solutions of fractional linear and nonlinear partial differential equations, for example, the (CLTM) [18,23], the modified Laplace transform for certain generalized fractional operators [24], fractional double Laplace transform method (FDLTM), and its properties [25]. This study aims to implement the conformable double Laplace transform method (CDLTM) to obtain the exact and approximate solutions of a class of CFPDEs. The remainder of this article is organized as follows: In Section 2, we introduce essential definitions and address some properties of the conformable double Laplace transform method (DLTM), as well as the single Laplace transform method (SLTM) and properties of conformable derivatives (CDs). In Section 3, we introduce the basic idea of a CDL and (CALM) for solving (PDEs). In Section 4, some applications of the offered method are used to gain the exact and approximate solutions of conformable fractional linear and nonlinear partial differential equations. Lastly, Section 5 contains the conclusion.

2. Conformable Double Laplace Transform Method (CDLTM) and Properties of Conformable Derivatives (CDs)

In this section, we present some definitions and the basic properties of the conformable double Laplace transform method (CDLTM), conformable single Laplace transform method (CSLTM), and the properties of conformable derivatives (CDs).

Definition 1 ([6,26])**.** *Let $f : (0, \infty) \longrightarrow \mathbb{R}$; thus, the conformable fractional derivatives (CFDs) of the function and $g\left(\frac{\mu^\alpha}{\alpha}\right)$ of order β and α are defined by*

$$\frac{d^\alpha}{d\mu^\alpha} g\left(\frac{\mu^\alpha}{\alpha}\right) = \lim_{\upsilon \longrightarrow 0} \frac{g\left(\frac{\mu^\alpha}{\alpha} + \upsilon\mu^{1-\alpha}\right) - g\left(\frac{\mu^\alpha}{\alpha}\right)}{\upsilon}, \tag{1}$$

where $\frac{\mu^\alpha}{\alpha} > 0, 0 < \alpha \leq 1$.

Definition 2 ([6,26])**.** *The conformable fractional partial derivatives (CFPDs) of order β of the functions $f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)$ are defined as follows:*

$$\frac{\partial^\beta}{\partial\eta^\beta} f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) = \lim_{\tau \longrightarrow 0} \frac{f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta} + \tau\eta^{1-\beta}\right) - f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\tau}, \tag{2}$$

where $\frac{\eta^\beta}{\beta} > 0, 0 < \beta \leq 1$.

The following theorem represents the relationship between the conformable partial derivatives (CPDs) and the first partial derivative.

Theorem 1 ([6,22])**.** *Let $\alpha, \beta \in (0, 1]$ and $f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)$ be α- and β-differentiable at a point $\frac{\mu^\alpha}{\alpha} > 0$ and $\frac{\eta^\beta}{\beta} > 0$. Thus,*

$$\frac{\partial^\alpha g\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\partial\mu^\alpha} = \mu^{1-\alpha} \frac{\partial g\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\partial\mu}, \tag{3}$$

and

$$\frac{\partial^\beta f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\partial\eta^\beta} = \eta^{1-\beta} \frac{\partial f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\partial\eta}. \tag{4}$$

Proof. By taking Definitions 1 and 2, and using $\omega = \upsilon\mu^{1-\alpha}$ in Equation (1), we obtain

$$\begin{aligned}
\frac{\partial^\alpha}{\partial\mu^\alpha} g\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) &= \lim_{\upsilon\longrightarrow 0} \frac{g\left(\frac{\mu^\alpha}{\alpha} + \upsilon\mu^{1-\alpha}, \frac{\eta^\beta}{\beta}\right) - g\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\upsilon} \\
&= \lim_{\omega\longrightarrow 0} \frac{g\left(\frac{\mu^\alpha}{\alpha} + \omega, \frac{\eta^\beta}{\beta}\right) - g\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\omega\mu^{\alpha-1}} \\
&= \mu^{1-\alpha} \lim_{\omega\longrightarrow 0} \frac{g\left(\frac{\mu^\alpha}{\alpha} + \omega, \frac{\eta^\beta}{\beta}\right) - g\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\omega} \\
&= \mu^{1-\alpha} \frac{\partial g\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\partial\mu}.
\end{aligned}$$

To prove Equation (3), take $\lambda = \tau\eta^{1-\beta}$ in Equation (4); thus,

$$\begin{aligned}
\frac{\partial^\beta f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\partial\eta^\beta} &= \eta^{1-\beta} \frac{\partial f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\partial\eta} \\
\frac{\partial^\beta}{\partial\eta^\beta} f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) &= \lim_{\tau\longrightarrow 0} \frac{f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta} + \tau\eta^{1-\beta}\right) - f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\tau} \\
&= \lim_{\lambda\longrightarrow 0} \frac{f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta} + \lambda\right) - f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\lambda\eta^{\beta-1}} \\
&= \eta^{1-\beta} \lim_{\lambda\longrightarrow 0} \frac{f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta} + \lambda\right) - f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\lambda} \\
&= \eta^{1-\beta} \frac{\partial f\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)}{\partial\eta}.
\end{aligned}$$

□

In the following example, we introduce some properties of the (CFDs) of certain functions.

Example 1. *Let $\alpha, \beta \in (0, 1]$, and a, b, k, ν, and $\lambda \in \mathbb{R}$; thus,*

1. $\frac{\partial^\beta}{\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\right)\left(\frac{\eta^\beta}{\beta}\right)^k = k\left(\frac{\mu^\alpha}{\alpha}\right)\left(\frac{\eta^\beta}{\beta}\right)^{k-1}$,
2. $\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\right)^a\left(\frac{\eta^\beta}{\beta}\right) = a\left(\frac{\mu^\alpha}{\alpha}\right)^{a-1}\left(\frac{\eta^\beta}{\beta}\right)$,
3. $\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\right)\left(\frac{\eta^\beta}{\beta}\right) = \left(\frac{\eta^\beta}{\beta}\right)$,
4. $\frac{\partial^\alpha}{\partial\mu^\alpha}\left[a\psi_1\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) + b\psi_2\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)\right]$
   $= a\left[\frac{\partial^\alpha}{\partial\mu^\alpha}\psi_1\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)\right] + b\left[\frac{\partial^\alpha}{\partial\mu^\alpha}\psi_2\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)\right]$,
5. $\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\right)^a\left(\frac{\eta^\beta}{\beta}\right)^b = a\left(\frac{\mu^\alpha}{\alpha}\right)^{a-1}\left(\frac{\eta^\beta}{\beta}\right)^b$,
6. $\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\left(\frac{\mu^\alpha}{\alpha}\right)^\nu\left(\frac{\eta^\beta}{\beta}\right)^\lambda\right) = \nu\lambda\left(\frac{\mu^\alpha}{\alpha}\right)^{\nu-\alpha}\left(\frac{\eta^\beta}{\beta}\right)^{\lambda-\beta}$,
7. $\frac{\partial^\alpha}{\partial\mu^\alpha}\left(e^{\upsilon\frac{\mu^\alpha}{\alpha}+\tau\frac{\eta^\beta}{\beta}}\right) = \upsilon\left(e^{\upsilon\frac{\mu^\alpha}{\alpha}+\tau\frac{\eta^\beta}{\beta}}\right)$,
8. $\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\cos\left(\pi\frac{\mu^\alpha}{\alpha}\right)\cos\left(\omega\frac{\eta^\beta}{\beta}\right)\right) = -\pi\sin\left(\pi\frac{\mu^\alpha}{\alpha}\right)\cos\left(\omega\frac{\eta^\beta}{\beta}\right)$,
9. $\frac{\partial^\beta}{\partial\eta^\beta}\left(\cos\left(\pi\frac{\mu^\alpha}{\alpha}\right)\sin\left(\omega\frac{\eta^\beta}{\beta}\right)\right) = \omega\cos\left(\pi\frac{\mu^\alpha}{\alpha}\right)\cos\left(\omega\frac{\eta^\beta}{\beta}\right)$.

Conformable Double Laplace Transform Method (CDLTM)

Definition 3 ([6])**.** *Let f be a real valued function. The conformable single Laplace transform method (CSLTM) is defined by*

$$\begin{aligned} £_{\eta}^{\beta}\left[f\left(\frac{\eta^{\beta}}{\beta}\right)\right] &= \int_{0}^{\infty} e^{-s\frac{\eta^{\beta}}{\beta}} f\left(\frac{\eta^{\beta}}{\beta}\right)\eta^{\beta-1}d\eta, \\ £_{\mu}^{\alpha}\left[f\left(\frac{\mu^{\alpha}}{\alpha}\right)\right] &= \int_{0}^{\infty} e^{-p\frac{\mu^{\alpha}}{\alpha}} f\left(\frac{\mu^{\alpha}}{\alpha}\right)\mu^{\alpha-1}d\mu. \end{aligned} \tag{5}$$

where p and s are complex variables of the conformable fractional Laplace transform method.

Definition 4 ([6])**.** *The CSLTM of a function with two variables with respect to μ and η of $\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)$ is denoted by*

$$\begin{aligned} £_{\eta}^{\beta}\left[\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)\right] &= \chi\left(\frac{\mu^{\alpha}}{\alpha},s\right) = \int_{0}^{\infty} e^{-s\frac{\eta^{\beta}}{\beta}}\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)d\eta, \\ £_{\mu}^{\alpha}\left[\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)\right] &= \chi\left(p,\frac{\eta^{\beta}}{\beta}\right) = \int_{0}^{\infty} e^{-p\frac{\mu^{\alpha}}{\alpha}}\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)d\mu. \end{aligned} \tag{6}$$

Definition 5 ([26])**.** *The CDLTM is defined by*

$$£_{\mu}^{\alpha}£_{\eta}^{\beta}\left[\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)\right] = \chi_{\alpha,\beta}(p,s) = \int_{0}^{\infty}\int_{0}^{\infty} e^{-p\frac{\mu^{\alpha}}{\alpha}-s\frac{\eta^{\beta}}{\beta}}\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)\eta^{\beta-1}\mu^{\alpha-1}d\eta d\mu, \tag{7}$$

where $£_{\mu}^{\alpha}£_{\eta}^{\beta}$ refers to the CDLTM, and $p,s\in\mathbb{C}, 0<\alpha,\beta\leq 1$ of the conformable integral with respect to μ and η, respectively.

Theorem 2 ([22])**.** *Let $\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)$ and $\Psi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)$ be two functions which use the CDLTM. Then,*

1.

$$£_{\mu}^{\alpha}£_{\eta}^{\beta}\left[\nu\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)+\upsilon\Psi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)\right] = \nu£_{\mu}^{\alpha}£_{\eta}^{\beta}\left[\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)\right]+\upsilon£_{\mu}^{\alpha}£_{\eta}^{\beta}\left[\Psi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)\right],$$

where $\nu,\upsilon\in\mathbb{R}$.

