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Some Aspects of Differential Topology of Subcartesian Spaces

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Abstract: In this paper, we investigate the differential topological properties of a large class of singular spaces: subcarteisan space. First, a minor further result on the partition of unity for differential spaces is derived. Second, the tubular neighborhood theorem for subcartesian spaces with constant structural dimensions is established. Third, the concept of Morse functions on smooth manifolds is generalized to differential spaces. For subcartesian space with constant structural dimension, a class of examples of Morse functions is provided. With the assumption that the subcartesian space can be embedded as a bounded subset of an Euclidean space, it is proved that any smooth bounded function on this space can be approximated by Morse functions. The infinitesimal stability of Morse functions on subcartesian spaces is studied. Classical results on Morse functions on smooth manifolds can be treated directly as corollaries of our results here.

Keywords: singular space; subcartesian space; partition of unity; tubular neighborhood; morse function; differential topology

1. Introduction

The framework of smooth manifolds has long been the core of differential geometry. However, in theoretical physics, there have been some objects that not possess smooth manifold structures. Hence, it is necessary to extend the framework of smooth manifolds to singular spaces, which admit certain basic geometric intuitions. There have been several different definitions which attempt to describe singular spaces, for example, Spallek's differentiable spaces [1], real algebraic varieties [2,3], orbifolds [4], diffeology [5], etc. Among them, Sikorski's [6] theory of differential spaces provides a framework of a large class of singular spaces by endowing the topological space S with a differential structure $C^\infty(S)$. Once a differential structure $C^\infty(S)$ is specified, we study geometric constructs on S in terms of their compatibility with $C^\infty(S)$.

It follows that an n-dimensional smooth manifold M can be treated as a differential space with a differential structure given by all smooth functions on it. Further, every point p on the manifold has a neighborhood U diffeomorphic to an open subset V of $\mathbb{R}^n$, by endowing U and V with differential structure generated by restrictions of smooth functions on M and $\mathbb{R}^n$, respectively, and by considering the diffeomorphism in the sense of differential space. If n is allowed to be an arbitrary non-negative integer depending on point p and V is allowed to be any subset in $\mathbb{R}^n$, it follows the concept of subcartesian space, as a special case of differential spaces.

The differential geometry of differential spaces was developed by Śniatycki et al. in recent years. There have been a lot of results on this topic [7–16], where the geometry of differential spaces, including their tangent and cotangent bundles, integration of vector fields, and distributions, are discussed. Detailed results are presented under the assumption that differential spaces are subcartesian. See [17] for a systematic review on this topic.

In this paper, we investigate the properties of subcartesian spaces from the perspective of differential topology. We will show that some important differential topological properties of smooth manifolds have solutions in subcartesian spaces.

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The first property we want to address is the partition of unity. The existence of a partition of unity on a smooth manifold is well-known. In case of differential spaces, as has been shown in [17], any locally compact, Hausdorff, and second countable differential space possesses a partition of unity. In this paper we first review the existing result and then present a minor further result on the partition of unity for differential spaces, which will be used in the following sections.

The second property we will investigate for subcartesian space is the tubular neighborhood property. As is well-known, any smooth manifold possesses a tubular neighborhood. In the case of subcartesian space, it can be assumed that the subcatesian space is with constant structural dimension, letting $i : S \to \mathbb{R}^m$ be an embedding, which is ensured in [14]. Then, the normal bundle N of S in $\mathbb{R}^m$ becomes a subcartesian space by endowing it with a proper differential structure. We first prove that there exists a local diffeomorphism between an open neighborhood of the zero section of the normal bundle N of S in $\mathbb{R}^m$ and a subset containing S of $\mathbb{R}^m$ with constant structural dimension m, where the open neighborhood of the zero section and the subset containing S of $\mathbb{R}^m$ are considered as differential subspace of N and $\mathbb{R}^m$, respectively. Further, by taking advantage of the partition of unity, we get a global tubular neighborhood: there exists a diffeomorphism between an open neighborhood of the zero section of the normal bundle N of S in $\mathbb{R}^m$ defined by $\Delta_\epsilon = \{\xi \in N||\xi| < \epsilon(\tau(\xi))\}$, and a subset containing S of $\mathbb{R}^m$ with a constant structural dimension m. We finally get a global result in the paper: there exists a diffeomorphism between the normal bundle N of S in $\mathbb{R}^m$ and a subset containing S of $\mathbb{R}^m$. Our results generalize the tubular neighborhood theorem in smooth manifolds to more general cases, i.e., subcartesian space with a constant structural dimension.

The third property we will investigate for subcartesian space is the Morse theory. In classical Morse theory [18], Morse functions on smooth manifolds are defined as smooth functions whose critical points are nondegenerate. In this paper, by taking advantage of the definition of derivation on differential space, we extend the definition of Morse functions on smooth manifolds to differential spaces. We then study some basic properties of Morse functions on subcartesian spaces. Precisely, by assuming a constant structural dimensional subcartesian space S, we will prove the following results in the paper:

1. Morse functions on S are plentiful. Let $i : S \to \mathbb{R}^m$ be an embedding. For almost all $p \in S$, the function L_p in S defined by $L_p(q) = ||p - q||^2$ is a Morse function on S;
2. Let $i : S \to \mathbb{R}^m$ be an embedding such that $i(S)$ is a bounded subset of $\mathbb{R}^m$. Then, any smooth bounded function on S can be approximated by Morse functions;
3. The set of critical points of a Morse function on S is discrete;
4. If S is compact, then the Morse functions are infinitesimal stable.

Consider smooth manifolds as subcartesian spaces with a differential structure defined by smooth functions on the manifolds. It follows immediately that the corresponding classical results on Morse functions on smooth manifolds [18] can be treated directly as corollaries of our results on subcartesian spaces here.

To the best of our knowledge, our work is the first attempt to initiate a systematic study of the differential topological properties of differential spaces.

The paper is organized as follows. In Section 2, some basic definitions and theorems on differential and subcartesian spaces, which will be used in our paper, are reviewed. We then define Morse functions on differential spaces. In Section 3, we present existing results and prove further results on the partition of unity for differential spaces. In Section 4, we investigate the tubular neighborhood property for subcartesian spaces with a constant structural dimension. In Section 5, we first provide examples of Morse functions on subcartesian spaces with a constant structural dimension, using which we then prove the approximation theorem for subcartesian spaces, which can be embedded as a bounded subset of $\mathbb{R}^m$. In Section 6, we study the stability of Morse functions on compact subcartesian spaces with constant structural dimensions. We present our conclusions in Section 7.

2. Differential Space and Subcartesian Space

Definition 1 ([17]). *A differential structure on a topological space S is a family $C^\infty(S)$ of real-valued functions on S satisfying the following conditions:*

1. *The family*
$$\{f^{-1}(I)|f \in C^\infty(S) \text{and } I \text{ is an open interval in } \mathbb{R}\}$$
is a sub-basis for the topology of S.
2. *If $f_1, \cdots, f_n \in C^\infty(S)$ and $F \in C^\infty(\mathbb{R}^n)$, then $F(f_1, \cdots, f_n) \in C^\infty(S)$.*
3. *If $f : S \to \mathbb{R}$ is a function such that, for every $x \in S$, there exist an open neighborhood U of x, and a function $f_x \in C^\infty(S)$ satisfying*
$$f_x|_U = f|_U,$$
then $f \in C^\infty(S)$. Here, the subscript vertical bar $|$ denotes a restriction. $(S, C^\infty(S))$ is said to be a differential space. Functions in $C^\infty(S)$ are called smooth functions on S.

