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Optimization in Symmetric Trees, Unicyclic Graphs, and Bicyclic Graphs with Help of Mappings Using Second Form of Generalized Power-Sum Connectivity Index

Muhammad Yasin Khan [1,†], Gohar Ali [1,*,†] and Ioan-Lucian Popa [2,3,*,†]

[1] Department of Mathematics, Islamia College Peshawar, Peshawar 25120, Khyber Pakhtunkhwa, Pakistan; yasin.student@icp.edu.pk
[2] Department of Computing, Mathematics and Electronics, 1 Decembrie 1918 University of Alba Iulia, 510009 Alba Iulia, Romania
[3] Faculty of Mathematics and Computer Science, Transilvania University of Brasov, Iuliu Maniu Street 50, 500091 Brasov, Romania
* Correspondence: gohar.ali@icp.edu.pk (G.A.); lucian.popa@uab.ro (I.-L.P.)
† These authors contributed equally to this work.

Abstract: The topological index (TI), sometimes referred to as the connectivity index, is a molecular descriptor calculated based on the molecular graph of a chemical compound. Topological indices (TIs) are numeric parameters of a graph used to characterize its topology and are usually graph-invariant. The generalized power-sum connectivity index (GPSCI) for the graph is Ω $Y_\alpha(\Omega) = \sum_{\zeta\varrho\in E(\Omega)}(d_\Omega(\zeta)^{d_\Omega(\zeta)} + d_\Omega(\varrho)^{d_\Omega(\varrho)})^\alpha$, while the second form of the GPSCI is defined as $Y_\beta(\Omega) = \sum_{\zeta\varrho\in E(\Omega)}(d_\Omega(\zeta)^{d_\Omega(\zeta)} \times d_\Omega(\varrho)^{d_\Omega(\varrho)})^\beta$.In this paper, we investigate Y_β in the family of trees, unicyclic graphs, and bicyclic graphs. We determine optimal graphs in the desired families for Y_β using certain mappings. For graphs with maximal values, two mappings are used, namely A and B, while for graphs with minimal values, mapping C and mapping D are considered.

Keywords: second form of generalized power-sum connectivity index; mappings; optimization; trees; unicyclic graphs; bicyclic graphs

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1. Introduction

A graph is a structure formed of edges and vertices. Edges are the connections between vertices with some properties. In symbolic form, we denote a graph by $\Omega = (V(\Omega), E(\Omega))$. If there is no confusion, then a graph is simply denoted by Ω. The number of vertices is termed as the order of the graph, while the total number of edges is the size of the graph. An edge between two vertices, ζ and ϱ, is denoted by $\zeta\varrho$. The total number of edges incident to a vertex ϱ in a graph Ω is referred to as the degree of ϱ, which is denoted by $d_\Omega(\varrho)$, if the considered graph is Ω. The neighbors of a vertex ϱ in Ω are the vertices connected to ϱ by means of an edge and are denoted by $N_\Omega(\varrho)$. More generally, we denote the set of all neighbors of a vertex ϱ by $N(\varrho)$. A network is a simple connected graph. A simple graph can be defined as a graph without loops or multiple edges between the same pair of vertices. The topological index (TI) is a transformation defined by $Top : \Omega \to R$, where R represents real numbers and Ω is a simple and finite graph. The transformation satisfies the property $Top(\Omega) = Top(H)$ if Ω and H are isomorphic graphs. The TI is a numerical value related to the chemical constitution, used to correlate chemical structure with various properties such as chemical reactivity, biological activity, and physical properties. Various tools, such as TIs, have been developed from graph theory and provided to chemists for analytical applications. Cheminformatics is a discipline formed of a combination of mathematics, chemistry, and information science. It investigates Quantitative Structure–Activity relationships (QSARs) and Qualitative Structure–Property relationships (QSPRs), which are applied to study the biological activities and properties of chemical compounds. In QSAR/QSPR studies, the physico-chemical properties and TIs, such as the Zagreb index, Szeged index, Wiener index, Randi ´c index and ABC index, are used to predict the bio-activity of various chemical compounds. "In graph theory, the structural formula of a chemical compound represents a molecular graph, where vertices represent atoms and edges represent chemical bonds". A molecular descriptor is a numerical value that represents the properties of a chemical graph. More specially, molecular descriptors represent the analyzed chemical graph, while topological descriptors describe its physico-chemical properties and structural features. TIs have various applications in the field of nano-biotechnology and QSAR/QSPR studies. The earliest TI was introduced by Wiener [1], which he named the path number, while studying the boiling point of Paraffin. Later, it was named the Wiener index [2].

In the last few decades, the field of optimization has played a key role in various daily life problems. This field is now the most powerful tool and technique for finding solutions to complicated problems. This field has become a major area of interest for researchers due to its ability to solve complex problems effectively. This field is now one of the most modern fields and is of great interest to mathematicians.There are a variety of fields where optimization plays a crucial role, including engineering, logistics, and machine learning. A lot of work is conducted in the stated fields, and some of this work is stated in the following cited article. In [3], the authors considered k-generalized quasi trees and found a closed result for the optimal values regarding the general power-sum connectivity index. In [4], the authors considered optimal values regarding forgotten TIs in terms of graph size. In [5], the authors investigated the molecular inter-connectivity index in polymers. In [6], the author considered trees, bicyclic graphs and unicyclic graphs for optimal values of Zagreb indices. In [7], the authors investigated KBSO indices for optimal values regarding high traffic load and minimal load by improving the physical layout. In [8], the authors investigated variable Zagreb indices, the variable sum-connectivity index, the variable geometric-arithmetic index and the the variable inverse sum indeg index for optimal values and found new bounds. In [9], the authors investigated Kulli–Basava indices in generalized form for quasi trees and found some optimal values and new bounds in k-generalized quasi trees. In [10], the authors investigated the Somber index for optimal values in trees with given parameters, like matching number, diameter, branching number, etc. In [11], the authors investigated exponential Zagreb indices for new bounds in unicyclic graphs, acyclic graphs and general graphs. In [12], the authors investigated the geometric arithmetic index for graphs and found optimal bounds. In [13], the authors investigated uphill Zagreb TIs for famous families of graph and developed exact new bounds. In [14], the authors investigated the variable sum exdeg index for bicyclic graphs with a given matching number. In [15], the authors investigated Zagreb indices for minimum and maximum values in k-apex trees; for further information, we refer the reader to [16–18].

In this work, we considered the second form of GPSCI for obtaining optimal graphs in the families of trees, unicyclic graphs and bicyclic graphs using some mappings. The second form of the generalized power-sum connectivity index is given and defined by:

$$Y_\beta(\Omega) = \sum_{\zeta\varrho \in E(\Omega)} (d_\Omega(\zeta)^{d_\Omega(\zeta)} \times d_\Omega(\varrho)^{d_\Omega(\varrho)})^\beta.$$

In this work, various results were compared using means plots. To better understand and implement the considered mappings, we provided graphs in the desired sections.

2. Mapping for Largest Values of Y_β

Mapping A: Let $\zeta\varrho \in E(\Omega)$ such that $d_\Omega(\varrho) \geq 2$ and $N(\zeta) = \{\varrho, w_1, w_2, w_3, \cdots, w_t\}$. Here $d(w_1) = d(w_2) = d(w_3) = \cdots = d(w_t) = 1$ and $\Omega' = G + \{\varrho w_1, \varrho w_2, \varrho w_3, \cdots, \varrho w_t\} - \{\zeta w_1, \zeta w_2, \zeta w_3, \cdots, \zeta w_t\}$. Refer to Figure 1 for an illustration.

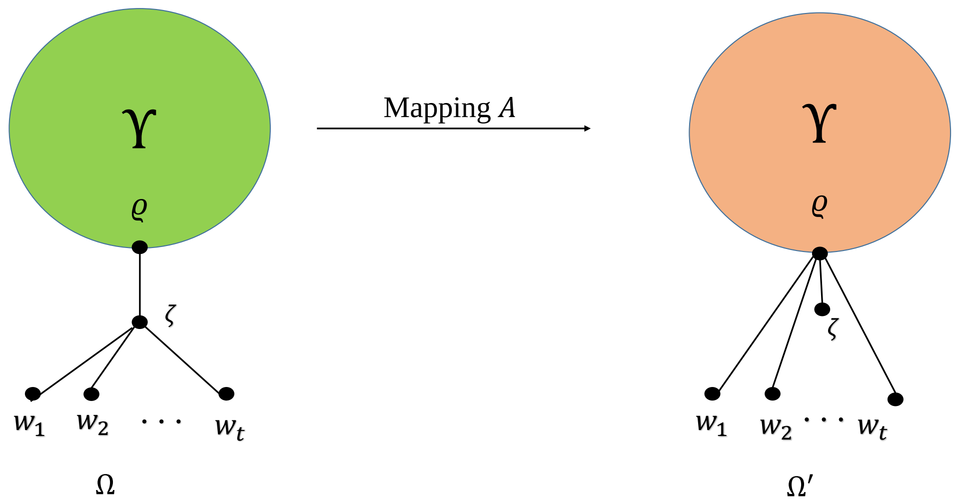

Figure 1. Diagram representing mapping A.

Lemma 1. *Let Ω' be obtained from Ω by means of mapping A and $d_\Omega(\varrho) \geq d_\Omega(\zeta)$; then,*

$$Y_\beta(\Omega') > Y_\beta(\Omega). \tag{1}$$

Proof. Let $\Upsilon = G - \{\zeta, w_1, w_2, w_3, \cdots, w_t\}$; the value of Y_β for Ω' is expressed as

$$\begin{aligned} Y_\beta(\Omega') = &\sum_{\delta\in N_\Upsilon(\varrho)} (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times d_{\Omega'}(\delta)^{d_{\Omega'}(\delta)})^\beta + (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times d_{\Omega'}(\zeta)^{d_{\Omega'}(\zeta)})^\beta \\ &+ (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times d_{\Omega'}(w_1)^{d_{\Omega'}(w_1)})^\beta + \cdots + (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times d_{\Omega'}(w_t)^{d_{\Omega'}(w_t)})^\beta. \\ &= \sum_{\delta\in N_\Upsilon(\varrho)} (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times d_{\Omega'}(\delta)^{d_{\Omega'}(\delta)})^\beta + (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times 1^1)^\beta + (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times 1^1)^\beta \\ &+ \cdots + (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times 1^1)^\beta, \\ &= \sum_{\delta\in N_\Upsilon(\varrho)} (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times d_{\Omega'}(\delta)^{d_{\Omega'}(\delta)})^\beta + (t+1)(d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times 1^1)^\beta. \end{aligned} \tag{2}$$

For Ω, the value of Y_β is given by

$$\begin{aligned} Y_\beta(\Omega) = &\sum_{\delta\in N_\Upsilon(\varrho)} (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(\zeta)^{d_\Omega(\zeta)})^\beta + (d_\Omega(\zeta)^{d_\Omega(\zeta)} \times d_\Omega(w_1)^{d_\Omega(w_1)})^\beta \\ &+ (d_\Omega(\zeta)^{d_\Omega(\zeta)} \times d_\Omega(w_2)^{d_\Omega(w_2)})^\beta + \cdots + (d_\Omega(\zeta)^{d_\Omega(\zeta)} \times d_\Omega(w_t)^{d_\Omega(w_t)})^\beta, \\ &= \sum_{\delta\in N_\Upsilon(\varrho)} (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(\zeta)^{d_\Omega(\zeta)})^\beta + t(d_\Omega(\zeta)^{d_\Omega(\zeta)} \times 1^1)^\beta. \end{aligned} \tag{3}$$

From (2) and (3), it follows that

$$\begin{aligned}
&\Upsilon_\beta(\Omega') - \Upsilon_\beta(\Omega) = \sum_{\delta \in N_\Upsilon(\varrho)} (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times d_{\Omega'}(\delta)^{d_{\Omega'}(\delta)})^\beta + (t+1)(d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times 1^1)^\beta \\
&- \sum_{\delta \in N_\Upsilon(\varrho)} (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta - (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(\zeta)^{d_\Omega(\zeta)})^\beta - t(d_\Omega(\zeta)^{d_\Omega(\zeta)} \times 1^1)^\beta, \\
&= \sum_{\delta \in N_\Upsilon(\varrho)} (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times d_{\Omega'}(\delta)^{d_{\Omega'}(\delta)})^\beta + 2(d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times 1^1)^\beta + (t-1)(d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times 1^1)^\beta \\
&- \sum_{\delta \in N_\Upsilon(\varrho)} (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta - (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(\zeta)^{d_\Omega(\zeta)})^\beta - t(d_\Omega(\zeta)^{d_\Omega(\zeta)} \times 1^1)^\beta, \\
&> 0.
\end{aligned} \tag{4}$$

For the inequality in (4), we consider

$$\begin{aligned}
&\sum_{\delta \in N_\Upsilon(\varrho)} (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times d_{\Omega'}(\delta)^{d_{\Omega'}(\delta)})^\beta > \sum_{\delta \in N_\Upsilon(\varrho)} (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta, \\
&2(d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)})^\beta \times 1^1) + (t-1)(d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times 1^1)^\beta > (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(\zeta)^{d_\Omega(\zeta)})^\beta + t(d_\Omega(\zeta)^{d_\Omega(\zeta)} \times 1^1)^\beta.
\end{aligned} \tag{5}$$

It follows that

$$\begin{aligned}
&d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} > d_\Omega(\varrho)^{d_\Omega(\varrho)}, d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} > d_\Omega(\zeta)^{d_\Omega(\zeta)}, \\
&\implies d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times 1 > d_\Omega(\varrho)^{d_\Omega(\varrho)}, d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times 1 > d_\Omega(\zeta)^{d_\Omega(\zeta)}, \\
&\implies 2(d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times 1)^\beta > (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(\zeta)^{d_\Omega(\zeta)})^\beta.
\end{aligned} \tag{6}$$

Using (5) and (6), we determine that $\Upsilon_\beta(\Omega') > \Upsilon_\beta(\Omega)$. □

Remark 1. *As a remark on mapping A, every tree can be changed to a star, and any unicyclic and bicyclic graph can be changed to a unicyclic and bicyclic graph in which all noncyclic edges have a degree of one.*

Mapping B: Let Ω be a graph and $\zeta, \varrho \in E(\Omega)$. Adjacent vertices with a degree of one to vertex ζ and ϱ are $u_1, u_2, \cdots, u_r$ and $\varphi_1, \varphi_2, \cdots, \varphi_t$, respectively. We obtain Ω' and Ω'', where $\Omega' = G + \{\varrho u_1, \varrho u_2, \cdots, \varrho u_r\} - \{\zeta u_1, \zeta u_2, \cdots, \zeta u_r\}$, and $\Omega'' = G - \{\varrho\varphi_1, \varrho\varphi_2, \cdots, \varrho\varphi_t\} + \{\zeta\varphi_1, \zeta\varphi_2, \cdots, \zeta\varphi_t\}$. Figure 2 demonstrates this.

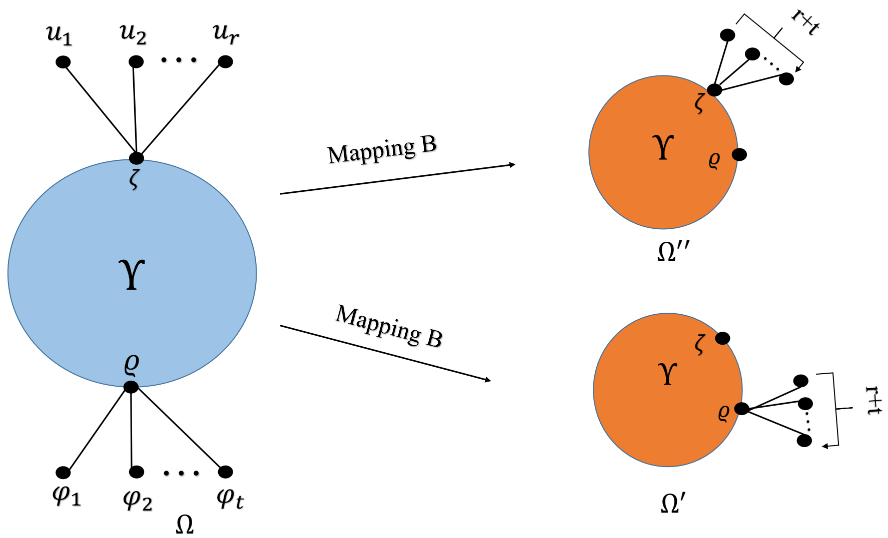

Figure 2. Diagram presenting mapping B.