2.

$$£_{\mu}^{\alpha}£_{\eta}^{\beta}\left(e^{-\sigma\frac{\mu^{\alpha}}{\alpha}-\omega\frac{\eta^{\beta}}{\beta}}\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)\right) = \chi(p+\sigma,s+\omega).$$

3.

$$£_{\mu}^{\alpha}£_{\eta}^{\beta}\left(\chi\left(\theta_1\frac{\mu^{\alpha}}{\alpha},\theta_2\frac{\eta^{\beta}}{\beta}\right)\right) = \frac{1}{\theta_1^{\alpha}}\frac{1}{\theta_2^{\beta}}\chi\left(\frac{p}{\theta_1},\frac{s}{\theta_2}\right).$$

Proof. (1) By employing the definition of the CDLTM, we get

$$\begin{aligned}
\pounds_\mu^\alpha \pounds_\eta^\beta\left[\nu\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)+\upsilon\Psi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right] &= \nu\pounds_\mu^\alpha \pounds_\eta^\beta\left[\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right]+\upsilon\pounds_\mu^\alpha \pounds_\eta^\beta\left[\Psi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right] \\
&= \int_0^\infty\int_0^\infty\left[\nu\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right]e^{-(p\mu+s\eta)}d\eta d\mu \\
&\quad +\int_0^\infty\int_0^\infty\left[\upsilon\Psi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right]e^{-(p\mu+s\eta)}d\eta d\mu \\
&= \nu\int_0^\infty\int_0^\infty\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)e^{-(p\mu+s\eta)}d\eta d\mu \\
&\quad +\upsilon\int_0^\infty\int_0^\infty\Psi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)e^{-(p\mu+s\eta)}d\eta d\mu \\
&= \nu\pounds_\mu^\alpha \pounds_\eta^\beta\left[\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right]+\upsilon\pounds_\mu^\alpha \pounds_\eta^\beta\left[\Psi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right].
\end{aligned}$$

(2) By taking the definition of the conformable double Laplace transform method (CDLTM), we obtain

$$\begin{aligned}
\pounds_\mu^\alpha \pounds_\eta^\beta\left(e^{-\sigma\frac{\mu^\alpha}{\alpha}-\omega\frac{\eta^\beta}{\beta}}\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right) &= \chi(p+\sigma,s+\omega) \\
&= \int_0^\infty\int_0^\infty\left[e^{-p\frac{\mu^\alpha}{\alpha}-s\frac{\eta^\beta}{\beta}}e^{-\sigma\frac{\mu^\alpha}{\alpha}-\omega\frac{\eta^\beta}{\beta}}\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right]d\eta d\mu \qquad (8) \\
&= \int_0^\infty e^{-p\frac{\mu^\alpha}{\alpha}-\sigma\frac{\mu^\alpha}{\alpha}}\left[\int_0^\infty e^{-s\frac{\eta^\beta}{\beta}-\omega\frac{\eta^\beta}{\beta}}\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)d\eta\right]d\mu.
\end{aligned}$$

Now, using the definition of the CALM to integrate inside the bracket in the above equation, we get

$$\int_0^\infty e^{-s\frac{\eta^\beta}{\beta}-\omega\frac{\eta^\beta}{\beta}}\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)d\eta=\chi(\mu,s+\omega), \qquad (9)$$

By substituting Equation (9) into Equation (8), one can get

$$\int_0^\infty e^{-(p+\sigma)\frac{\mu^\alpha}{\alpha}}\chi(\mu,s+\omega)d\mu=\chi(p+\sigma,s+\omega).$$

Similarly, we can prove (3). □

Definition 6 ([5]). *The CDLTM of the conformable partial derivatives of orders α-th and β-th* $\frac{\partial^\alpha\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)}{\partial\mu^\alpha}$, $\frac{\partial^\beta\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)}{\partial\eta^\beta}$, $\frac{\partial^{2\alpha}\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)}{\partial\mu^{2\alpha}}$ *and* $\frac{\partial^{2\beta}\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)}{\partial\eta^{2\beta}}$ *is given by*

$$\begin{aligned}
\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\partial^\alpha\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)}{\partial\mu^\alpha}\right] &= p\chi(p,s)-\chi(0,s), \\
\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\partial^\beta\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)}{\partial\eta^\beta}\right] &= s\chi(p,s)-\chi(p,0), \qquad (10)
\end{aligned}$$

and the CDLTM for the first- and second-order fractional derivatives is given by

$$\pounds_{\mu}^{\alpha}\pounds_{\eta}^{\beta}\left[\frac{\partial^{2\alpha}\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)}{\partial\mu^{2\alpha}}\right] = p^{2}\chi(p,s)-p\chi(0,s)-\pounds_{\eta}^{\beta}\left[\frac{\partial^{\alpha}\chi\left(0,\frac{\eta^{\beta}}{\beta}\right)}{\partial\mu^{\alpha}}\right], \quad (11)$$

$$\pounds_{\mu}^{\alpha}\pounds_{\eta}^{\beta}\left[\frac{\partial^{2\beta}\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)}{\partial\eta^{2\beta}}\right] = s^{2}\chi(p,s)-s\chi(p,0)-\pounds_{\mu}^{\alpha}\left[\frac{\partial^{\beta}\chi\left(\frac{\mu^{\alpha}}{\alpha},0\right)}{\partial\eta^{\beta}}\right], \quad (12)$$

where $\frac{\partial^{\alpha}\chi}{\partial\mu^{\alpha}}$, $\frac{\partial^{\beta}\chi}{\partial\eta^{\beta}}$, $\frac{\partial^{2\alpha}\chi}{\partial\mu^{2\alpha}}$ and $\frac{\partial^{2\beta}\chi}{\partial\eta^{2\beta}}$ represent the α-th- and β-th-order conformable fractional partial derivatives.

In the next theorem, we generalize the conformable fractional double Laplace transform method (CFDLTM) for m,n-time conformable fractional derivatives.

Theorem 3 ([22,26])**.** *Let $0<\alpha,\beta\leq 1$ and $m,n\in\mathbb{N}$ such that $\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)\in\mathbb{C}^{\hbar}(\mathbb{R}^{+}\times\mathbb{R}^{+})$, $\hbar=\max(m,n)$. Also, let the CFLTM of the functions $\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)$, $\frac{\partial^{(m)\alpha}}{\partial\mu^{(m)\alpha}}\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)$ and $\frac{\partial^{(n)\beta}}{\partial\eta^{(n)\beta}}\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)$ exist. Then,*

$$\pounds_{\mu}^{\alpha}\pounds_{\eta}^{\beta}\left[\frac{\partial^{(m)\alpha}}{\partial\mu^{(m)\alpha}}\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)\right] = p^{m}\chi_{\alpha,\beta}(p,s)-p^{m-1}\chi_{\beta}(0,s) -\sum_{i=1}^{m-1}p^{m-1-i}\pounds_{\eta}^{\beta}\left[\frac{\partial^{(i)\alpha}}{\partial\mu^{(i)\alpha}}\chi\left(0,\frac{\eta^{\beta}}{\beta}\right)\right], \quad (13)$$

and

$$\pounds_{\mu}^{\alpha}\pounds_{\eta}^{\beta}\left[\frac{\partial^{(n)\beta}}{\partial\eta^{(n)\beta}}\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)\right] = s^{n}\chi_{\alpha,\beta}(p,s)-s^{n-1}\chi_{\alpha}(p,0) -\sum_{j=1}^{n-1}s^{n-1-j}\pounds_{\mu}^{\alpha}\left[\frac{\partial^{(j)\beta}}{\partial\eta^{(j)\beta}}\chi\left(\frac{\mu^{\alpha}}{\alpha},0\right)\right], \quad (14)$$

where $\frac{\partial^{(m)\alpha}}{\partial\mu^{(m)\alpha}}\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)$ and $\frac{\partial^{(n)\beta}}{\partial\eta^{(n)\beta}}\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)$ denote the m,n-time conformable fractional derivatives of function $\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)$ with order α and β, respectively.

In the following example, we calculate the CFDLTM for certain functions:

Example 2. *Let $\alpha,\beta\in(0,1]$, and ω,v,ν,c,d and $\sigma\in\mathbb{R}$; thus,*

1. $\pounds_{\mu}^{\alpha}\pounds_{\eta}^{\beta}(\omega)=\frac{\omega}{ps}$,
2. $\pounds_{\mu}^{\alpha}\pounds_{\eta}^{\beta}\left[\nu\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)+v\Psi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)\right]=\nu\pounds_{\mu}^{\alpha}\pounds_{\eta}^{\beta}\left[\chi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)\right]+v\pounds_{\mu}^{\alpha}\pounds_{\eta}^{\beta}\left[\Psi\left(\frac{\mu^{\alpha}}{\alpha},\frac{\eta^{\beta}}{\beta}\right)\right]$,
3. $\pounds_{\mu}^{\alpha}\pounds_{\eta}^{\beta}\left[\sin\left(c\frac{\mu^{\alpha}}{\alpha}+d\frac{\eta^{\beta}}{\beta}\right)\right]=\frac{pd+sc}{(p^{2}+c^{2})(s^{2}+d^{2})}$,
4. $\pounds_{\mu}^{\alpha}\pounds_{\eta}^{\beta}\left[\cos\left(c\frac{\mu^{\alpha}}{\alpha}+d\frac{\eta^{\beta}}{\beta}\right)\right]=\frac{ps-cd}{(p^{2}+c^{2})(s^{2}+d^{2})}$,
5. $\pounds_{\mu}^{\alpha}\pounds_{\eta}^{\beta}\left(e^{\sigma\frac{\mu^{\alpha}}{\alpha}+\omega\frac{\eta^{\beta}}{\beta}}\right)=\frac{1}{(p-\sigma)(s-\omega)}$,
6. $\pounds_{\mu}^{\alpha}\pounds_{\eta}^{\beta}\left[\left(\frac{\mu^{\alpha}}{\alpha}\right)^{h}\left(\frac{\eta^{\beta}}{\beta}\right)^{k}\right]=\frac{h!\,k!}{p^{h+1}\,s^{k+1}}=\frac{\Gamma(h+1)\,\Gamma(k+1)}{p^{h+1}\,s^{k+1}}$, *where* $h,k>-1$,
7. $\pounds_{\mu}^{\alpha}\pounds_{\eta}^{\beta}\left[\left(\frac{\mu^{m\alpha}}{\alpha}\right)\left(\frac{\eta^{n\beta}}{\beta}\right)e^{\frac{\mu^{\alpha}}{\alpha}+\frac{\eta^{\beta}}{\beta}}\right]=\frac{m!n!}{(p-1)^{m+1}(s-1)^{n+1}}$, $m,n\in\mathbb{N}$.

