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Existence and Uniqueness of a Solution of a Boundary Value Problem Used in Chemical Sciences via a Fixed Point Approach

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Abstract: In this paper, we present Proinov-type fixed point theorems in the setting of bi-polar metric spaces and fuzzy bi-polar metric spaces. Fuzzy bi-polar metric spaces with symmetric property extend classical metric spaces to address dual structures and uncertainty, ensuring consistency and balance. We provide different concrete conditions on the real-valued functions $\Omega, \Pi : (0, \infty) \to \mathbb{R}$ for the existence of fixed points via the (Ω, Π)-contraction in bi-polar metric spaces. Further, we define real-valued functions $\Omega, \Pi : (0, 1] \to \mathbb{R}$ to obtain fixed point theorems in fuzzy bi-polar metric spaces. We apply (Ω, Π) fuzzy bi-polar version of a Banach fixed point theorem to show the existence of solutions. Furthermore, we provide some non-trivial examples to show the validity of our results. In the end, we find the existence and uniqueness of a solution of integral equations and boundary value problem used in chemical sciences by applying main results.

Keywords: fuzzy bi-polar metric space; fixed point; existence and uniqueness; boundary value problem

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1. Introduction

A point g is a fixed point of a self-mapping of $A(g)$ if $A(g) = g$. In 1960, the notion of continuous t-norm was presented by Schweizer and Sklar [1]. In 1965, Zadeh [2] presented the notion of a fuzzy set. Fuzzy sets extend fixed point theory to handle uncertainty and imprecision, enabling the analysis of systems with vague or incomplete information. They facilitate the generalization of classical fixed point results to fuzzy settings, broadening their applicability to real-world problems in optimization, decision making, and dynamic systems. First, Karamosil and Michlek [3] established the notion of fuzzy metric space (FMS). Gregori and Sapena [4] presented a fuzzy contractive mapping and proved some fixed point theorems in the context of the Karamosil and Michlek FMS. In 2008, Mihet [5] established some fixed point results by using Ψ-contractive mappings in non-Archimedean FMS. A novel family of contractions was introduced Hierro et al. [6] in the framework of non-Archimedean FMSs, offering a significant advancement in the field. The primary advantage of this family lies in its incorporation of general auxiliary functions, enhancing its flexibility and applicability across diverse mathematical and practical contexts.

Zhou et al. [7] presented some auxiliary functions in FMS and established a novel family of contractions based on Proinov-type contractions and proved some fixed point theorems. Sessa et al. [8] established some fixed point theorems by using the fuzzy orthogonal contraction $\alpha\Gamma - F$ and provided an application for non-linear equations. Ishtiaq et al. [9] proved some fixed point theorems using different types of interpolative contraction mappings.

In 2020, Afshari et al. [10] initiated some fixed point results by using (α, Ψ)-contractive mappings in b-metric spaces. In 2020 Proinov [11] obtained fixed point results in a metric space (MS) by using generalized contractive mappings. Then, in 2021, Alqahtani et al. [12] determined some fixed point theorems by modifying Proinov-type [11] fixed point results using certain conditions to the corresponding contraction. Hiero et al. [13] proved several fixed point theorems by utilizing multi-parametric contractions and related Hardy Rogers-type fixed point theorem.

Mutlu and Gurdal [14] presented a new idea of bi-polar metric space (in short, BMS) and proved several fixed point results. They used E and F two nonempty sets and define a mapping $\alpha : E \times F \to \mathbb{R}^+$, where $\mathbb{R}^+$. Khajasteh et al. [15] presented some simulation functions and proved various fixed-point results in the setting of MS. Semet et al. [16] presented some fixed point results for $\alpha - \Psi$ contractive mappings in complete MS. Murthy et al. [17] proved some common fixed point theorems for BMS by using Meir-Keeler type contractions. Prasad [18] established several common fixed point theorems in BMS by using covariant mappings. Jahangeer et al. [19] proved several best proximity point theorems in BMS by using certain interpolative contractions. Bartwal et al. [20] established a new idea of fuzzy bi-polar metric space (in short, FBMS) and proved some fixed point theorems. An FMS uses a fuzzy membership value to describe the degree of closeness between two points, with larger values indicating greater proximity. It involves a single non-empty set with parameter $t > 0$ and is mostly used to deal with ambiguity in distance measurements. In an FBM space, two different non-empty sets with parameter $t > 0$ are used to represent dual features evaluations of distance. Meanwhile, FMSs are simpler and widely used in applications like image processing and fixed point theory. FBMSs are more complex and suited for scenarios involving bi-polar evaluations, such as satisfaction/dissatisfaction or attraction/repulsion in decision making and psychology. Beg et al. [21] proved several common coupled fixed point results in the setting of FBMSs. Ramalingam et al. [22] used the triangular property of a fuzzy bi-polar b-metric space to derive fixed point theorems without continuity, expanding on previously proven results. Bi-polar metric spaces provide a dual framework for evaluating relationships, and fuzzy bi-polar metric spaces extend this framework to uncertain environments using fuzzified metrics. Symmetry in both settings ensures balanced and consistent evaluation of distances, making these spaces suitable for a wide range of theoretical and applied problems.

Motivated from the above discussion, we prove some fixed point results in the context of BMS and FBMS. We introduce the L family of functions in the setting of BMS and FBMS. Further, we provide some corollaries and remarks which relate our results to the existing ones in the literature. We divide this paper into four parts. The first part is dedicated for basic definitions and results from the existing literature. In the second part, we present some lemmas, propositions and fixed point results in the setting of BMS and FBMS with several non-trivial examples. In the third part, we find the existence and uniqueness of a solution of a boundary value problem by applying the main result. In the fourth part, we provide a conclusion of our work.

2. Preliminaries

This section contains several definitions and results from the existing literature.

Definition 1 ([14]). *Suppose E and F are nonempty sets and let a mapping $\alpha : E \times F \to \mathbb{R}^+$ be a function, where $\mathbb{R}^+$ denotes the set of non-negative real numbers. Then, (E, F, α) is said to be a BMS if it fulfills the following conditions:*

(bp1) If $\alpha(g, j) = 0$, then $g = j$ for all $(g, j) \in E \times F$;

(bp2) If $g = j$, then $\alpha(g, j) = 0$ for all $(g, j) \in E \times F$;

(bp3) $\alpha(g, j) = \alpha(j, g)$ for all $g, j \in E \cap F$;

(bp4) $\alpha(g_1, j_2) \leq \alpha(g_1, j_1) + \alpha(g_2, j_1) + \alpha(g_2, j_2)$ for all $g_1, g_2 \in E$ and $j_1, j_2 \in F$.

Definition 2 ([14]). *Let (E, F, α) be a BMS.*

(i) A sequence (g_n, j_n) on the set $E \times F$ is called a bi-sequence on (E, F, α).

(ii) If both (g_n) and (j_n) are convergent, then the bi-sequence is said to be convergent. If both sequences (g_n) and (j_n) both converges to the same point $s \in E \cap F$, then the bi-sequence is said to be bi-convergent.

(iii) A bi-sequence (g_n, j_n) on (E, F, α) is called a Cauchy bi-sequence if, for every $\epsilon > 0$, there exists a number $n_0 \in \mathbb{N}$, such that for all positive integers $n, m \geq n_0$, $\alpha(g_n, j_m) < \epsilon$.