Lemma 2. *Let Ω' and Ω'' be obtained from Ω using mapping B; then, either $Y_\beta(\Omega) < Y_\beta(\Omega')$ or $Y_\beta(\Omega) < Y_\beta(\Omega'')$.*

Proof. Let $Y = G - \{\varphi_1, \varphi_2, \cdots, \varphi_t, u_1, u_2, \cdots, u_r\}$; further, we consider the following cases:
Let $d_Y(\zeta) = \theta$ and $d_Y(\varrho) = \phi$.
Case 1. If ζ and ϱ are not neighbors in Ω, then it follows that

$$\begin{aligned} Y_\beta(\Omega) = & \sum_{\delta y \in E(\Omega - \{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} (d_\Omega(\zeta)^{d_\Omega(\zeta)} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\ & + \sum_{\delta \in N_Y(\varrho)} (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_Y(\delta)^{d_Y(\delta)})^\beta + (d_\Omega(\zeta)^{d_\Omega(\zeta)} \times 1)^\beta + (d_\Omega(\zeta)^{d_\Omega(\zeta)} \times 1)^\beta + \cdots \\ & + (d_\Omega(\zeta)^{d_\Omega(\zeta)} \times 1)^\beta + (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times 1)^\beta + (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times 1)^\beta + \cdots + (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times 1)^\beta, \\ = & \sum_{\delta y \in E(\Omega - \{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} ((\theta + r)^{\theta + r} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\ & + \sum_{\delta \in N_Y(\varrho)} ((\phi + t)^{\phi + t} \times d_Y(\delta)^{d_Y(\delta)})^\beta + r((\theta + r)^{\theta + r} \times 1)^\beta + t((\phi + t)^{\phi + t} \times 1)^\beta. \end{aligned} \tag{7}$$

For Ω', it follows that

$$\begin{aligned} Y_\beta(\Omega') = & \sum_{\delta y \in E(\Omega - \{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} (d_Y(\zeta)^{d_Y(\zeta)} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\ & + \sum_{\delta \in N_Y(\varrho)} (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times d_Y(\delta)^{d_Y(\delta)})^\beta + (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times 1)^\beta + \cdots + (d_{\Omega'}(\varrho)^{d_{\Omega'}(\varrho)} \times 1)^\beta, \\ = & \sum_{\delta y \in E(\Omega - \{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} (\theta^\theta \times d_Y(\delta)^{d_Y(\delta)})^\beta \\ & + \sum_{\delta \in N_Y(\varrho)} ((\phi + r + t)^{\phi + r + t} \times d_Y(\delta)^{d_Y(\delta)})^\beta + (r + t)((\phi + r + t)^{\phi + r + t} \times 1)^\beta \end{aligned} \tag{8}$$

For Ω'', it follows that

$$\begin{aligned} Y_\beta(\Omega'') = & \sum_{\delta y \in E(\Omega - \{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} (d_{\Omega''}(\zeta)^{d_{\Omega''}(\zeta)} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\ & + \sum_{\delta \in N_Y(\varrho)} (d_Y(\varrho)^{d_Y(\varrho)} \times d_Y(\delta)^{d_Y(\delta)})^\beta + (d_{\Omega''}(\zeta)^{d_{\Omega''}(\zeta)} \times 1)^\beta + \cdots + (d_{\Omega''}(\zeta)^{d_{\Omega''}(\varrho)} \times 1)^\beta, \\ = & \sum_{\delta y \in E(\Omega - \{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} ((\theta + r + t)^{\theta + r + t} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\ & + \sum_{\delta \in N_Y(\varrho)} (\phi^\phi \times d_Y(\delta)^{d_Y(\delta)})^\beta + (r + t)((\theta + r + t)^{\theta + r + t} \times 1)^\beta. \end{aligned} \tag{9}$$

Let $\Delta_1 = Y_\beta(\Omega') - Y_\beta(\Omega)$ and $\Delta_2 = Y_\beta(\Omega'') - Y_\beta(\Omega)$; we consider the following:

$$
\begin{aligned}
\Delta_1 =& \sum_{\delta y \in E(\Omega - \{\zeta, \varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} (\theta^\theta \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
&+ \sum_{\delta \in N_Y(\varrho)} ((\phi + r + t)^{\phi+r+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta + (r+t)((\phi + r + t)^{\phi+r+t} \times 1)^\beta \\
&- \sum_{\delta y \in E(\Omega - \{\zeta, \varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta - \sum_{\delta \in N_Y(\zeta)} ((\theta + r)^{\theta+r} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
&- \sum_{\delta \in N_Y(\varrho)} ((\phi + t)^{\phi+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta - r((\theta + r)^{\theta+r} \times 1)^\beta - t((\phi + t)^{\phi+t} \times 1)^\beta, \\
&= \sum_{\delta \in N_Y(\zeta)} (\theta^\theta \times d_Y(\delta)^{d_Y(\delta)})^\beta + \sum_{\delta \in N_Y(\varrho)} ((\phi + r + t)^{\phi+r+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
&+ (r+t)((\phi + r + t)^{\phi+r+t} \times 1)^\beta - \sum_{\delta \in N_Y(\zeta)} ((\theta + r)^{\theta+r} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
&- \sum_{\delta \in N_Y(\varrho)} ((\phi + t)^{\phi+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta - r((\theta + r)^{\theta+r} \times 1)^\beta - t((\phi + t)^{\phi+t} \times 1)^\beta
\end{aligned} \tag{10}
$$

Here $\Delta_1 > 0$ if

$$
\begin{aligned}
&\sum_{\delta \in N_Y(\zeta)} (\theta^\theta \times d_Y(\delta)^{d_Y(\delta)})^\beta + \sum_{\delta \in N_Y(\varrho)} ((\phi + r + t)^{\phi+r+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
&\quad + (r+t)((\phi + r + t)^{\phi+r+t} \times 1)^\beta \\
&> [\sum_{\delta \in N_Y(\zeta)} ((\theta + r)^{\theta+r} \times d_Y(\delta)^{d_Y(\delta)})^\beta + \sum_{\delta \in N_Y(\varrho)} ((\phi + t)^{\phi+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
&\quad + r((\theta + r)^{\theta+r} \times 1)^\beta + t((\phi + t)^{\phi+t} \times 1)^\beta], \\
&\implies \sum_{\delta \in N_Y(\zeta)} ((\theta + r + t)^{(\theta+r+t)} \times d_Y(\delta)^{d_Y(\delta)})^\beta + \sum_{\delta \in N_Y(\varrho)} ((\phi + r + t)^{\phi+r+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
&\quad + (r+t)((\phi + r + t)^{\phi+r+t} \times 1)^\beta \\
&> [\sum_{\delta \in N_Y(\zeta)} ((\theta + r)^{\theta+r} \times d_Y(\delta)^{d_Y(\delta)})^\beta + \sum_{\delta \in N_Y(\varrho)} ((\phi + t)^{\phi+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
&\quad + r((\theta + r)^{\theta+r} \times 1)^\beta + t((\phi + t)^{\phi+t} \times 1)^\beta].
\end{aligned} \tag{11}
$$

If $\Delta_1 > 0$ then, $\Delta_2 < 0$, we proceed as follows:

$$\begin{aligned}
\Delta_2 = & \sum_{\delta y \in E(\Omega-\{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} ((\theta+r+t)^{\theta+r+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
& + \sum_{\delta \in N_Y(\varrho)} (\phi^\phi \times d_Y(\delta)^{d_Y(\delta)})^\beta + (r+t)((\theta+r+t)^{\theta+r+t} \times 1)^\beta \\
& - \sum_{\delta y \in E(\Omega-\{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} ((\theta+r)^{\theta+r} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
& + \sum_{\delta \in N_Y(\varrho)} ((\phi+t)^{\phi+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta + r((\theta+r)^{\theta+r} \times 1)^\beta + t((\phi+t)^{\phi+t} \times 1)^\beta, \\
& = \sum_{\delta \in N_Y(\zeta)} ((\theta+r+t)^{\theta+r+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta + \sum_{\delta \in N_Y(\varrho)} (\phi^\phi \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
& + (r+t)((\theta+r+t)^{\theta+r+t} \times 1)^\beta - \sum_{\delta \in N_Y(\zeta)} ((\theta+r)^{\theta+r} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
& - \sum_{\delta \in N_Y(\varrho)} ((\phi+t)^{\phi+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta - r((\theta+r)^{\theta+r} \times 1)^\beta - t((\phi+t)^{\phi+t} \times 1)^\beta, \\
& < \sum_{\delta \in N_Y(\zeta)} ((\theta+r+t)^{\theta+r+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta + \sum_{\delta \in N_Y(\varrho)} (\phi^\phi \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
& + (r+t)((\theta+r+t)^{\theta+r+t} \times 1)^\beta - \sum_{\delta \in N_Y(\zeta)} ((\theta+r+t)^{(\theta+r+t)} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
& - \sum_{\delta \in N_Y(\varrho)} ((\phi+r+t)^{\phi+r+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta - (r+t)((\phi+r+t)^{\phi+r+t} \times 1)^\beta \\
& = \sum_{\delta \in N_Y(\varrho)} (\phi^\phi \times d_Y(\delta)^{d_Y(\delta)})^\beta - \sum_{\delta \in N_Y(\varrho)} ((\phi+r+t)^{\phi+r+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
& < 0.
\end{aligned} \tag{12}$$

Case 2. If $\zeta, \varrho \in V(\Omega)$ are adjacent, then

$$\begin{aligned}
Y_\beta(\Omega) = & \sum_{\delta y \in E(\Omega-\{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} ((\theta+r)^{\theta+r} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
& + \sum_{\delta \in N_Y(\varrho)} ((\phi+t)^{\phi+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta + ((\theta+r)^{\theta+r} \times 1)^\beta + \cdots + ((\theta+r)^{\theta+r} \times 1)^\beta \\
& + ((\phi+t)^{\phi+t} \times 1)^\beta + \cdots + ((\phi+t)^{\phi+t} \times 1)^\beta - ((\theta+r)^{\theta+r} \times (\phi+t)^{\phi+t})^\beta, \\
& = \sum_{\delta y \in E(\Omega-\{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} ((\theta+r)^{\theta+r} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\
& + \sum_{\delta \in N_Y(\varrho)} ((\phi+t)^{\phi+t} \times d_Y(\delta)^{d_Y(\delta)})^\beta + r((\theta+r)^{\theta+r} \times 1)^\beta + t((\phi+t)^{\phi+t} \times 1)^\beta \\
& - ((\theta+r)^{\theta+r} \times (\phi+t)^{\phi+t})^\beta.
\end{aligned} \tag{13}$$

For Ω', it follows that

$$\begin{aligned} Y_\beta(\Omega') = &\sum_{\delta y \in E(\Omega - \{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} (\theta^\theta \times d_Y(\delta)^{d_Y(\delta)})^\beta \\ &+ \sum_{\delta \in N_Y(\varrho)} ((\phi + r + t)^{\phi + r + t} \times d_Y(\delta)^{d_Y(\delta)})^\beta + ((\phi + r + t)^{(\phi + r + t)} \times 1)^\beta + \cdots \\ &+ ((\phi + r + t)^{\phi + r + t} \times 1)^\beta - (\theta^\theta \times (\phi + r + t)^{(\phi + r + t)})^\beta, \\ = &\sum_{\delta y \in E(\Omega - \{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} ((\theta)^\theta \times d_Y(\delta)^{d_Y(\delta)})^\beta \\ &+ \sum_{\delta \in N_Y(\varrho)} ((\phi + r + t)^{\phi + r + t} \times d_Y(\delta)^{d_Y(\delta)})^\beta + (r + t)((\phi + r + t)^{(\phi + r + t)} \times 1)^\beta \\ &- (\theta^\theta \times (\phi + r + t)^{(\phi + r + t)})^\beta. \end{aligned} \tag{14}$$

For Ω'', we consider the following:

$$\begin{aligned} Y_\beta(\Omega'') = &\sum_{\delta y \in E(\Omega - \{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} ((\theta + r + t)^{(\theta + r + t)} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\ &+ \sum_{\delta \in N_Y(\varrho)} (\phi^\phi \times d_Y(\delta)^{d_Y(\delta)})^\beta + ((\theta + r + t)^{(\theta + r + t)} \times 1)^\beta + \cdots \\ &+ ((\theta + r + t)^{\theta + r + t} \times 1)^\beta - ((\theta + r + t)^{(\theta + r + t)} \times \phi^\phi)^\beta, \\ = &\sum_{\delta y \in E(\Omega - \{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} ((\theta + r + t)^{(\theta + r + t)} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\ &+ \sum_{\delta \in N_Y(\varrho)} (\phi^\phi \times d_Y(\delta)^{d_Y(\delta)})^\beta + (r + t)((\theta + r + t)^{(\theta + r + t)} \times 1)^\beta - ((\theta + r + t)^{(\theta + r + t)} \times \phi^\phi)^\beta. \end{aligned} \tag{15}$$

Consider $\Delta_1 = Y_\beta(\Omega') - Y_\beta(\Omega)$ and $\Delta_2 = Y_\beta(\Omega'') - Y_\beta(\Omega)$. For Δ_1, it follows that

$$\begin{aligned} \Delta_1 = &\sum_{\delta y \in E(\Omega - \{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta + \sum_{\delta \in N_Y(\zeta)} ((\theta)^\theta \times d_Y(\delta)^{d_Y(\delta)})^\beta \\ &+ \sum_{\delta \in N_Y(\varrho)} ((\phi + r + t)^{\phi + r + t} \times d_Y(\delta)^{d_Y(\delta)})^\beta + (r + t)((\phi + r + t)^{(\phi + r + t)} \times 1)^\beta \\ &- (\theta^\theta \times (\phi + r + t)^{(\phi + r + t)})^\beta \\ &- \sum_{\delta y \in E(\Omega - \{\zeta,\varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^\beta - \sum_{\delta \in N_Y(\zeta)} ((\theta + r)^{\theta + r} \times d_Y(\delta)^{d_Y(\delta)})^\beta \\ &- \sum_{\delta \in N_Y(\varrho)} ((\phi + t)^{\phi + t} \times d_Y(\delta)^{d_Y(\delta)})^\beta - r((\theta + r)^{\theta + r} \times 1)^\beta - t((\phi + t)^{\phi + t} \times 1)^\beta \\ &+ ((\theta + r)^{\theta + r} \times (\phi + t)^{\phi + t})^\beta \end{aligned}$$

$$
\begin{aligned}
\Delta_1 = & \sum_{\delta \in N_Y(\zeta)} (\theta^{\theta} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} + \sum_{\delta \in N_Y(\varrho)} ((\phi + r + t)^{\phi+r+t} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} \\
& + (r+t)((\phi + r + t)^{(\phi+r+t)} \times 1)^{\beta} - (\theta^{\theta} \times (\phi + r + t)^{(\phi+r+t)})^{\beta} \\
& - \sum_{\delta \in N_Y(\zeta)} ((\theta + r)^{\theta+r} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} - \sum_{\delta \in N_Y(\varrho)} ((\phi + t)^{\phi+t} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} \\
& - r((\theta + r)^{\theta+r} \times 1)^{\beta} - t((\phi + t)^{\phi+t} \times 1)^{\beta} + ((\theta + r)^{\theta+r} \times (\phi + t)^{\phi+t})^{\beta}, \\
= & \sum_{\delta \in N_Y(\zeta)} (\theta^{\theta} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} + \sum_{\delta \in N_Y(\varrho)} ((\phi + r + t)^{\phi+r+t} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} \\
& + (r+t)((\phi + r + t)^{(\phi+r+t)} \times 1)^{\beta} + ((\theta + r)^{\theta+r} \times (\phi + t)^{\phi+t})^{\beta} \\
& - (\theta^{\theta} \times (\phi + r + t)^{(\phi+r+t)})^{\beta} - \sum_{\delta \in N_Y(\zeta)} ((\theta + r)^{\theta+r} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} \\
& - \sum_{\delta \in N_Y(\varrho)} ((\phi + t)^{\phi+t} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} - r((\theta + r)^{\theta+r} \times 1)^{\beta} - t((\phi + t)^{\phi+t} \times 1)^{\beta}.
\end{aligned}
\tag{16}
$$

If $\Delta_1 > 0$, it follows that

$$
\begin{aligned}
& \sum_{\delta \in N_Y(\zeta)} (\theta^{\theta} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} + \sum_{\delta \in N_Y(\varrho)} ((\phi + r + t)^{\phi+r+t} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} \\
& + (r+t)((\phi + r + t)^{(\phi+r+t)} \times 1)^{\beta} + ((\theta + r)^{\theta+r} \times (\phi + t)^{\phi+t})^{\beta} \\
& > -(\theta^{\theta} \times (\phi + r + t)^{(\phi+r+t)})^{\beta} - \sum_{\delta \in N_Y(\zeta)} ((\theta + r)^{\theta+r} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} \\
& - \sum_{\delta \in N_Y(\varrho)} ((\phi + t)^{\phi+t} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} - r((\theta + r)^{\theta+r} \times 1)^{\beta} - t((\phi + t)^{\phi+t} \times 1)^{\beta}.
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow & \sum_{\delta \in N_Y(\zeta)} ((\theta + r + t)^{(\theta+r+t)} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} + \sum_{\delta \in N_Y(\varrho)} ((\phi + r + t)^{\phi+r+t} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} \\
& + (r+t)((\phi + r + t)^{(\phi+r+t)} \times (\theta + r + t)^{(\theta+r+t)})^{\beta} + ((\theta + r)^{\theta+r} \times (\phi + r + t)^{\phi+r+t})^{\beta} \\
& > -(\theta^{\theta} \times (\phi + r + t)^{(\phi+r+t)})^{\beta} - \sum_{\delta \in N_Y(\zeta)} ((\theta + r)^{\theta+r} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} \\
& - \sum_{\delta \in N_Y(\varrho)} ((\phi + t)^{\phi+t} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} - r((\theta + r)^{\theta+r} \times 1)^{\beta} - t((\phi + t)^{\phi+t} \times 1)^{\beta}.
\end{aligned}
\tag{17}
$$