3. Analysis of the Method (CDLTM)

This part illustrates the suggested approach (CFDLTM) and (CSLTM). We consider the general form of linear and nonlinear CFPDEs in the following form:

$$\frac{\partial^{2\beta}}{\partial\eta^{2\beta}}\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)+\mathcal{L}\left(\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right)+\mathcal{N}\left(\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right)=k\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right), \tag{15}$$

with initial conditions

$$\chi\left(\frac{\mu^\alpha}{\alpha},0\right)=k_1\left(\frac{\mu^\alpha}{\alpha}\right),\quad \frac{\partial^\beta\chi\left(\frac{\mu^\alpha}{\alpha},0\right)}{\partial\eta^\beta}=k_2\left(\frac{\mu^\alpha}{\alpha}\right). \tag{16}$$

where $\frac{\partial^{2\beta}}{\partial\eta^{2\beta}}\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)$, denote the time β is an order of a conformable fractional derivative of function $\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)$, and the linear operator is defined as $\mathcal{L}$, the symbol $\mathcal{N}$ denotes the nonlinear operator, and $k\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)$ is a known analytical function, and k_1,k_2 are given functions. To solve Equation (15), the following steps are needed:

First step: Using the CDLTM for Equation (15) and CSLTM for Equation (16), respectively, we will gain

$$\begin{aligned}&s^2\pounds_\mu^\alpha\pounds_\eta^\beta\left[\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right]-sk_1(p)-k_2(p)+\pounds_\mu^\alpha\pounds_\eta^\beta\left[L\left(\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right)\right]\\&+\pounds_\mu^\alpha\pounds_\eta^\beta\left[\mathcal{N}\left(\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right)\right]=\pounds_\mu^\alpha\pounds_\eta^\beta\left[k\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right],\end{aligned} \tag{17}$$

and

$$\chi(p,0)=k_1(p),\quad \text{and}\quad \pounds_\mu^\alpha\left(D_\eta^\beta\chi\left(\frac{\mu^\alpha}{\alpha},0\right)\right)=k_2(p). \tag{18}$$

By substituting Equation (18) into Equation (17), we gain

$$\begin{aligned}\chi(p,s)\;=\;&\frac{k_1(p)}{s}+\frac{k_2(p)}{s^2}+\pounds_\mu^\alpha\pounds_\eta^\beta\left[\frac{k(p,s)}{s^2}\right]-\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left[L\left(\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right)\right]\\&-\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left[N\left(\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right)\right].\end{aligned} \tag{19}$$

Second step: By operating the inverse CDLTM for Equation (19), one can get

$$\begin{aligned}\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\;=\;&k_1\left(\frac{\mu^\alpha}{\alpha}\right)+\frac{\eta^\beta}{\beta}k_2\left(\frac{\mu^\alpha}{\alpha}\right)+\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta(k(p,s))\right]\\&-\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left[L\left(\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right)\right]\right]\\&-\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left[\mathcal{N}\left(\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right)\right]\right].\end{aligned} \tag{20}$$

The nonlinear term $\mathcal{N}\left(\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right)$ is expressed as

$$\mathcal{N}\left(\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right)=\sum_{j=0}^{\infty}A_j. \tag{21}$$

The few terms of the Adomian polynomials for A_j are denoted by

$$A_j=\frac{1}{j!}\left[\frac{d^j}{d\lambda^j}\left[\mathcal{N}\sum_{i=0}^{\infty}\left(\lambda^j\chi_j\right)\right]\right]_{\lambda=0},\quad j=0,1,2,\ldots \tag{22}$$

Third step: Now, we apply the homotopy perturbation method (HPM), as follows:

$$\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)=\sum_{j=0}^{\infty}q^j\chi_j\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right),$$

and we get

$$\begin{aligned}\sum_{i=0}^{\infty}q^j\chi_j\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) \quad = \quad & k_1\left(\frac{\mu^\alpha}{\alpha}\right)+\frac{\eta^\beta}{\beta}k_2\left(\frac{\mu^\alpha}{\alpha}\right)\\ & +q\left[\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta(k(p,s))\right]\right]\\ & -q\left[\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left[L\left(\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right)\right]\right]\right]\\ & -q\left[\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta[A_j]\right]\right].\end{aligned} \tag{23}$$

Fourth step: Matching the coefficient of the power of q, the following approximation is obtained:

$$q^0:\chi_0\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)=k_1\left(\frac{\mu^\alpha}{\alpha}\right)+\frac{\eta^\beta}{\beta}k_2\left(\frac{\mu^\alpha}{\alpha}\right).$$

The subsequent terms are

$$\begin{aligned} q^1 \quad : \quad & \chi_1\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)=\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta(k(p,s))\right]\\ & -\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left[L\left(\chi_0\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right)+A_0\right]\right]\\ q^2 \quad : \quad & \chi_2\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)=-\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left[L\left(\chi_1\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right)+A_1\right]\right]\\ q^3 \quad : \quad & \chi_3\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)=-\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left[L\left(\chi_2\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)\right)+A_2\right]\right].\\ & \vdots \end{aligned} \tag{24}$$

Therefore, the exact solution of Equation (15) is given by

$$\chi=\chi_0+\chi_1+\chi_2+\chi_3+\ldots. \tag{25}$$

4. Application

This part of this paper offers some well-known partial differential equations in the sense of conformable fractional derivatives.

Example 3. *Consider a singular conformable fractional Boussinesq's equation in one dimension [6] as follows:*

$$\frac{\partial^{2\beta}}{\partial\eta^{2\beta}}\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)=\left(\frac{\mu^\alpha}{\alpha}\right)^2\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\partial^\alpha\chi}{\partial\mu^\alpha}\frac{\partial^{2\alpha}\chi}{\partial\mu^{2\alpha}}\right)-\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\partial^{2\alpha}\chi}{\partial\mu^{2\alpha}}\right)^2-\chi, \tag{26}$$

with the initial conditions

$$\chi\left(\frac{\mu^\alpha}{\alpha},0\right)=0,\qquad\frac{\partial^\beta\chi\left(\frac{\mu^\alpha}{\alpha},0\right)}{\partial\eta^\beta}=\left(\frac{\mu^\alpha}{\alpha}\right)^2. \tag{27}$$

By utilizing the CDLTM for Equation (26) and CSLTM for Equation (27), respectively, we yield

$$s^2 \pounds_\mu^\alpha \pounds_\eta^\beta \left[\chi\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)\right] - s\chi\left(\frac{\mu^\alpha}{\alpha}, 0\right) - \chi\left(\frac{\mu^\alpha}{\alpha}, 0\right) = \pounds_\mu^\alpha \pounds_\eta^\beta \left[\left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\partial^\alpha}{\partial \mu^\alpha}\left(\frac{\partial^\alpha \chi}{\partial \mu^\alpha}\frac{\partial^{2\alpha}\chi}{\partial \mu^{2\alpha}}\right)\right] - \pounds_\mu^\alpha \pounds_\eta^\beta \left[\left(\frac{\mu^\alpha}{\alpha}\right)^2 \left(\frac{\partial^{2\alpha}\chi}{\partial \mu^{2\alpha}}\right)^2 + \chi\right], \tag{28}$$

After an assessment, Equation (28) can be simplified to

$$\chi(p,s) = \frac{1}{s}\pounds_\mu^\alpha\left(\chi\left(\frac{\mu^\alpha}{\alpha},0\right)\right) - \frac{1}{s^2}\pounds_\mu^\alpha\left(\chi\left(\frac{\mu^\alpha}{\alpha},0\right)\right) + \frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\partial^\alpha}{\partial \mu^\alpha}\left(\frac{\partial^\alpha \chi}{\partial \mu^\alpha}\frac{\partial^{2\alpha}\chi}{\partial \mu^{2\alpha}}\right)\right] - \frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\partial^{2\alpha}\chi}{\partial \mu^{2\alpha}}\right)^2 + \chi\right], \tag{29}$$

By substituting Equation (27) into Equation (29), we have

$$\chi(p,s) = \frac{2!}{p^3 s^2} + \frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\partial^\alpha}{\partial \mu^\alpha}\left(\frac{\partial^\alpha \chi}{\partial \mu^\alpha}\frac{\partial^{2\alpha}\chi}{\partial \mu^{2\alpha}}\right) - \left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\partial^{2\alpha}\chi}{\partial \mu^{2\alpha}}\right)^2 - \chi\right). \tag{30}$$

By applying the inverse CDLTM to Equation (30), we obtain

$$\chi\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) = \left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\eta^\beta}{\beta} + \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\partial^\alpha}{\partial \mu^\alpha}\left(\frac{\partial^\alpha \chi}{\partial \mu^\alpha}\frac{\partial^{2\alpha}\chi}{\partial \mu^{2\alpha}}\right)\right] - \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\partial^{2\alpha}\chi}{\partial \mu^{2\alpha}}\right)^2 - \chi\right)\right]. \tag{31}$$

Now, we apply the homotopy perturbation method (HPM), $\chi\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) = \sum_{j=0}^\infty q^j \chi_j\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right)$*, and we get*

$$\sum_{j=0}^\infty q^j \chi_j\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) = \left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\eta^\beta}{\beta} + q\left[\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\partial^\alpha}{\partial \mu^\alpha}\sum_{j=0}^\infty q^j\left(\frac{\partial^\alpha \chi_j}{\partial \mu^\alpha}\frac{\partial^{2\alpha}\chi_j}{\partial \mu^{2\alpha}}\right)\right)\right]\right] - q\left[\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2 \sum_{j=0}^\infty q^j\left(\frac{\partial^{2\alpha}\chi_j}{\partial \mu^{2\alpha}}\right)^2 - \sum_{j=0}^\infty P^j \chi_j\right)\right]\right]. \tag{32}$$

The nonlinear terms $\frac{\partial^\alpha \chi_j}{\partial \mu^\alpha}\frac{\partial^{2\alpha}\chi_j}{\partial \mu^{2\alpha}}, \left(\frac{\partial^{2\alpha}\chi_j}{\partial \mu^{2\alpha}}\right)^2$ *are defined by*

$$\frac{\partial^\alpha \chi_j}{\partial \mu^\alpha}\frac{\partial^{2\alpha}\chi_j}{\partial \mu^{2\alpha}} = \sum_{j=0}^\infty A_j, \quad \text{and} \quad \left(\frac{\partial^{2\alpha}\chi_j}{\partial \mu^{2\alpha}}\right)^2 = \sum_{j=0}^\infty B_j\,. \tag{33}$$