Example 1. *Let S be an arbitrary set endowed with the trivial topology, i.e., the empty set, and S are the only open sets. Let differential structure $C^\infty(S)$ be defined as the set of all constant functions on S. $(S, C^\infty(S))$ is a differential space.*

Definition 2 ([17]). *Let S_1 and S_2 be two differential spaces. A map $\phi : S_1 \to S_2$ is smooth if $\phi^* f = f \circ \phi \in C^\infty(S_1)$ for each $f \in C^\infty(S_2)$. A map ϕ between differential spaces is a diffeomorphism if it is smooth, invertible, and its inverse is smooth.*

Let $\mathcal{F}$ be a family of real-valued functions on S. Endow S with the topology generated by a subbasis
$$\{f^{-1}(I)|f \in \mathcal{F} \text{ and } I \text{ is an open interval in } \mathbb{R}\}.$$

We can construct a differential structure on S as follows.

Define $C^\infty(S)$ by requiring that $h \in C^\infty(S)$ if, for each $x \in S$, there exist an open subset U of S, functions $f_1, \cdots, f_n \in \mathcal{F}$, and $F \in C^\infty(\mathbb{R}^n)$ such that
$$h|_U = F(f_1, \cdots, f_n)|_U.$$

Clearly, $\mathcal{F} \subseteq C^\infty(S)$. It is proved in [17] that $C^\infty(S)$ defined here is a differential structure on S. We refer to it as the differential structure on S generated by $\mathcal{F}$.

Let $(S, C^\infty(S))$ be a differential space, and let $T \subseteq S$ be a subset of S endowed with the subspace topology. Let
$$S(T) = \{f|_T | f \in C^\infty(S)\}.$$

Proposition 1 ([17]). *The family of functions $S(T)$ of restrictions to $T \subseteq S$ of smooth functions on S generates a differential structure $C^\infty(T)$ on T such that the differential-space topology of S coincides with its subspace topology. In this differential structure, the inclusion map $i : T \to S$ is smooth.*

Definition 3 ([17]). *A differential space $(S, C^\infty(S))$ is said to be subcartesian if every point p of S has a neighborhood U diffeomorphic to a subset of some Cartesian space $\mathbb{R}^n$, where $(U, \Phi, \mathbb{R}^n)$ is a local chart of p, and $\Phi : U \to \Phi(U) \subseteq \mathbb{R}^n$ is the diffeomorphism.*

Example 2. *Let S be any subset of $\mathbb{R}^n$ and $(S, C^\infty(S))$ be the differential subspace of $\mathbb{R}^n$. $(S, C^\infty(S))$ is a subcartesian space.*

Example 3. *Let S be a smooth manifold and $C^\infty(S)$ be the set of all smooth functions on S. $(S, C^\infty(S))$ is a subcartesian space.*

In the following, we restrict our attention to locally compact, Hausdorff, second countable subcartesian spaces. Based the above assumptions, the existence of a partition of unity on a subcartesian space is ensured. This will be detailed in the following section. Note that it follows directly from Definition 3 that a subcartesian space must be Hausdorff. Moreover, it follows from Definition 3 and Condition 1 of Definition 1 that a subcartesian space must be locally compact. Thus, we only need the following assumption.

Assumption 1. *All subcartesian spaces considered here are second countable.*

Definition 4 ([17]). *Let $(S, C^\infty(S))$ be a differential space. A derivation of $C^\infty(S)$ is a linear map*

$$\begin{aligned} X: \quad & C^\infty(S) \to C^\infty(S) \\ & f \to X(f), \end{aligned}$$

which satisfies Leibniz's rule

$$X(f_1 f_2) = X(f_1) f_2 + f_1 X(f_2)$$

for every $f_1, f_2 \in C^\infty(S)$.

We denote by $\mathrm{Der}C^\infty(S)$ the space of derivations of $C^\infty(S)$. This has the structure of Lie algebra, with the Lie bracket $[X_1, X_2]$ defined by

$$[X_1, X_2](f) = X_1(X_2(f)) - X_2(X_1(f))$$

for every $X_1, X_2 \in \mathrm{Der}C^\infty(S)$ and $f \in C^\infty(S)$.

Definition 5 ([17]). *Let $(S, C^\infty(S))$ be a differential space. A derivation of $C^\infty(S)$ at $x \in S$ is a linear map $v : C^\infty(S) \to \mathbb{R}$ such that*

$$v(f_1 f_2) = v(f_1) f_2(x) + f_1(x) v(f_2)$$

for every $f_1, f_2 \in C^\infty(S)$.

We denote by $Der_x(C^\infty(S))$ the space of derivations of $C^\infty(S)$ at $x \in S$.

We interpret derivations of $C^\infty(S)$ at $x \in S$ as tangent vectors to S at x. The set of all derivations of $C^\infty(S)$ at x is denoted by $T_x S$ and is called the tangent space to S at x.

If X is a derivation of $C^\infty(S)$ then, for every $x \in S$, we have a derivation $X(x)$ of $C^\infty(S)$ at x given by

$$X(x) : C^\infty(S) \to \mathbb{R} : f \to X(x) f = (Xf)(x). \tag{1}$$

The derivation (1) is called the value of X at x. Clearly, the derivation X is uniquely determined by the collection $\{X(x) | x \in S\}$ of its values at all points in S.

Let S be a differential subspace of $\mathbb{R}^n$. Let $N(s)$ denote the ideal of functions in $C^\infty(\mathbb{R}^n)$ that vanish identically on S:

$$N(s) = \{F \in C^\infty(\mathbb{R}^n) | F|_S = 0\}.$$

Proposition 2 ([17]). *A smooth vector field Y on $\mathbb{R}^n$ restricts to a derivation of $C^\infty(S)$ if $Y(F) \in N(S)$ for every $F \in N(S)$.*

Definition 6 ([17]). *Let $(S, C^\infty(S))$ be a differential space. Point $x \in S$ is called a critical point of $f \in C^\infty(S)$ if $X(f) = 0$ for each $X \in Der_x(C^\infty(S))$.*

If x is a critical point of $f \in C^\infty(S)$, then consider the smooth distribution on S defined by $\mathbb{T}S = \mathrm{span}_\mathbb{R}\{X | X \in \mathrm{Der}(C^\infty(S))\}$. We can define a bilinear symmetric functional f_x^{**}

on $\mathbb{T}S(x)$, called the Hessian of f at x, as follows. Let $v, w \in \mathbb{T}S(x)$. Then, there exist $V, W \in \mathrm{Der}(C^\infty(S))$, such that $V(x) = v, W(x) = w$. We define

$$\begin{aligned} f_x^{**} &: \mathbb{T}S(x) \times \mathbb{T}S(x) \to \mathbb{R} \\ f_x^{**}(v, w) &= VW(f)(x). \end{aligned} \tag{2}$$

f_x^{**} is well-defined. Let $\tilde{V}, \tilde{W} \in \mathrm{Der}(C^\infty(S))$, such that $\tilde{V}(x) = v, \tilde{W}(x) = w$. We have

$$\begin{aligned} \tilde{V}\tilde{W}(f)(x) &= \tilde{V}(x)\tilde{W}f = V(x)\tilde{W}(f) = V\tilde{W}(f)(x) = [V, \tilde{W}](f)(x) + \tilde{W}V(f)(x) \\ &= 0 + W(x)V(f) = WV(f)(x) = [W, V]f(x) + VW(f)(x) \\ &= VW(f)(x), \end{aligned} \tag{3}$$

where $[W, V]f(x) = 0$ because $[W, V] \in \mathrm{Der}(C^\infty(S))$, and x is a critical point of f. From (3), we also know that f_x^{**} is symmetrical and bilinear.