If condition (17) holds, then

$$
\begin{aligned}
\Delta_2 = & \sum_{\delta y \in E(\Omega - \{\zeta, \varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^{\beta} + \sum_{\delta \in N_Y(\zeta)} ((\theta + r + t)^{(\theta+r+t)} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} \\
& + \sum_{\delta \in N_Y(\varrho)} (\phi^{\phi} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} + (r+t)((\theta + r + t)^{(\theta+r+t)} \times 1)^{\beta} + \cdots \\
& - ((\theta + r + t)^{(\theta+r+t)} \times \phi^{\phi})^{\beta} \\
& - \sum_{\delta y \in E(\Omega - \{\zeta, \varrho\})} (d_Y(\delta)^{d_Y(\delta)} \times d_Y(y)^{d_Y(y)})^{\beta} - \sum_{\delta \in N_Y(\zeta)} ((\theta + r)^{\theta+r} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} \\
& - \sum_{\delta \in N_Y(\varrho)} ((\phi + t)^{\phi+t} \times d_Y(\delta)^{d_Y(\delta)})^{\beta} - r((\theta + r)^{\theta+r} \times 1)^{\beta} - t((\phi + t)^{\phi+t} \times 1)^{\beta} \\
& + ((\theta + r)^{\theta+r} \times (\phi + t)^{\phi+t})^{\beta}.
\end{aligned}
$$

$$\begin{aligned}\Delta_2 = &\sum_{\delta\in N_Y(\zeta)}((\theta+r+t)^{(\theta+r+t)}\times d_Y(\delta)^{d_Y(\delta)})^\beta + \sum_{\delta\in N_Y(\varrho)}(\phi^\phi\times d_Y(\delta)^{d_Y(\delta)})^\beta\\ &+(r+t)((\theta+r+t)^{(\theta+r+t)}\times 1)^\beta + ((\theta+r)^{\theta+r}\times(\phi+t)^{\phi+t})^\beta\\ &-((\theta+r+t)^{(\theta+r+t)}\times\phi^\phi)^\beta - \sum_{\delta\in N_Y(\zeta)}((\theta+r)^{\theta+r}\times d_Y(\delta)^{d_Y(\delta)})^\beta\\ &-\sum_{\delta\in N_Y(\varrho)}((\phi+t)^{\phi+t}\times d_Y(\delta)^{d_Y(\delta)})^\beta - r((\theta+r)^{\theta+r}\times 1)^\beta - t((\phi+t)^{\phi+t}\times 1)^\beta.\end{aligned}$$

$$\begin{aligned}\Delta_2 < &\sum_{\delta\in N_Y(\zeta)}((\theta+r+t)^{(\theta+r+t)}\times d_Y(\delta)^{d_Y(\delta)})^\beta + \sum_{\delta\in N_Y(\varrho)}(\phi^\phi\times d_Y(\delta)^{d_Y(\delta)})^\beta\\ &+(r+t)((\theta+r+t)^{(\theta+r+t)}\times 1)^\beta + ((\theta+r)^{\theta+r}\times(\phi+t)^{\phi+t})^\beta\\ &-\sum_{\delta\in N_Y(\zeta)}((\theta+r+t)^{(\theta+r+t)}\times d_Y(\delta)^{d_Y(\delta)})^\beta - \sum_{\delta\in N_Y(\varrho)}((\phi+r+t)^{\phi+r+t}\times d_Y(\delta)^{d_Y(\delta)})^\beta\\ &-(r+t)((\phi+r+t)^{(\phi+r+t)}\times(\theta+r+t)^{(\theta+r+t)})^\beta - ((\theta+r)^{\theta+r}\times(\phi+r+t)^{\phi+r+t})^\beta,\\ &<0.\end{aligned}$$

Hence, it follows that if $\Delta_1 > 0$, then $\Delta_2 < 0$. □

Remark 2. *Using mapping B repeatedly, any unicyclic graph can be turned to a unicyclic graph in which all the edges with a degree of* 1 *are connected directly to a unique vertex. Any graph with two cycles can be changed to a bicyclic graph in which all the pendent edges are attached to a unique vertex.*

3. Graphs Having Largest Value of $Y_\beta(\Omega)$

Let U_n^k be a unicyclic graph obtained from C_k, where $n-k$ pendent edges are attached to the same vertex on C_k by means of Lemmas 1 and 2. It follows that Theorem 1 is true.

Theorem 1. *Let Ω be a unicyclic graph of order n and girth k. If $G \neq U_n^k$; then, $Y_\beta(\Omega) < Y_\beta(U_n^k)$.*

Consider a bicyclic graph of size $n+1$ and order n and denoted by $(n, n+1)$. Let $\mathbb{G}(n, n+1)$ be a set of simple connected graphs of order n and size $n+1$. For any $G \in \mathbb{G}(n, n+1)$, consider two cycles, C_θ and C_ϕ. All $(n, n+1) - graphs$ are divided into three classes as follows.

(1) $\mathbb{A}(\theta, \phi)$ is a set containing $G \in \mathbb{G}(n, n+1)$ in which C_θ and C_ϕ have a common vertex.
(2) $\mathbb{B}(\theta, \phi)$ is a set containing $G \in \mathbb{G}(n, n+1)$ in which C_θ and C_ϕ have no vertices in common.
(3) $\mathbb{C}(\theta, \phi, l)$ is a set containing $G \in \mathbb{G}(n, n+1)$ in which C_θ and C_ϕ have a path of length l in common.

Induced sub-graphs of $G \in \mathbb{A}(\theta, \phi)$, $\mathbb{B}(\theta, \phi)$, and $\mathbb{C}(\theta, \phi, l)$) are shown in Figure 3i–iii.

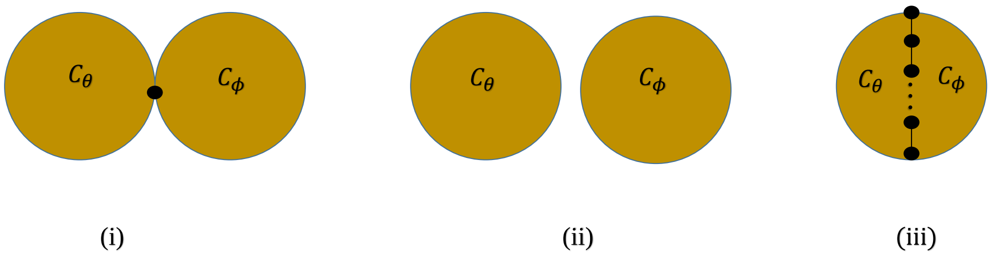

Figure 3. Diagram presenting bicyclic graphs. (**i**) Bicyclic graph having a common vertex, (**ii**) Bi cyclic graphs having no vertex in common, (**iii**) Bicyclic graphs having a path in common.

4. Largest Value of Y_β in $\mathbb{A}(\theta,\phi)$

This section is devoted to the investigation of bicyclic graphs for the greatest value of Y_β in $\mathbb{A}(\theta,\phi)$. Let $S_n(\theta,\phi)$ belong to $\mathbb{A}(\theta,\phi)$, where $n+1-\theta-\phi$ vertices with a degree of one are connected to a vertex which is common between C_θ and C_ϕ. Figure 4 demonstrates this.

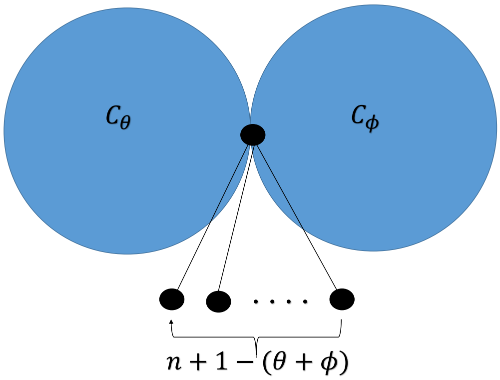

Figure 4. Diagram presenting bicyclic graph $S_n(\theta,\phi)$.

Theorem 2. *In $\mathbb{A}(\theta,\phi)$, the graph $S_n(\theta,\phi)$ has greatest value of Y_β.*

Proof. If we use mapping A and mapping B on Ω, we obtain Ω', where all edges not on the cycles are attached to vertex ϱ. Using Lemmas 1 and 2, we obtain $Y_\beta(\Omega') > Y_\beta(\Omega)$ with equality if and only if all the edges not on the cycles are attached to the same vertex in Ω. If $\Omega' \ncong S_n(\theta,\phi)$, then vertices ζ and ϱ are different and ζ is the common vertex of C_θ and C_ϕ. Let ϱ be on C_θ; we consider the following cases:

Case 1. If vertices ζ and ϱ are not neighbors, then it follows that

$$\begin{aligned}
&Y_\beta(S_n(\theta,\phi)) = ((n+5-\theta-\phi)^{(n+5-\theta-\phi)} \times 1)^\beta + ((n+5-\theta-\phi)^{(n+5-\theta-\phi)} \times 1)^\beta \\
&+ \cdots + ((n+5-\theta-\phi)^{(n+5-\theta-\phi)} \times 1)^\beta + 4((n+5-\theta-\phi)^{(n+5-\theta-\phi)} \times 2^2)^\beta \\
&+ (\theta-2)(2^2 \times 2^2)^\beta + (\phi-2)(2^2 \times 2^2)^\beta, \\
&= (n+1-\theta-\phi)((n+5-\theta-\phi)^{(n+5-\theta-\phi)} \times 1)^\beta + 4((n+5-\theta-\phi)^{(n+5-\theta-\phi)} \times 2^2)^\beta \\
&+ (\theta-2)(16)^\beta + (\phi-2)(16)^\beta.
\end{aligned}$$

For Ω', it follows that

$$\begin{aligned}
&Y_\beta(\Omega') = ((n+3-\theta-\phi)^{(n+3-\theta-\phi)} \times 1)^\beta + ((n+3-\theta-\phi)^{(n+3-\theta-\phi)} \times 1)^\beta \\
&+ \cdots + ((n+3-\theta-\phi)^{(n+3-\theta-\phi)} \times 1)^\beta + 2((n+3-\theta-\phi)^{(n+3-\theta-\phi)} \times 2^2)^\beta \\
&+ (\theta-4)(2^2 \times 2^2)^\beta + (\phi-2)(2^2 \times 2^2)^\beta + 4(4^4 \times 2^2)^\beta, \\
&= (n+1-\theta-\phi)((n+3-\theta-\phi)^{(n+3-\theta-\phi)} \times 1)^\beta + 2((n+3-\theta-\phi)^{(n+3-\theta-\phi)} \times 2^2)^\beta \\
&+ (\theta-4)(16)^\beta + (\phi-2)(16)^\beta + 4(1024)^\beta.
\end{aligned}$$

Consider the following difference:

$$\begin{aligned}
Y_\beta(S_n(\theta,\phi)) - Y_\beta(\Omega^{'}) &= (n+1-\theta-\phi)((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 1)^\beta + 4((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 2^2)^\beta \\
&+ (\theta-2)(16)^\beta - (n+1-\theta-\phi)((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 1)^\beta - 2((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 2^2)^\beta \\
&- (\theta-4)(16)^\beta - 4(1024)^\beta, \\
&= (n+1-\theta-\phi)((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 1)^\beta + 2((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 2^2)^\beta \\
&+ (\theta-2)(16)^\beta + 2((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 2^2)^\beta - (n+1-\theta-\phi)((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 1)^\beta \\
&- 2((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 2^2)^\beta - (\theta-4)(16)^\beta - 4(1024)^\beta > 0.
\end{aligned} \tag{18}$$

The difference is positive, while the equality holds if and only if $\Omega^{'} \cong S_n(\theta,\phi)$.

Case 2. If the vertices ζ and ϱ are adjacent in $\Omega^{'}$, then

$$\begin{aligned}
Y_\beta(\Omega^{'}) &= ((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 1^1)^\beta + ((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 1^1)^\beta \\
&+ \cdots + ((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 1^1)^\beta + ((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 2^2)^\beta \\
&+ ((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 4^4)^\beta + 3(4^4\times 2^2) + (\theta-3)(2^2\times 2^2)^\beta + (\phi-2)(2^2\times 2^2)^\beta, \\
&= (n+1-\theta-\phi)((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 1^1)^\beta + ((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 4)^\beta \\
&+ ((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 256)^\beta + 3(1024)^\beta + (\theta-3)(16)^\beta + (\phi-2)(16)^\beta.
\end{aligned} \tag{19}$$

Now, consider the following difference:

$$\begin{aligned}
Y_\beta(S_n(\theta,\phi)) - Y_\beta(\Omega^{'}) &= (n+1-\theta-\phi)((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 1)^\beta + 4((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 2^2)^\beta \\
&+ (\theta-2)(16)^\beta + (\phi-2)(16)^\beta \\
&- (n+1-\theta-\phi)((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 1^1)^\beta - ((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 4)^\beta \\
&- ((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 256)^\beta - 3(1024)^\beta - (\theta-3)(16)^\beta - (\phi-2)(16)^\beta, \\
&= (n+1-\theta-\phi)((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 1)^\beta + ((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 2^2)^\beta \\
&+ 2((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 2^2)^\beta + (\theta-2)(16)^\beta + ((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 2^2)^\beta \\
&- (n+1-\theta-\phi)((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 1^1)^\beta - ((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 4)^\beta \\
&- ((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 256)^\beta - (\theta-3)(16)^\beta - 3(1024)^\beta > 0.
\end{aligned} \tag{20}$$

The difference is positive with equality if and only if $\Omega^{'} \cong S_n(\theta,\phi)$. □

For $\theta \geq 3$ and $\phi \geq 3$, according to the previous Theorem, it follows that $S_n(\theta,\phi)$ is the unique graph with the greatest value of Y_β in $\mathbb{A}(\theta,\phi)$.

Lemma 3. *(i) If* $3 < \theta$*, then* $Y_\beta(S_n(\theta-1,\phi)) > Y_\beta(S_n(\theta,\phi))$*;*
(ii) If $3 < \phi$*, then* $Y_\beta(S_n(\theta,\phi-1)) > Y_\beta(S_n(\theta,\phi))$*.*

Proof. (i) We have

$$\begin{aligned}
Y_\beta(S_n(\theta,\phi)) &= (n+1-\theta-\phi)((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 1)^\beta + 4((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 2^2)^\beta \\
&+ (\theta-2)(16)^\beta + (\phi-2)(16)^\beta.
\end{aligned}$$

If we replace θ with $\theta-1$, we obtain

$$
\begin{aligned}
Y_\beta(S_n(\theta-1,\phi)) &= (n+1-\theta+1-\phi)((n+5-\theta+1-\phi)^{(n+5-\theta+1-\phi)}\times 1)^\beta \\
&+4((n+5-\theta+1-\phi)^{(n+5-\theta+1-\phi)}\times 2^2)^\beta+(\theta-1-2)(16)^\beta+(\phi-2)(16)^\beta, \\
&=(n+2-\theta-\phi)((n+6-\theta-\phi)^{(n+6-\theta-\phi)}\times 1)^\beta \\
&+4((n+6-\theta-\phi)^{(n+6-\theta-\phi)}\times 2^2)^\beta+(\theta-3)(16)^\beta+(\phi-2)(16)^\beta
\end{aligned}
$$

Consider the difference

$$
\begin{aligned}
Y_\beta(S_n(\theta-1,\phi))-Y_\beta(S_n(\theta,\phi)) &= (n+2-\theta-\phi)((n+6-\theta-\phi)^{(n+6-\theta-\phi)}\times 1)^\beta \\
&+4((n+6-\theta-\phi)^{(n+6-\theta-\phi)}\times 2^2)^\beta+(\theta-3)(16)^\beta+(\phi-2)(16)^\beta \\
-(n+1-\theta-\phi)((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 1)^\beta &- 4((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 2^2)^\beta \\
&-(\theta-2)(16)^\beta-(\phi-2)(16)^\beta>0.
\end{aligned}
$$

Hence, $Y_\beta(S_n(\theta-1,\phi))>Y_\beta(S_n(\theta,\phi))$.
As in (i), we obtain the following for (ii):

$$
\begin{aligned}
Y_\beta(S_n(\theta,q-1))-Y_\beta(S_n(\theta,\phi)) &= (n+2-\theta-\phi)((n+6-\theta-\phi)^{(n+6-\theta-\phi)}\times 1)^\beta \\
&+4((n+6-\theta-\phi)^{(n+6-\theta-\phi)}\times 2^2)^\beta+(\theta-2)(16)^\beta+(q-3)(16)^\beta \\
-(n+1-\theta-\phi)((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 1)^\beta &- 4((n+5-\theta-\phi)^{(n+5-\theta-\phi)}\times 2^2)^\beta \\
&-(\theta-2)(16)^\beta-(\phi-2)(16)^\beta>0.
\end{aligned}
$$

Hence, $Y_\beta(S_n(\theta,q-1))>Y_\beta(S_n(\theta,\phi))$. □

From Theorem 2 and Lemma 3, it follows that Theorem 3 is true.

Theorem 3. *For any $\theta \geq 3$ and $\phi \geq 3$, Y_β is $S_n(3,3)$ in $\mathbb{A}(\theta,\phi)$ and is the unique graph with greatest value of Y_β.*

5. Largest Value of Y_β in $\mathbb{B}(\theta,\phi)$

In this section, we investigate Y_β in $\mathbb{B}(\theta,\phi)$. Suppose that $T_n^r(\theta,\phi)$ is a graph with order n and size $n+1$, which is obtained by connecting C_θ and C_ϕ with a path of length r. The remaining $n+1-\theta-\phi-r$ edges are connected to the common vertex of the path and cycle C_θ (see Figure 5a). Similarly the graph $T_n^r(\phi,\theta)$ is shown in Figure 5b, while $T_n(\theta,\phi)$ is a graph of order n and size $n+1$ and is obtained by connecting C_θ and C_ϕ with a path $\zeta\varrho w$ of length 2; all the others are attached to ϱ. Figure 5c demonstrates this.

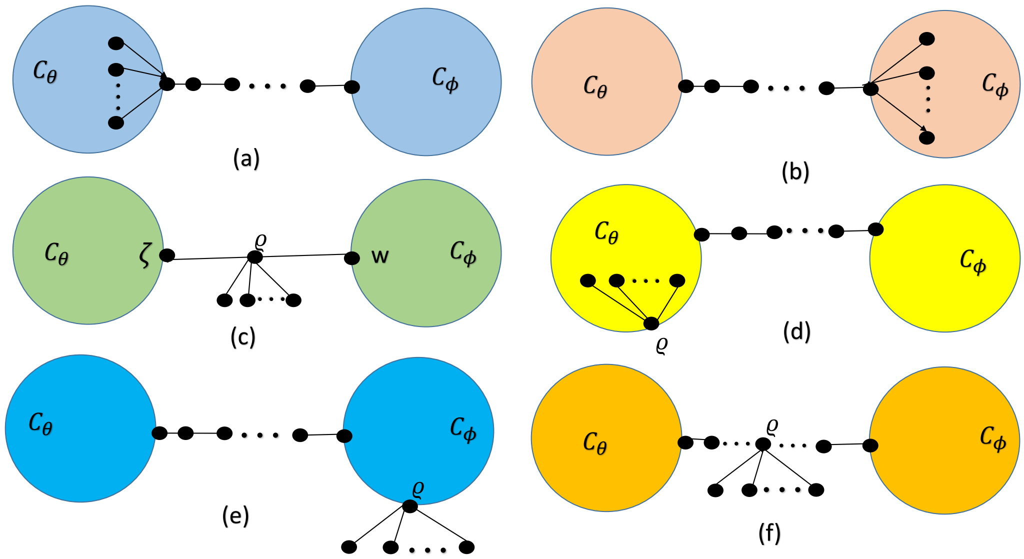

Figure 5. (**a**) The graph $T_n^r(\theta,\phi)$; (**b**) the graph $T_n^r(\phi,\theta)$; (**c**) the graph $T_n(\theta,\phi)$. (**d**) All the pendant edges are connected to vertex ϱ on the cycle C_θ. (**e**) All the pendant edges are connected to vertex ϱ on the cycle C_ϕ; (**f**) All the pendant edges are connected to vertex ϱ on the cycle on the path.