The Adomian polynomials of A_j*, and* B_j *are defined by*

$$A_j = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j}\left[\sum_{i=0}^\infty\left(\lambda^j \chi_j\right)\right]\right]_{\lambda=0}, \tag{34}$$

and

$$B_j = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j}\left[\sum_{i=0}^\infty\left(\lambda^j \chi_j\right)\right]\right]_{\lambda=0}, \tag{35}$$

Moreover, the few components of the Adomian polynomials of Equation (26) are given as follows:

$$\begin{aligned} A_0 &= (\chi_0)_\mu(\chi_0)_{\mu\mu}, \\ A_1 &= (\chi_0)_\mu(\chi_1)_{\mu\mu} + (\chi_1)_\mu(\chi_0)_{\mu\mu}, \\ A_2 &= (\chi_0)_\mu(\chi_2)_{\mu\mu} + (\chi_1)_\mu(\chi_1)_{\mu\mu} + (\chi_2)_\mu(\chi_0)_{\mu\mu}, \\ A_3 &= (\chi_0)_\mu(\chi_3)_{\mu\mu} + (\chi_1)_\mu(\chi_2)_{\mu\mu} + (\chi_2)_\mu(\chi_1)_{\mu\mu} + (\chi_3)_\mu(\chi_0)_{\mu\mu}, \end{aligned} \tag{36}$$

and

$$\begin{aligned} B_0 &= (\chi_0^2)_{\mu\mu}, \\ B_1 &= 2(\chi_0)_{\mu\mu}(\chi_1)_{\mu\mu}, \\ B_2 &= 2(\chi_0)_{\mu\mu}(\chi_2)_{\mu\mu} + (\chi_1^2)_{\mu\mu}, \\ B_3 &= 2(\chi_0)_{\mu\mu}(\chi_3)_{\mu\mu} + (\chi_3)_{\mu\mu}(\chi_0)_{\mu\mu}, \end{aligned} \tag{37}$$

By matching the coefficient of the same power of q, the following approximation is given:

$$q^0 : \chi_0\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) = \left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\eta^\beta}{\beta},$$

and

$$\begin{aligned} q^1 &: \chi_1\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) = \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\partial^\alpha}{\partial \mu^\alpha} A_0 - \left(\frac{\mu^\alpha}{\alpha}\right)^2 B_0 - \chi_0\right)\right], \\ q^2 &: \chi_2\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) = \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\partial^\alpha}{\partial \mu^\alpha} A_1 - \left(\frac{\mu^\alpha}{\alpha}\right)^2 B_1 - \chi_1\right)\right], \\ q^3 &: \chi_3\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) = \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\partial^\alpha}{\partial \mu^\alpha} A_2 - \left(\frac{\mu^\alpha}{\alpha}\right)^2 B_2 - \chi_2\right)\right], \\ &\vdots \end{aligned} \tag{38}$$

Then, we have

$$\chi_0\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) = \left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\eta^\beta}{\beta}, \tag{39}$$

The first iteration is given by

$$\begin{aligned} \chi_1\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) &= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2 (A_0) - \left(\frac{\mu^\alpha}{\alpha}\right)^2 (B_0) - \chi_0\right)\right] \\ &= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2 ((\chi_0)_\mu(\chi_0)_{\mu\mu}) - \left(\frac{\mu^\alpha}{\alpha}\right)^2 (\chi_0^2)_{\mu\mu} - \chi_0\right)\right] \\ &= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left(-\left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\eta^\alpha}{\alpha}\right)\right] \\ &= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\left(-\frac{2!}{p^3 s^2}\right)\right] \\ &= -\frac{1}{6}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^3, \end{aligned}$$

At j $= 1$*, we have*

$$\begin{aligned}
\chi_2\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) &= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2(A_1)-\left(\frac{\mu^\alpha}{\alpha}\right)^2(B_1)-\chi_1\right)\right]\\
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2\frac{\partial}{\partial\mu^\alpha}((\chi_0)_\mu(\chi_1)_{\mu\mu}+(\chi_1)_\mu(\chi_0)_{\mu\mu})\right)\right]\\
&\quad -\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2(2(\chi_0)_{\mu\mu}(\chi_1)_{\mu\mu})-\chi_1\right)\right]\\
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left(\frac{1}{6}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^3\right)\right]\\
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\left(\frac{1}{6}\frac{2!3!}{p^3s^4}\right)\right]=\frac{1}{120}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^5,
\end{aligned}$$

Similarly, in the case of j $= x2$*, we have*

$$\begin{aligned}
\chi_3\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) &= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2(A_2)-\left(\frac{\mu^\alpha}{\alpha}\right)^2(B_2)-\chi_2\right)\right]\\
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2\frac{\partial}{\partial\mu^\alpha}((\chi_0)_\mu(\chi_2)_{\mu\mu}+(\chi_1)_\mu(\chi_1)_{\mu\mu}+(\chi_2)_\mu(\chi_0)_{\mu\mu})\right)\right]\\
&\quad -\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(2(\chi_0)_{\mu\mu}(\chi_2)_{\mu\mu}-(\chi_1^2)_{\mu\mu}\right)+\chi_2\right)\right]\\
&= -\pounds_p^{-1}\pounds\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left(\frac{1}{120}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^5\right)\right]\\
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\left(-\frac{1}{120}\frac{2!5!}{p^3s^6}\right)\right]\\
&= -\frac{1}{5040}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^7,
\end{aligned}$$

In the same way, at j $= 3$*, we have*

$$\begin{aligned}
\chi_4\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) &= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2(A_3)-\left(\frac{\mu^\alpha}{\alpha}\right)^2(B_3)-\chi_3\right)\right]\\
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2\frac{\partial}{\partial\mu^\alpha}\begin{pmatrix}(\chi_0)_\mu(\chi_3)_{\mu\mu}+(\chi_1)_\mu(\chi_2)_{\mu\mu}\\+(\chi_2)_\mu(\chi_1)_{\mu\mu}+(\chi_3)_\mu(\chi_0)_{\mu\mu}\end{pmatrix}\right)\right]\\
&\quad \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left(-\left(\frac{\mu^\alpha}{\alpha}\right)^2(2(\chi_0)_{\mu\mu}(\chi_3)_{\mu\mu}+(\chi_3)_{\mu\mu}(\chi_0)_{\mu\mu})-\chi_3\right)\right]\\
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left(\frac{1}{5040}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^7\right)\right]\\
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{5040}\frac{2!7!}{p^3s^{10}}\right]\\
&= \frac{1}{362880}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^9.
\end{aligned}$$

and so on. The solution of this series is denoted by

$$\sum_{j=0}^{\infty} \chi_j\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) = \chi_0 + \chi_1 + \chi_2 + \chi_3 + \chi_4 + \ldots,$$

Therefore,

$$\begin{aligned}
\chi\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) &= \left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\eta^\beta}{\beta} - \frac{1}{6}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^3 + \frac{1}{120}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^5 \\
&\quad -\frac{1}{5040}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^7 + \frac{1}{362880}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^9 - \ldots \\
&= \left(\frac{\mu^\alpha}{\alpha}\right)^2 \frac{\eta^\beta}{\beta} - \frac{1}{3!}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^3 + \frac{1}{5!}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^5 \\
&\quad -\frac{1}{7!}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^7 + \frac{1}{9!}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\alpha}{\alpha}\right)^9 - \ldots \\
&= \left(\frac{\mu^\alpha}{\alpha}\right)^2\left[\frac{\eta^\beta}{\beta} - \frac{1}{3!}\left(\frac{\eta^\alpha}{\alpha}\right)^3 + \frac{1}{5!}\left(\frac{\eta^\alpha}{\alpha}\right)^5 - \frac{1}{7!}\left(\frac{\eta^\alpha}{\alpha}\right)^7 + \frac{1}{9!}\left(\frac{\eta^\alpha}{\alpha}\right)^9 - \ldots\right] \\
&= \left(\frac{\mu^\alpha}{\alpha}\right)^2 \sin\left(\frac{\eta^\beta}{\beta}\right),
\end{aligned} \tag{40}$$

Therefore, the solution of Equation (26) can be written in the form

$$\chi\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) = \left(\frac{\mu^\alpha}{\alpha}\right)^2 \sin\left(\frac{\eta^\beta}{\beta}\right), \tag{41}$$

Additionally, to obtain the exact solution, assume $\alpha = \beta = 1$; thus, we have

$$\chi(\mu, \eta) = \mu^2 \sin(\eta).$$

Figure 1 shows the exact solution at $\alpha = \beta = 1$ and the approximate solution for different values of $\alpha = \beta = 0.90, 0.70, 0.50$ and 0.30. Refer to Example 3.

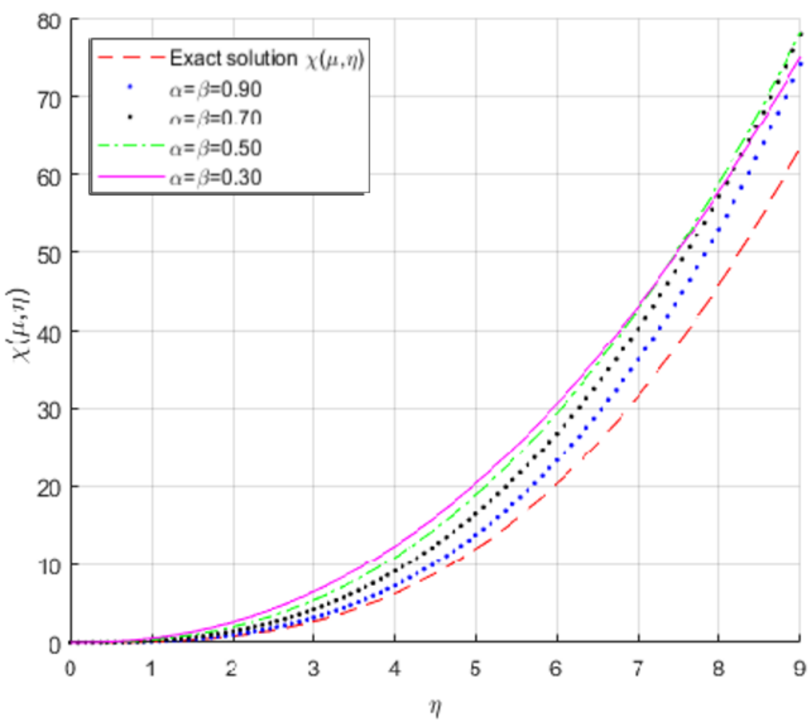

Figure 1. The exact and approximation solutions of $\chi(\mu, \eta)$.

Table 1 shows the numerical solution for different values of α and β for the function $\chi(\mu,\eta)$.

Table 1. Numerical solution for different values of α and β for the function $\chi(\mu,\eta)$.