Definition 7. *Let $(S, C^\infty(S))$ be a differential space. Point $x \in S$ is called a nondegenerate critical point of $f \in C^\infty(S)$ if x is a critical point of f, such that f_x^{**} is nondegenerate.*

Definition 8. *Let $(S, C^\infty(S))$ be a differential space. A smooth function $f \in C^\infty(S)$ is said to be a Morse function if each critical point $x \in S$ of f is nondegenerate.*

Remark 1. *A Morse function on a smooth manifold is defined as a smooth function whose critical points are nondegenerate. It is natural to generalize the concept of critical points on smooth manifolds to the case of differential space by using the definition of derivation on differential spaces. To define the nondegenerate property we need to restrict to $\mathbb{T}S$ instead of TS. We know $\mathbb{T}S$ and TS coincide when S is a subcartesian with constant structural dimension.*

We have the following definition of structural dimension for subcartesian space.

Definition 9 ([17])**.** *Let S be a subcartesian space. The structural dimension of S at a point $x \in S$ is the smallest integer n such that for some open neighborhood $U \subseteq S$ of x, there is a diffeomorphism of U onto a subset $V \subseteq \mathbb{R}^n$. The structural dimension of S is the smallest integer n such that for every point $x \in S$, the structural dimension n_x of S at x satisfies $n_x \leq n$.*

Theorem 1 ([17])**.** *For a subcartesian space S, the structural dimension at x is equal to $\dim T_x S$.*

We have the embedding theorem for subcartesian space.

Theorem 2 ([14])**.** *Let S be a subcartesian space with structural dimension n. Then, there exists a proper embedding map $\Psi : S \to \mathbb{R}^m$, where $m \geq 2n + 1$.*

The subcartesian space is said to be with constant structural dimension if the structural dimension of each $x \in S$ is the same.

Example 4. *The Koch curve is a subset K of $\mathbb{R}^2$ defined as follows. The set $K_0 = \{(0,0), (1,0)\}$ consists of the end points of the line segment $C_0 = [0,1] \times \{0\} \in \mathbb{R}^2$. Construct a set C_1 by removing the middle third from the segment C_0, replacing it with two equal segments that would form an equilateral triangle with the removed piece. The resulting four-sided zigzag has vertices $K_1 = \{(0,0), (0, \frac{1}{3}), (\frac{1}{2}, \frac{\sqrt{3}}{6}), (\frac{2}{3}, 0), (1,0)\}$. Next, construct a set C_2 by applying the same construction to each line segment of the set C_1. We denote the set of vertices of C_2 by K_2. Continuing in this way, we obtain a sequence of piecewise linear sets C_n and the sets K_n of their vertices. Let K_∞ be the union of all sets K_n, i.e., $K_\infty = \cup_{n=0}^{\infty} K_n$. The Koch curve K is the topological closure of K_∞. Since K is a closed subset of $\mathbb{R}^2$, its differential structure $C^\infty(K)$ consists of the restrictions to K of smooth functions on $\mathbb{R}^2$. We can show that $\dim T_x K = 2$ for each $x \in K$. Hence, K is a subcartesian space with constant structural dimension.*

We make the following assumption.

Assumption 2. *All subcartesian spaces considered here have constant structural dimensions.*

Lemma 1. *Let S be a subcartesian space with a constant structural dimension n and $\Phi : S \to \mathbb{R}^m$ be a smooth map. Let $\mathfrak{O}$ be an open cover of S. Then, there exist locally finite open covers $(U_j)_{j\in\mathbb{Z}_{>0}}, (V_j)_{j\in\mathbb{Z}_{>0}}, (W_j)_{j\in\mathbb{Z}_{>0}}$, such that $cl(U_j) \subseteq V_j, cl(V_j) \subseteq W_j$, $cl(W_j)$ is compact for each $j > 0$, where $(W_j, \mathbb{R}^n, \phi_j)$ is a local chart of S and $\mathfrak{W} = \{W_j\} \prec \mathfrak{O}$. Furthermore, there exists a smooth extension $\tilde{\Phi}$ of Φ on U_j; that is, $\tilde{\Phi} \circ \phi_j | U_j = \Phi | U_j$.*

Proof. The proof follows by replacing $(2) p \in W \subseteq (G_{h+1}/cl(G_{h-2})) \cap \mathscr{V}$ in the proof of Lemma 3.3 in [14] with $(2) p \in W \subseteq (G_{h+1}/cl(G_{h-2})) \cap \mathscr{V} \cap Q$, where $Q \in \mathfrak{O}$ is an open subset containing p, and by replacing f, Φ, and n_j in the proof of Lemma 3.3 in [14] with Φ, $\tilde{\Phi}$, and n. □

In the remaining part of this section, we will show that the subcartesian space S with structural dimension is a metric space.

Definition 10. *A smooth Riemannian metric on a subcartesian space is a symmetric positive definite bilinear form $g(x)$ in $T_x S$ for each $x \in S$, such that for each smooth section σ of TS, the function $g(x)(\sigma(x), \sigma(x)) \in C^\infty(S)$.*

Theorem 3 ([19])**.** *Let S be a subcartesian space with structural dimension n. Then, there exists a smooth Riemannian metric on S.*

Definition 11. *Let S be a subcartesian space with a constant structural dimension. Given two points $p, q \in S$, the distance $d(p, q)$ is defined by $d(p, q)$ = infimum of the lengths of all curves $\gamma_{p,q}$, where $\gamma_{p,q}$ is a piecewise differentiable curve joining p to q.*

Proposition 3. *With the distance d, the subcartesian S with constant structural dimension is a metric space.*

(1) $d(p, x) \leq d(p, q) + d(q, x)$ for $p, q, x \in S$;
(2) $d(p, q) = d(q, p)$;
(3) $d(p, q) \geq 0$ and $d(p, q) = 0$ if and only if $p = q$.

Proof. We only need to show that if $d(p, q) = 0$, then $p = q$. Assume that p, q are two distinct points. It follows that there is a normal ball $B_r(p)$ (which is diffeomorphic to a subset V of $T_p S$ with $g(p)(v, v) < r^2$, for $v \in V$) that does not contain q. Since $d(p, q) = 0$, there exists a curve c joining p and q of length less than r. Hence, the segment of c must contain in $B_r(p)$; hence, c cannot join p and q. This makes a contradiction.