Theorem 4. *Let $G \in \mathbb{B}(\theta,\phi)$, and let cycles C_θ and C_ϕ be connected by a path with a length of r; then,*

$$\begin{aligned} &Y_\beta(T_n^r(\theta,\phi)) \geq Y_\beta(\Omega) \text{ with equality if and only, if } G \cong T_n^r(\theta,\phi) \\ &Y_\beta(T_n^r(\phi,\theta)) \geq Y_\beta(\Omega) \text{ with equality if and only, if } G \cong T_n^r(q,\theta). \end{aligned} \tag{21}$$

Proof. Let $P = \varphi_1\varphi_2\cdots\varphi_t\varphi_{r+1}$ be the shortest path between C_θ and C_ϕ in Ω. Let φ_1 be a vertex which is common in C_θ and P. Let φ_{r+1} be a common vertex of C_ϕ and P. By mapping A and B on Ω, we obtain Ω', as given in Figure 5, in which the edges not on the cycles with a degree of 1 are attached to ϱ. Using Lemmas 1 and 2, we obtain $Y_\beta(\Omega') > Y_\beta(\Omega)$, with equality if and only if all the edges not on the cycle are pendant and attached to the same vertex in Ω. Now, we have two cases:

Case 1. If ϱ lies on C_θ, vertices ϱ and φ_1 are not adjacent (see Figure 5d); then,

$$\begin{aligned} Y_\beta(\Omega') &= ((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta + \cdots + ((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta \\ &+ 2((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 2^2)^\beta + (3^3 \times d(\varphi_2)^{d(\varphi_2)})^\beta + (\theta-4)(2^2\times 2^2)^\beta \\ &+ (\phi-2)(2^2\times 2^2)^\beta + (r-2)(2^2\times 2^2)^\beta + 3(2^2\times 3^3)^\beta + 2(2^2\times 3^3)^\beta, \\ = (n+1-\theta-\phi-r)&((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta + 2((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 4)^\beta \\ &+ (27\times d(\varphi_2)^{d(\varphi_2)})^\beta + (16)^8(\theta-4) + (16)^\beta(\phi-2) + 8^\beta(r-2) + 5(108)^\beta. \end{aligned} \tag{22}$$

Similarly for $Y_\beta(T_n^r(\theta,\phi))$, it follows that

$$\begin{aligned} Y_\beta(T_n^r(\theta,\phi)) &= (n+1-\theta-\phi-r)((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 1)^\beta \\ &+ 2((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 2^2)^\beta + ((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times d(\varphi_2)^{d(\varphi_2)})^\beta \\ &+ (r-2)(2^2\times 2^2)^\beta + (\theta-2)(2^2\times 2^2)^\beta + (\phi-2)(2^2\times 2^2)^\beta + 3(2^2\times 3^3)^\beta, \\ &= (n+1-\theta-\phi-r)((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 1)^\beta \\ &+ 2((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 4)^\beta + ((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times d(\varphi_2)^{d(\varphi_2)})^\beta \\ &+ (16)^\beta(r-2) + (16)^\beta(\theta-2) + (16)^\beta(\phi-2) + 3(108)^\beta. \end{aligned} \tag{23}$$

Consider the following difference:

$$\begin{aligned}
&Y_\beta(T_n^r(\theta,\phi)) - Y_\beta(\Omega^{'}) = (n+1-\theta-\phi-r)((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 1)^\beta \\
&+ 2((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 4)^\beta + ((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times d(\varphi_2)^{d(\varphi_2)})^\beta \\
&+ (16)^\beta(r-2) + (16)^\beta(\theta-2) + (16)^\beta(\phi-2) + 3(108)^\beta \\
&- (n+1-\theta-\phi-r)((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta - 2((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 4)^\beta \\
&- (27 \times d(\varphi_2)^{d(\varphi_2)})^\beta - (16)^\beta(\theta-4) - (16)^\beta(\phi-2) - (16)^\beta(r-2) - 5(108)^\beta, \\
&= (n+1-\theta-\phi-r)((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 1)^\beta \\
&+ 2((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 4)^\beta + ((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times d(\varphi_2)^{d(\varphi_2)})^\beta \\
&- (n+1-\theta-\phi-r)((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta - 2((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 4)^\beta \\
&- (27 \times d(\varphi_2)^{d(\varphi_2)})^\beta + (16)^\beta(\theta-2) + 3(108)^\beta - (16)^\beta(\theta-4) - 5(108)^\beta \geq 0.
\end{aligned} \tag{24}$$

Hence, $Y_\beta(T_n^r(\theta,\phi)) - Y_\beta(\Omega^{'}) > 0$, with equality iff $\Omega^{'} \cong T_n^r(\theta,\phi)$. The inequality can be observed in Figure 6.

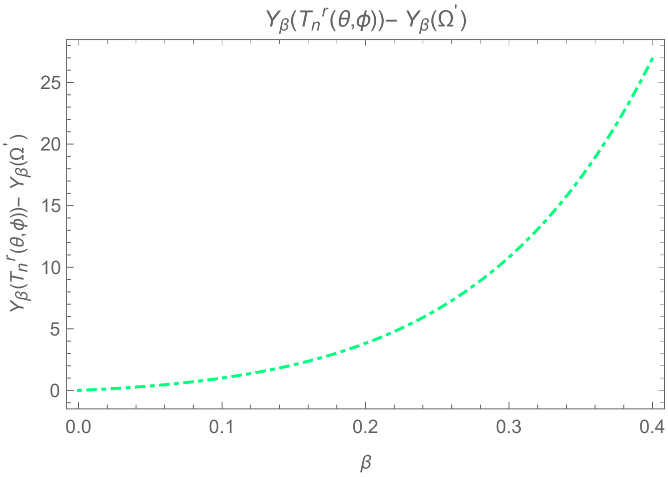

Figure 6. Plot for $Y_\beta(T_n^r(\theta,\phi)) - Y_\beta(\Omega^{'})$, where φ_1 and ϱ are not adjacent.

If ϱ and φ_1 are adjacent in $\Omega^{'}$, then

$$\begin{aligned}
&Y_\beta(\Omega^{'}) = ((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta + \cdots + ((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta \\
&+ ((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 2^2)^\beta + ((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 3^3)^\beta \\
&+ (3^3 \times d(\varphi_2)^{d(\varphi_2)})^\beta + 4(2^2 \times 3^3)^\beta + (\theta-3)(2^2 \times 2^2)^\beta + (\phi-2)(2^2 \times 2^2)^\beta + (r-2)(2^2 \times 2^2)^\beta, \\
&= (n+1-\theta-\phi-r)((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta + ((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 4)^\beta \\
&+ ((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 27)^\beta + (3^3 \times d(\varphi_2)^{d(\varphi_2)})^\beta + 4(108)^\beta + (\theta-3)(16)^\beta \\
&+ (\phi-2)(16)^\beta + (r-2)(16)^\beta.
\end{aligned} \tag{25}$$

Consider $Y_\beta(T_n^r(\theta,\phi)) - Y_\beta(\Omega^{'})$ in the following:

$$\begin{aligned}
&Y_\beta(T_n^r(\theta,\phi)) - Y_\beta(\Omega^{'}) = (n+1-\theta-\phi-r)((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 1)^\beta \\
&+ ((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 4)^\beta + ((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 4)^\beta \\
&+ ((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times d(\varphi_2)^{d(\varphi_2)})^\beta + (16)^\beta(r-2) + (16)^\beta(\theta-2) + (16)^\beta(\phi-2) + 3(108)^\beta \\
&- (n+1-\theta-\phi-r)((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta - ((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 4)^\beta \\
&- ((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 27)^\beta - (3^3 \times d(\varphi_2)^{d(\varphi_2)})^\beta - (\theta-3)(16)^\beta - (\phi-2)(16)^\beta - (r-2)(16)^\beta \\
&- 4(108)^\beta, \\
&= (n+1-\theta-\phi-r)((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 1)^\beta \\
&+ ((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 4)^\beta + ((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 4)^\beta \\
&+ ((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times d(\varphi_2)^{d(\varphi_2)})^\beta + (16)^\beta(\theta-2) + 3(108)^\beta \\
&- (n+1-\theta-\phi-r)((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta - ((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 4)^\beta \\
&- ((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 27)^\beta - (3^3 \times d(\varphi_2)^{d(\varphi_2)})^\beta - (\theta-3)(16)^\beta - 4(108)^\beta
\end{aligned} \quad (26)$$

Hence, $Y_\beta(T_n^r(\theta,\phi)) - Y_\beta(\Omega^{'}) > 0$, with equality iff $\Omega^{'} \cong (T_n^r(\theta,\phi))$. The difference can be observed in Figure 7.

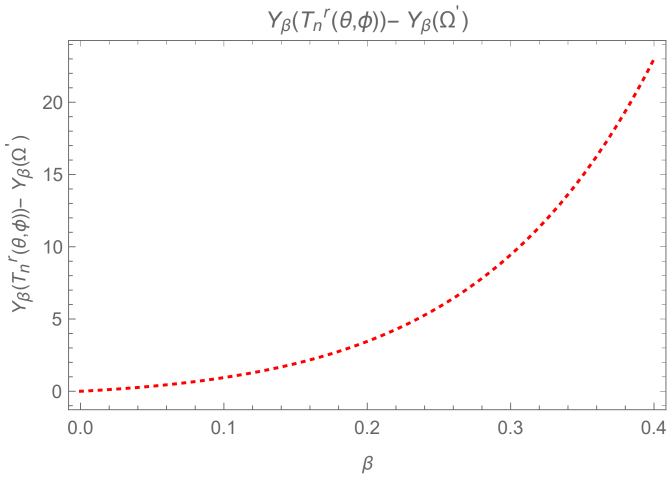

Figure 7. Plot for $Y_\beta(T_n^r(\theta,\phi)) - Y_\beta(\Omega^{'})$, where φ_1 and ϱ are adjacent

Case 2. If vertex ϱ is on C_ϕ, then the proof is similar to **Case 1**. Figure 5e demonstrates this.

Case 3. If ϱ lies on P (Figure 5f), and if $\Omega^{'} \ncong T_n(\theta,\phi)$, then $3 \le r$, and the following is true:

If $r > t > 2$, then $3 < r$, and we have

$$\begin{aligned}
&Y_\beta(T_n(\theta,\phi)) = ((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 1)^\beta + \cdots + ((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 1)^\beta \\
&+ 2((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 3^3)^\beta + 4(2^2 \times 3^3)^\beta + (\theta-2)(2^2 \times 2^2)^\beta + (\phi-2)(2^2 \times 2^2)^\beta, \\
&= (n-1-\theta-\phi)((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 1)^\beta \ + 2((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 3^3)^\beta \\
&+ 4(108)^\beta + (\theta-2)(16)^\beta + (\phi-2)(16)^\beta.
\end{aligned} \quad (27)$$

For $\Omega^{'}$, we obtain

$$
\begin{aligned}
Y_\beta(\Omega') &= ((n+1-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta + \cdots + ((n+1-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta \\
&+ 2((n+1-\theta-\phi-r)^{(n+1-\theta-\phi-r)} \times 2^2)^\beta + (\theta-2)(2^2 \times 2^2)^\beta + (\phi-2)(2^2 \times 2^2)^\beta \\
&+ (r-4)(2^2 \times 2^2)^\beta + 6(2^2 \times 3^3), \\
&= (n+1-\theta-\phi-r)((n+1-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta + 2((n+1-\theta-\phi-r)^{(n+1-\theta-\phi-r)} \times 4)^\beta \\
&+ (\theta-2)(16)^\beta + (\phi-2)(16)^\beta + (r-4)(16)^\beta + 6(108)^\beta.
\end{aligned}
\tag{28}
$$

Consider the following difference

$$
\begin{aligned}
Y_\beta(T_n(\theta,\phi)) - Y_\beta(\Omega') &= (n-1-\theta-\phi)((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 1)^\beta \\
&+ 2((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 3^3)^\beta + (\theta-2)(16)^\beta + (\phi-2)(16)^\beta + 4(108)^\beta \\
&- (n+1-\theta-\phi-r)((n+1-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta - 2((n+1-\theta-\phi-r)^{(n+1-\theta-\phi-r)} \times 4)^\beta \\
&- (\theta-2)(16)^\beta - (\phi-2)(16)^\beta - (r-4)(16)^\beta - 6(108)^\beta, \\
&= (n-1-\theta-\phi)((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 1)^\beta + 2((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 3^3)^\beta + 4(108)^\beta \\
&- (n+1-\theta-\phi-r)((n+1-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta - 2((n+1-\theta-\phi-r)^{(n+1-\theta-\phi-r)} \times 4)^\beta \\
&- (r-4)(16)^\beta - 6(108)^\beta.
\end{aligned}
\tag{29}
$$

Figure 8 illustrates the difference in (29).

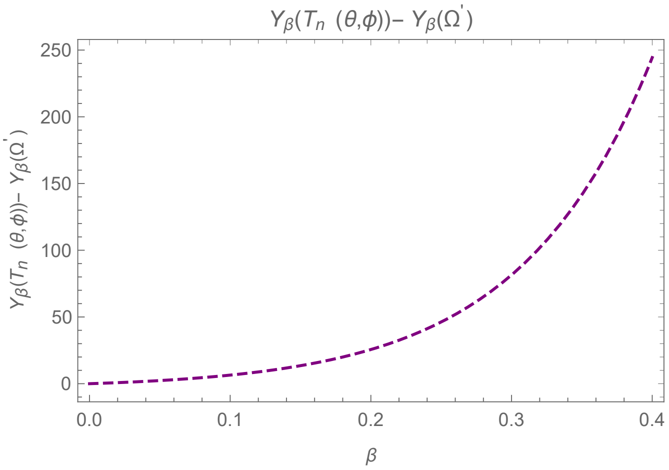

Figure 8. Plot for $Y_\beta(T_n(\theta,\phi)) - Y_\beta(\Omega')$, where $3 < r$.

If $t = r$ or $t = 2$, then for Ω', we obtain

$$
\begin{aligned}
Y_\beta(\Omega') &= 5(2^2 \times 3^3)^\beta + (\theta-2)(2^2 \times 2^2)^\beta + (\phi-2)(2^2 \times 2^2)^\beta \\
&+ (3^3 \times (n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)})^\beta + (2^2 \times (n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)})^\beta \\
&+ ((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta + \cdots + ((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta \\
&(r-3)(2^2 \times 2^2)^\beta \\
&= (\theta-2)(2^2 \times 2^2)^\beta + (\phi-2)(2^2 \times 2^2)^\beta + (3^3 \times (n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)})^\beta \\
&+ (2^2 \times (n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)})^\beta + (n+1-\theta-\phi-r)((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta \\
&+ (r-3)(2^2 \times 2^2)^\beta + 5(108)^\beta, \\
&> 0.
\end{aligned}
\tag{30}
$$

We consider the following difference in this case:

$$
\begin{aligned}
&Y_\beta(T_n(\theta,\phi)) - Y_\beta(\Omega^{'}) = (n-1-\theta-\phi)((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 1)^\beta \\
&+ 2((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 27)^\beta + (\theta-2)(16)^\beta + (\phi-2)(16)^\beta + 4(108)^\beta \\
&- (27 \times (n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)})^\beta - (4 \times (n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)})^\beta \\
&- (n+1-\theta-\phi-r)((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta - (\theta-2)(16)^\beta - (\phi-2)(16)^\beta \\
&- (r-3)(16)^\beta - 5(108)^\beta \\
&= (n-1-\theta-\phi)((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 1)^\beta + 2((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 27)^\beta \\
&- (27 \times (n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)})^\beta - (4 \times (n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)})^\beta \\
&- (n+1-\theta-\phi-r)((n+3-\theta-\phi-r)^{(n+3-\theta-\phi-r)} \times 1)^\beta - (r-3)(16)^\beta - (108)^\beta, \\
&> 0.
\end{aligned}
\tag{31}
$$

The above last difference can be observed in Figure 9. □

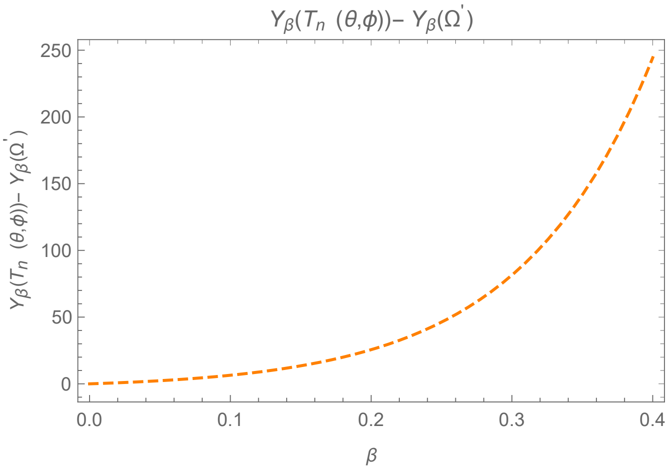

Figure 9. Plot for $Y_\beta(T_n(\theta,\phi)) - Y_\beta(\Omega^{'})$, where $t = r$ or $t = 2$.