Definition 3 ([14]). *Let (E, F, α) be a BMS. A left sequence (g_n) converges to a right point j (symbolically $(g_n) \to j$ or $\lim_n g_n = j$) if and only if $\epsilon > 0$ exists as $n_0 \in \mathbb{N}$ such that $\alpha(g_n, j) < \epsilon$ for all $n \geq n_0$. Similarly, a right sequence (j_n) converges to a right point g (symbolically $(j_n) \to g$ or $\lim_n j_n = g$) if and only if $\epsilon > 0$ exists $n_0 \in \mathbb{N}$ such that $\alpha(g, j_n) < \epsilon$ for all $n \geq n_0$. When $(g_n \to j$ or $\lim_n g_n = j$ for a BMS (E, F, α), without exact information about the domain of the sequence, this means that (g_n) is a left sequence and j is a right point, or (g_n) a right sequence and j is a left point.*

Definition 4 ([14]). *A BMS is called complete if every Cauchy bi-sequence in this space is convergent.*

Theorem 1 ([14]). *Let (E, F, α) be a complete BMS and a contraction $A : (E, F, \alpha) \rightrightarrows (E, F, \alpha)$ (here, "$\rightrightarrows$" shows the covariant map). Then, the function $A : E \cup F \to E \cup F$ has a unique fixed point.*

Definition 5 ([1]). *A binary operation $\circ : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm (ctn) if it satisfies the following conditions:*
(T1) $g_1 \circ g_2 = g_2 \circ g_1$ and $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g$ for all $g_1, g_2, g_3 \in [0, 1]$;
(T2) $\circ$ is continuous;
(T3) $g \circ 1 = g$ for all $g \in [0, 1]$;
(T4) $g_1 \circ g_2 \leq g_3 \circ g_4$ when $g_1 \leq g_3$ and $g_2 \leq g_4$, with $g_1, g_2, g_3, g_4 \in [0, 1]$.

Definition 6 ([20]). *Suppose $E \neq \Phi$ and $\circ$ is a ctn. A mapping $\alpha : E \times E \times (0, \infty) \to [0, 1]$ is called a fuzzy b-metric if it satisfies the following axioms for all $g_1, g_2, g_3 \in E$ and $p, \nu > 0$:*

(A1) $\alpha(g_1, g_2, p) > 0$;

(A2) $\alpha(g_1, g_2, p) = 1$ if and only if $g_1 = g_2$;

(A3) $\alpha(g_1, g_2, p) = \alpha(g_2, g_1, p)$;

(A4) $\alpha(g_1, g_3, p + \nu) \geq \alpha(g_1, g_2, p) \circ \alpha(g_2, g_3, \nu)$;

(A5) $\alpha(g_1, g_2, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Then, $(E, \alpha, \circ)$ is called an FMS.

Example 1. *Let $E \neq \Phi$ and define a mapping $\alpha : E \times E \times (0, \infty) \to [0, 1]$ by $\alpha(g_1, g_2, p) = \frac{p}{p + |g_1 - g_2|}$. Then, $(E, \alpha, \circ)$ is an FMS with ctn $\alpha \circ \beta = \alpha\beta$.*

Definition 7 ([20])**.** *Suppose* $E, F \neq \Phi$ *and* $\circ$ *is a ctn. A mapping* $\alpha : E \times F \times (0, \infty) \to [0, 1]$ *is said to be fuzzy bi-polar metric if it satisfies the below axioms for all* $p, \nu, \lambda > 0$:

(A1) $\alpha(g, j, p) > 0$ *for all* $(g, j) \in E \times F$;

(A2) $\alpha(g, j, p) = 1$ *if and only if* $g = j$ *for* $g \in E$ *and* $j \in F$;

(A3) $\alpha(g, j, p) = \alpha(j, g, p)$ *for all* $g, j \in E \cap F$;

(A4) $\alpha(g_1, j_2, p + \nu + \lambda) \geq \alpha(g_1, j_1, p) \circ \alpha(g_2, j_1, \nu) \circ (g_2, j_2, \lambda)$ *for all* $g_1, g_2 \in E$ *and* $j_1, j_2 \in F$;

(A5) $\alpha(g, j, \cdot) : [0, \infty) \to [0, 1]$ *is left continuous;*

(A6) $\alpha(g, j, \cdot)$ *is non-decreasing for all* $g \in E$ *and* $j \in F$.

Then, $(E, F, \alpha, \circ)$ *is called an FBMS.*

Example 2. *For all* $g \in E$, $j \in F$ *and* $p > 0$, *define a mapping* $\alpha : E \times F \times (0, \infty) \to [0, 1]$ *by*

$$\alpha(g, j, p) = e^{-\frac{|g-j|}{p}}.$$

Then, $(E, F, \alpha, \circ)$ *is an FBMS with ctn* $a \circ b = ab$.

Definition 8 ([20])**.** *Suppose* $(E, F, \alpha, \circ)$ *is an FBMS. The points belonging to* E, F *and* $E \cap F$ *are named as left, right, and central points respectively, and sequences belonging to* E, F *and* $E \cap F$ *are called left, right, and central sequences, respectively.*

Lemma 1 ([20])**.** *Suppose* $(E, F, \alpha, \circ)$ *is an FBMS such that*

$$\alpha(g, j, lp) \geq \alpha(g, j, p)$$

for all $g \in E$, $j \in F$, $p > 0$, *and* $l \in (0, 1)$. *Then,* $g = j$.

Definition 9 ([20])**.** *Suppose* $(E, F, \alpha, \circ)$ *is an FBMS. A sequence* $\{g_n\} \in E$ *converges to a right point* j *if and only if for each* $\epsilon > 0$ *and* $p > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $\alpha(g_n, j, p) > 1 - \epsilon$ *for all* $n \geq n_0$. *Similarly, a right sequence* $\{j_n\}$ *converges to a left point* g *if and only if, for each* $\epsilon > 0$ *and* $p > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $\alpha(g, j_n, p) > 1 - \epsilon$ *for all* $n \geq n_0$.

Definition 10 ([20])**.** *Suppose* $(E, F, \alpha, \circ)$ *is an FBMS; then,*

(i) A sequence $(g_n, j_n) \in E \times F$ *is called a bi-sequence on* $(E, F, \alpha, \circ)$

(ii) If both sequences (g_n) *and* (j_n) *converge, then the sequence* $(g_n, s_n) \in E \times F$ *is said to be bi-convergent. If both* (g_n) *and* (j_n) *converge to the same center point, the bi-sequence* (g_n, j_n) *is said to be bi-convergent.*

(iii) A bi-sequence (g_n, j_n) *on* $(E, F, \alpha, \circ)$ *is called a Cauchy bi-sequence if, for each* $\epsilon > 0$, *there exists a* $n_0 \in \mathbb{N}$, *such that for every positive integer* $n, m \geq n_0$,$(n, m \in \mathbb{N})$ $\alpha(g_n, j_m, p) > 1 - \epsilon$ *for each* $p > 0$, *i.e., a bi-sequence* (g_n, j_n) *is said to be a Cauchy bi-sequence if* $\alpha(g_n, j_m, p) \to 1$ *as* $n, m \to \infty$ *for all* $p > 0$.

Definition 11 ([20])**.** *The FBMS* $(E, F, \alpha, \circ)$ *is known as complete if every Cauchy bi-sequence in* $E \times F$ *is convergent in it.*

Proposition 1 ([20])**.** *In FBMS, every bi-convergent bi-sequence is a Cauchy bi-sequence.*

Lemma 2 ([20])**.** *Suppose* $(E, F, \alpha, \circ)$ *is an FBMS. If* $g \in E \cap F$ *is a limit of the sequence, then it is a unique limit of the sequence.*

Theorem 2 ([20])**.** *Let* $(E, F, \alpha, \circ)$ *be a complete FBMS such that* $\lim_{p \to \infty} \alpha(g, j, p) = 1$ *for all* $g \in E$, $j \in F$. *Then,* $A : E \cup F \to E \cup F$ *is a mapping verifying*

(i) $A(E) \subseteq E$ *and* $A(F) \subseteq F$;
(ii) $\alpha(A(g), A(j), lp) \geq \alpha(g, j, p)$ *for all* $g \in E$, $j \in F$, $p > 0$, *where* $l \in (0, 1)$.
Then, A has a unique fixed point.