The remaining item follows on directly from the definition of $d(p, q)$. □

3. Partition of Unity

Definition 12. *A countable partition of unity on a differential space S is a countable family of functions $\{f_i\} \in C^\infty(S)$:*

(a) The collection of their supports is locally finite.
(b) $f_i(x) \geq 0$ for each i and each $x \in S$.
(c) $\sum_{i=1}^{\infty} f_i(x) = 1$ for each $x \in S$.

The following theorem in [17] establishes the existence of a partition of unity for locally compact, second countable Hausdorff differential spaces.

Theorem 4 ([17])**.** *Let S be a differential space with differential structure $C^\infty(S)$, and let $\{U_\alpha\}$ be an open cover of S. If S is Hausdorff, locally compact, and second countable, then there exists a* *countable partition of unity* $\{f_i\} \in C^\infty(S)$ *subordinate to* $\{U_\alpha\}$*, such that the support of each* f_i *is compact.*

We present a minor further result on the partition of unity for differential spaces, which will be used in the following section.

Lemma 2. *Let* S *be a Hausdorff, locally compact, and second countable differential space with differential structure* $C^\infty(S)$*. Let* $F \subseteq S$ *be a non-empty closed subset and* $G \subseteq S$ *be an open subset such that* $F \subseteq G$*. Then, there exists a smooth function* $g \in C^\infty(S)$*, such that* $F \subseteq \{p \in S | g(p) = 1\} \subseteq suppg \subseteq G$*.*

Proof. Let $H = S \setminus F$. Then, $\{S \setminus F, G\}$ is an open cover of S. It follows from Theorem 4 that there exists a countable partition of unity $\{f_i\} \in C^\infty(S)$ subordinate to $\{S \setminus F, G\}$, such that the support of each f_i is compact.

Define $g = \sum_{suppf_i \subseteq G} f_i$. Since the collection of the supports of $\{f_i\}$ is locally finite, it follows from condition 3 in Definition 1 that $g \in C^\infty(S)$. And, we have

$$F = S \setminus H \subseteq \{p \in S | g(p) = 1\} \subseteq suppg \subseteq G.$$

Then, the result follows immediately. □

Corollary 1. *Let* S *be a Hausdorff, locally compact, and second countable differential space with differential structure* $C^\infty(S)$*. Let* $\{G_i\}$ *be a family of locally finite open subsets. Let* $K_i \subseteq G_i$ *be compact, such that* $\cup_i K_i = S$*. Then, there exists a family of smooth functions* $\{v_i\}$*, such that*

(1) $0 \leq v_i \leq 1, \sum_i v_i = 1$;
(2) $K_i \subseteq suppv_i \subseteq G_i$.

Proof. It follows from Lemma 2 that there exists $\mu_i \in C^\infty(S)$, such that $K_i \subseteq \{p \in S | \mu_i(p) = 1\} \subseteq supp\mu_i \subseteq G_i$.

Since $\{G_i\}$ is locally finite, it follows that $\sum_j \mu_j < +\infty$. Further, $\sum_j \mu_j \in C^\infty(S)$. On the other hand, since $\cup_i K_i = S$, it follows that $\sum_j \mu_j \geq 1$.

Define $v_i = \mu_i / \sum_j \mu_j$. $v_i \in C^\infty(S)$. It follows immediately that $\{v_i\}$ satisfies conditions (1) and (2). □

4. Tubular Neighborhoods

Let S be a subcartesian space with a constant structural dimension n. From Theorem 2, we know that there exists a proper embedding map $i : S \to \mathbb{R}^m$, where $m \geq 2n + 1$.

Define $N \subseteq S \times \mathbb{R}^m$ by

$$N = \{(q, v) | q \in S, v \,\text{perpendicular to}\, i_*(T_q S) \,\text{at}\, q\}. \tag{4}$$

Denote by $\tau : N \to S$ the projection $\pi(q, v) = q$. The differential structural $C^\infty(N)$ of N is generated by the family of functions $\{f \circ \tau, df | f \in C^\infty(\mathbb{R}^m)\}$.

Since S is a subcartesian space with a constant structural dimension n, it follows that $\dim T_q S = n$ for each $q \in N$. Hence, the dimension of the linear space $Q = \{v | v \,\text{perpendicular to}\, i_*(T_q S) \,\text{at}\, q\}$ is $m - n$ for each $q \in S$. $(N, \tau, S, \mathbb{R}^{m-n})$ is a vector bundle on S, where $\tau : N \to S$ is a smooth map and $\tau^{-1}(U)$ is diffeomorphic to $U \times \mathbb{R}^{m-n}$, where U is an open subset of S. Hence, N is a subcartesian space with a constant structural dimension $n + (m - n) = m$.

Lemma 3 ([20]). *Let* X, Y *be metric spaces.* X *is locally compact and second countable. Let* A *be a closed subset of* X*. Assume that the continuous map* $\psi : X \to Y$ *satisfies that*

(1) $\psi : X \to Y$ *is a local homeomorphism;*
(2) $\psi|_A$ *is an injection.*

Then, there exists an open neighborhood G of A in X and an open neighborhood $H = \psi(G)$ of $B = \psi(A)$ in Y, such that $\psi|_G$ is a homeomorphism from G to H.

Let $\psi : N \to \mathbb{R}^m$ be defined by $\psi((q, v)) = i(q) + v$.

Lemma 4. *There exists an open neighborhood G of the zero section Z of N, such that $\psi|_G : G \to \psi(G)$ is a diffeomorphism between the subcartesian space G and $\psi(G)$.*

Proof. For any $q \in S$ and $0_q \in Z, \psi(0_q) = q \in S$ consider $(d\psi)_{0_q} : T_{0_q}N \to \mathbb{R}^m = i_*(T_qS) \oplus (i_*(T_qS))^\perp$. Due to the local product property of $(N, \tau, S, \mathbb{R}^{m-n})$, we have $T_{0_q}N = T_{0_q}Z \oplus T_{0_q}N_q$. Since $\psi|_Z : Z \to N$ is a diffeomorphism, we have $(d\psi)_{0_q}|(T_{0_q}Z) : T_{0_q}Z \to T_qS$ is an linear isomorphism. Furthermore, $(d\psi)_{0_q}(T_{0_q}N_q) : T_{0_q}N_q \to (T_qS)^\perp$ is an linear isomorphism. Hence, $(d\psi)_{0_q}(T_{0_q}N) : T_{0_q}N \to (T_qS)^\perp \oplus T_qS = \mathbb{R}^m$ is a linear isomorphism. Since N is a subcartesian space with constant structural dimension m, let $(U, \phi, \mathbb{R}^m)$ be a local chart of N; then, ψ can be locally extended to be a smooth map $\tilde{\psi}$ from an open subset of $\mathbb{R}^m$ to $\mathbb{R}^m$. Since $(d\psi)_{0_q} : T_{0_q}N \to \mathbb{R}^m$ is a linear isomorphism, it follows that $(d\tilde{\psi})_0$ is a linear isomorphism. Hence, $\tilde{\psi}$ is a local diffeomorphism around 0, which yields that ψ is a local diffeomorphism around 0_q. Since q is arbitrary, we get that there exists an open neighborhood X of zero section Z of N, such that $\psi|_X : X \to \psi(X)$ is a local diffeomorphism. On the other hand, $\psi : Z \to S$ is a diffeomorphism.