Lemma 4. $Y_\beta(T_n(\theta,\phi)) \leq Y_\beta(T_n(3,3))$ *with equality if and only if* $\theta = 3 = \phi$.

Proof. As we have

$$
\begin{aligned}
Y_\beta(T_n(\theta,\phi)) &= (n-1-\theta-\phi)((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 1)^\beta + 2((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 3^3)^\beta \\
&+ 4(108)^\beta + (\theta-2)(16)^\beta + (\phi-2)(16)^\beta.
\end{aligned}
\tag{32}
$$

for $Y_\beta(T_n(3,3))$, we obtain;

$$
\begin{aligned}
Y_\beta(T_n(3,3)) &= 4(3^3 \times 2^2)^\beta + 2(2^2 \times 2^2)^\beta + 2((n-5)^{(n-5)} \times 3^3)^\beta + (n-7)((n-5)^{(n-5)} \times 1)^\beta, \\
&= 2(16)^\beta + 4(108)^\beta + 2((n-5)^{(n-5)} \times 1)^\beta + (n-7)((n-5)^{(n-5)} \times 1)^\beta.
\end{aligned}
\tag{33}
$$

We thus obtain the following difference:

$$
\begin{aligned}
&Y_\beta(T_n(3,3)) - Y_\beta(T_n(\theta,\phi)) = 4(108)^\beta + 2(16)^\beta + 2((n-5)^{(n-5)} \times 27)^\beta + (n-7)((n-5)^{(n-5)} \times 1)^\beta \\
&- (n-1-\theta-\phi)((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 1)^\beta + 2((n+1-\theta-\phi)^{(n+1-\theta-\phi)} \times 3^3)^\beta, \\
&+ 4(108)^\beta + (\theta-2)(16)^\beta + (\phi-2)(16)^\beta \\
&\geq 0.
\end{aligned}
\tag{34}
$$

Figure 10 illustrates the inequality. The equality holds if and only if $\theta = 3 = q$. Figure 11 demonstrates this. □

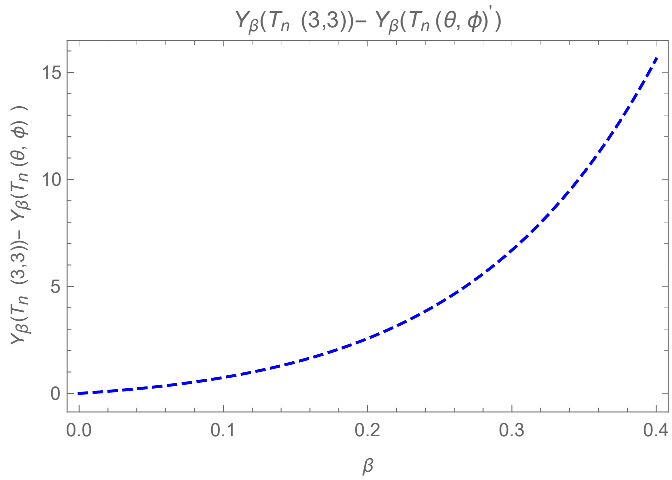

Figure 10. Plot for $Y_\beta(T_n(3,3)) - Y_\beta(T_n(\theta,\phi))$.

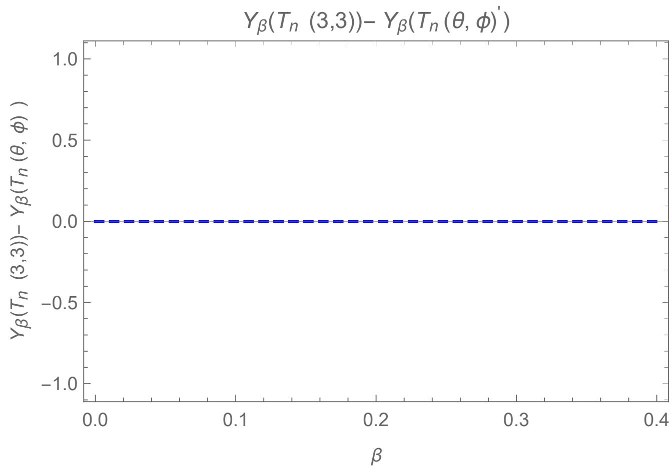

Figure 11. Plot for $Y_\beta(T_n(3,3)) - Y_\beta(T_n(\theta,\phi))$, where $\theta = 3 = q$.

Lemma 5. *For* $2 \leq r$, $Y_\beta(T_n^{r-1}(\theta,\phi)) > Y_\beta(T_n^r(\theta,\phi))$.

Proof. Consider the following:

If $r > 2$, then $Y_\beta(T_n^r(\theta,\phi))$ is applied in the following:

$$Y_\beta(T_n^r(\theta,\phi)) = (n+1-\theta-\phi-r)((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 1)^\beta$$
$$+ 2((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 4)^\beta + ((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times d(\varphi_2)^{d(\varphi_2)})^\beta$$
$$+ (16)^\beta(r-2) + (16)^\beta(\theta-2) + (16)^\beta(\phi-2) + 3(108)^\beta.$$

Here, $d(\varphi_2) = 2$; then, it follows that

$$\begin{aligned} Y_\beta(T_n^r(\theta,\phi)) &= (n+1-\theta-\phi-r)((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 1)^\beta \\ &+ 2((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 4)^\beta + ((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 4)^\beta \\ &+ (16)^\beta(r-2) + (16)^\beta(\theta-2) + (16)^\beta(\phi-2) + 3(108)^\beta. \end{aligned} \tag{35}$$

If we plug r by $r-1$ into (35), we obtain

$$Y_\beta(T_n^{r-1}(\theta,\phi)) = (n+2-\theta-\phi-r)((n+5-\theta-\phi-r)^{(n+5-\theta-\phi-r)} \times 1)^\beta$$
$$+ 2((n+5-\theta-\phi-r)^{(n+5-\theta-\phi-r)} \times 4)^\beta + ((n+5-\theta-\phi-r)^{(n+5-\theta-\phi-r)} \times 4)^\beta$$
$$+ (16)^\beta(r-3) + (16)^\beta(\theta-2) + (16)^\beta(\phi-2) + 3(108)^\beta.$$

Consider the following difference:

$$Y_\beta(T_n^{r-1}(\theta,\phi)) - Y_\beta(T_n^{r}(\theta,\phi)) = (n+2-\theta-\phi-r)((n+5-\theta-\phi-r)^{(n+5-\theta-\phi-r)} \times 1)^\beta$$
$$+ 2((n+5-\theta-\phi-r)^{(n+5-\theta-\phi-r)} \times 4)^\beta + ((n+5-\theta-\phi-r)^{(n+5-\theta-\phi-r)} \times 4)^\beta$$
$$+ (16)^\beta(r-3) + (16)^\beta(\theta-2) + (16)^\beta(\phi-2) + 3(108)^\beta$$
$$- (n+1-\theta-\phi-r)((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 1)^\beta$$
$$- 2((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 4)^\beta - ((n+4-\theta-\phi-r)^{(n+4-\theta-\phi-r)} \times 4)^\beta$$
$$- (16)^\beta(r-2) - (16)^\beta(\theta-2) - (16)^\beta(\phi-2) - 3(108)^\beta.$$

Figure 12 demonstrates this.

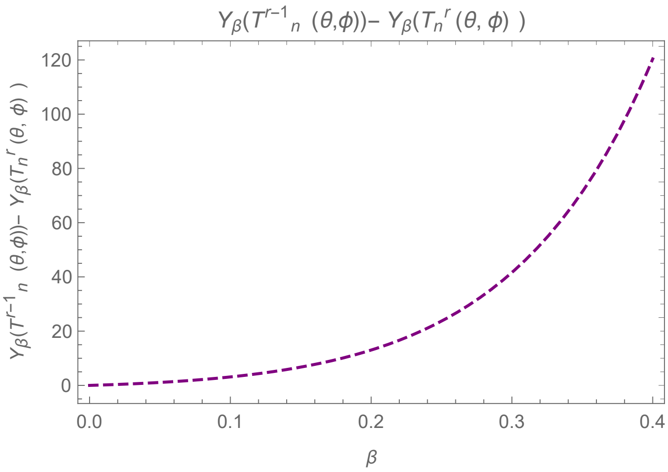

Figure 12. Plot for $Y_\beta(T_n^{r-1}(\theta,\phi)) - Y_\beta(T_n^{r-1}(\theta,\phi))$, where $r > 2$.

If $r = 2$, then it follows that

$$Y_\beta(T_n^{r}(\theta,\phi)) = (n-1-\theta-\phi)((n+2-\theta-\phi)^{(n+2-\theta-\phi)} \times 1)^\beta$$
$$+ 2((n+2-\theta-\phi)^{(n+2-\theta-\phi)} \times 4)^\beta + ((n+2-\theta-\phi)^{(n+2-\theta-\phi)} \times 4)^\beta$$
$$+ (16)^\beta(2-2) + (16)^\beta(\theta-2) + (16)^\beta(\phi-2) + 3(108)^\beta.$$

Similarly,

$$Y_\beta(T_n^{r-1}(\theta,\phi)) = (n-\theta-\phi)(1 \times (n-\theta-\phi+3)^{(n-\theta-\phi+3)})^\beta + 2(2^2 \times (n+3-\theta-\phi)^{(n+3-\theta-\phi)})^\beta$$
$$+ (3^3 \times (n+3-\theta-\phi)^{(n+3-\theta-\phi)})^\beta + (\theta-2)(2^2 \times 2^2)^\beta + (\phi-2)(2^2 \times 2^2)^\beta + 2(2^2 \times 3^3)^\beta,$$
$$= (n-\theta-\phi)(1 \times (n-\theta-\phi+3)^{(n-\theta-\phi+3)})^\beta + 2(4 \times (n+3-\theta-\phi)^{(n+3-\theta-\phi)})^\beta$$
$$+ (27 \times (n+3-\theta-\phi)^{(n+3-\theta-\phi)})^\beta + (\theta-2)(16)^\beta + (\phi-2)(16)^\beta + 2(108)^\beta.$$

Consider the difference $Y_\beta(T_n^{r-1}(\theta,\phi)) - Y_\beta(T_n^{r}(\theta,\phi))$, as in the following:

$$Y_\beta(T_n^{r-1}(\theta,\phi)) - Y_\beta(T_n^{r}(\theta,\phi)) = (n-\theta-\phi)(1\times(n-\theta-\phi+3)^{(n-\theta-\phi+3)})^\beta$$
$$+2((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 4)^\beta + ((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 27)^\beta$$
$$+(\theta-2)(16)^\beta + (\phi-2)(16)^\beta + 2(108)^\beta - (n-1-\theta-\phi)((n+2-\theta-\phi)^{(n+2-\theta-\phi)}\times 1)^\beta$$
$$-2((n+2-\theta-\phi)^{(n+2-\theta-\phi)}\times 4)^\beta - ((n+2-\theta-\phi)^{(n+2-\theta-\phi)}\times 4)^\beta$$
$$-(16)^\beta(2-2) - (16)^\beta(\theta-2) - (16)^\beta(\phi-2) - 3(108)^\beta,$$
$$> 0.$$

The above difference can be observed in Figure 13. □

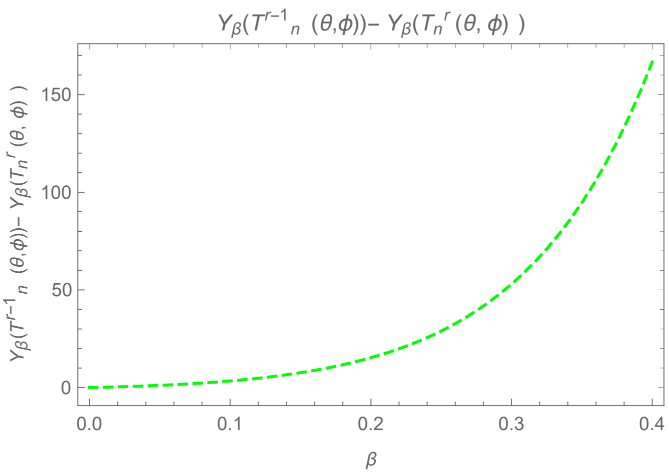

Figure 13. Plot for $Y_\beta(T_n^{r-1}(\theta,\phi)) - Y_\beta(T_n^{r-1}(\theta,\phi))$, where $r = 2$.

Lemma 6. *$Y_\beta(T_n^1(3,3)) \geq Y_\beta(T_n^1(\theta,\phi))$ with equality if and only if $\theta = 3 = q$.*

Proof. We have

$$Y_\beta(T_n^1(\theta,\phi)) = (n-\theta-\phi)(1\times(n-\theta-\phi+3)^{(n-\theta-\phi+3)})^\beta + 2((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 4)^\beta$$
$$+((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 27)^\beta + (\theta-2)(16)^\beta + (\phi-2)(16)^\beta + 2(108)^\beta. \tag{36}$$

If we plug $\theta = 3 = q$ into (36), we obtain

$$Y_\beta(T_n^1(3,3)) = (n-6)((n-3)^{(n-3)}\times 1)^\beta + 2((n-3)^{(n-3)}\times 4)^\beta + ((n-3)^{(n-3)}\times 27)^\beta$$
$$+(\theta-2)(16)^\beta + (\phi-2)(16)^\beta + 2(108)^\beta.$$

We consider the following difference:

$$Y_\beta(T_n^1(3,3)) - Y_\beta(T_n^1(\theta,\phi)) = (n-6)((n-3)^{(n-3)}\times 1)^\beta + 2((n-3)^{(n-3)}\times 4)^\beta + ((n-3)^{(n-3)}\times 27)^\beta$$
$$-(n-\theta-\phi)(1\times(n-\theta-\phi+3)^{(n-\theta-\phi+3)})^\beta - 2((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 4)^\beta$$
$$-((n+3-\theta-\phi)^{(n+3-\theta-\phi)}\times 27)^\beta \geq 0. \tag{37}$$

Figures 14 and 15 illustrate this, with equality if and only if $\theta = 3 = q$. □

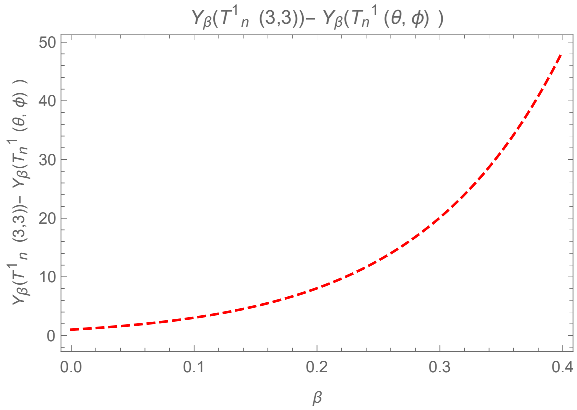

Figure 14. Plot for $Y_\beta(T_n^1(3,3)) - Y_\beta(T_n^1(\theta,\phi))$, where θ and ϕ, and both are not 3.

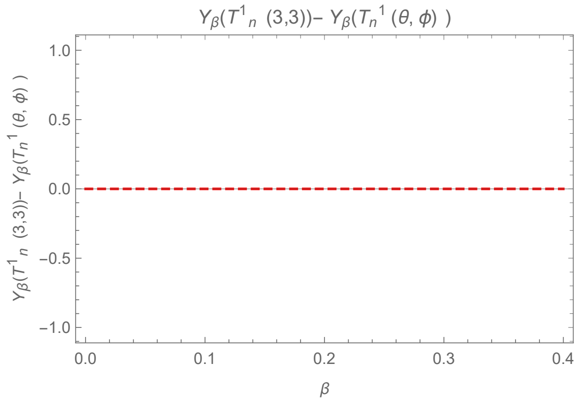

Figure 15. Plot for $Y_\beta(T_n^1(3,3)) - Y_\beta(T_n^1(\theta,\phi))$, where $\theta = 3 = \phi$.

It is not difficult to verify that $Y_\beta(T_n^1(3,3)) > Y_\beta(T_n(3,3))$. Further, it follows that Theorem 5 is true:

Theorem 5. *In $\mathbb{B}(\theta,\phi)$, $T_n^1(3,3)$ is the unique tree having greatest value of Y_β for all $\phi, \theta \geq 3$.*

6. Largest Value of Y_β in $\mathbb{C}(\theta,\phi,l)$

$\mathbb{C}(\theta,\phi,l)$ represents the bicyclic graph of cycles C_θ and C_ϕ. C_θ and C_ϕ are connected by a path of length l. Suppose $L_n^l(\theta,\phi)$ is obtained from Figure 3iii by connecting $n + l + 1 - \theta - \phi$ vertices to the vertex of degree 3, as shown in Figure 16i.

Theorem 6. *Let $G \in \mathbb{C}(\theta,\phi,l)$; then, $Y_\beta(\Omega) \leq Y_\beta(\Upsilon)$ with equality if and only if $G \cong \Upsilon$.*

Proof. If we apply mappings A and B to Ω, we obtain Ω', in which all the pendant edges not on the cycles are connected to v_0, i.e, Ω' is one (Figure 16). Using Lemmas 1 and 2, $Y_\beta(\Omega') \geq Y_\beta(\Omega)$ with equality if and only if all the edges not on the cycle are pendant edges attached to the same vertex in Ω.

Let $Q_1 = \varrho x_{t-1}x_{t-2}\cdots x_2x_1\zeta$ be the common path of C_θ and C_ϕ in Ω_1 (Figure 16). $Q_2 = \varrho y_r y_{r-1}\cdots y_2y_1\zeta$ and $Q_3 = \varrho z_t z_{t-1}\cdots z_2z_1\zeta$ are other paths from ζ to ϱ on the cycles C_θ and C_ϕ, respectively; $r = \theta - l - 1$, $t = \phi - l - 1$, $0 \leq r$, $0leqt$, $1 \leq l$, and $3 \leq l + r + t$.