μ	η	$\alpha=\beta=0.90$	$\alpha=\beta=0.95$	$\alpha=\beta=0.99$	Exact S.	Abs. Error
0.25	0.2	0.015115	0.013713	0.012668	0.012417	2.51×10^{-4}
	0.4	0.027551	0.025904	0.024679	0.024339	3.41×10^{-4}
	0.6	0.038149	0.036729	0.035579	0.035290	2.89×10^{-4}
	0.8	0.046979	0.045934	0.045059	0.044835	2.24×10^{-4}
0.75	0.2	0.136036	0.123418	0.114011	0.111752	2.26×10^{-3}
	0.4	0.247607	0.233141	0.221832	0.219048	2.78×10^{-3}
	0.6	0.343342	0.330557	0.320210	0.317611	2.59×10^{-3}
	0.8	0.422808	0.413407	0.405529	0.403551	1.97×10^{-3}

Example 4. *Consider a singular conformable fractional thermo-elasticity-coupled system in one dimension [27], as follows:*

$$\begin{aligned}
&\frac{\partial^{2\beta}}{\partial\eta^{2\beta}}\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)-\frac{\alpha}{\mu^\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\chi}{\partial\mu^\alpha}\right)-\frac{\alpha}{\mu^\alpha}\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\chi}{\partial\mu^\alpha}\right)+\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\Psi}{\partial\mu^\alpha}\\
=\;&2\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)-4\frac{\eta^\beta}{\beta}-4,\\
&\frac{\partial^{\beta}}{\partial\eta^{\beta}}\Psi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)-\frac{\alpha}{\mu^\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\Psi}{\partial\mu^\alpha}\right)-\frac{\alpha}{\mu^\alpha}\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\Psi}{\partial\mu^\alpha}\right)+\frac{\mu^\alpha}{\alpha}\frac{\partial^{\alpha+\beta}\chi}{\partial\mu^\alpha\partial\eta^\beta}\\
=\;&\left(\frac{\mu^\alpha}{\alpha}\right)^2\cos\left(\frac{\eta^\beta}{\beta}\right)-4\sin\left(\frac{\eta^\beta}{\beta}\right)-4\cos\left(\frac{\eta^\beta}{\beta}\right)+2\left(\frac{\mu^\alpha}{\alpha}\right)^2,
\end{aligned}\tag{42}$$

Subject the initial condition as follows:

$$\chi\left(\frac{\mu^\alpha}{\alpha},0\right)=0,\quad\frac{\partial^\beta\chi\left(\frac{\mu^\alpha}{\alpha},0\right)}{\partial\eta^\beta}=\left(\frac{\mu^\alpha}{\alpha}\right)^2,\quad\Psi\left(\frac{\mu^\alpha}{\alpha},0\right)=0,\tag{43}$$

By using the CDLTM for Equation (42) and CSLTM for Equation (43), respectively, one can get

$$\begin{aligned}
\chi(p,s)\;=\;&\frac{1}{s}\chi(p,0)+\frac{1}{s^2}\chi(p,0)+\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\chi}{\partial\mu^\alpha}\right)\right]\\
&+\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\chi}{\partial\mu^\alpha}\right)-\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\Psi}{\partial\mu^\alpha}\right]\\
&+\frac{1}{s^2}\pounds_\mu^\alpha\pounds_\eta^\beta\left[2\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)-4\frac{\eta^\beta}{\beta}-4\right],
\end{aligned}\tag{44}$$

and

$$\begin{aligned}
&\Psi(p,s)=\frac{1}{s}\Psi(p,0)+\frac{1}{s}\pounds_\mu^\alpha\pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\Psi}{\partial\mu^\alpha}\right)\right]\\
&+\frac{1}{s}\pounds_\mu^\alpha\pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\Psi}{\partial\mu^\alpha}\right)\right]-\frac{1}{s}\pounds_\mu^\alpha\pounds_\eta^\beta\left[\frac{\mu^\alpha}{\alpha}\frac{\partial^{\alpha+\beta}\chi}{\partial\mu^\alpha\partial\eta^\beta}\right]\\
&+\frac{1}{s}\pounds_\mu^\alpha\pounds_\eta^\beta\left[\left(\frac{\mu^\alpha}{\alpha}\right)^2\cos\left(\frac{\eta^\beta}{\beta}\right)-4\sin\left(\frac{\eta^\beta}{\beta}\right)-4\cos\left(\frac{\eta^\beta}{\beta}\right)+2\left(\frac{\mu^\alpha}{\alpha}\right)^2\right],
\end{aligned}\tag{45}$$

By substituting Equation (43) into Equations (44) and (45), we have

$$\begin{aligned}\chi(p,s) &= \frac{2!}{p^3 s^2} + \frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha \chi}{\partial\mu^\alpha}\right)\right] \\ &+\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha \chi}{\partial\mu^\alpha}\right) - \frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha \Psi}{\partial\mu^\alpha}\right] \\ &+\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[2\left(\frac{\mu^\alpha}{\alpha}\right)^2 \sin\left(\frac{\eta^\beta}{\beta}\right) - 4\frac{\eta^\beta}{\beta} - 4\right],\end{aligned} \tag{46}$$

and

$$\begin{aligned}&\Psi(p,s) = \frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha \Psi}{\partial\mu^\alpha}\right)\right] \\ &+\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha \Psi}{\partial\mu^\alpha}\right)\right] - \frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\mu^\alpha}{\alpha}\frac{\partial^{\alpha+\beta}\chi}{\partial\mu^\alpha\partial\eta^\beta}\right] \\ &+\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\left(\frac{\mu^\alpha}{\alpha}\right)^2 \cos\left(\frac{\eta^\beta}{\beta}\right) - 4\sin\left(\frac{\eta^\beta}{\beta}\right) - 4\cos\left(\frac{\eta^\beta}{\beta}\right) + 2\left(\frac{\mu^\alpha}{\alpha}\right)^2\right].\end{aligned} \tag{47}$$

By using the inverse CDLTR to Equations (46) and (47), we get

$$\begin{aligned}\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) &= \left(\frac{\mu^\alpha}{\alpha}\right)^2\frac{\eta^\beta}{\beta} + \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha \chi}{\partial\mu^\alpha}\right)\right]\right] \\ &+\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha \chi}{\partial\mu^\alpha}\right) - \frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha \Psi}{\partial\mu^\alpha}\right]\right] \\ &+\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[2\left(\frac{\mu^\alpha}{\alpha}\right)^2 \sin\left(\frac{\eta^\beta}{\beta}\right) - 4\frac{\eta^\beta}{\beta} - 4\right]\right],\end{aligned} \tag{48}$$

and

$$\begin{aligned}&\Psi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) = \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha \Psi}{\partial\mu^\alpha}\right)\right]\right] \\ &+\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha \Psi}{\partial\mu^\alpha}\right)\right] - \frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\mu^\alpha}{\alpha}\frac{\partial^{\alpha+\beta}\chi}{\partial\mu^\alpha\partial\eta^\beta}\right]\right] \\ &+\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\frac{\mu^\alpha}{\alpha}\right)^2 \cos\left(\frac{\eta^\beta}{\beta}\right) - 4\sin\left(\frac{\eta^\beta}{\beta}\right) - 4\cos\left(\frac{\eta^\beta}{\beta}\right) + 2\left(\frac{\mu^\alpha}{\alpha}\right)^2\right].\end{aligned} \tag{49}$$

Now, we apply the homotopy perturbation method (HPM) and get

$$\begin{aligned}\sum_{j=0}^{\infty} q^j \chi_j\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) &= \left(\frac{\mu^\alpha}{\alpha}\right)^2\frac{\eta^\beta}{\beta} + q\left[\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\sum_{j=0}^{\infty} q^j \chi_{j\mu}\right)\right]\right]\right] \\ &+q\left[\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\sum_{j=0}^{\infty} q^j \chi_j\right) - \frac{\mu^\alpha}{\alpha}\sum_{j=0}^{\infty} q^j \Psi_{j\mu}\right]\right]\right] \\ &+q\left[\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[2\left(\frac{\mu^\alpha}{\alpha}\right)^2 \sin\left(\frac{\eta^\beta}{\beta}\right) - 4\frac{\eta^\beta}{\beta} - 4\right]\right]\right],\end{aligned} \tag{50}$$

and

$$
\begin{aligned}
\sum_{j=0}^{\infty} q^j \Psi_j\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) &= q\left[\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\sum_{j=0}^{\infty} q^j \Psi_{j\mu}\right)\right]\right]\right] \\
&+q\left[\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\sum_{j=0}^{\infty} q^j \Psi_{j\mu}\right)\right]\right]\right] \\
&-q\left[\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\mu^\alpha}{\alpha}\sum_{j=0}^{\infty} q^j \Psi_{j\mu\eta}\right]\right]\right] \\
&+q\left[\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\frac{\mu^\alpha}{\alpha}\right)^2\cos\left(\frac{\eta^\beta}{\beta}\right)-4\sin\left(\frac{\eta^\beta}{\beta}\right)\right]\right] \\
&-q\left[\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[4\cos\left(\frac{\eta^\beta}{\beta}\right)-2\left(\frac{\mu^\alpha}{\alpha}\right)^2\right]\right]\right],
\end{aligned} \tag{51}
$$

By matching the coefficients of the same powers of q, we have

$$
\begin{aligned}
\chi_0 &= \left(\frac{\mu^\alpha}{\alpha}\right)^2\frac{\eta^\beta}{\beta}, \\
\Psi_0 &= 0,
\end{aligned} \tag{52}
$$