Since N is a subcartesian space with a constant structural dimension, it follows from Proposition 3 that N is a metric space; hence, X is a metric space as an open subset of N. Then, it follows from Lemma 3 that there exists an open neighborhood $G \subseteq X$ of Z and an open neighborhood $H = \psi(G)$ of S in $\psi(X)$, such that $\psi|_G : G \to H$ is a homeomorphism. Since $\psi|_X : X \to \psi(X)$ is a local diffeomorphism, it follows immediately that $\psi|_G$ is a diffeomorphism. This completes the proof of the lemma. □

Consider the vector bundle $(N, \tau, S, \mathbb{R}^{m-n})$ on S. Due to the local trivial property of the vector bundle together with existence of a partition of unity on S, there exists a smooth Riemannian metric on $(N, \tau, S, \mathbb{R}^{m-n})$.

Lemma 5. *Let β be a smooth Riemannian metric on $(N, \tau, S, \mathbb{R}^{m-n})$. Let Z be a zero section of N and G be an open neighborhood of Z. Then, there exists a smooth function $\epsilon > 0$ on S, such that*

$$\Delta_\epsilon = \{\xi \in N | |\xi| < \epsilon(\tau(\xi))\} \subseteq G,$$

where $|\cdot|$ is the norm determined by the Riemannian metric β.

Proof. We first claim that for any $q \in S$, there exist an open neighborhood Q of q on S and $\delta > 0$, such that

$$\{\xi \in \tau^{-1}(Q) | |\xi| < \delta\} \subseteq G.$$

Consider the local trivial neighborhood U of q. Then, there exists a diffeomorphism $h : \tau^{-1}(U) \to U \times \mathbb{R}^{m-n}$. Since $h(\tau^{-1}(U) \cap G)$ is an open neighborhood of Z in $U \times \mathbb{R}^{m-n}$ and since S is locally compact, it follows that there exist an open neighborhood $Q \subseteq U$ of q, where $Q \subseteq cl(Q) \subseteq U$ and $cl(Q)$ are compact and $\gamma > 0$, such that

$$\{(x, v) \in cl(Q) \times \mathbb{R}^{m-n} | ||v|| < \gamma\} \subseteq h(\tau^{-1}(U) \cap G).$$

Denote $\lambda(x, v) = |h^{-1}(x, v)|$ for all $(x, v) \in cl(Q) \times \mathbb{R}^{m-n}$. Since $cl(Q) \times S^{m-n-1}$ is compact, there exists $\mu > 0$, such that $\lambda(x, v) \geq \mu$ for any $(x, v) \in cl(Q) \times S^{m-n-1}$. Hence, $\lambda(x, v) \geq \mu||v||$ for any $(x, v) \in cl(Q) \times \mathbb{R}^{m-n}$. Let $\delta = \mu\gamma$. We have $\lambda(x, v) = |h^{-1}(x, v)| < \delta$, which yields that $||v|| < \gamma$. Hence, we have

$$h(\{\xi \in \tau^{-1}(Q) | |\xi| < \delta\}) \subseteq \{(x, v) \in cl(Q) \times \mathbb{R}^{m-n} | ||v|| < \gamma\} \subseteq h(\tau^{-1}(U) \cap G).$$

It follows immediately that $\{\xi \in \tau^{-1}(Q) || \xi| < \delta\} \subseteq G$.

Hence, there exist an open cover $\mathfrak{O} = \{Q\}$ of S and a family $\{\delta_Q\}$, such that $\{\xi \in \tau^{-1}(Q) || \xi| < \delta_Q\} \subseteq G$.

It follows from Lemma 1 that there exist locally finite open covers

$$(U_j)_{j \in \mathbb{Z}_{>0}}, (V_j)_{j \in \mathbb{Z}_{>0}}, (W_j)_{j \in \mathbb{Z}_{>0}},$$

such that $cl(U_j) \subseteq V_j, cl(V_j) \subseteq W_j$, and $cl(W_j)$ is compact for each $j > 0$, where $\mathfrak{W} = \{W_j\} \prec \mathfrak{O}$. For each W_j, there exists $\delta_j > 0$, such that $\{\xi \in \tau^{-1}(W_j) || \xi| < \delta_j\} \subseteq G$.

We claim that there exists a smooth function $\epsilon > 0$ on S, such that $\epsilon(x) < \delta_k$ for any $x \in U_k, k \in \mathbb{N}$. It follows from Corollary 1 that there exists partition of unity $\{\lambda_i\} \subseteq C^\infty(S)$, such that $U_k \subseteq \{x \in S | \lambda_k(x) > 0\} \subseteq supp\lambda_k \subseteq V_k, k = 1, 2, \cdots$.

Given $j \in \mathbb{N}$, $\epsilon_j = min\{\delta_k | cl(V_k) \cap cl(V_j) \neq \emptyset, k \in \mathbb{N}\}, j = 1, 2 \cdots$. Let $\epsilon(x) = \sum_{j=1}^{\infty} \epsilon_j \lambda_j(x)$. For $x \in U_k$, we have $\epsilon(x) < \sum_{clV_j \cap cl(v_k) \neq \emptyset} \epsilon_j \lambda_j(x) \leq \delta_k \sum_{j=1}^{\infty} \lambda_j(x) = \delta_k$.

Hence, we have proved that

$$\Delta_\epsilon = \{\xi \in N || \xi < \epsilon(\tau(\xi))\} \subseteq G. \tag{5}$$

□

We have the following tubular neighborhood theorem for subcartesian space.

Theorem 5. *Let S be a subcartesian space with constant structural dimensions. Let $i : S \to \mathbb{R}^m$ be an embedding. Let N be defined by*

$$N = \{(q, v) | q \in S, v \, \text{perpendicular to} \, i_*(T_q S) \, \text{at} \, q\}.$$

N is a subcartesian space with a constant structural dimension m. Let $\tau : N \to S$ be the projection. Let $\psi : N \to \mathbb{R}^m$ be defined by $\psi((q, v)) = i(q) + v$. There exists a smooth function $\epsilon > 0$ on S, such that $\psi|_{\Delta_\epsilon} : \Delta_\epsilon \to \psi(\Delta_\epsilon)$ is a diffeomorphism between the subcartesian space Δ_ϵ and $\psi(\Delta_\epsilon)$, where $\Delta_\epsilon = \{\xi \in N || |\xi|| < \epsilon(\tau(\xi))\}$ and $|| \cdot ||$ is the Euclidean norm in $\mathbb{R}^m$. Further, define $\rho = \tau \circ \psi^{-1}$. Then, $\rho : \psi(\Delta_\epsilon) \to S$ is a smooth map satisfying that $\rho(x) = x$ for any $x \in S$. Furthermore, $\rho \circ \psi|_{\Delta_\epsilon} = \tau|_{\Delta_\epsilon}$. $\psi(\Delta_\epsilon)$ is said to be a tubular neighborhood of S in $\mathbb{R}^m$, and $\rho : \psi(\Delta_\epsilon) \to S$ is said to be the contraction map of the tubular neighborhood.