Proposition 2 ([7]). *Let* $\{g_n\}$ *be a Picard sequence in FMS* $(E, \alpha, \circ)$ *such that* $\lim_{n\to\infty} \alpha(g_n, g_{n+1}, p) = 1$ *if for each sequence* $\{g_n\} \subseteq E$ *for all* $p > 0$. *If there are* $m_0, n_0 \in \mathbb{N}$ *such that* $m_0 < n_0$ *and* $g_{m_0} = g_{n_0}$, *then there is* $s_0 \in \mathbb{N}$ *and* $g^* \in E$ *such that* $g_n = g^*$ *for all* $n \geq s_0$. *In such a case,* g^* *is a fixed point of the self-mapping for which* $\{g_n\}$ *is a Picard sequence.*

Proposition 3 ([7]). *Every Picard sequence is either infinite or almost periodic.*

Proposition 4 ([7]). *We say that an FMS verifies the property if for each* $\{g_n\} \subseteq E$, *which is not Cauchy sequence and holds* $\lim_{n\to\infty} \alpha(g_n, g_{n+1}, p) = 1$ *for all* $p > 0$, *there are* $\epsilon_0 \in (0, 1)$ *and* $p_0 > 0$ *and two partial subsequences* $\{g_{m_k}\}$ *and* $\{g_{n_k}\}$ *of* $\{g_n\}$, *such that, for all* $k \in \mathbb{N}$, *the following is fulfilled:*

$$k < m_k < n_k < m_{k+1} \text{ and}$$

$$\alpha(g_{m_k}, g_{n_k - 1}, p_0) > 1 - \epsilon_0 \geq \alpha(g_{m_k}, g_{n_k}, p_0),$$

$$\lim_{n\to\infty} \alpha(g_{m_k}, g_{n_k}, p_0) = \lim_{n\to\infty} \alpha(g_{m_{k-1}}, g_{n_k - 1}, p_0) = 1 - \epsilon_0.$$

Definition 12 ([7]). *Let* $(E, \alpha, \circ)$ *be an FMS. We denote, by L, the family of pairs* (Ω, Π) *of the functions* $\Omega, \Pi : (0, 1] \to \mathbb{R}$ *fulfills the following axioms:*
(p1) Ω *is non-decreasing;*
(p2) $\Pi(g) > \Omega(g)$ *for any* $g \in (0, 1)$;
(p3) $\lim_{g\to H^-} \inf \Pi(g) > \lim_{g\to H^-} \Omega(g)$ *for any* $H \in (0, 1)$;
(p4) If $p \in (0, 1]$ *is such that* $\Omega(p) \geq \Pi(1)$, *then* $p = 1$.

Example 3.
(1) $\Omega(g) = g$ *and* $\Pi(g) = \sqrt{g}$ *for all* $g \in (0, 1]$.
(2) $\Omega(g) = \frac{1}{\ln g}$ *and* $\Pi(g) = \frac{1}{\ln g^2}$ *for all* $g \in (0, 1]$.
(3) $\Omega(g) = \frac{1}{2^{\ln 2g}}$ *and* $\Pi(g) = \frac{1}{2^{\ln g}}$ *for all* $g \in (0, 1]$.

3. Main Results

This section contains several fixed point results in the setting of BMS and FBMS.

3.1. Fixed Point Theorems for (Ω, Π)-Contractions in BMS

We provide a fixed point theorem for a self-mapping A on a complete BMS (in short, CBMS) (E, F, α) satisfying a contractive condition

$$\Omega(\alpha(Ag, Aj)) \leq \Pi(\alpha(g, j)) \text{ for all } (g, j) \in E \times F \text{ and } \alpha(Ag, Aj) > 0, \quad (1)$$

where $\Omega, \Pi : (0, \infty) \to \mathbb{R}$ are two function such that $\Pi(g) < \Omega(g)$ for $g > 0$.

Lemma 3. *Let* (E, F, α) *be a BMS and* $\{g_n, j_n\}$ *be a bi-sequence in* $E \cup F$ *which is not Cauchy bi-sequence and* $\lim_{n\to\infty} \alpha(g_n, j_{n+1}) = 0$. *Then, there exists* $\epsilon > 0$ *and two bi-subsequences* $\{g_{n_k}\}$ *and* $\{g_{m_k}\}$ *of* $\{g_n\}$ *and* $\{j_{n_k}\}$, $\{j_{m_k}\}$ *of* $\{j_n\}$ *such that*

$$\lim_{k\to\infty} \alpha(g_{n_k+1}, j_{m_k+1}) = \epsilon, \quad (2)$$

$$\lim_{k\to\infty} \alpha(g_{n_k}, j_{m_k}) = \lim_{k\to\infty} \alpha(g_{n_k+1}, j_{m_k}) = \lim_{k\to\infty} \alpha(g_{n_k}, j_{m_k+1}) = \epsilon. \quad (3)$$

Proof. Since $\{g_n, j_n\}$ is not a Cauchy bi-sequence and $\lim_{n\to\infty} \alpha(g_n, j_n) = 0$, there exists for $\epsilon > 0$ and $n_0 \geq 1$ such that for each $n > n_0$ there exists $n, m > n_0$ such that $n \geq m$

$$\alpha(g_{n+1}, j_{m+1}) > \epsilon \text{ and } \alpha(g_{n+1}, j_n) \leq \epsilon.$$

Thus, we can make two subsequences $\{j_{n_k}\}$ and $\{j_{m_k}\}$ of $\{j_n\}$, such that

$$\alpha\left(g_{n_k+1}, j_{m_k+1}\right) > \epsilon \text{ and } \alpha\left(g_{n_k+1}, j_{m_k}\right) \leq \epsilon.$$

From these inequalities and triangular inequality, we obtain

$$\epsilon < \alpha\left(g_{n_k+1}, j_{m_k+1}\right) \leq \alpha\left(g_{n_k+1}, j_{n_k}\right) + \alpha\left(j_{n_k}, j_{m_k}\right) + \alpha\left(j_{m_k}, j_{m_k+1}\right) \leq \epsilon\alpha\left(j_{m_k}, j_{m_k+1}\right).$$

By the Sandwich theorem, we get (2). Furthermore, we have

$$\alpha\left(g_{n_k+1}, j_{m_k+1}\right) - \alpha\left(j_{m_k+1}, j_{m_k}\right) \leq \alpha\left(g_{n_k+1}, j_{n_k}\right) + \alpha\left(j_{n_k}, j_{m_k}\right) \leq 2\epsilon,$$

which implies the second limit (3). From the following two inequalities,

$$\alpha\left(g_{n_k+1}, j_{m_k+1}\right) - \alpha\left(j_{n_k}, j_{n_k+1}\right) \leq \alpha\left(j_{n_k}, j_{m_k}\right) + \alpha\left(g_{n_k+1}, j_{m_k+1}\right) \leq \epsilon\alpha\left(j_{n_k}, j_{n_k+1}\right),$$

$$\epsilon - \alpha\left(g_{n_k}, j_{n_k+1}\right) < \alpha\left(g_{n_k}, j_{m_k+1}\right) + \alpha\left(j_{n_k}, j_{m_k}\right) \leq \alpha\left(j_{n_k+1}, j_{m_k+1}\right) + \alpha\left(j_{n_k}, j_{n_k+1}\right),$$

we deduce the first and third limits in (3). □

Lemma 4. *Let* $\Omega : (0, \infty) \to \mathbb{R}$*. Then, Conditions (i), (ii) and (iii) are equivalent, as follows:*
(i) $\inf_{g>\epsilon} \Omega(g) > -\infty$ *for each* $\epsilon > 0$;
(ii) $\lim_{g\to\epsilon} \inf \Omega(g) > -\infty$ *for each* $\epsilon > 0$;
(iii) $\lim_{n\to\infty} \Omega(g_n) = -\infty \implies \lim_{n\to\infty} g_n = 0$.

Proof. (i)⇒(ii): Suppose that Condition (i) is satisfied and $\inf_{g>\epsilon} \Omega(g) = S$ for some $\epsilon > 0$. Then, $\Omega(g) \geq S$ for each $g > \epsilon$. However, $\liminf_{g\to\epsilon} \Omega(g) \geq S$, i.e., Condition (ii) holds.