If $uv \in E(\Omega_1)$, where $d_{\Omega_1}(u) = d_{\Omega_1}(v) = 2$, we obtain Ω_1' by deleting edge uv and add a new pendant edge $uu' = e'$ to ζ, then, $Y_\beta(\Omega_1') > Y_\beta(\Omega_1)$, as $(d_{\Omega_1}(u)^{d_{\Omega_1}(u)} \times d_{\Omega_1}(v)^{d_{\Omega_1}(v)})^\beta = (2^2 \times 2^2)^\beta = 8^\beta$ and $(d_{\Omega_1}(\zeta)^{d_{\Omega_1}(\zeta)} \times d_{\Omega_1}(\zeta')^{d_{\Omega_1}(\zeta')})^\beta > 8^\beta$. Hence, $Y_\beta(\Upsilon) \geq Y_\beta(\Omega_1)$ with equality if and only if $\Omega_1 \cong \Upsilon$.

Similarly, if there are two edges in the graph Ω_2, where the degree of their end vertices is 2, we obtain Ω_2' by removing these edges and add new edges to vertex x_i; then,

$\Upsilon_\beta(\Omega_2') \geq \Upsilon_\beta(\Omega_2)$; therefore, $\Upsilon_\beta(\Omega_1) \leq \Upsilon_\beta(\Omega_1)$ and $\Upsilon_\beta(\Omega_2) \leq \Upsilon_\beta(\Omega_4)$. Based on this, it is not difficult to show that $\Upsilon_\beta(\Upsilon) > \Upsilon_\beta(\Omega_3)$ and $\Upsilon_\beta(\Upsilon) > \Upsilon_\beta(\Omega_4)$. $\square$

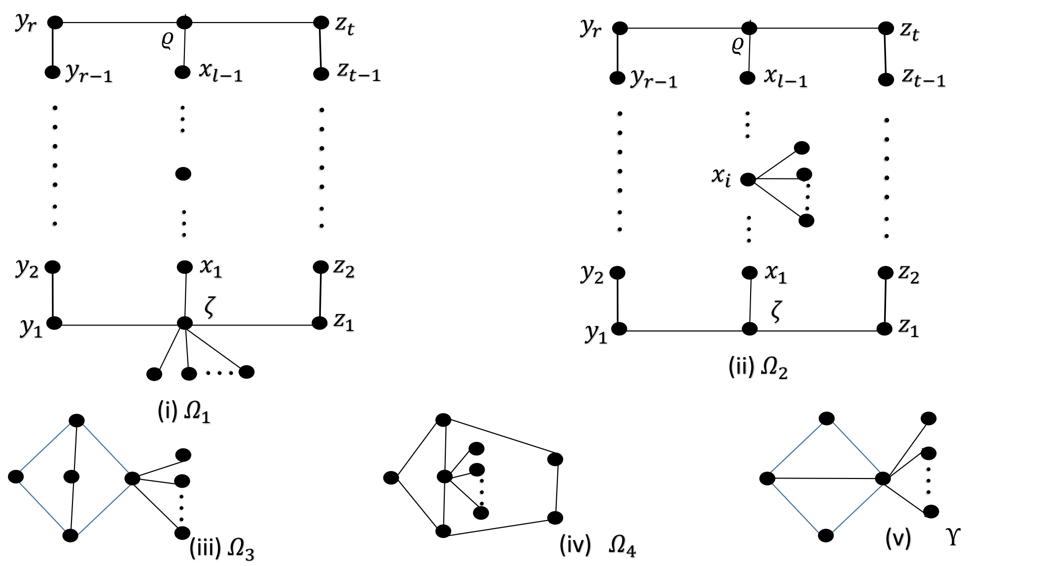

Figure 16. The graphs **(i)** Ω_1, **(ii)** Ω_2, **(iii)** Ω_3, **(iv)**Ω_4, **(v)** Υ.

Theorem 7. *In all bicyclic graphs,* Υ *is the unique graph with the greatest value of* Υ_β*.*

Proof. Using Theorems 3, 5 and 6, we compare the value of Υ_β for $S_n(3,3)$, $T_n^1(3,3)$, and Υ. Hence, it follows that

$$\Upsilon_\beta(T_n^1(3,3)) < \Upsilon_\beta(S_n(3,3)) < \Upsilon_\beta(\Upsilon).$$

Figures 17 and 18 illustrate this inequality, where

$$\Upsilon_\beta(T_n^1(3,3)) = (n-6)((n-3)^{(n-3)} \times 1)^\beta + 2((n-3)^{(n-3)} \times 4)^\beta + ((n-3)^{(n-3)} \times 27)^\beta + (\theta-2)(16)^\beta + (\phi-2)(16)^\beta + 2(108)^\beta.$$

$$\Upsilon_\beta(S_n(3,3)) = 2(16)^\beta + 4((n-\theta-\phi+5)^{(n-\theta-\phi+5)} \times 4)^\beta + (n-\theta-\phi+1)(1 \times (n-\theta-\phi+5)^{n-\theta-\phi+5})^\beta.$$

$$\Upsilon_\beta(\Upsilon) = 2(108)^\beta + 2((n+2+3-\theta-\phi)^{(n+2+3-\theta-\phi)} \times 4)^\beta + ((n-\theta-\phi+2+3)^{n-\theta-\phi+2+3} \times 27)^\beta + (n-\theta-\phi+2)((n-\theta-\phi+2+3)^{n-\theta-\phi+2+3} \times 1)^\beta.$$

$\square$

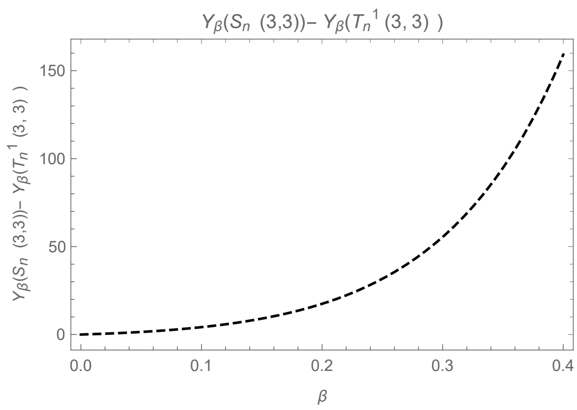

Figure 17. Sketch for $\Upsilon_\beta(S_n(3,3)) - \Upsilon_\beta(T_n^1(3,3))$.

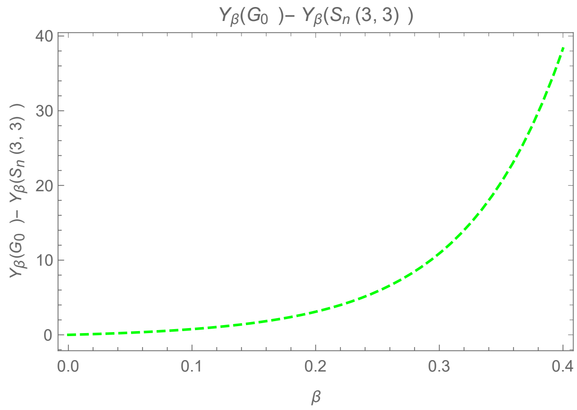

Figure 18. Plot for $Y_\beta(\Upsilon) - Y_\beta(S_n(3,3))$.

7. Mappings for Decreasing Y_β

In this section, we provide a mapping which is used for decreasing the value of Y_β; the mapping is given in the following.

Mapping C: Let Ω be a simple and connected graph other than P_1, and select vertex ζ from $V(\Omega)$. Suppose Ω_1 is the graph in which we mark ζ with φ_k of the path $\varphi_1\varphi_2\varphi_3\cdots\varphi_n$, where $1 < k < n$. Let Ω_2 be obtained from Ω_1 by deleting $\varphi_{k-1}\varphi_k$ and adding $\varphi_n\varphi_{k-1}$. Figure 19 demonstrates this.

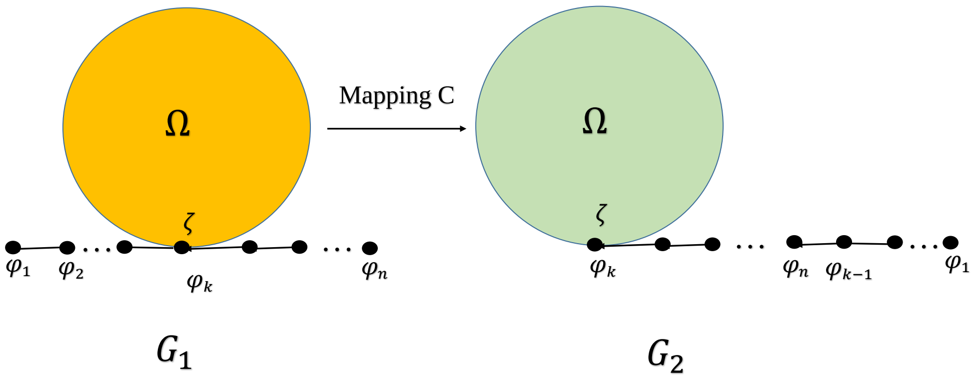

Figure 19. Diagram for mapping C.

Lemma 7. *Let Ω_1 and Ω_2 be as in Figure 19; then, $Y_\beta(\Omega_2) < Y_\beta(\Omega_1)$.*

Proof. Using the definition of Y_β, we obtain

$$Y_\beta(\Omega_1) = \sum_{\delta\in N_\Omega(\zeta)} ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times (d_{\Omega_1}(\varphi_{k-1}))^{d_{\Omega_1}(\varphi_{k-1})})^\beta$$
$$+ ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times (d_{\Omega_1}(\varphi_{k+1}))^{d_{\Omega_1}(\varphi_{k+1})})^\beta + ((d_{\Omega_1}(\varphi_{n-1}))^{d_{\Omega_1}(\varphi_{n-1})} \times (d_{\Omega_1}(\varphi_n))^{d_{\Omega_1}(\varphi_n)})^\beta.$$

For Ω_2, it follows that

$$Y_\beta(\Omega_2) = \sum_{\delta\in N_\Omega(\zeta)} ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta + ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times (d_{\Omega_2}(\varphi_{k+1}))^{d_{\Omega_2}(\varphi_{k+1})})^\beta$$
$$+ ((d_{\Omega_2}(\varphi_{n-1}))^{d_{\Omega_2}(\varphi_{n-1})} \times (d_{\Omega_2}(\varphi_n))^{d_{\Omega_2}(\varphi_n)})^\beta + ((d_{\Omega_2}(\varphi_n))^{d_{\Omega_2}(\varphi_n)} \times (d_{\Omega_2}(\varphi_{k-1}))^{d_{\Omega_2}(\varphi_{k-1})})^\beta.$$

Let us investigate the cases in the following.

Case 1: If $k = 2$ and $n = 3$, then

$$Y_\beta(\Omega_1) = \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+2)^{d_\Omega(\zeta)+2} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times d_{\Omega_1}(\varphi_1)^{d_{\Omega_1}(\varphi_1)})^\beta$$
$$+ ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times d_{\Omega_1}(\varphi_3)^{d_{\Omega_1}(\varphi_3)})^\beta,$$
$$= \sum_{\delta\in N_\Omega(\zeta)} ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 1)^\beta$$
$$+ ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 1)^\beta.$$

For Ω_2, we obtain

$$Y_\beta(\Omega_2) = \sum_{\delta\in N_\Omega(\zeta)} ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta + ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times d_{\Omega_2}(\varphi_3)^{d_{\Omega_2}(\varphi_3)})^\beta$$
$$+ ((d_{\Omega_2}(\varphi_3))^{d_{\Omega_2}(\varphi_3)} \times d_{\Omega_2}(\varphi_1)^{d_{\Omega_2}(\varphi_1)})^\beta,$$
$$= \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta + ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times 4)^\beta \times (4)^\beta.$$

Consider $Y_\beta(\Omega_1) - Y_\beta(\Omega_2)$ as given below:

$$Y_\beta(\Omega_1) - Y_\beta(\Omega_2 = \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+2)^{d_\Omega(\zeta)+2} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 1)^\beta$$
$$+ ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 1)^\beta - \sum_{\delta\in N_\Omega(\zeta)} ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta$$
$$- ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times 4)^\beta - (4)^\beta > 0.$$

So $Y_\beta(\Omega_1) > Y_\beta(\Omega_2)$.
Case 2: If $k = 2$ and $n > 3$, then we obtain

$$Y_\beta(\Omega_1) = \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+2)^{d_\Omega(\zeta)+2} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times d_{\Omega_1}(\varphi_1)^{d_{\Omega_1}(\varphi_1)})^\beta$$
$$+ ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times d_{\Omega_1}(\varphi_3)^{d_{\Omega_1}(\varphi_3)})^\beta + ((d_{\Omega_1}(\varphi_{n-1}))^{d_{\Omega_1}(\varphi_{n-1})} \times (d_{\Omega_1}(\varphi_n))^{d_{\Omega_1}(\varphi_n)})^\beta,$$
$$= \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+2)^{d_\Omega(\zeta)+2} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 1)^\beta$$
$$+ ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 4)^\beta + (4 \times 1)^\beta,$$
$$= \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+2)^{d_\Omega(\zeta)+2} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 4)^\beta$$
$$+ ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 1)^\beta + (4)^\beta.$$

For $Y_\beta(\Omega_2)$, we obtain

$$Y_\beta(\Omega_1) = \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta + ((1+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times d_{\Omega_1}(\varphi_3)^{d_{\Omega_1}(\varphi_3)})^\beta$$
$$+ ((d_{\Omega_2}(u_{n-1}))^{d_{\Omega_2}(\varphi_{n-1})} \times d_{\Omega_2}(\varphi_n)^{d_{\Omega_2}(\varphi_n)})^\beta + ((d_{\Omega_2}(\varphi_n))^{d_{\Omega_2}(\varphi_n)} \times (d_{\Omega_2}(\varphi_1))^{d_{\Omega_2}(\varphi_1)})^\beta,$$
$$= \sum_{\delta\in N_\Omega(\zeta)} ((1+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta + ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times 4)^\beta$$
$$+ (4 \times 4)^\beta + (4 \times 1)^\beta,$$
$$= \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta + ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times 4)^\beta + (16)^\beta + (4)^\beta.$$

The difference in $Y_\beta(\Omega_1)$ and $Y_\beta(\Omega_2)$ is considered in the following:

$$
\begin{aligned}
Y_\beta(\Omega_1) - Y_\beta(\Omega_2) = &\sum_{\delta\in N_\Omega(\zeta)} ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 4)^\beta \\
&+ ((2+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times 1)^\beta + (4)^\beta - \sum_{\delta\in N_\Omega(\zeta)} ((1+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times (d_\Omega(\delta))^{d_\Omega(\delta)})^\beta \\
&((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times 4)^\beta - (16)^\beta - (4)^\beta, \\
&> 0.
\end{aligned}
$$

Using term-by-term comparison, it follows that $Y_\beta(\Omega_1) > Y_\beta(\Omega_2)$.

Case 3: If $k > 2$ and $n = k+1$, then

$$
\begin{aligned}
Y_\beta(\Omega_1) = &\sum_{\delta\in N_\Omega(\zeta)} ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times d_{\Omega_1}(\varphi_{k+1})^{d_{\Omega_1}(\varphi_{k+1})})^\beta \\
&+ ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times d_{\Omega_1}(\varphi_{k-1})^{d_{\Omega_1}(\varphi_{k-1})})^\beta, \\
= &\sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+2)^{d_\Omega(\zeta)+2} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 4)^\beta \\
&+ ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 4)^\beta
\end{aligned}
$$

For Ω_2, it follows that

$$
\begin{aligned}
Y_\beta(\Omega_2) = &\sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times d_{\Omega_2}(\varphi_{k+1})^{d_{\Omega_2}(\varphi_{k+1})})^\beta \\
&+ ((d_{\Omega_2}(\varphi_{k+1}))^{d_{\Omega_2}(\varphi_{k+1})} \times d_{\Omega_2}(\varphi_{k-1})^{d_{\Omega_2}(\varphi_{k-1})})^\beta, \\
= &\sum_{\delta\in N_\Omega(\zeta)} ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times 4)^\beta \\
&+ (4\times 4)^\beta, \\
= &\sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times 4)^\beta + (16)^\beta.
\end{aligned}
$$

We consider the difference in $Y_\beta(\Omega_1)$ and $Y_\beta(\Omega_2)$ in the following:

$$
\begin{aligned}
Y_\beta(\Omega_1) - Y_\beta(\Omega_2) = &\sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+2)^{d_\Omega(\zeta)+2} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+2)^{d_\Omega(\zeta)+2} \times 4)^\beta \\
&+ ((d_\Omega(\zeta)+2)^{d_\Omega(\zeta)+2} \times 4)^\beta - \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta \\
&- ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times 4)^\beta - (16)^\beta > 0.
\end{aligned}
$$

Using term-by-term comparison, we obtain $Y_\beta(\Omega_1) - Y_\beta(\Omega_2) > 0$; hence, $Y_\beta(\Omega_1) > Y_\beta(\Omega_2)$.