The above result can be extended to the following global result.

Theorem 6. *Let S be a subcartesian space with constant structural dimensions. Let $i : S \to \mathbb{R}^m$ be an embedding. Let N be defined by*

$$N = \{(q, v) | q \in S, v \, \text{perpendicular to} \, i_*(T_q S) \, \text{at} \, q\}.$$

N is a subcartesian space with a constant structural dimension m. Let $\tau : N \to S$ be the projection. Then, there exists a diffeomorphism $\omega : N \to \omega(N) \subseteq \mathbb{R}^m$, such that $\omega(0_x) = x$ for any $x \in S$. Further, there exists a smooth contraction map $\rho : \omega(N) \to S$, such that $\rho \circ \omega = \tau$.

Proof. It follows from Theorem 5 that there exists a smooth function $\epsilon > 0$ on S, such that $\psi : \Delta_\epsilon \to \psi(\Delta_\epsilon) \subseteq \mathbb{R}^m$ is a diffeomorphism. Furthermore, there exists a contraction map $\rho : \psi(\Delta_\epsilon) \to S$, such that $\rho \circ \psi|_{\Delta_\epsilon} = \tau|_{\Delta_\epsilon}$.

Define a smooth map $\theta : N \to \Delta_\epsilon$ by $\theta(\xi) = \frac{\epsilon(\tau(\xi))}{\sqrt{1+||\xi||^2}}\xi$. We claim that θ is a diffeomorphism. Consider the smooth map $\gamma : \Delta_\epsilon \to N$ by $\gamma(\eta) = \frac{\eta}{\sqrt{\epsilon(\tau(\eta))^2 - ||\eta||^2}}$. It follows that $\gamma \circ \theta = Id : N \to N$ and $\theta \circ \gamma = Id : \Delta_\epsilon \to \Delta_\epsilon$. Hence, θ is an bijection. Since both θ and γ are smooth, it follows immediately that θ is a diffeomorphism.

Let $\omega = \psi \circ \theta : N \to \phi(\Delta_\epsilon)$. Then, ω is a diffeomorphism, which satisfies that $\omega(0_x) = \psi \circ \theta(0_x) = \psi(0_x) = x$ for all $x \in S$. Furthermore, $\rho \circ \omega = \rho \circ \psi \circ \theta = \tau \circ \theta = \tau$. This completes the proof of the theorem. □

5. Approximating Bounded Smooth Functions by Morse Functions on Subcartesian Spaces

Let S be a subcartesian space with constant structural dimension n embedded in $\mathbb{R}^m$, i.e., $i : S \to \mathbb{R}^m$. Let $p \in \mathbb{R}^m$. Define the function $L_p : S \to \mathbb{R}$ by

$$L_p(q) = ||p - q||^2. \tag{6}$$

It will be proven that for almost all p, the function L_p is a Morse function on S.

From the above section, we know that N defined by (4) is a subcartesian space with a constant structural dimension m.

Consider $\psi : N \to \mathbb{R}^m$ is $\psi(q, v) = q + v$.

Definition 13. *$e \in \mathbb{R}^m$ is a focal point of (S, q) if $e = q + v$, where $(q, v) \in N$ and $\ker(d\psi)_{(q,v)} \neq 0$. Point e is a focal point of S if e is a focal point of (S, q) for some $q \in S$.*

Theorem 7. *Let S be a subcartesian space with constant structural dimension n and let $\Phi : S \to \mathbb{R}^n$ be smooth. The image of the set of the points where $d\Phi$ is singular has measure 0 in $\mathbb{R}^n$.*

Proof. It follows from Lemma 1 that there exist an open cover $(U_j)_{j \in Z>0}$ and a local chart $(U_j, \mathbb{R}^k, \phi_j)$ for each j, such that there exists a smooth extension $\tilde{\Phi}$ of Φ on U_j; that is, $\tilde{\Phi} \circ \phi_j | U_j = \Phi | U_j$.

Since the structural dimension of S is n, it follows that the set of points on U_j where $d\Phi$ is singular is the same as the set of points on U_j, where $d(\tilde{\Phi} \circ \phi_j)$ is singular. From Sard's Theorem we know that the image of the set of points where $d\tilde{\Phi}$ is singular has measure 0 in $\mathbb{R}^n$. It follows that the image of the set of points on U_j where $d\Phi$ is singular has measure 0 in $\mathbb{R}^n$. Then, the image of the set of the points where $d\Phi$ is singular is a union of countable sets, where each set has measure 0 in $\mathbb{R}^n$. Hence, the image of the set of the points where $d\Phi$ is singular has measure 0 in $\mathbb{R}^n$. □

Corollary 2. *For almost all $x \in \mathbb{R}^m$, the point x is not a focal point of S.*

Proof. The point x is a focal point of S if and only if x is in the image of the set of points, where $d\psi$ is singular. The result follows from Theorem 7. □

Let $q \in S$ with $(u_1, \cdots, u_n)$ being local coordinates for q. Then, the inclusion $i : S \to \mathbb{R}^m$ can be locally extended to be a smooth map $x = (x_1(u_1, \cdots, u_n), \cdots, x_m(u_1, \cdots, u_n))$.

Define the matrices associated with the coordinate system by

$$(g_{ij}) = ((\frac{\partial x}{\partial u_i})^T \frac{\partial x}{\partial u_j}).$$

Consider the vector $\frac{\partial^2 x}{\partial u_i \partial u_j}$. Let v be a unit vector that is perpendicular to $i_*(T_q S)$. Define the vector l_{ij} to be the normal component of $\frac{\partial^2 x}{\partial u_i \partial u_j}$. Given any unit vector v, which is normal to S at q, we have the matrix

$$(v^T \frac{\partial^2 x}{\partial u_i \partial u_j}) = (v^T l_{ij}).$$

The coordinates $(u_1, \cdots, u_n)$ can be chosen such that (g_{ij}) evaluated at q is the identity matrix. Then, the eigenvalues of the matrix $(v^T l_{ij})$ are called the principal curvature

$K_1, \cdots, K_n$ of S at q in the normal direction v. $K_1^{-1}, \cdots, K_n^{-1}$ are called principle radii of curvature. If the matrix $(v^T l_{ij})$ is singular, one or more of the K_i will be zero; hence, the corresponding K_i^{-1} will not be defined.

Now consider the normal line $l = q + tv$.