(ii) →(iii): Suppose that Condition (ii) is satisfied and $\lim_{n\to\infty} \Omega(g_n) = -\infty$ for a sequence $(g_n) \subseteq (0, \infty)$. Suppose that (g_n) does not converge to 0. Then, there exists $\epsilon > 0$ and a subsequence (g_{n_k}) such that $g_{n_k} > \epsilon$ for every $k \geq 1$. Since $\lim_{n\to\infty} \Omega(g_n) = -\infty$ implies $\lim_{k\to\infty} \Omega(g_{n_k}) = -\infty$ also for $\lim_{k\to\infty} \Omega(j_{n_k}) = -\infty$, we conclude that $\lim_{n\to\epsilon} \Omega(g) = -\infty$, which is a contradiction to Condition (ii). Hence, $\lim_{n\to\infty} g_n = 0$, that is, (iii) is satisfied.

(iii)⇒(i): Suppose that Condition (iii) is satisfied. Suppose that a $\inf_{g>\epsilon} \Omega(g) = -\infty$ for some $\epsilon > 0$. Then, there exists a subsequence $(g_n) \subseteq (0, \infty)$ such that $(g_n) > \epsilon$ for each $n \geq 1$ and $\lim_{n\to\infty} \Omega(g_n) = -\infty$. From Condition (iii), we obtain that $\lim_{n\to\infty} g_n = 0$, which contradicts $(g_n) > \epsilon$. That is, Condition (i) is satisfied. □

Lemma 5. *Suppose* $\Pi : (0, \infty) \to \mathbb{R}$*. Then, Condition (i)* $\implies$ *Condition (ii), where*
(i) $\lim_{n\to\infty} \Pi(g_n) = 0$ *implies* $\lim_{n\to\infty} g_n = 0$;
(ii) $\lim_{g\to\epsilon} \inf \Pi(g) > 0$ *for every* $\epsilon > 0$.

Proof. Let Condition (i) be fulfilled and $\liminf_{g\to\epsilon} \Pi(g) = 0$ for some $\epsilon > 0$. Then, there exists a sequence $(g_n) \subseteq (0, \infty)$ such that $g_{n,} \to \epsilon$ and $\Pi(g_n) \to 0$. From (i), we have $g_n \to 0$, which is a contradiction. □

Definition 13. *Let $E, F \neq \Phi$. A self-mapping A on BMS (E, F, α) is said to be asymptotically regular (in short, ATR) at a point $g \in E$ and $j \in F$ if*

$$\lim_{n\to\infty} \alpha(A^n g, A^n j) = 0.$$

Lemma 6. *Suppose (E, F, α) is BMS and let $A : E \cup F \to E \cup F$ be a mapping, where the functions $\Omega, \Pi : (0, +\infty) \to \mathbb{R}$ are such that*

(i) $\inf_{g>\epsilon} \Omega(g) > -\infty$ for any $\epsilon > 0$;

Let one of the following be satisfied:

(ii) Ω is non-decreasing and $\lim_{g\to\epsilon} \sup \Pi(g) < \Omega(\epsilon)$ for any $\epsilon > 0$;

(iii) if $(\Omega(g_n))$ and $\Pi((g_n))$ are convergent sequences with the same limit and $(\Omega(g_n))$ is strictly decreasing, then $\lim_{n\to\infty} g_n = 0$. Then, A is an ATR.

Proof. Put $g_n = A^n g$ and $J_n = A^n j$ and $H_n = \alpha(g_n, j_n)$. We examine that $H_n \to 0$ every $n \geq 0$. If $H_n = 0$ for some $n \geq 0$, then it is obvious. Applying inequality (1) with $s = g_n$ and $e = j_n$, and taking into account Condition (i), we obtain

$$\Omega(H_{n+1}) \leq \Pi(H_n) < \Omega(H_n). \tag{4}$$

Suppose that Condition (ii) holds. Then, by utilizing (4), that is, $H_{n+1} < H_n$ for each $n \geq 0$, (H_n) is a strictly decreasing and positive sequence. Therefore, there exists $H \geq 0$ such that $H_n \to H$ as $n \to \infty$. Now, we investigate that $H = 0$. Suppose that $H > 0$. Letting $n \to \infty$ in (4), we deduce

$$\Omega(H) = \lim_{n\to\infty} \Omega(H_{n+1}) \leq \lim_{n\to\infty} \sup \Pi(H_n) \leq \lim_{l\to H} \sup \Pi(l),$$

which is a contradiction of Condition (ii). That is, $H = 0$.

Let Condition (iv) be satisfied. By utilizing (4), the sequence $\Omega(H_n)$ is strictly decreasing. We take $(\Omega(H_n))$ as not bounded below. Then, we apply Condition (i) and Lemma 4, that is $H_n \to 0$, as $n \to \infty$. Now let $(\Omega(H_n))$ be bounded below. Then, $(\Omega(H_n))$ is a convergent bi-sequence. From (4), $(\Pi(H_n))$ is also a convergent bi-sequence with the same limit. Therefore, by utilizing Condition (iv), that is, $H_n \to 0$ as $n \to \infty$. □

Lemma 7. *Suppose that (E, F, α) is BMS and let $A : E \cup F \to E \cup F$ be a mapping verifying (1), with the functions $\Omega, \Pi : (0, +\infty) \to \mathbb{R}$ which fulfills at least one of the following:*

(i) $\lim_{g\to\epsilon^+} \sup \Pi(g) < \Omega(\epsilon)$ for any $\epsilon > 0$;

(ii) $\lim \sup_{g\to\epsilon} \Pi(g) < \lim \inf_{g\to\epsilon+} \Omega(g)$ for any $\epsilon > 0$;

If A is an ATR at a point $g \in E$ and $j \in F$, then $(A^n g, A^n j)$ is a Cauchy bi-sequence.

Proof. Suppose A is an ATR mapping at an element $s \in E \cap F$. Let bi-sequence $(A^n g, A^n j)$, which is not Cauchy bi-sequence. Set $g_n = A^n g$ and $j_n = A^n j$ for every $n \geq 0$.

Suppose that Ω, Π verify Condition (i). By applying Lemma 3, there exist $\epsilon > 0$ and two subsequences (j_{n_k}) and (j_{m_k}) of (j_n) such that limits (2) and (3) fulfilled. By utilizing (2), $\lim_{k\to\infty} \alpha(g_{n_k+1}, j_{m_k}) > \epsilon$ for each $k \geq 1$. By using (1) with $g = g_{n_k}$ and $j = j_{m_k}$, we obtain

$$\Omega(\alpha(g_{n_k+1}, j_{n_k+1})) \leq \Pi(\alpha(g_{n_k}, j_{n_k})) \tag{5}$$

for all $k \geq 1$. We set $b_k = \alpha(g_{n_k+1}, j_{n_k+1})$ and $c_k = \alpha(g_{n_k}, j_{n_k})$. Then, from (5), we obtain

$$\Omega(b_k) \leq \Pi(c_k). \tag{6}$$

Hence, taking into account Lemma (6) (i), we get

$$\Omega(b_k) \leq \Pi(c_k) < \Omega(c_k). \tag{7}$$

For this and the monotonicity of Ω, we obtain $b_k < c_k$. Then, we use (2) and (3), that is, $b_k \to \epsilon$ and $c_k \to \epsilon$. Taking the superior limit in (6) as $k \to \infty$, we obtain

$$\Omega(\epsilon) = \lim_{k\to\infty} \Omega(a_k) \leq \lim_{k\to\infty} \sup \Pi(c_k) \leq \lim_{g\to\epsilon} \sup \Pi(g)$$

which contradicts Condition (i).