Case 4: If $n > k+1$ and $k > 2$, for Ω_1 and Ω_2 we have

$$
\begin{aligned}
Y_\beta(\Omega_1) = & \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta)+2)^{d_\Omega(\zeta)+2} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times d_{\Omega_1}(\varphi_{k-1})^{d_{\Omega_1}(\varphi_{k-1})})^\beta \\
& + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times d_{\Omega_1}(\varphi_{k+1})^{d_{\Omega_1}(\varphi_{k+1})})^\beta + ((d_{\Omega_1}(\varphi_{n-1}))^{d_{\Omega_1}(\varphi_{n-1})} \times d_{\Omega_1}(\varphi_n)^{d_{\Omega_1}(\varphi_n)})^\beta, \\
& = \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta)+2)^{d_\Omega(\zeta)+2} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 4)^\beta \\
& + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 4)^\beta + (4 \times 1)^\beta, \\
& = \sum_{\delta \in N_\Omega(\zeta)} ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 4)^\beta \\
& + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 4)^\beta + (4)^\beta.
\end{aligned}
$$

For $Y_\beta(\Omega_2)$, we obtain

$$
\begin{aligned}
Y_\beta(\Omega_2) = & \sum_{\delta \in N_\Omega(\zeta)} ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times d_{\Omega_2}(\varphi_{k+1})^{d_{\Omega_2}(\varphi_{k+1})})^\beta \\
& + ((d_{\Omega_2}(\varphi_{n-1}))^{d_{\Omega_2}(\varphi_{n-1})} \times d_{\Omega_2}(\varphi_n)^{d_{\Omega_2}(\varphi_n)})^\beta + ((d_{\Omega_2}(\varphi_n))^{d_{\Omega_2}(\varphi_n)} \times d_{\Omega_2}(\varphi_{k-1})^{d_{\Omega_2}(\varphi_{k-1})})^\beta, \\
& = \sum_{\delta \in N_\Omega(\zeta)} ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times 4)^\beta \\
& + (4 \times 4)^\beta + (4 \times 4)^\beta, \\
= & \sum_{\delta \in N_\Omega(\zeta)} ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times 4)^\beta + (16)^\beta + (16)^\beta.
\end{aligned}
$$

Consider $Y_\beta(\Omega_1) - Y_\beta(\Omega_2)$, as given below:

$$
\begin{aligned}
Y_\beta(\Omega_1) = & \sum_{\delta \in N_\Omega(\zeta)} ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 4)^\beta \\
& + ((2+d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 4)^\beta + (4)^\beta \\
& - \sum_{\delta \in N_\Omega(\zeta)} ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta - ((1+d_\Omega(\zeta))^{1+d_\Omega(\zeta)} \times 4)^\beta - (16)^\beta - (16)^\beta, \\
& > 0.
\end{aligned}
$$

Hence, $Y_\beta(\Omega_1) - Y_\beta(\Omega_2) > 0$. For the last two terms, it follows that $((2 + d_\Omega(\zeta))^{2+d_\Omega(\zeta)} \times 4)^\beta + (4)^\beta - (16)^\beta - (16)^\beta > 0$. □

Remark 3. *By using mapping **C** repeatedly, any tree attached to a graph can be transformed into a path as in Figure 20. The value of Y_β decreases.*

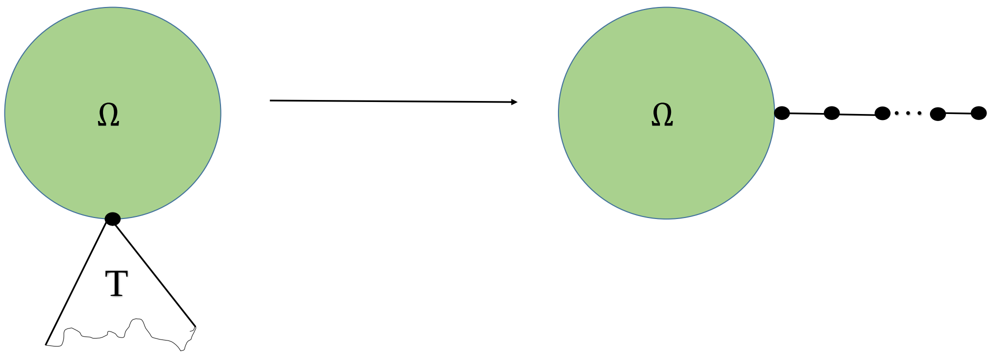

Figure 20. Diagram for remark using mapping C.

Mapping D: Consider graph Ω and $\zeta, \varrho \in E(\Omega)$, and let Ω_1 be obtained by marking ζ with u_0, ϱ with v_0 for the two paths $u_0u_1 \cdots u_{r-1}u_r$ and $v_0\varphi_1 \cdots \varphi_{t-1}\varphi_t$, respectively. Suppose Ω_2 is obtained from Ω_1 by deleting u_0u_1 and add $\varphi_t u_1$ (Figure 21).

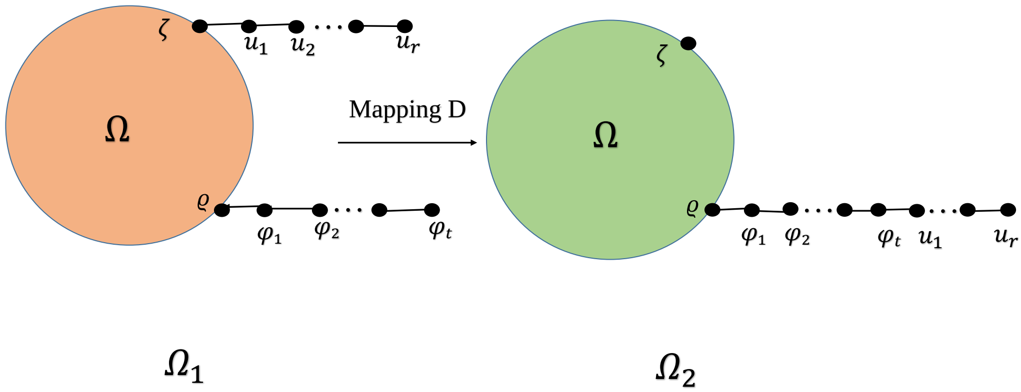

Figure 21. Diagram for mapping D.

Lemma 8. *Consider Ω_1 and Ω_2 from Figure 21 with $1 < d_\Omega(\varrho) \le d_\Omega(\zeta)$, $r \ge 1$, and $t \ge 0$; then, it follows that*

(1) If $t > 0$, then $Y_\beta(\Omega_1) > Y_\beta(\Omega_2)$;
(2) If $t = 0$ and $\sum_{y \in N_\Omega(\varrho)-\{\zeta\}} d_\Omega(y) < \sum_{\delta \in N_\Omega(\zeta)-\{\varrho\}} d_\Omega(\delta)$, then $Y_\beta(\Omega_1) > Y_\beta(\Omega_2)$.

Proof. (1) Here $d_\Omega(\zeta) > 1$ and $t > 0$, we consider;

Case 1. If $t = 1$ and $r = 1$, then;

$$Y_\beta(\Omega_1) = \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_{\Omega_1}(u_1)^{d_{\Omega_1}(u_1)})^\beta$$
$$+ ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times d_{\Omega_1}(\varphi_1)^{d_{\Omega_1}(\varphi_1)})^\beta,$$
$$= \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)\times 1} + 1)^\beta$$
$$+ ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times 1)^\beta.$$

For $Y_\beta(\Omega_2)$, it follows that

$$Y_\beta(\Omega_2) = \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times d_{\Omega_2}(\varphi_1)^{d_{\Omega_2}(\varphi_1)})^\beta$$
$$+ ((d_{\Omega_2}(\varphi_1))^{d_{\Omega_2}(\varphi_1)} \times d_{\Omega_2}(u_1)^{d_{\Omega_2}(u_1)})^\beta,$$
$$= \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times 4)^\beta + (4 \times 1)^\beta,$$
$$= \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times 4)^\beta + (4)^\beta.$$

Consider $Y_\beta(\Omega_1) - Y_\beta(\Omega_2)$, as given by

$$Y_\beta(\Omega_1) - Y_\beta(\Omega_2) = \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times 1)^\beta$$
$$+ ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times 1)^\beta - \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta - ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times 4)^\beta$$
$$- (4)^\beta$$
$$> 0, \qquad \because d_\Omega(\zeta) \ge d_\Omega(\varrho).$$

Hence, $Y_\beta(\Omega_1) > Y_\beta(\Omega_2)$.

Case 2. If $t > 1$ and $r = 1$, then

$$\begin{aligned}
Y_\beta(\Omega_1) = &\sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_{\Omega_1}(u_1)^{d_{\Omega_1}(u_1)})^\beta \\
&+ ((d_{\Omega_1}(\varphi_{t-1}))^{d_{\Omega_1}(\varphi_{t-1})} \times d_{\Omega_1}(\varphi_t)^{d_{\Omega_1}(\varphi_t)})^\beta, \\
&= \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times 1)^\beta + (4\times 1)^\beta \\
&= \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times 1)^\beta + (4)^\beta
\end{aligned}$$

For $Y_\beta(\Omega_2)$, we obtain

$$\begin{aligned}
Y_\beta(\Omega_2) = &\sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_{\Omega_2}(\varphi_{t-1}))^{d_{\Omega_2}(\varphi_{t-1})} \times d_{\Omega_2}(\varphi_t)^{d_{\Omega_2}(\varphi_t)})^\beta \\
&+ ((d_{\Omega_2}(\varphi_t))^{d_{\Omega_2}(\varphi_t)} \times d_{\Omega_2}(u_1)^{d_{\Omega_2}(u_1)})^\beta, \\
&= \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + (4\times 4)^\beta + (4\times 1)^\beta, \\
&= \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + (16)^\beta + (4)^\beta.
\end{aligned}$$

Consider $Y_\beta(\Omega_1) - Y_\beta(\Omega_2)$, as given by

$$\begin{aligned}
Y_\beta(\Omega_1) - Y_\beta(\Omega_2) = &\sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times 1)^\beta + (4)^\beta \\
&- \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta - (16)^\beta - (4)^\beta, \\
&> 0.
\end{aligned}$$

So $Y_\beta(\Omega_1) > Y_\beta(\Omega_2)$.

Case 3. If $t = 1$ and $1 < r$, then

$$\begin{aligned}
Y_\beta(\Omega_1) = &\sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_{\Omega_1}(u_1)^{d_{\Omega_1}(u_1)})^\beta \\
&+ ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times d_{\Omega_1}(\varphi_1)^{d_{\Omega_1}(\varphi_1)})^\beta, \\
&= \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times 4)^\beta \\
&+ ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times 1)^\beta.
\end{aligned}$$

For $Y_\beta(\Omega_2)$, it follows that

$$\begin{aligned}
Y_\beta(\Omega_2) = &\sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times d_{\Omega_2}(\varphi_1)^{d_{\Omega_2}(\varphi_1)})^\beta \\
&+ ((d_{\Omega_2}(\varphi_1))^{d_{\Omega_2}(\varphi_1)} \times d_{\Omega_2}(u_1)^{d_{\Omega_2}(u_1)})^\beta, \\
&= \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times 4)^\beta + (4\times 4)^\beta, \\
&= \sum_{\delta\in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times 4)^\beta + (16)^\beta.
\end{aligned}$$

Consider $Y_\beta(\Omega_1) - Y_\beta(\Omega_1)$, as given by

$$
\begin{aligned}
Y_\beta(\Omega_1) - Y_\beta(\Omega_2) = & \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times 4)^\beta \\
& + ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times 1)^\beta - \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta \\
& - ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times 4)^\beta - (16)^\beta, \\
& > 0.
\end{aligned}
$$

So $Y_\beta(\Omega_1) > Y_\beta(\Omega_2)$.

Case 4. If $t > 1$ and $1 < r$, then

$$
\begin{aligned}
Y_\beta(\Omega_1) = & \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_{\Omega_1}(u_1)^{d_{\Omega_1}(u_1)})^\beta \\
& + ((d_{\Omega_1}(\varphi_{t-1}))^{d_{\Omega_1}(\varphi_{t-1})} \times d_{\Omega_1}(\varphi_t)^{d_{\Omega_1}(\varphi_t)})^\beta, \\
= & \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times 4)^\beta + (4 \times 1)^\beta, \\
= & \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times 4)^\beta + (4)^\beta.
\end{aligned}
$$

For $Y_\beta(\Omega_2)$, we obtain

$$
\begin{aligned}
Y_\beta(\Omega_2) = & \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_{\Omega_2}(\varphi_{t-1}))^{d_{\Omega_2}(\varphi_{t-1})} \times d_{\Omega_2}(\varphi_t)^{d_{\Omega_2}(\varphi_t)})^\beta \\
& + ((d_{\Omega_2}(\varphi_t))^{d_{\Omega_2}(\varphi_t)} \times d_{\Omega_2}(u_1)^{d_{\Omega_2}(u_1)})^\beta, \\
= & \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + (4 \times 4)^\beta + (4 \times 1)^\beta, \\
= & \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + (16)^\beta + (4)^\beta.
\end{aligned}
$$

Consider the following:

$$
\begin{aligned}
Y_\beta(\Omega_1) - Y_\beta(\Omega_2) = & \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times 4)^\beta + (4)^\beta \\
& - \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + (16)^\beta + (4)^\beta, \\
& > 0.
\end{aligned}
$$

Hence, $Y_\beta(\Omega_1) - Y_\beta(\Omega_2) > 0$ so $Y_\beta(\Omega_1) > Y_\beta(\Omega_2)$.

(2). If $t = 0$ and $\sum_{y \in N_\Omega(\varrho)-\{\zeta\}} d_\Omega(y) < \sum_{\delta \in N_\Omega(\zeta)-\{\varrho\}} d_\Omega(\delta)$, if ζ and ϱ are adjacent vertices, then

$$
\begin{aligned}
Y_\beta(\Omega_1) = & \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_{\Omega_1}(u_1)^{d_{\Omega_1}(u_1)})^\beta \\
& + \sum_{y \in N_\Omega(\varrho)} (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(y)^{d_\Omega(y)})^\beta.
\end{aligned}
$$

For $Y_\beta(\Omega_2)$, we obtain

$$Y_\beta(\Omega_2) = \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times d_{\Omega_2}(u_1)^{d_{\Omega_2}(u_1)})^\beta + \sum_{y \in N_\Omega(\varrho)} ((d_\Omega(\varrho)+1)^{(d_\Omega(\varrho)+1)} \times d_\Omega(y)^{d_\Omega(y)})^\beta.$$

Consider$Y_\beta(\Omega_1) - Y_\beta(\Omega_2)$, as given by

$$\begin{aligned} Y_\beta(\Omega_1) - Y_\beta(\Omega_2) = & \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_{\Omega_1}(u_1)^{d_{\Omega_1}(u_1)})^\beta \\ & + \sum_{y \in N_\Omega(\varrho)} (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(y)^{d_\Omega(y)})^\beta - \sum_{\delta \in N_\Omega(\zeta)} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta. \\ & - ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times d_{\Omega_2}(u_1)^{d_{\Omega_2}(u_1)})^\beta - \sum_{y \in N_\Omega(\varrho)} ((d_\Omega(\varrho)+1)^{(d_\Omega(\varrho)1)} \times d_\Omega(y)^{d_\Omega(y)})^\beta, \\ & > 0. \qquad \because d_{\Omega_1}(u_1) = d_{\Omega_2}(u_1) \end{aligned}$$

Hence, $Y_\beta(\Omega_1) > Y_\beta(\Omega_2)$.
If ζ and ϱ are adjacent, then we have

$$Y_\beta(\Omega_1) = \sum_{\delta \in N_\Omega(\zeta)-\{\varrho\}} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_{\Omega_1}(u_1)^{d_{\Omega_1}(u_1)})^\beta + \sum_{y \in N_\Omega(\varrho)-\{\zeta\}} (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(y)^{d_\Omega(y)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\varrho)^{d_\Omega(\varrho)})^\beta.$$

For $Y_\beta(\Omega_2)$, we obtain

$$Y_\beta(\Omega_2) = \sum_{\delta \in N_\Omega(\zeta)-\{\varrho\}} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times d_{\Omega_2}(u_1)^{d_{\Omega_2}(u_1)})^\beta + \sum_{y \in N_\Omega(\varrho)-\{\zeta\}} ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times d_\Omega(y)^{d_\Omega(y)})^\beta + (d_\Omega(\zeta)^{d_\Omega(\zeta)} \times (d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1})^\beta.$$

Consider $Y_\beta(\Omega_1) - Y_\beta(\Omega_2)$, as given by

$$\begin{aligned} & Y_\beta(\Omega_1) - Y_\beta(\Omega_2) = \\ & \sum_{\delta \in N_\Omega(\zeta)-\{\varrho\}} ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_{\Omega_1}(u_1)^{d_{\Omega_1}(u_1)})^\beta \\ & + \sum_{y \in N_\Omega(\varrho)-\{\zeta\}} (d_\Omega(\varrho)^{d_\Omega(\varrho)} \times d_\Omega(y)^{d_\Omega(y)})^\beta + ((d_\Omega(\zeta)+1)^{d_\Omega(\zeta)+1} \times d_\Omega(\varrho)^{d_\Omega(\varrho)})^\beta \\ & - \sum_{\delta \in N_\Omega(\zeta)-\{\varrho\}} ((d_\Omega(\zeta))^{d_\Omega(\zeta)} \times d_\Omega(\delta)^{d_\Omega(\delta)})^\beta - ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times d_{\Omega_2}(u_1)^{d_{\Omega_2}(u_1)})^\beta \\ & - \sum_{y \in N_\Omega(\varrho)-\{\zeta\}} ((d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1} \times d_\Omega(y)^{d_\Omega(y)})^\beta - (d_\Omega(\zeta)^{d_\Omega(\zeta)} \times (d_\Omega(\varrho)+1)^{d_\Omega(\varrho)+1})^\beta, \\ & > 0. \qquad \because d_{\Omega_1}(u_1) = d_{\Omega_2}(u_1) \end{aligned}$$

Hence, $Y_\beta(\Omega_1) - Y_\beta(\Omega_2) > 0$, so $Y_\beta(\Omega_1) > Y_\beta(\Omega_2)$. □

Remark 4. *Using mapping **C** and mapping **D** continuously, every tree can be transformed into a path, any unicyclic graph can be transformed into a unicyclic graph such that the path is connected to a cycle, and every bicyclic graph can be changed into a bicyclic graph cycle in which the path is connected to one of the graphs provided in Figure 22 (Lemma 8(i)). Generally, any bicyclic graph can be changed into a bicyclic graph where the path is connected to a vertex of degree 2 (Lemma 8(ii)), and the value of Y_β decreases.*

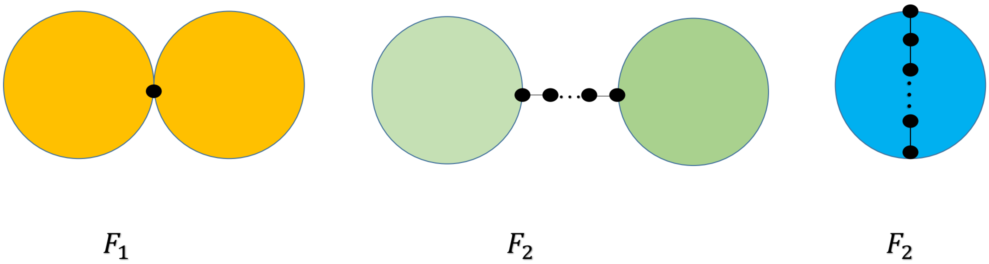

Figure 22. Diagram for F_1, F_2, and F_3.