Lemma 6. *The focal points of (S, q) along l are precisely the points $q + K_i^{-1}v$, where $1 \le i \le n$, $K_i \ne 0$. Thus, there are at most n focal points of (S, q) along l, each being counted with its proper multiplicity.*

Proof. Choose $m - n$ vector fields $w_i(u_1, u_2, \cdots, u_n)$, which are unit vectors orthogonal to each other and to $i_*(TS)$. We can introduce local coordinates $(u_1, \cdots, u_n, t_1, \cdots, t_{m-n})$ for N, which corresponds to the point $(x(u_1, \cdots, u_n), \sum_{i=1}^{m-n} t^i w_i(u_1, \cdots, u_n)) \in N$. Then, the map $\psi : N \to \mathbb{R}^n$ has the local coordinate expression

$$(u_1, \cdots, u_n, t_1, \cdots, t_{m-n}) \to x(u_1, \cdots, u_n) + \sum_{i=1}^{m-n} t^i w_i(u_1, \cdots, u_n).$$

Since S has constant structural dimension n, we have $\{\frac{\partial}{\partial u_i}, \frac{\partial}{\partial t_i}\}$ span TN around $\tau^{-1}(q)$. Hence, we have

$$d\psi(\frac{\partial}{\partial u_i}) = \frac{\partial x}{\partial u_i} + \sum_{j=1}^{m-n} t_j \frac{\partial w_j}{\partial u_i},$$
$$d\psi(\frac{\partial}{\partial t_i}) = w_i. \tag{7}$$

Taking the inner products of these vectors with the basis vector $\frac{\partial x}{\partial u_i}, w_j$, we then get the following matrix:

$$\begin{pmatrix} ((\frac{\partial x}{\partial u_i})^T \frac{\partial x}{\partial u_j} + \sum_{l=1}^{m-n} t_l (\frac{\partial w_l}{\partial u_i})^T \frac{\partial x}{\partial u_j}) & (\sum_{l=1}^{m-n} t_l (\frac{\partial w_l}{\partial u_i})^T w_j) \\ 0 & \text{identity matrix} \end{pmatrix},$$

since $\frac{\partial x}{\partial u_i}, w_j$ are orthogonal.

Since

$$0 = \frac{\partial}{\partial u_i}(w_l^T \frac{\partial x}{\partial u_j}) = (\frac{\partial w_l}{\partial u_i})^T \frac{\partial x}{\partial u_j} + w_l^T \frac{\partial^2 x}{\partial u_i \partial u_j},$$

we have $((\frac{\partial x}{\partial u_i})^T \frac{\partial x}{\partial u_j} + \sum_{l=1}^{m-n} t_l (\frac{\partial w_l}{\partial u_i})^T \frac{\partial x}{\partial u_j}) = (g_{ij} - \sum_{l=1}^{m-n} t_l w_l^T l_{ij})) = (g_{ij} - tv^T l_{ij})$.

Since (g_{ij}) evaluated at q is the identity matrix, it follows that the above matrix is singular at (q, tv) if and only if $t = K_i^{-1}$, where K_i the principal curvature of S at q in the normal direction v.

Since $q + tv$ is the focal point of (S, q) if and only if $d\psi$ is singular at (q, tv) if and only if the above matrix is singular at (q, tv), the result follows immediately. □

Now, to fix $p \in \mathbb{R}^m$, let us study the function L_p defined above.

$$L_p(x(u_1, \cdots, u_k)) = ||x(u_1, \cdots, u_k) - p||^2 = x^T x - 2x^T p + p^T p.$$

We have

$$(L_p)_* \frac{\partial}{\partial u_i} = 2(\frac{\partial x}{\partial u_i})^T (x - p).$$

Hence, q is a critical point of L_p if and only if $q - p$ is normal to $i_*(T_q S)$ at q.

Since $\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j} L_p = 2((\frac{\partial x}{\partial u_i})^T \frac{\partial x}{\partial u_j} + (\frac{\partial^2 x}{\partial u_i \partial u_j})^T (x - p))$. It follows from the proof of Lemma 6 that $(\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j} L_p)$ is singular at q if and only if $p = q + tv$, where v is unit vector normal to $i_*(T_qS)$ at q and $t = K_i$, where K_i is the principal curvature of S at q in the normal direction v.

Lemma 7. *The point $q \in S$ is a degenerate critical point of L_p if and only if p is a focal point of (S, q).*

Theorem 8. *For almost all $p \in \mathbb{R}^m$, the function $L_p : S \to \mathbb{R}$ has no degenerate critical point.*

Proof. The result follows from Lemma 7 and Corollary 2. □

Theorem 9. *Assume that S can be embedded as a bounded subset of $\mathbb{R}^m$. Let $f \in C^\infty(S)$ be bounded. Then, for any $\epsilon > 0$, there exists a Morse function $g \in C^\infty(S)$, such that*

$$|g(y) - f(y)| < \epsilon,$$

for any $y \in S$.

Proof. Let $h : S \to \mathbb{R}^m$ be the bounded embedding, with the first coordinate h_1 being precisely the given smooth function f. Let c be a large number. Choose a point

$$p = (-c + \epsilon_1, \epsilon_2, \cdots, \epsilon_m)$$

close to $(-c, 0, \cdots, 0) \in \mathbb{R}^m$, such that the function $L_p : S \to \mathbb{R}$ is a Morse function and let

$$g(x) = \frac{L_p(x) - c^2}{2c}.$$

g is a Morse function, and by computation we have

$$g(x) = f(x) + \sum_{i=1}^{m} \frac{h_i(x)^2}{2c} - \sum_{i=1}^{m} \epsilon_i \frac{h_i(x)}{c} + \sum_{i=1}^{m} \frac{\epsilon_i^2}{2c} - \epsilon_1.$$

Since $h_i, i = 1, \cdots, m$ is bounded, choose c to be sufficiently large and ϵ_i to be sufficiently small; then,

$$|g(y) - f(y)| < \epsilon$$

for any $y \in S$. This completes the proof. □

6. Infinitesimal Stability of Morse Functions on Subcartesian Spaces

In this section, we study the stability of Morse functions on a subcartesian space S with constant structural dimensions. See [21] for a systematic treatment on stability theory of Morse functions on smooth manifolds.

Lemma 8 ([21])**.** *Let f be a smooth function on $\mathbb{R}^n$ with $f(0) = 0$. Then,*

$$f(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \cdots, x_n),$$

where g_i are smooth functions on $\mathbb{R}^n$, such that $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Lemma 9 ([21])**.** *Let p be a non-degenerate critical point for $f \in C^\infty(\mathbb{R}^n)$. Then, there is a local coordinate system $(y_1, \cdots, y_n)$ in a neighborhood U of p with $y_i(p) = 0$ for all i, such that the identity*

$$f(y_1, \cdots, y_n) = f(p) - (y_1)^2 - ... - (y_\lambda)^2 + (y_{\lambda+l})^2 + \cdots + (y_n)^2$$

holds throughout U.