Assume that Ω and Π verify Condition (ii). By utilizing Lemma 3, there exist $\epsilon > 0$ and two bi-subsequences of $\{g_{n_k}\}$ and $\{g_{m_k}\}$ of $\{g_n\}$ and $\{j_{n_k}\}$ and $\{j_{m_k}\}$ of $\{j_n\}$ such that the limits (2) and (3) hold. From (6), we conclude that

$$\alpha\big(Ag_{n_k}, Aj_{m_k}\big) = \alpha\big(g_{n_k+1}, j_{m_k+1}\big) > 0$$

for every k. By applying (1) with $g = g_{n_k}$ and $j = j_{m_k}$, (5) is satisfied for every k. Again, take $b_k = \alpha\big(g_{n_k+1}, j_{m_k+1}\big)$ and $c_k = \alpha\big(g_{n_k}, j_{m_k}\big)$. Then, (5) takes from (6). It follows from (2) and (3) that $b_k \to \epsilon$ and $c_k \to \epsilon$. From (5), we obtain

$$\lim_{g\to\epsilon} \inf \Omega(g) \leq \lim_{k\to\infty} \inf \Omega(b_k) \leq \lim_{k\to\infty} \sup \Pi(c_k) \leq \lim_{g\to\epsilon} \sup \Omega(g),$$

which is a contradiction. However, $(A^n g, A^n j)$ is a Cauchy bi-sequence. □

Definition 14. *A self-mapping A on a BMS (E, F, α) is called a closed graph if*

$$Graph(A) = \{(g, j) \in E \cup F \times E \cup F : j = Ag\}$$

is closed in its product $E \times F$ topology. Also, A has a closed graph if and only if, for each bi-sequence (g_n) and (j_n) in $E \cup F$ such that $g_n \to s$ and $j_n \to s$ as $n \to \infty$, we have $j = Ag$.

Lemma 8. *Let (E, F, α) be a BMS and let a mapping $A : E \cup F \to E \cup F$ verify (1), with the function $\Omega, \Pi : (0, \infty) \to \mathbb{R}$ which fulfills at least one of the following:*

(i) A has a closed graph;

any $g > 0$;

(ii) $\limsup_{g\to 0} \Pi(g) < \liminf_{g\to\epsilon} \Omega(g)$ for any $\epsilon > 0$.

If $\lim_{n\to\infty}(A^n g, A^n j) = \zeta$ for some $g, j \in E \cup F$, then ζ is a fixed point of A.

Proof. If we take (ii), then the proof of (i) is obvious. Suppose $g_n = A^n g$. If $\alpha(Ag_n, A\zeta) = 0$ for each n, then

$$\alpha(\zeta, A\zeta) \leq \alpha(\zeta, Ag_n) + \alpha(Ag_n, g_{n+1}) + \alpha(Ag_n, A\zeta) = \alpha(\zeta, g_{n+1}).$$

Since A has a closed graph, by letting $n \to \infty$, we obtain $\alpha(\zeta, A\zeta) \leq 0$, which implies $\alpha(\zeta, A\zeta) = 0$. This means that $\zeta = A\zeta$. This means that $\zeta \in E \cap F$ is a fixed point of A. Let $\alpha(Ag_n, A\zeta) > 0$ be satisfied for every n. Then, by utilizing (1) with $g = g_n$ and $j = \zeta$, we obtain

$$\Omega(\alpha(Ag_n, A\zeta)) \leq \Pi(\alpha(g_n, \zeta)) \tag{8}$$

which hold for the values of n.

Let Ω and Π satisfy (1). Then, it follows from (8) that

$$\Omega(\alpha(Ag_n, A\zeta)) \leq \Pi(\alpha(g_n, \zeta)) < \Omega(\alpha(g_n, \zeta)) \tag{9}$$

This implies $\alpha(Ag_n, A\zeta) < \alpha(g_n, \zeta)$. Taking the limit as $n \to \infty$, we obtain $\alpha(\zeta, A\zeta) \le 0$. This means that ζ is an fixed point of A.

Suppose that Ω and Π fulfill Condition (iii). By utilizing (8), we deduce

$$\Omega(b_n) \le \Pi(c_n) \tag{10}$$

for each n, where $b_n = \alpha(g_{n+1}, A\zeta)$ and $c_n = \alpha(g_n, \zeta)$. Obviously, $b_n \to \epsilon$ and $c_n \to 0$ as $n \to \infty$, where $\epsilon = \alpha(\zeta, A\zeta)$. By applying (10), we obtain

$$\lim_{g\to\epsilon} \inf \Omega(g) \le \lim_{n\to\infty} \inf \Omega(b_n) \le \lim_{n\to\infty} \sup \Pi(c_n) \le \lim_{g\to 0} \sup \Pi(g).$$

If we take $\epsilon > 0$, then it is a contradiction to Condition (ii). Therefore, $\alpha(\zeta, A\zeta) = 0$. This means that ζ is a fixed point of A. □

Theorem 3. *Assume that (E, F, α) is a BMS and that a mapping $A : E \cup F \to E \cup F$ satisfies (1), with the functions $\Omega, \Pi : (0, \infty) \to \mathbb{R}$, fulfilling at least one of the following:*

(i) Ω is non-decreasing;

(ii) $\limsup_{g\to\epsilon} \Pi(g) < \Omega(\epsilon)$ for any $\epsilon > 0$.

Then, A has a unique fixed point $\zeta \in E \cap F$ and the iterative bi-sequence $(A^n g, A^n j)$ converges to ζ for every $g \in E$ and $j \in F$.

Proof. Let $g_0 \in E$ and $j_0 \in F$. For each $n \in \mathbb{N}$, define $A(g_n) = g_{n+1}$ and $A(j_n) = j_{n+1}$. Then, (g_n, j_n) is a bi-sequence in (E, F, α). By applying Conditions (i) and (ii) and Lemma 6, A is asymptotically regular. Also, by utilizing Conditions (i) and (ii) and Lemma 7, the bi-sequence $(A^n g, A^n j)$ is a Cauchy bi-sequence. However, if (E, F, α) is complete, then the bi-sequence converges to a point $\zeta \in E \cap F$. From Condition (i) and Lemma 8, it is clear that ζ is a fixed point of A. The uniqueness of the fixed point is easy to determine using (1). □

Remark 1. *If $\Omega(g) = g$, and $\Pi(g) = \lambda g$, where $0 \le \lambda < 1$, then Theorem 3 is reduced to a Banach contraction principle.*

Example 4. *Let $E = [0, 1]$ and $F = [1, 2]$ be equipped with $\alpha(g, j) = \frac{|g-j|}{1+|g-j|}$ for all $g \in E$ and $j \in F$. Then, (E, F, α) is a complete bi-polar metric space. Define $A : E \cup F \to E \cup F$ by*

$$A(s) = \frac{s+4}{5}$$

for all $s \in E \cup F$. Now, define the function $\Omega, \Pi : (0, \infty) \to \mathbb{R}$ by

$$\Omega(r) = r \qquad \Pi(r) = \frac{r}{2} \text{ for all } r \in (0, \infty).$$

Now, we have to show that A satisfies (1). Therefore,

$$
\begin{aligned}
\Omega(\alpha(A(g),A(j))) &= \Omega\left(\alpha\left(\frac{g+4}{5},\frac{j+4}{5}\right)\right)\\
&= \Omega\left(\frac{\mid \frac{g+4}{5}-\frac{j+4}{5}\mid}{1+\mid \frac{g+4}{5}-\frac{j+4}{5}\mid}\right)\\
&= \frac{\mid \frac{g+4}{5}-\frac{j+4}{5}\mid}{1+\mid \frac{g+4}{5}-\frac{j+4}{5}\mid}\\
&= \frac{\mid \frac{g}{5}-\frac{j}{5}\mid}{1+\mid \frac{g}{5}-\frac{j}{5}\mid}\\
&= \frac{\mid g-j\mid}{5+\mid g-j\mid}\\
&\leqslant \frac{1}{2}\left(\frac{\mid g-j\mid}{1+\mid g-j\mid}\right)\\
&= \Pi(\alpha(g,j)).
\end{aligned}
$$

Hence, all the other conditions of Theorem 3 hold. Moreover, 1 *is the fixed point of* A*.*

Now, we see that without (Ω,Π), $g=0$ and $j=1$ do not hold; thus, we take $\lambda=\frac{1}{6}$

$$
\begin{aligned}
(\alpha(A(0),A(1))) &\leq \lambda\alpha(0,1)\\
\alpha\left(\frac{4}{5},1\right) &\leq \lambda\alpha(0,1)\\
0.1667 &\leq \frac{1}{5}(0.5)\\
0.1667 &\leq 0.1,
\end{aligned}
$$

which is a contradiction. Hence, it does not hold without (Ω,Π).