Lemma 9. *Consider the graph Ω_1 given in Figure 23 if the path $x_1, x_2, \cdots, x_k$ with $1 < k$ is connected to vertex x_1. Suppose Ω_2 is obtained from Ω_1 by removing ϱx_1 and add ϱx_k; then, $Y_\beta(\Omega_1) > Y_\beta(\Omega_2)$.*

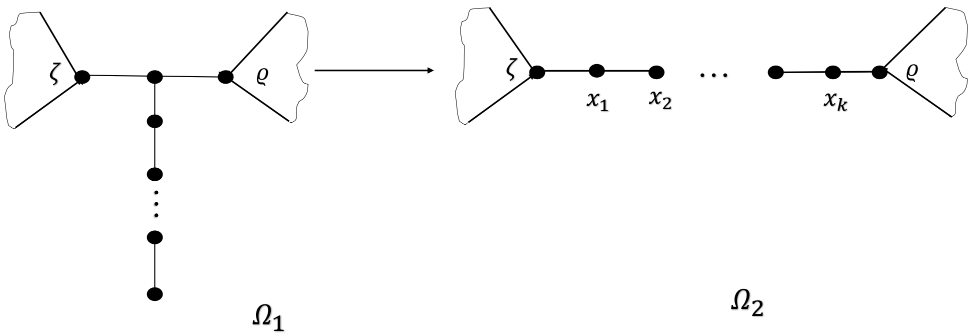

Figure 23. Diagram for Lemma 9.

Proof. If $k = 2$, then x_k and x_1 are different; it follows that

$$\begin{aligned} Y_\beta(\Omega_1) &= (d(\zeta)^{d(\zeta)} \times d_{\Omega_1}(x_1)^{d_{\Omega_1}(x_1)})^\beta + (d_{\Omega_1}(x_1)^{d_{\Omega_1}(x_1)} \times d_{\Omega_1}(x_2)^{d_{\Omega_1}(x_2)})^\beta \\ &+ (d_{\Omega 1}(x_1)^{d_{\Omega_1}(x_1)} \times d_{\Omega_1}(\varrho)^{d_{\Omega_1}(\varrho)})^\beta, \\ &= (d(\zeta)^{d(\zeta)} \times 27)^\beta + (27)^\beta + (27 \times d_{\Omega_1}(\varrho)^{d_{\Omega_1}(\varrho)})^\beta. \end{aligned}$$

For $Y_\beta(\Omega_2)$, we obtain

$$\begin{aligned} Y_\beta(\Omega_2) &= (d(\zeta)^{d(\zeta)} \times d_{\Omega_2}(x_1)^{d_{\Omega_2}(x_1)})^\beta + (d_{\Omega_2}(x_1)^{d_{\Omega_2}(x_1)} \times d_{\Omega_2}(x_2)^{d_{\Omega_2}(x_2)})^\beta \\ &+ (d_{\Omega 2}(x_2)^{d_{\Omega_2}(x_2)} \times d_{\Omega_2}(\varrho)^{d_{\Omega_2}(\varrho)})^\beta, \\ &= (d(\zeta)^{d(\zeta)} \times 4)^\beta + (16)^\beta + (4 \times d_{\Omega_2}(\varrho)^{d_{\Omega_2}(\varrho)})^\beta. \end{aligned}$$

Consider $Y_\beta(\Omega_1) - Y_\beta(\Omega_2)$, as given by

$$\begin{aligned} Y_\beta(\Omega_1) - Y_\beta(\Omega_2) &= (d(\zeta)^{d(\zeta)} \times 27)^\beta + (27)^\beta + (27 \times d_{\Omega_1}(\varrho)^{d_{\Omega_1}(\varrho)})^\beta \\ &- (d(\zeta)^{d(\zeta)} \times 4)^\beta - (16)^\beta - (4 \times d_{\Omega_2}(\varrho)^{d_{\Omega_2}(\varrho)})^\beta, \\ &> 0. \end{aligned}$$

For $k > 2$, it follows that;

$$\begin{aligned} Y_\beta(\Omega_1) &= (d(\zeta)^{d(\zeta)} \times d_{\Omega_1}(x_1)^{\Omega_1(x_1)})^\beta + (d_{\Omega_1}(x_1)^{d_{\Omega_1}(x_1)} \times d_{\Omega_1}(x_2)^{d}_{\Omega_1}(x_2))^\beta \\ &+ (d_{\Omega 1}(x_1)^{d_{\Omega_1}(x_1)} \times d_{\Omega_1}(\varrho)^{d_{\Omega_1}(\varrho)})^\beta + (d_{\Omega_1}(x_{k-1})^{d_{\Omega_1}(x_{k-1})} \times d_{\Omega_1}(x_k)^{d_{\Omega_1}(x_k)})^\beta, \\ &= (d(\zeta)^{d(\zeta)} \times 27)^\beta + (108)^\beta + (27 \times d_{\Omega_1}(\varrho)^{d_{\Omega_1}(\varrho)})^\beta + (4)^\beta. \end{aligned}$$

For $Y_\beta(\Omega_2)$, we obtain

$$\begin{aligned} Y_\beta(\Omega_2) &= (d(\zeta)^{d(\zeta)} \times d_{\Omega_2}(x_1)^{\Omega_2(x_1)})^\beta + (d_{\Omega_2}(x_1)^{d_{\Omega_2}(x_1)} \times d_{\Omega_2}(x_2)^{d}_{\Omega_2}(x_2))^\beta \\ &+ (d_{\Omega_2}(x_k)^{d_{\Omega_2}(x_k)} \times d_{\Omega_2}(\varrho)^{d_{\Omega_2}(\varrho)})^\beta + (d_{\Omega 2}(x_{k-1})^{d_{\Omega_2}(x_{k-1})} \times d_{G2}(x_k)^{d_{\Omega_2}(x_k)})^\beta, \\ &= (d(\zeta)^{d(\zeta)} \times 4)^\beta + (16)^\beta + (4 \times d_{\Omega_2}(\varrho)^{d_{\Omega_2}(\varrho)})^\beta + (16)^\beta. \end{aligned}$$

Consider $Y_\beta(\Omega_1) - Y_\beta(\Omega_2)$, as given by

$$\begin{aligned} Y_\beta(\Omega_1) - Y_\beta(\Omega_2) &= (d(\zeta)^{d(\zeta)} \times 27)^\beta + (27 \times d_{\Omega_1}(\varrho)^{d_{\Omega_1}(\varrho)})^\beta - (d(\zeta)^{d(\zeta)} \times 4)^\beta - (4 \times d_{\Omega_2}(\varrho)^{d_{\Omega_2}(\varrho)})^\beta \\ &+ (108)^\beta + (4)^\beta - 2(16)^\beta, \\ &> 0. \end{aligned}$$

Hence, $Y_\beta(\Omega_1) - Y_\beta(\Omega_2) > 0$ in both cases if $k = 2$ and $k > 2$. This confirms that $Y_\beta(\Omega_1) > Y_\beta(\Omega_2)$. □

8. Smallest Value of Y_β in Unicyclic Graphs, Trees, and Bicyclic Graphs

Here, we found unicyclic graphs, trees, and bicyclic graphs with the smallest value of Y_β; using Lemma 7, we obtain the following:

Theorem 8. *Consider a T on n vertices; if T is not P_n, then $Y_\beta(T_n) > Y_\beta(P_n)$.*

Let U_n^k be a unicyclic graph which is obtained by connecting a path to C_k of length $n-k$. According to Lemmas 7 and 8, it follows that

Theorem 9. *Let Ω be a unicyclic graph of girth k and order n; if Ω is not U_n^k, then $Y_\beta(F_n)^k < Y_\beta(\Omega)$.*

Using Lemma 9, it follows that

Theorem 10. *C_n is the unique graph on n vertices with smallest value for Y_β.*

Let F_1, F_2, and F_3 be the graphs in Figure 22; from Lemmas 7 and 9, the bicyclic graph with a minimum value of Y_β is one from F_1, F_2, and F_3, where

$$\begin{aligned} Y_\beta(F_1) &= 4(4 \times 256)^\beta + (n-3)(4 \times 4)^\beta, \\ &= (n-3)(16)^\beta + 4(1024)^\beta. \end{aligned}$$

.

$$\begin{aligned} Y_\beta(F_2) = Y_\beta(F_3) &= 4(4 \times 27)^\beta + (27 \times 27)^\beta + (n-4)(4 \times 4)^\beta, \\ &= (n-4)(16)^\beta + 4(108)^\beta + (729)^\beta, \quad \text{if } \zeta \text{ and } \varrho \text{ are adjacent.} \end{aligned}$$

$$\begin{aligned} Y_\beta(F_2) = Y_\beta(F_3) &= 6(4 \times 27)^\beta + (n-5)(4 \times 4)^\beta, \\ &= (n-5)(16)^\beta + 6(108)^\beta, \quad \text{if } \zeta \text{ and } \varrho \text{ are not adjacent.} \end{aligned}$$

The difference is considered below:

$$Y_\beta(F_1) - Y_\beta(F_2) = (16)^\beta + 4(1024)^\beta - 4(108)^\beta - (729)^\beta > 0. \quad (38)$$

The inequality in (38) can be observed in Figure 24.

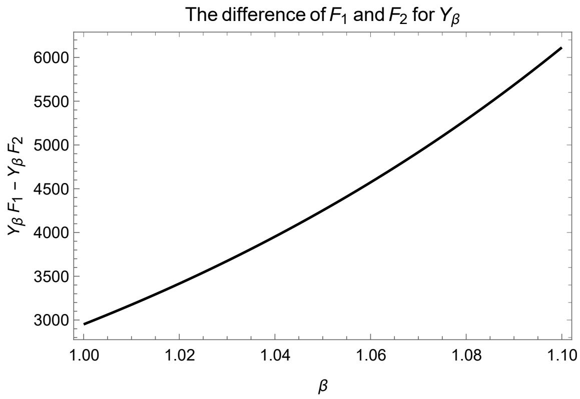

Figure 24. Plot for $Y_\beta(F_1) - Y_\beta(F_2)$.

$$Y_\beta(F_1) - Y_\beta(F_3) = 2(16)^\beta + 4(1024)^\beta - 6(108)^\beta > 0. \tag{39}$$

$$Y_\beta(F_2) - Y_\beta(F_3) = (16)^\beta + 4(108)^\beta + (729)^\beta - 6(108)^\beta > 0. \tag{40}$$

The inequalities in (39) and (40) can be observed in Figures 25 and 26.

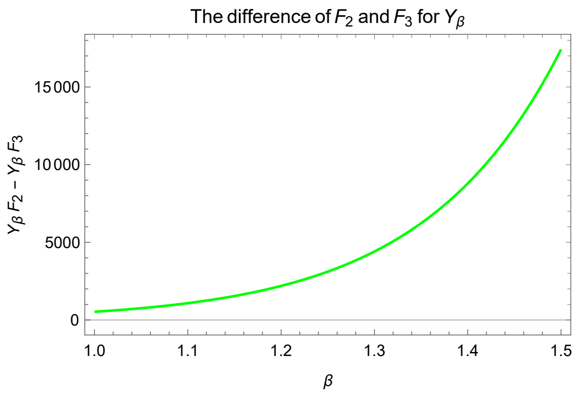

Figure 25. Plot for $Y_\beta(F_2) - Y_\beta(F_3)$.

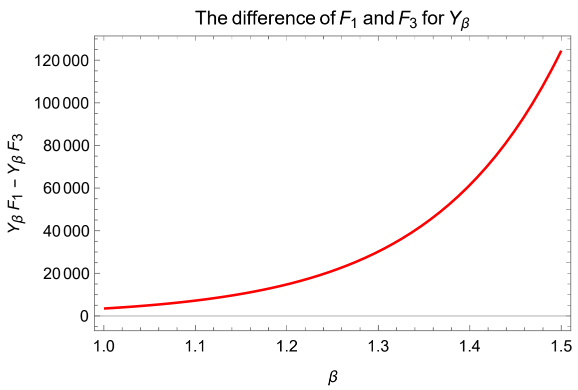

Figure 26. Plot for $Y_\beta(F_2) - Y_\beta(F_3)$.

From the above investigation, the following is true:

Theorem 11. *Consider a family Ω containing bicyclic graphs on n vertices; then, the greatest value of Y_β is for F_1.*

Theorem 12. *Consider a family Ω containing bicyclic graphs on n vertices; then, F_3 is the graph with the smallest value for Y_β. Here vertices with a degree of 3 are not adjacent, i.e., $Y_\beta(F_3) < Y_\beta(F_2) < Y_\beta(F_1)$.*

9. Conclusions

Topological indices are numerical values associated with a graph and remain unchanged for isomorphic graphs. The field of indices is of considerable interest to researchers. In this article, we considered the second form of GPSCI and investigated its use on trees, unicyclic graphs, and bicyclic graphs. In the desired families, graphs with optimal values were found by means of mapping. Two mappings were considered for maximum values, and two mappings were considered for minimum values. The considered bi-cyclic graphs are of three types: cycles with no vertices in common, cycles with one vertex in common, and cycles with a path in common.

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References

1. Wiener, H. Structural determination of paraffin boiling points. *J. Am. Chem. Soc.* **1947**, *69*, 17–20. [CrossRef]
2. Deza, M.; Fowler, P.W.; Rassat, A. KM Rogers Fullerenes as tilings of surfaces. Report LIENS 99-4, Ecole Normale Superieure, Paris 1999. *J. Chem. Inf. Comput. Sci.* **2000**, *403*, 550558.
3. Rui, C.C.; Ali, G.; Rahmat, G.; Khan, M.Y.; Semanicova-Fenovcikova, A.; Liu, J.-B. Investigation of General Power Sum-Connectivity Index for Some Classes of Extremal Graphs. *Complexity* **2021**, *2021*, 6623277.
4. Che, Z.; Chen, Z. Lower and upper bounds of the forgotten topological index. *MATCH Commun. Math. Comput. Chem.* **2016**, *76*, 635–648.
5. Camarda, K.V.; Maranas, C.D. Optimization in polymer design using connectivity indices. *Ind. Eng. Chem. Res.* **1999**, *38*, 1884–1892. [CrossRef]
6. Deng, H. A unified approach to the extremal Zagreb indices for trees, unicyclic graphs and bicyclic graphs. *MATCH Commun. Math. Comput. Chem.* **2007**, *57*, 597–616.
7. Hamid, K.; Iqbal, M.W.; Ashraf, M.U.; Alghamdi, A.M.; Bahaddad, A.A.; Almarhabi, K.A. Optimized Evaluation of Mobile Base Station by Modern Topological Invariants. *Comput. Mater. Contin.* **2023**, *74*, 363–378. [CrossRef]
8. Sigarreta, J.M. Mathematical properties of variable topological indices. *Symmetry* **2020**, *13*, 43. [CrossRef]
9. Afridi, S.; Khan, M.Y.; Ali, G.; Ali, M.; Nurhidayat, I.; Arefin, M.A. Sharp Bounds of Kulli–Basava Indices in Generalized Form for k-Generalized Quasi Trees. *J. Math.* **2023**, *2023*, 7567411. [CrossRef]
10. Chen, H.; Li, W.; Wang, J. Extremal values on the Sombor index of trees. *MATCH Commun. Math. Comput. Chem.* **2022**, *87*, 23–49. [CrossRef]
11. Akgunes, N.; Aydin, B. Introducing new exponential Zagreb indices for graphs. *J. Math.* **2021**, *2021*, 6675321. [CrossRef]
12. Martínez-Pérez, A.; Rodríguez, J.M. New lower bounds for the geometric-arithmetic index. *arXiv* **2017**, arXiv:1703.05524.
13. Saleh, A.; Bazhear, S.; Muthana, N. On the Uphill Zagreb Indices of Graphs. *Int. J. Anal. Appl.* **2022**, *20*, 6. [CrossRef]
14. Rizwan, M.; Bhatti, A.A.; Javaid, M.; Bonyah, E. [Retracted] Extremal Values of Variable Sum Exdeg Index for Conjugated Bicyclic Graphs. *J. Chem.* **2021**, *2021*, 4272208. [CrossRef]
15. Akhter, N.; Jamil, M.K.; Tomescu, I. Extremal first and second Zagreb indices of apex trees. *UPB Sci. Bull. Ser. A Appl. Math. Phys.* **2016**, *78*, 221–230.
16. Wang, M.; Xiang, D.; Wang, S. Connectivity and diagnosability of leaf-sort graphs. *Parallel Process. Lett.* **2020**, *30*, 2040004. [CrossRef]
17. Wang, M.; Xiang, D.; Qu, Y.; Li, G. The diagnosability of interconnection networks. *Discret. Appl. Math.* **2024**, *357*, 413–428. [CrossRef]
18. Wang, M.; Wang, S. Connectivity and diagnosability of center k-ary n-cubes. *Discret. Appl. Math.* **2021**, *294*, 98–107. [CrossRef]