The first iteration is given by

$$
\begin{aligned}
\chi_1 &= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left(2\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)-4\frac{\eta^\beta}{\beta}-4\right)\right] \\
&+\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\Psi_{0\mu}\right)\right]\right] \\
&+\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\chi_0\right)-\frac{\mu^\alpha}{\alpha}\Psi_{0\mu}\right]\right] \\
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left(2\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)-4\frac{\eta^\beta}{\beta}-4+4\frac{\eta^\beta}{\beta}+4\right)\right] \\
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left(2\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)\right)\right] \\
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{4}{p^3s^2(s^2+1)}\right] = \pounds_p^{-1}\pounds_s^{-1}\left[\frac{4}{p^3s^2}-\frac{4}{p^3(s^2+1)}\right] \\
&= 2\left(\frac{\mu^\alpha}{\alpha}\right)^2\frac{\eta^\beta}{\beta}-2\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_1 &= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2\cos\left(\frac{\eta^\beta}{\beta}\right)-4\sin\left(\frac{\eta^\beta}{\beta}\right)\right)\right] \\
&-\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left(4\cos\left(\frac{\eta^\beta}{\beta}\right)-2\left(\frac{\mu^\alpha}{\alpha}\right)^2\right)\right] \\
&+\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\frac{\alpha}{\mu^\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\Psi}{\partial\mu^\alpha}\right)\right)\right] \\
&+\pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\frac{\alpha}{\mu^\alpha}\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\Psi}{\partial\mu^\alpha}\right)-\frac{\mu^\alpha}{\alpha}\frac{\partial^{\alpha+\beta}\chi}{\partial\mu^\alpha\partial\eta^\beta}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left(\left(\frac{\mu^\alpha}{\alpha}\right)^2\cos\left(\frac{\eta^\beta}{\beta}\right)-4\sin\left(\frac{\eta^\beta}{\beta}\right)-4\cos\left(\frac{\eta^\beta}{\beta}\right)\right)\right]\\
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\left(\frac{2s}{p^3(s^2+1)}-\frac{4}{p(s^2+1)}-\frac{4s}{p(s^2+1)}\right)\right]\\
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{2}{p^3(s^2+1)}-\frac{4}{ps}+\frac{4s}{p(s^2+1)}-\frac{4}{p(s^2+1)}\right]\\
&= \left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)+4\cos\left(\frac{\eta^\beta}{\beta}\right)-4\sin\left(\frac{\eta^\beta}{\beta}\right)-4,
\end{aligned}
$$

and

$$
\begin{aligned}
\chi_2 &= \pounds_p^{-1}\pounds_s^{-1}\left(\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\chi_1}{\partial\mu^\alpha}\right)\right]\right)\\
&\quad+\pounds_p^{-1}\pounds_s^{-1}\left(\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\chi_1}{\partial\mu^\alpha}\right)-\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\Psi_1}{\partial\mu^\alpha}\right]\right)\\
&= \pounds_p^{-1}\pounds_s^{-1}\left(\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[8\frac{\eta^\beta}{\beta}-8\sin\left(\frac{\eta^\beta}{\beta}\right)-8\cos\left(\frac{\eta^\beta}{\beta}\right)+8-2\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)\right]\right)\\
&= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{8}{ps^4}+\frac{8}{ps^3}-\frac{8}{ps^2}+\frac{8}{p(s^2+1)}-\frac{8}{ps}+\frac{8s}{p(s^2+1)}-\frac{4}{p^3s^2}+\frac{4}{p^3(s^2+1)}\right]\\
&= \frac{4}{3}\left(\frac{\eta^\beta}{\beta}\right)^3+4\left(\frac{\eta^\beta}{\beta}\right)^2-8\frac{\eta^\beta}{\beta}+8\sin\left(\frac{\eta^\beta}{\beta}\right)\\
&\quad+8\cos\left(\frac{\eta^\beta}{\beta}\right)-2\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\beta}{\beta}\right)+2\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)-8,
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_2 &= \pounds_p^{-1}\pounds_s^{-1}\left(\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\chi_1}{\partial\mu^\alpha}\right)\right]\right)\\
&\quad+\pounds_p^{-1}\pounds_s^{-1}\left(\frac{1}{s^2}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\frac{\alpha}{\mu^\alpha}\frac{\partial^{\alpha+\beta}}{\partial\mu^\alpha\partial\eta^\beta}\left(\frac{\mu^\alpha}{\alpha}\frac{\partial^\alpha\chi_1}{\partial\mu^\alpha}\right)-\frac{\mu^\alpha}{\alpha}\frac{\partial^{\alpha+\beta}\chi_1}{\partial\mu^\alpha\partial\eta^\beta}\right]\right)\\
&= \pounds_p^{-1}\pounds_s^{-1}\left(\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[4\sin\left(\frac{\eta^\beta}{\beta}\right)+4\cos\left(\frac{\eta^\beta}{\beta}\right)\right]\right)\\
&\quad-\pounds_p^{-1}\pounds_s^{-1}\left(\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[4\left(\frac{\mu^\alpha}{\alpha}\right)^2-4\left(\frac{\mu^\alpha}{\alpha}\right)^2\cos\left(\frac{\eta^\beta}{\beta}\right)\right]\right)\\
&= \pounds_p^{-1}\pounds_s^{-1}\left(\frac{4}{ps}-\frac{4s}{p(s^2+1)}+\frac{4}{p(s^2+1)}-\frac{8}{p^3s^2}+\frac{8}{p^3(s^2+1)}\right)\\
&= 4-4\cos\left(\frac{\eta^\beta}{\beta}\right)+4\sin\left(\frac{\eta^\beta}{\beta}\right)-4\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\beta}{\beta}\right)+4\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right),
\end{aligned}
$$

In a similar way, we obtain

$$
\begin{aligned}
\chi_3\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) &= -\frac{4}{3}\left(\frac{\eta^\beta}{\beta}\right)^3-4\left(\frac{\eta^\beta}{\beta}\right)^2-\frac{4}{3}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\beta}{\beta}\right)^3-8\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)\\
&\quad+8\left(\frac{\eta^\beta}{\beta}\right)-8\sin\left(\frac{\eta^\beta}{\beta}\right)-8\cos\left(\frac{\eta^\beta}{\beta}\right)+\frac{4}{3}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\beta}{\beta}\right)^3,
\end{aligned}
$$

and

$$\Psi_3\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) = -8\left(\frac{\eta^\beta}{\beta}\right)^2-16\left(\frac{\eta^\beta}{\beta}\right)+16\sin\left(\frac{\eta^\beta}{\beta}\right)+4\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\beta}{\beta}\right) -4\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)-16\cos\left(\frac{\eta^\beta}{\beta}\right)+16.$$

Therefore, the series of solutions is denoted by

$$\begin{aligned}\sum_{j=0}^{\infty}\chi_j\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) &= \chi_0+\chi_1+\chi_2+\chi_3+\ldots\\ \sum_{j=0}^{\infty}\Psi_j\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) &= \Psi_0+\Psi_1+\Psi_2+\Psi_3+\ldots,\end{aligned} \tag{53}$$

Hence,

$$\begin{aligned}\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) =& \left(\frac{\mu^\alpha}{\alpha}\right)^2\frac{\eta^\beta}{\beta}+2\left(\frac{\mu^\alpha}{\alpha}\right)^2\frac{\eta^\beta}{\beta}-2\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)\\ &+\frac{4}{3}\left(\frac{\eta^\beta}{\beta}\right)^3+4\left(\frac{\eta^\beta}{\beta}\right)^2-8\frac{\eta^\beta}{\beta}+8\sin\left(\frac{\eta^\beta}{\beta}\right)\\ &+8\cos\left(\frac{\eta^\beta}{\beta}\right)-2\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\beta}{\beta}\right)+2\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)-8\\ &-\frac{4}{3}\left(\frac{\eta^\beta}{\beta}\right)^3-4\left(\frac{\eta^\beta}{\beta}\right)^2-\frac{4}{3}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\beta}{\beta}\right)^3-8\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)\\ &+8\left(\frac{\eta^\beta}{\beta}\right)-8\sin\left(\frac{\eta^\beta}{\beta}\right)-8\cos\left(\frac{\eta^\beta}{\beta}\right)+\frac{4}{3}\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\beta}{\beta}\right)^3\end{aligned} \tag{54}$$

and

$$\begin{aligned}\Psi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) =& \left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)+4\cos\left(\frac{\eta^\beta}{\beta}\right)-4\sin\left(\frac{\eta^\beta}{\beta}\right)-4\\ &+4-4\cos\left(\frac{\eta^\beta}{\beta}\right)+4\sin\left(\frac{\eta^\beta}{\beta}\right)-4\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\beta}{\beta}\right)+4\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)\\ &-8\left(\frac{\eta^\beta}{\beta}\right)^2-16\left(\frac{\eta^\beta}{\beta}\right)+16\sin\left(\frac{\eta^\beta}{\beta}\right)+4\left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\beta}{\beta}\right)\\ &-4\left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right)-16\cos\left(\frac{\eta^\beta}{\beta}\right)+16.\end{aligned} \tag{55}$$

By simplifying Equations (54) and (55), we gain the solution of Equation (42), as follows:

$$\begin{aligned}\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) &= \left(\frac{\mu^\alpha}{\alpha}\right)^2\left(\frac{\eta^\beta}{\beta}\right),\\ \Psi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) &= \left(\frac{\mu^\alpha}{\alpha}\right)^2\sin\left(\frac{\eta^\beta}{\beta}\right).\end{aligned} \tag{56}$$

By taking $\alpha=\beta=1$, the exact solution becomes

$$\begin{aligned}\chi(\mu,\eta) &= \mu^2\eta,\\ \Psi(\mu,\eta) &= \mu^2\sin(\eta).\end{aligned}$$

Figure 2 shows that the exact solution at $\alpha = \beta = 1$ and the approximate solution for a different value of α and β as $\alpha = \beta = 0.90, 0.70, 0.50$ and 0.30. Refer to Example 4.

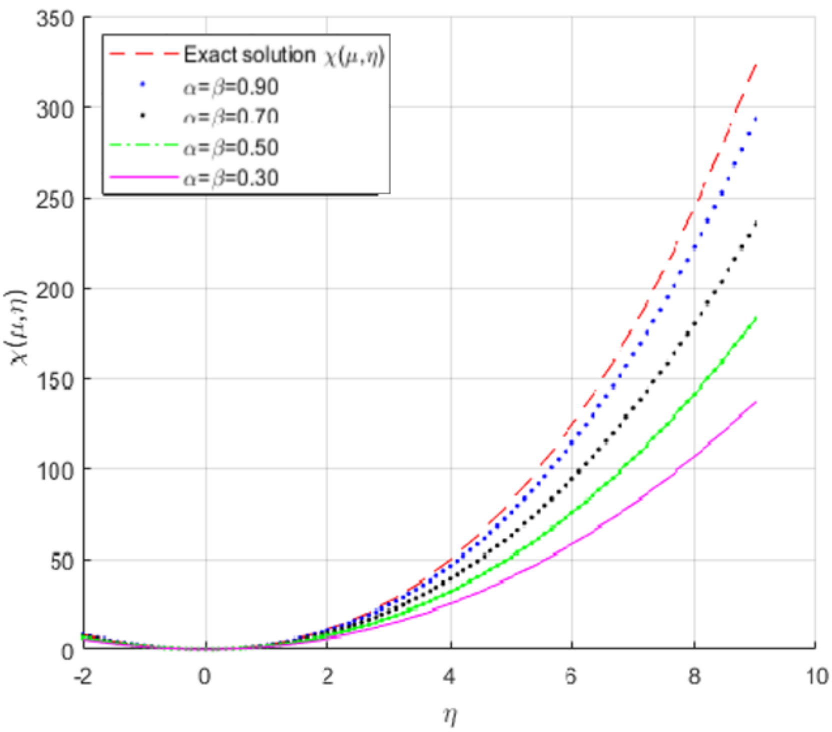

Figure 2. The exact and approximation solutions of $\chi(\mu, \eta)$.