Theorem 4. *Assume that* (E,F,α) *is a BMS and that* $A: E\cup F\to E\cup F$ *is a mapping that satisfies (1), with the functions* $\Omega,\Pi:(0,\infty)\to\mathbb{R}$*, verifying at least one from the following:*

(i) $\inf_{g>\epsilon}\Omega(g)>-\infty$ *for any* $\epsilon>0$*;*

(ii) If $(\Omega(g_n))$ *and* $\Pi((g_n))$ *are convergent sequences with the same limit and* $(\Omega(g_n))$ *is strictly decreasing, then* $g_n\to 0$ *as* $n\to\infty$*;*

(iii) A has closed graph or $\lim_{g\to 0}\Pi(g)<\lim_{g\to\epsilon}\inf\Omega(g)$ *for any* $\epsilon>0$*.*

Then, A has a unique fixed point $\zeta\in E\cap F$ *and the iterative bi-sequence* $(A^n g, A^n j)$ *converges to* ζ *for every* $g\in E$ *and* $j\in F$*.*

Proof. Let $g_0\in E$ and $j_0\in F$. For each $n\in\mathbb{N}$, define $A(g_n)=g_{n+1}$ and $A(j_n)=j_{n+1}$. Then, (g_n,j_n) is a bi-sequence in (E,F,α). By applying Conditions (i) and (ii) and Lemma 6, A is asymptotically regular at g and j. By using Condition (iii) and Lemma 7, it is clear that bi-sequence $(A^n g, A^n j)$ is Cauchy. Therefore, $(A^n g, A^n j)$ converges to a point $\zeta\in E\cap F$. Moreover, by applying Condition (iii) and Lemma 8, it is clear that ζ is a fixed point of A. The uniqueness is easy to determine by using (1). □

Remark 2. *If* $\Omega(g)=g$*, and* $\Pi(g)=\lambda g$*, where* $0\leq\lambda<1$*, then Theorem 4 is reduced to Banach contraction principle.*

3.2. Fixed Point Theorem for (Ω,Π)*-Contraction in Fuzzy Bi-Polar Metric Spaces*

In this part, we prove fixed point theorems in FBMS.

Proposition 5. *Let $\{g_n, j_n\}$ be a Picard bi-sequence in FBMS $(E, F, \alpha, \circ)$ such that $\lim_{n\to\infty} \alpha(g_n, j_n, p) = 1$ if the bi-sequence $(g_n, j_n) \subseteq E \cup F$ for all $p > 0$. If there are $m_0, n_0 \in \mathbb{N}$ such that $m_0 < n_0$ and $g_{m_0} = j_{n_0}$, then there is $s_0 \in \mathbb{N}$ and $s^* \in E \cap F$ such that $\lim_{n\to\infty} \alpha(g_n, j_n, p) = s^*$. In such a case, s^* is a fixed point of the self-mapping for which $\{g_n, j_n\}$ is a Picard bi-sequence.*

Lemma 9. *We say that an FBMS sequence is not Cauchy if for each $(g_n, j_n) \subseteq E \cup F$, which is not a Cauchy sequence and holds $\lim_{n\to\infty} \alpha(g_n, j_n, p) = 1$ for each $p > 0$, there are $\epsilon_0 \in (0, 1)$ and $p_0 > 0$ and two partial bi-subsequences, two bi-subsequences $\{g_{n_k}\}$ and $\{g_{m_k}\}$ of $\{g_n\}$ and $\{j_{n_k}\}$ and $\{j_{m_k}\}$ of $\{j_n\}$ such that for each $k \in \mathbb{N}$, the following holds:*

$$k < m_k < n_k < m_{k+1} \text{ and}$$

$$\alpha\big(g_{m_k}, j_{n_k-1}, p_0\big) > 1 - \epsilon_0 \geq \alpha\big(g_{m_k}, j_{n_k}, p_0\big),$$
$$\lim_{n\to\infty} \alpha\big(g_{m_k}, j_{n_k}, p_0\big) = \lim_{n\to\infty} \alpha\big(g_{m_k}, j_{n_k-1}, p_0\big) = 1 - \epsilon_0.$$

Theorem 5. *Suppose that $(E, F, \alpha, \circ)$ is a complete FBMS and suppose that $A : E \cup F \to E \cup F$ is a mapping for which there exists $(\Omega, \Pi) \in L$ such that*

$$\Omega(\alpha(Ag, Aj, p)) \geq \Pi(\alpha(g, j, p)) \text{ for all } g \in E \text{ and } j \in F \text{ with } Ag \neq Aj \text{ and } p > 0. \tag{11}$$

Then, each iterative Picard bi-sequence $\{A^n g, A^n g\}_{n\in\mathbb{N}}$ is bi-convergent to the unique fixed point of A.

Proof. Choose $g_0 \in E$ and $j_0 \in F$ and suppose that $A(g_n) = g_{n+1}$ and $A(j_n) = j_{n+1}$ $\forall$ $n \in \mathbb{N} \cup \{0\}$. Then, we obtain (g_n, j_n) as a bi-sequence on FBMS $(E, F, \alpha, \circ)$. Now, we have

$$\Omega(\alpha(g_1, j_1, p)) = \Omega(\alpha(Ag_0, Aj_0, p)) \geq \Pi(\alpha(g_0, j_0, p))$$

for all $p > 0$ and $n \in \mathbb{N}$. By continuing this process, we obtain

$$\Omega(\alpha(g_{n+1}, j_{n+1}, p)) = \Omega(\alpha(Ag_n, Aj_n, p)) \geq \Pi(\alpha(g_n, j_n, p)) \tag{12}$$

for all $p > 0$ and $n \in \mathbb{N}$. For the next proof, we use six steps to prove the statement.

Step 1. For all $p > 0$, the bi-sequence $\{\alpha(g_n, j_n, p)\}_{n\in\mathbb{N}} \subseteq (0, 1]$ is non-decreasing.

Let $p > 0$ be arbitrary. We consider two cases depending on $\alpha(g_{n+1}, j_{n+1}, p) = 1$ or $\alpha(g_{n+1}, j_{n+1}, p) < 1$.

• If $\alpha(g_{n+1}, j_{n+1}, p) = 1$, then

$$\Omega(\alpha(g_{n+1}, j_{n+1}, p)) \geq \Pi(\alpha(g_n, j_n, p)) = \Pi(1).$$

In such a case, Property (p4) leads to

$$\alpha(g_{n+1}, j_{n+1}, p) = \alpha(g_n, j_n, p) = 1$$

Specifically,

$$\alpha(g_{n+1}, j_{n+1}, p) \geq \alpha(g_n, j_n, p).$$

• If $\alpha(g_n, j_n, p) \in (0, 1)$, then (12) and Property (p2) certify that

$$\Omega(\alpha(g_{n+1}, j_{n+1}, p)) \geq \Pi(\alpha(g_n, j_n, p)) > \Omega(\alpha(g_n, j_n, p)).$$