Lemma 10. *Let $(S, C^\infty(S))$ be a subcartesian space with constant structural dimension. Let $f \in C^\infty(S)$. Let $x \in S$ be a nondegenerate critical point of f. Then, there is a local coordinate system $(\phi, U, \mathbb{R}^n)$ of x with local coordinate system $(y^1, \cdots, y^n)$ in $\mathbb{R}^n$, such that f has a smooth extension $\tilde{f}$ on $\mathbb{R}^n$*

$$\tilde{f}(y_1, \cdots, y_n) = f(p) - (y_1)^2 - \cdots - (y_\lambda)^2 + (y_{\lambda+1})^2 + \cdots + (y_n)^2. \tag{8}$$

Proof. Let $(\phi, U, \mathbb{R}^n)$ be a local coordinate system of x, such that $\phi(x) = 0$. Let $\tilde{f}$ be a smooth extension of f. Hence, $\tilde{f}(0) = f(p)$. Since S has constant structural dimension and x is a nondegenerate critical point of f, it follows that 0 is a nondegenerate critical point of $\tilde{f}$. Then, the result follows immediately from Lemma 9. □

Corollary 3. *The set of critical points of a Morse function on S is discrete.*

Proof. The result follows from Lemma 10 directly. □

Definition 14. *Let S_1, S_2 be two subcartesian spaces. Let $\Phi : S_1 \to S_2$ be smooth.*

(a) *Let $\pi_{S_2} : TS_2 \to S_2$ be the canonical projection, and let $w : S_1 \to TS_2$ be smooth. Then, w is a derivation along Φ if $\pi_{S_2} \circ w = \Phi$. Let $C^\infty_\Phi(S_1, S_2)$ denote the set of derivation along Φ.*

(b) *Φ is infinitesimally stable if for every w, a derivation along Φ, there is a derivation s on S_1 and a derivation t on S_2 such that*

$$w = (d\Phi)s + t \circ \Phi.$$

Theorem 10. *Let $(S, C^\infty(S))$ be a subcartesian space with constant structural dimensions. Assume that S is compact. Let $f \in C^\infty(S)$ be a Morse function, all of whose critical values are distinct, i.e., if p and q are distinct critical points of f in S, then $f(p) \neq f(q)$. Then, f is infinitesimal stable.*

Proof. Let $w : S \to \mathbb{R} \times \mathbb{R}$ be a derivation along f. Then $w(x) = (f(x), \bar{w}(x))$ for every $x \in S$, where $\bar{w} \in C^\infty(S)$. Let s be a derivation of S. Then, $df(s)(x) = (f(x), s(f)(x))$. Let t be a vector field on $\mathbb{R}$. Then, $t \circ f(x) = (f(x), \bar{t}(f(x)))$, where $\bar{t} \in C^\infty(\mathbb{R})$. The condition of infinitesimal stability reduces in this case to the following: for every $w \in C^\infty(S)$, there exists a derivation s of S and a function $t \in C^\infty(\mathbb{R})$ such that

$$w = df(s) + t \circ f. \tag{9}$$

We now show how to solve the above equation. Since S is compact, it follows that there is only a finite number of critical points of f. Since all the critical values of f are distinct, we choose $t \in C^\infty(\mathbb{R})$, such that $t(f(x)) = w(x)$ for every critical point x of f. To solve (9), it is sufficient to solve

$$w = df(s), \tag{10}$$

where $w \in C^\infty(S)$ satisfies that $w(x) = 0$ for x being critical point of f. We now construct s.

Around each point p in S, choose an open neighborhood U_p with local coordinates $(U_p, \Phi_p, \mathbb{R}^n)$, such that both f and w have smooth extensions $\tilde{f}$ and $\tilde{w}$ on $\mathbb{R}^n$ with $\tilde{f} \circ \Phi = f$ and $\tilde{w} \circ \Phi = w$.

(a) If p is a regular point, choose U_p so small that $(df)_q \neq 0$ for every $q \in U_p$. Choose a derivation s^p on U_p, such that $(df)(s^p) \neq 0$ on U_p.

(b) If p is a critical point, then $\tilde{f} = c + \epsilon_1 x_1^2 + \cdots + \epsilon_n x_n^2$, where $\epsilon_1, \cdots, \epsilon_n = \pm 1$. $w(p) = 0$, and since $\tilde{w}(0) = 0$, it follows from Lemma 8 that $\tilde{w} = \sum_{i=1}^n h_i(x) x_i$, where $h_i, i = 1, \cdots, n$ are smooth functions on $\mathbb{R}^n$.

The collection $\{U_p\}_{p \in S}$ forms an open covering of S. Since S is compact, there exists a finite subcovering $U_1, \cdots, U_m$ corresponding to $p_l, \ldots, p_m$. Let $\rho_1, \ldots, \rho_m$ be a partition of unity subordinate to this covering. Choose derivations s_i on S $(1 \leq i \leq m)$ as follows:

(a) if p_i is a regular point, then let

$$s_i(x) = \begin{cases} \frac{w(x)\rho_i(x)s^{p_i}(x)}{df(s^{p_i})(x)}, x \in U_{p_i} \\ 0, x \in S \backslash U_{p_i}. \end{cases}$$

(b) If p_i is a critical point, let $\tilde{s}_i = \sum_{i=1}^{n} \frac{\epsilon_i h_i}{2} \frac{\partial}{\partial x_i}$ on $\mathbb{R}^n$. Since S has a constant structural dimension n, it follows from Proposition 2 that $\tilde{s}_i$ defines a derivation $\hat{s}_i$ on U_i, since for any $f \in N(S)$, $\frac{\partial}{\partial x_i}(f) \in N(S)$; otherwise, $f^{-1}(0)$ has a dimension less than n. Let $s_i = \rho_i \hat{s}_i$.

If p_i is a regular point, then

$$s_i(f) = \begin{cases} \frac{w\rho_i s^{p_i}(f)}{df(s^{p_i})} = w\rho_i, x \in U_i \\ 0, x \in S \backslash U_i. \end{cases}$$

If p_i is a singular point, then

$$s_i(f) = \begin{cases} \rho_i \sum_{j=1}^{n} \frac{\epsilon_j h_j}{2} \frac{\partial}{\partial x_j}(c + \epsilon_1 x_1^2 + \cdots + \epsilon_n x_n^2) = \rho_i \sum_{j=1}^{n} h_j x_j = \rho_i w, x \in U_i \\ 0, x \in S \backslash U_i. \end{cases}$$

Let $s = s_1 + \cdots + s_n$. It follows that $s(f) = \rho_1 w + \cdots + \rho_n w = \sum_{i=1}^{n} \rho_i w = w$. Hence, (10) is solved. The result follows immediately. □

7. Conclusions

In this paper, we have initiated a study of the differential topological properties for a subclass of singular space, subcartesian space. The purpose of our study was to discover important and interesting problems on smooth manifolds with solutions in subcartesian spaces when working in the framework of subcartesian spaces. Along this line, we mainly studied three aspects of differential topological properties for subcartesian spaces.

The first property concerns the partition of unity. The existence of a partition of unity on a differential space was already proven in the existing literature. After reviewing the result, we presented a minor further result on this point for differential space.

The second property we studied was the tubular neighborhood property, which is well-known for a smooth manifold. We established the tubular neighborhood theorem for subcartesian spaces with constant structural dimensions, both locally and globally.

The third property we studied is the Morse theory on subcartesian spaces. By taking advantage of the definition of derivations, we defined Morse functions on differential spaces. For a subcartesian space S with constant structural dimensions, we provided a class of examples of Morse functions, showing that Morse functions are plentiful. Further, we assumed that S admits a bounded embedding in some Euclidean space, showing that bounded smooth functions on S can be approximated by Morse functions. We proved that the set of critical points of any Morse function is discrete on S. Further, if S is compact, we proved that the Morse functions are infinitesimal stable.

In the future, we would like to conduct a further study of the Morse theory on subcartesian space and obtain more results on the differential topological properties of subcartesian spaces.

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