Figure 3 shows the exact and approximate solutions of $\Psi(\mu, \eta)$ for Example 4 for different values of $\alpha = \beta = 0.90, 0.70, 0.50$ and 0.30.

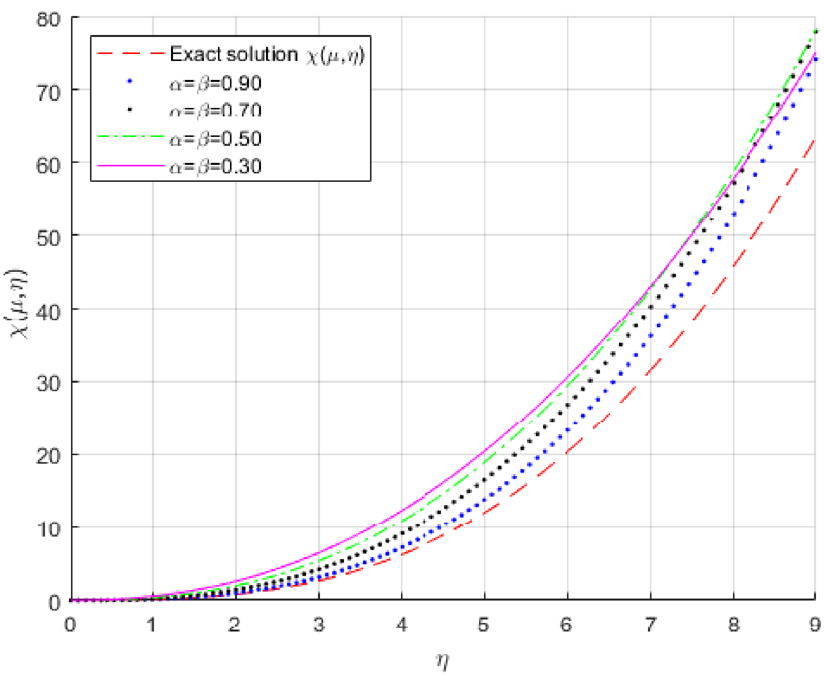

Figure 3. The exact and approximation solutions of $\Psi(\mu, \eta)$.

Table 2 shows the numerical solution for different values of α and β for the function $\chi(\mu, \eta)$. Table 3 shows the numerical solution for different values of α and β for the function $\Psi(\mu, \eta)$.

Table 2. Numerical solution for different values of α and β for function $\chi(\mu, \eta)$.

μ	η	$\alpha = \beta = 0.90$	$\alpha = \beta = 0.95$	$\alpha = \beta = 0.99$	Exact S.	Abs. Error
0.25	0.2	0.015115	0.013713	0.012668	0.012417	2.51×10^{-4}
	0.4	0.027551	0.025904	0.024679	0.024339	3.41×10^{-4}
	0.6	0.038149	0.036729	0.035579	0.035290	2.89×10^{-4}
	0.8	0.046979	0.045934	0.045059	0.044835	2.24×10^{-4}
0.75	0.2	0.136036	0.123418	0.114011	0.111752	2.26×10^{-3}
	0.4	0.247607	0.233141	0.221832	0.219048	2.78×10^{-3}
	0.6	0.343342	0.330557	0.320210	0.317611	2.59×10^{-3}
	0.8	0.422808	0.413407	0.405529	0.403551	1.97×10^{-3}

Table 3. Numerical solution for different values of α and β for the function $\Psi(\mu, \eta)$.

μ	η	$\alpha = \beta = 0.90$	$\alpha = \beta = 0.95$	$\alpha = \beta = 0.99$	Exact S.	Abs. Error
0.25	0.2	0.015266	0.013826	0.0127562	0.01250	2.56×10^{-4}
	0.4	0.028418	0.026709	0.0253362	0.02500	3.36×10^{-4}
	0.6	0.041034	0.039260	0.0378505	0.03750	3.51×10^{-4}
	0.8	0.035161	0.051599	0.0503224	0.05000	3.22×10^{-4}
0.75	0.2	0.137398	0.124431	0.1148060	0.11250	2.31×10^{-3}
	0.4	0.256393	0.240384	0.2280258	0.22500	3.026×10^{-3}
	0.6	0.369309	0.253340	0.3406547	0.33750	3.155×10^{-3}
	0.8	0.478447	0.464392	0.4529014	0.45000	2.90×10^{-3}

Example 5. *Consider a conformable fractional of a partial differential equation in one dimension [28], as follows:*

$$\frac{\partial^{\beta}}{\partial \eta^{\beta}}\chi\left(\frac{\mu^{\alpha}}{\alpha}, \frac{\eta^{\beta}}{\beta}\right) - \frac{\partial^{2\alpha+\beta}\chi}{\partial \mu^{2\alpha}\partial \eta^{\beta}} + \frac{\partial^{\alpha}\chi}{\partial \mu^{\alpha}} + \chi\frac{\partial^{\alpha}\chi}{\partial \mu^{\alpha}} - 3\frac{\partial^{\alpha}\chi}{\partial \mu^{\alpha}}\frac{\partial^{2\alpha}\chi}{\partial \mu^{2\alpha}} - \chi\frac{\partial^{3\alpha}\chi}{\partial \mu^{3\alpha}} = 0, \tag{57}$$

The initial condition is

$$\chi\left(\frac{\mu^{\alpha}}{\alpha}, 0\right) = -\frac{\mu^{\alpha}}{\alpha}. \tag{58}$$

In order to implement our method for Equations (57) and (58), we must have

$$\chi_0\left(\frac{\mu^{\alpha}}{\alpha}, \frac{\eta^{\beta}}{\beta}\right) = -\frac{\mu^{\alpha}}{\alpha}, \tag{59}$$

and

$$\chi_{j+1}\left(\frac{\mu^{\alpha}}{\alpha}, \frac{\eta^{\beta}}{\beta}\right) = £_p^{-1}£_s^{-1}\left(\frac{1}{s}£_{\mu}^{\alpha}£_{\eta}^{\beta}\left[\frac{\partial^{2\alpha+\beta}\chi}{\partial \mu^{2\alpha}\partial \eta^{\beta}} - \frac{\partial^{\alpha}\chi}{\partial \mu^{\alpha}} - \chi\frac{\partial^{\alpha}\chi}{\partial \mu^{\alpha}} + 3\frac{\partial^{\alpha}\chi}{\partial \mu^{\alpha}}\frac{\partial^{2\alpha}\chi}{\partial \mu^{2\alpha}} + \chi\frac{\partial^{3\alpha}\chi}{\partial \mu^{3\alpha}}\right]\right). \tag{60}$$

The nonlinear terms $\chi\frac{\partial^{\alpha}\chi_0}{\partial \mu^{\alpha}}$, $\frac{\partial^{\alpha}\chi_0}{\partial \mu^{\alpha}}\frac{\partial^{2\alpha}\chi_0}{\partial \mu^{2\alpha}}$ *and* $\chi_0\frac{\partial^{3\alpha}\chi_0}{\partial \mu^{3\alpha}}$ *are defined by*

$$\chi\frac{\partial^{\alpha}\chi_0}{\partial \mu^{\alpha}} = \sum_{j=0}^{\infty} A_j, \quad \frac{\partial^{\alpha}\chi_0}{\partial \mu^{\alpha}}\frac{\partial^{2\alpha}\chi_0}{\partial \mu^{2\alpha}} = \sum_{j=0}^{\infty} B_j, \text{ and } \chi_0\frac{\partial^{3\alpha}\chi_0}{\partial \mu^{3\alpha}} = \sum_{j=0}^{\infty} C_j, \tag{61}$$

We have a few terms of the Adomian polynomials for A_j, B_j *and* C_j *that are denoted by*

$$A_j = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j}\left[\sum_{i=0}^{\infty}\left(\lambda^j \chi_j\right)\right]\right]_{\lambda=0}, \tag{62}$$

and

$$B_j = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j}\left[\sum_{i=0}^{\infty}\left(\lambda^j \chi_j\right)\right]\right]_{\lambda=0}, \tag{63}$$

and

$$C_j = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j}\left[\sum_{i=0}^{\infty}\left(\lambda^j \chi_j\right)\right]\right]_{\lambda=0}, \tag{64}$$

The few components of the Adomian polynomials of Equation (57) are given as follows:

$$\begin{aligned} A_0 &= \chi_0\chi_{0\mu}, \\ A_1 &= \chi_0\chi_{1\mu} + \chi_1\chi_{0\mu}, \\ A_2 &= \chi_0\chi_{2\mu} + \chi_1\chi_{1\mu} + \chi_2\chi_{0\mu}, \\ A_3 &= \chi_0\chi_{3\mu} + \chi_1\chi_{2\mu} + \chi_2\chi_{1\mu} + \chi_3\chi_{0\mu}, \end{aligned} \tag{65}$$

$$\begin{aligned} B_0 &= \chi_{0\mu}\chi_{0\mu\mu}, \\ B_1 &= \chi_{0\mu}\chi_{1\mu\mu} + \chi_{1\mu}\chi_{0\mu\mu}, \\ B_2 &= \chi_{0\mu}\chi_{2\mu\mu} + \chi_{1\mu}\chi_{1\mu\mu} + \chi_{2\mu}\chi_{0\mu\mu}, \\ B_3 &= \chi_{0\mu}\chi_{3\mu\mu} + \chi_{1\mu}\chi_{2\mu\mu} + \chi_{2\mu}\chi_{1\mu\mu} + \chi_{3\mu}\chi_{0\mu\mu}, \end{aligned} \tag{66}$$

and

$$\begin{aligned} C_0 &= \chi_0\chi_{0\mu\mu\mu}, \\ C_1 &= \chi_0\chi_{1\mu\mu\mu} + \chi_1\chi_{0\mu\mu\mu}, \\ C_2 &= \chi_0\chi_{2\mu\mu\mu} + \chi_1\chi_{1\mu\mu\mu} + \chi_2\chi_{0\mu\mu\mu}, \\ C_3 &= \chi_0\chi_{3\mu\mu\mu} + \chi_1\chi_{2\mu\mu\mu} + \chi_2\chi_{1\mu\mu\mu} + \chi_3\chi_{0\mu\mu\mu}, \end{aligned} \tag{67}$$

where $j \geq 0$*, and the first iteration is given by*

$$\begin{aligned} \chi_1\left(\frac{\mu^\alpha}{\alpha}, \frac{\eta^\beta}{\beta}\right) &= \pounds_p^{-1}\pounds_s^{-1}\left(\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left[\chi_{0\mu\mu\eta} - \chi_{0\mu} - A_0 + 3B_0 + C_0\right]\right) \\ &= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\pounds_\mu^\alpha \pounds_\eta^\beta\left(1 - \frac{\mu^\alpha}{\alpha}\right)\right] \\ &= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{s}\left(\frac{1}{ps} - \frac{1}{p^2 s}\right)\right] \\ &= \pounds_p^{-1}\pounds_s^{-1}\left[\frac{1}{ps^2} - \frac{1}{p^2 s^2}\right] \\ &= \left(1 - \frac{\mu^\alpha}{\alpha}\right)\frac{\eta^\beta}{\beta}, \end{aligned}$$