As Ω is non-decreasing according to Property (p1), then

$$\alpha(g_{n+1}, j_{n+1}, p) \geq \alpha(g_n, j_n, p).$$

We proved that the bi-sequence $\{\alpha(g_n, j_n, p)\}_{n\in\mathbb{N}} \subseteq (0,1]$ is non-decreasing. This property allows us to define the function $\tau : (0,\infty) \to (0,1]$ as

$$\tau(p) = \lim_{n\to\infty} \alpha(g_n, j_n, p) \text{ for all } p > 0.$$

Step 2. $\tau(p) = 1$ for all $p > 0$

Suppose $p > 0$. If there is $n_0 \in \mathbb{N}$ such that $\alpha(g_{n_0}, j_{n_0}, p) = 1$, then $\alpha\big(g_{n_0+1}, j_{n_0+1}, p\big) \geq \alpha(g_{n_0}, j_{n_0}, p) = 1$, so, $\alpha(g_{n_0}, j_{n_0}, p) = 1$. In this case, by induction, we can check that $\alpha(g_{n_0}, j_{n_0}, p) = 1$ for all $n \geq n_0$, which implies that $\tau(p) = \lim_{n\to\infty} \alpha(g_n, j_n, p) = 1$. Next, suppose that

$$\tau(p) = \lim_{n\to\infty} \alpha(g_n, j_n, p) < 1 \text{ for all } n \in \mathbb{N}.$$

In this case, (12) and Property (p2) confirm that

$$\Omega(\alpha(g_{n+1}, j_{n+1}, p)) \geq \Pi(\alpha(g_n, j_n, p)) > \Omega(\alpha(g_n, j_n, p)). \tag{13}$$

As Ω is non-decreasing, then

$$\alpha(g_{n+1}, j_{n+1}, p) > \alpha(g_n, j_n, p) \text{ for all } n \in \mathbb{N}.$$

In order to prove that $\tau(p) = 1$, suppose, by contradiction, that $\tau(p) < 1$. In such a case,

$$0 < \alpha(g_n, j_n, p) < \alpha(g_{n+1}, j_{n+1}, p) < \tau(p) < 1 \text{ for all } n \in \mathbb{N}.$$

Consider

$$\lim_{n\to\infty} \alpha(g_n, j_n, p) = \lim_{n\to\infty} \alpha(g_{n+1}, j_{n+1}, p) = \tau(p);$$

thus, it follows that

$$\lim_{n\to\infty} \Omega(\alpha(g_n, j_n, p)) = \lim_{n\to\infty} \Omega(\alpha(g_{n+1}, j_{n+1}, p)) = \lim_{c\to\tau(p)^-} \Omega(c).$$

This limit exists and is finite because Ω is well defined on $(0,1]$ and is non-decreasing on $(0,1)$. By letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} \Pi(\alpha(g_n, j_n, p)) = \lim_{c\to\tau(p)^-} \Omega(c).$$

However, this contradicts Property (p3) because

$$\lim_{c\to\tau(p)^-} \Omega(c) = \lim_{n\to\infty} \Pi(\alpha(g_n, j_n, p)) \geq \lim_{c\to\tau(p)^-} \inf \Pi(r) > \lim_{c\to\tau(p)^-} \Omega(c).$$

This contraction shows that $\tau(p) = 1$ for all $p > 0$, which completes Step 2 and proves that

$$\lim_{n\to\infty} \alpha(g_n, j_n, p) = 1 \text{ for all } p > 0. \tag{14}$$

Step 3. The bi-sequence (g_n, j_n) is either almost constant or infinite, and in this last case,

$$Ag_{n_1} \neq Aj_{n_1} \text{ for all } n_1, n_2 \in \mathbb{N} \text{ such that } n_1 \neq n_2. \tag{15}$$

If we suppose that $g_{n_1} = j_{n_1}$ for $n_1, n_2 \in \mathbb{N}$ and we consider (14), Proposition 5 guarantees that the bi-sequence (g_n, j_n) is almost constant. This means that there are $n_0 \in \mathbb{N}$ and $s \in E \cap F$. Therefore, s is a fixed point of A, and we are finished. Oppositely, let $g_{n_1} \neq j_{n_1}$ for any $n_1, n_2 \in \mathbb{N}$ such that $n_1 \neq n_2$. For the second case, by continuing the process, we can see that (15) is satisfied.

Step 4. We claim that (g_n, j_n) is a Cauchy bi-sequence.

Contrarily, assume that (g_n, j_n) is not a Cauchy bi-sequence. Hence, there are $H_0 \in (0,1)$, $p_0 > 0$, and two partial bi-subsequences $\{g_{n_k}\}$ and $\{g_{m_k}\}$ of $\{g_n\}$ and $\{j_{n_k}\}$ and $\{j_{m_k}\}$ of $\{j_n\}$, such that for each $k \in \mathbb{N}$, the following holds:

$$k < m_k < n_k < m_{k+1} \text{ and}$$

$$\alpha(g_{m_k}, j_{n_k-1}, p_0) > H_0 \geq \alpha(g_{m_k}, j_{n_k}, p_0), \tag{16}$$

and

$$\lim_{k\to\infty} \alpha(g_{m_k}, j_{n_k}, p_0) = \lim_{k\to\infty} \alpha(g_{m_k}, j_{n_k-1}, p_0) = H_0. \tag{17}$$

Since $\lim_{n\to\infty} \alpha(g_{m_k}, j_{n_k}, p_0) = H_0 < 1$, there is $n_0 \in \mathbb{N}$ such that

$$\alpha(g_{m_k}, j_{n_k}, p_0) < 1 \text{ for all } n \geq n_0.$$

Assume that

$$\alpha(g_{m_k}, j_{n_k}, p_0) < 1 \text{ for all } n \in \mathbb{N}. \tag{18}$$

By applying (11), Property (p2), and (18), we conclude that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \Omega(\alpha(g_{m_k}, j_{n_k}, p_0)) &= \Omega(\alpha(Ag_{m_k-1}, Aj_{n_k-1}, p_0)) \\ &\geq \Pi(\alpha(g_{m_k-1}, j_{n_k-1}, p_0)) \\ &> \Omega(\alpha(g_{m_k-1}, j_{n_k-1}, p_0)). \end{aligned}$$

In particular,

$$\Omega(\alpha(g_{m_k}, j_{n_k}, p_0)) = \Omega(\alpha(Ag_{m_k-1}, Aj_{n_k-1}, p_0)) > \Omega(\alpha(g_{m_k-1}, j_{n_k-1}, p_0)). \tag{19}$$

Since Ω is non-decreasing, then

$$\alpha(g_{m_k}, j_{n_k}, p_0) > \alpha(g_{m_k-1}, j_{n_k-1}, p_0),$$

With (19), we obtain

$$\alpha(g_{m_k-1}, j_{n_k-1}, p_0) < \alpha(g_{m_k}, j_{n_k}, p_0) \leq H_0 < 1 \text{ for all } n \in \mathbb{N}. \tag{20}$$

Using (19) and (20), we conclude that

$$\lim_{k\to\infty} \Omega(\alpha(g_{m_k}, j_{n_k}, p_0)) = \lim_{k\to\infty} \Omega(\alpha(g_{m_k-1}, j_{n_k-1}, p_0)) = \lim_{c\to H_0^-} \Omega(c).$$

If $n \to \infty$ in (19), it follows that

$$\lim_{k\to\infty} \Pi(\alpha(g_{m_k-1}, j_{n_k-1}, p_0)) = \lim_{c\to H_0^-} \Omega(c).$$

However, this is a contradiction to Property (p3) because

$$\lim_{c \to H_0^-} \Omega(c) = \lim_{k \to \infty} \Pi\big(\alpha\big(g_{m_k - 1}, j_{n_k - 1}, p_0\big)\big) \geq \lim_{p \to H_0^-} \inf \Pi(p) > \lim_{c \to H_0^-} \Omega(c).$$

which is contradiction, stating that (g_n, j_n) is a Cauchy bi-sequence.

As $(E, F, \alpha, \circ)$ is a complete FBMS, there is $s \in E \cap F$ such that (g_n, j_n) is bi-convergent to s. As stated in Proposition 1, the bi-sequence $\big(\hbar_p, \ell_p\big)$ is bi-convergent.

As bi-sequence (g_n, j_n) is bi-convergent, then $\exists\, s \in E \cap F$, which is a limit of both sequences $\{g_n\}$ and $\{j_n\}$. Using Lemma 2, we have a unique limit for both bi-sequences $\{g_n\}$ and $\{j_n\}$, that is,

$$\lim_{n \to \infty} (g_n, s, p) = 1 \text{ for all } p > 0.$$

Step 5. An element $s \in E \cap F$ is a fixed point of A.