Hence,

$$
\begin{aligned}
\chi_2\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) &= \pounds_p^{-1}\pounds_s^{-1}\left(\frac{1}{s}\pounds_\mu^\alpha\pounds_\eta^\beta\left[\chi_{1\mu\mu\eta}-\chi_{1\mu}-A_1+3B_1+C_1\right]\right)\\
&= \pounds_p^{-1}\pounds_s^{-1}\left(\frac{1}{s}\pounds_\mu^\alpha\pounds_\eta^\beta\left[\begin{array}{c}\chi_{1\mu\mu\eta}-\chi_{1\mu}-\chi_0\chi_{1\mu}-\chi_1\chi_{0\mu}+3\chi_{0\mu}\chi_{1\mu\mu}\\ +3\chi_{1\mu}\chi_{0\mu\mu}+\chi_0\chi_{1\mu\mu\mu}+\chi_1\chi_{0\mu\mu\mu}\end{array}\right]\right)\\
&= \pounds_p^{-1}\pounds_s^{-1}\left(\frac{1}{s}\pounds_\mu^\alpha\pounds_\eta^\beta\left[2\frac{\eta^\beta}{\beta}-2\frac{\mu^\alpha}{\alpha}\frac{\eta^\beta}{\beta}\right]\right)\\
&= \pounds_p^{-1}\pounds_s^{-1}\left(\frac{2}{ps^3}-\frac{2}{p^2s^3}\right)\\
&= \left(1-\frac{\mu^\alpha}{\alpha}\right)\left(\frac{\eta^\beta}{\beta}\right)^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\chi_3\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) &= \pounds_p^{-1}\pounds_s^{-1}\left(\frac{1}{s}\pounds_\mu^\alpha\pounds_\eta^\beta\left[\chi_{2\mu\mu\eta}-\chi_{2\mu}-A_2+3B_2+C_2\right]\right)\\
&= \pounds_p^{-1}\pounds_s^{-1}\left(\frac{1}{s}\pounds_\mu^\alpha\pounds_\eta^\beta\left[\begin{array}{c}\chi_{1\mu\mu\eta}-\chi_{1\mu}-\chi_0\chi_{1\mu}-\chi_1\chi_{0\mu}+3\chi_{0\mu}\chi_{1\mu\mu}\\ +3\chi_{1\mu}\chi_{0\mu\mu}+\chi_0\chi_{1\mu\mu\mu}+\chi_1\chi_{0\mu\mu\mu}\end{array}\right]\right)\\
&= \pounds_p^{-1}\pounds_s^{-1}\left(\frac{1}{s}\pounds_\mu^\alpha\pounds_\eta^\beta\left[3\left(\frac{\eta^\beta}{\beta}\right)^2-3\frac{\mu^\alpha}{\alpha}\left(\frac{\eta^\beta}{\beta}\right)^2\right]\right)\\
&= \pounds_p^{-1}\pounds_s^{-1}\left(\frac{6}{ps^4}-\frac{6}{p^2s^4}\right)\\
&= \left(1-\frac{\mu^\alpha}{\alpha}\right)\left(\frac{\eta^\beta}{\beta}\right)^3,
\end{aligned}
$$

The approximate solution of Equation (57) is given by

$$
\begin{aligned}
\sum_{j=0}^{\infty}\chi_j\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) &= \chi_0+\chi_1+\chi_2+\chi_3+\cdots\\
\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right) &= -\frac{\mu^\alpha}{\alpha}+\left(1-\frac{\mu^\alpha}{\alpha}\right)\frac{\eta^\beta}{\beta}+\left(1-\frac{\mu^\alpha}{\alpha}\right)\left(\frac{\eta^\beta}{\beta}\right)^2\\
&\quad+\left(1-\frac{\mu^\alpha}{\alpha}\right)\left(\frac{\eta^\beta}{\beta}\right)^3+\left(1-\frac{\mu^\alpha}{\alpha}\right)\left(\frac{\eta^\beta}{\beta}\right)^4+\ldots\\
&= -\frac{\mu^\alpha}{\alpha}\left[1+\frac{\eta^\beta}{\beta}+\left(\frac{\eta^\beta}{\beta}\right)^2+\left(\frac{\eta^\beta}{\beta}\right)^3+\left(\frac{\eta^\beta}{\beta}\right)^4+\ldots\right]\\
&\quad+\frac{\eta^\beta}{\beta}\left[1+\frac{\eta^\beta}{\beta}+\left(\frac{\eta^\beta}{\beta}\right)^2+\left(\frac{\eta^\beta}{\beta}\right)^3+\left(\frac{\eta^\beta}{\beta}\right)^4+\ldots\right]\\
&= \left[-\frac{\mu^\alpha}{\alpha}+\frac{\eta^\beta}{\beta}\right]\left[1+\frac{\eta^\beta}{\beta}+\left(\frac{\eta^\beta}{\beta}\right)^2+\left(\frac{\eta^\beta}{\beta}\right)^3+\left(\frac{\eta^\beta}{\beta}\right)^4+\ldots\right],
\end{aligned}
$$

Hence, the conformable solution becomes

$$
\chi\left(\frac{\mu^\alpha}{\alpha},\frac{\eta^\beta}{\beta}\right)=\frac{-\frac{\mu^\alpha}{\alpha}+\frac{\eta^\beta}{\beta}}{\alpha-\frac{\eta^\beta}{\beta}} \tag{68}
$$

By taking $\alpha = \beta = 1$, the exact solution of Equation (57) is given by

$$\chi(\mu, \eta) = \frac{-\mu + \eta}{1 - \eta}.$$

Figure 4 shows the exact solution at $\alpha = \beta = 1$ and the approximate solution for different values of α and β is as follows: $\alpha = \beta = 0.90, 0.70, 0.50$ and 0.30. Refer to Example 5.

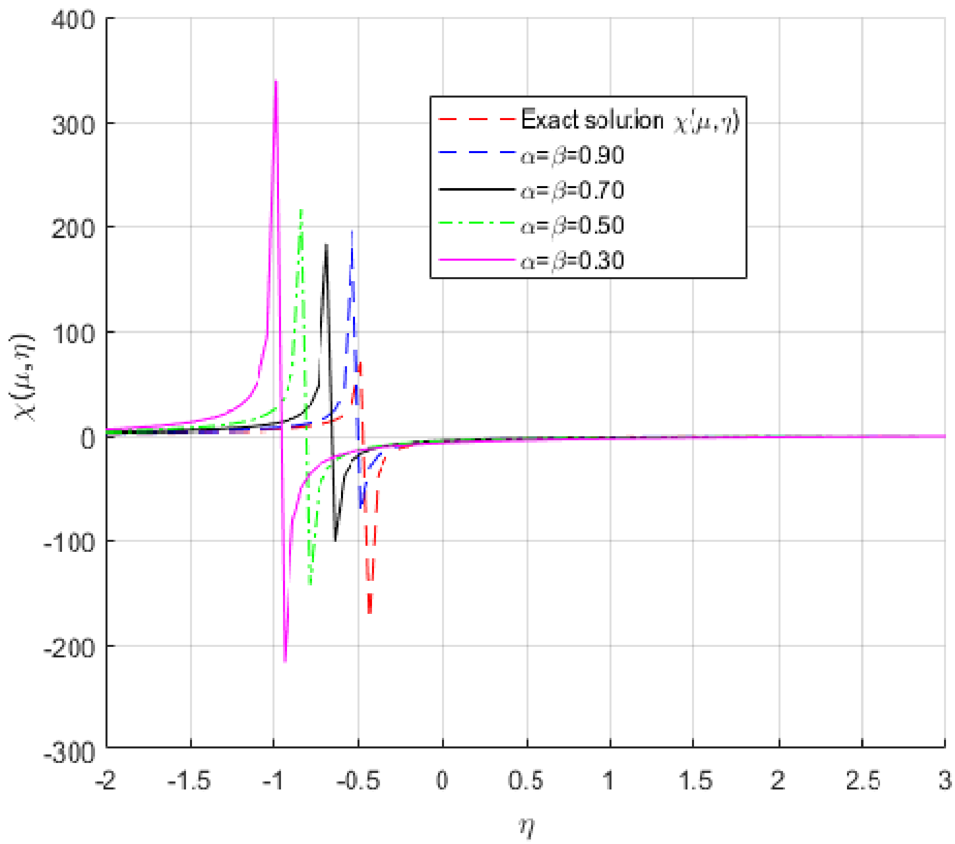

Figure 4. The exact and approximation solutions of $\chi(\mu, \eta)$.

Table 4 shows the numerical solution for different values of α and β for the function $\chi(\mu, \eta)$.

Table 4. The numerical solution for different values of α and β for the function $\chi(\mu, \eta)$.

μ	η	$\alpha = \beta = 0.90$	$\alpha = \beta = 0.95$	$\alpha = \beta = 0.99$	Exact S.	Abs. Error
0.25	0.2	−0.007591	−0.036973	−0.057671	−0.06250	4.83×10^{-3}
	0.4	0.3781972	0.3097008	0.2613081	0.25000	1.13×10^{-1}
	0.6	1.1837141	1.0169983	0.9016628	0.87500	$\times 10^{-4}$
	0.8	4.0191766	3.3001129	2.8492753	2.75000	2.51×10^{-4}
0.75	0.2	−0.669197	−0.678989	−0.685890	−0.6857	4.81×10^{-3}
	0.4	−0.450600	−0.563433	−0.579564	−0.5833	3.77×10^{-3}
	0.6	−0.272096	−0.327667	−0.366112	−0.3750	8.89×10^{-3}
	0.8	0.6730589	0.4333709	0.2830918	0.2500	$\times 10^{-4}$

5. Conclusions

In the current study, we have successfully applied the method that combined the conformable double Laplace transform method (CDLTM) and the homotopy perturbation method (HPM) to obtain the exact solutions of conformable fractional partial differential equations (FPDEs). The results show that the CDLTM is suitable, effective, useful, reliable, and sufficient for acquiring the exact solutions of CFPDEs. Moreover, three examples were discussed to confirm our method. In the future, we will apply this method to solve some mathematical problems related to physics and engineering that involve singularity.

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