Oppositely, suppose that s is not a fixed point of A. As the bi-sequence (g_n, j_n) is infinite, then there is $n_0 \in \mathbb{N}$ such that $g_n \neq As$ and $As \neq g_n$ for every $n \geq n_0$. Assume that

$$g_n \neq As \text{ and } Ag_n \neq As \text{ for each } n \in \mathbb{N}.$$

Condition (4) of the theorem leads to

$$\Omega(\alpha(g_{n+1}, As, p)) \geq \Pi(\alpha(Ag_n, As, p)) > \Omega(\alpha(g_n, s, p)),$$

for each $n \in \mathbb{N}$ and each $p > 0$. Now, we investigate that $\alpha(g_{n+1}, As, p) \geq \alpha(g_n, s, p)$ by discussing two cases, as follows:

• If $\alpha(g_n, s, p) = 1$, then

$$\Omega(\alpha(g_{n+1}, As, p)) \geq \Pi(\alpha(Ag_n, As, p)) = \Pi(1).$$

By assumption, Property (p4) guarantees that $\alpha(g_{n+1}, As, p) = \alpha(g_n, As, p) = 1$.

In particular, $\alpha(g_{n+1}, As, p) \geq \alpha(g_n, As, p)$;

• If $\alpha(g_n, s, p) < 1$, then

$$\Omega(\alpha(g_{n+1}, As, p)) \geq \Pi(\alpha(Ag_n, As, p)) > \Omega(\alpha(g_n, s, p)),$$

as Ω is non-decreasing, we conclude that $\alpha(g_{n+1}, As, p) > \alpha(Ag_n, As, p)$.

In both cases, we checked that

$$\alpha(g_n, As, p) \leq \alpha(g_{n+1}, As, p) \text{ for all } n \in \mathbb{N} \text{ and all } p > 0,$$

which means that the bi-sequence (g_n, j_n) also converges to As. The uniqueness of the limit of a bi-convergent sequence in an FBMS demonstrates that $As = s$.

Step 6. The mapping A has a unique fixed point in $(E, F, \alpha, \circ)$.

Finally, suppose that $s_1, s_2 \in E \cap F$ are two distinct fixed points of A. Since $As_1 \neq As_2$, then, for all $p > 0$,

$$\Omega(\alpha(s_1, s_2, p)) = \Omega(\alpha(As_1, As_2, p)) \geq \Pi(\alpha(s_1, s_2, p)).$$

If we suppose that $\alpha(s_1, s_2, p) < 1$ for some $p > 0$, then

$$\Omega(\alpha(s_1, s_2, p)) \geq \Pi(\alpha(s_1, s_2, p)) > \Omega(\alpha(s_1, s_2, p)),$$

which is a contradiction. Hence, $\alpha(s_1, s_2, p) = 1$ for all $p > 0$. Therefore, A has a unique limit. □

Example 5. *Let $E = [0,1]$ and $F = [1,2]$ be equipped with $\alpha(g,j,p) = e^{-\frac{|g-j|}{p}}$ for all $g \in E$ and $j \in F$. Then, $(E, F, \alpha, \circ)$ is a complete FBMS. Define $A : E \cup F \to E \cup F$ by*

$$A(s) = \frac{s+4}{5}$$

for all $s \in E \cup F$. Now, define the function $\Omega, \Pi : (0, \infty) \to \mathbb{R}$ by

$$\Omega(r) = \frac{1}{2^{\ln 2r}} \qquad \Pi(r) = \frac{1}{2^{\ln r}} \text{ for all } r \in (0,1].$$

Now, we have to show that A satisfies (11). Therefore, we consider $g = 0$ and $j = 1, p = 1$ as follows:

$$\begin{aligned} \Omega(\alpha(A(0), A(1), 1)) &\geq \Pi(\alpha(0,1,1)) \\ \Omega\left(\alpha\left(\frac{4}{5}, 1\right)\right) &\geq \Pi(\alpha(0,1,1)) \\ \Omega(0.8187) &\geq \Pi(0.3678) \\ 2.0280 &\geq 2.0001. \end{aligned}$$

Hence, all the other conditions of Theorem 5 hold. The fixed point of A is 0. This is similar to other cases.

Now, we see that without (Ω, Π), $g = 0$ and $j = 1$ do not hold. Therefore, we take $\lambda = 0.1$

$$\begin{aligned} (\alpha(A(0), A(1), (0.1)(1))) &\geq \alpha(0,1,1) \\ \alpha\left(\frac{4}{5}, 1, (0.1)\right) &\geq \alpha(0,1,1) \\ 0.1353 &\geq 0.3678, \end{aligned}$$

which is a contradiction. Hence, it does not hold without (Ω, Π).

Corollary 1. *Suppose $(E, F, \alpha, \circ)$ is a complete FBMS, and suppose that $A : E \cup F \to E \cup F$ is a mapping for which there exists $(\Omega, \Pi) \in L$, such that*

$$\Omega(\alpha(Ag.Aj, p)) \geq \Pi(\alpha(g, j, p)) \text{ for all } g \in E \text{ and } j \in F \text{ and } p > 0.$$

Then, each iterative Picard bi-sequence $(A^n g, A^n g)_{n \in \mathbb{N}}$ is bi-convergent to the unique fixed point of A for every initial condition $g_0 \in E$ and $j_0 \in F$.

Proof. The proof of Corollary 1 is taken as being the same as the proof of Theorem 5. □

Corollary 2. *Suppose that $(E, F, \alpha, \circ)$ is a complete FBMS and $A : E \cup F \to E \cup F$ is a mapping for which there exists $(\Omega, \Pi) : (0,1) \to \mathbb{R}$, such that*

$$\Omega(\alpha(Ag.Aj, p)) \geq \Pi(\alpha(g, j, p)) \text{ for all } g \in E \text{ and } j \in F \text{ with } Ag \neq Aj \text{ and } p > 0.$$

Suppose that Ω and Π verify the following assumptions:

($p'1$) Ω is non-decreasing;
($p'2$) $\Pi(g) > \Omega(g)$ for any $g \in (0,1)$;
($p'3$) $\lim_{s \to H^-} \inf \Pi(g) > \lim_{s \to H^-} \Pi(g)$ for any $H \in (0,1)$;
($p'4$) $\Pi(1) \geq \sup(\{\Pi(g) : g \in (0,1)\})$.

Then, each iterative Picard bi-sequence $(A^n g, A^n g)_{n\in\mathbb{N}}$ is bi-convergent to the unique fixed point of A for every initial condition $g_0 \in E$ and $j_0 \in F$.

Proof. By applying Property (p2), Condition $\left(p'4\right)$ implies Property (p4). Suppose that $g \in (0,1]$ such that $\Omega(g) \geq \Pi(1)$. To prove that $g = 1$, we suppose the opposite of $g < 1$, i.e.,

$$\Omega(g) \geq \Pi(1) \geq \sup(\{\Pi(g) : g \in (0,1)\}) > \Pi(g).$$

However, $\Omega(g) \geq \Pi(g)$ contradicts Property (p2). Therefore, $g = 1$. Hence, the remaining proof is the same as the proof of Theorem 5. □

Corollary 3. *Suppose that $(E, F, \alpha, \circ)$ is a complete FBMS, and suppose that $A : E \cup F \to E \cup F$ is a mapping, such that*

$$\alpha(Ag.Aj, p) \geq \sqrt{\alpha(g, j, p)} \text{ for all } g \in E \text{ and } j \in F \text{ with } Ag \neq Aj \text{ and } p > 0.$$

Then, each iterative Picard bi-sequence $\{A^n g, A^n g\}_{n\in\mathbb{N}}$ is bi-convergent to the unique fixed point of A for every initial condition $g_0 \in E$ and $j_0 \in F$.

Proof. If we take $\Omega(g) = g$ and $\Pi(g) = \sqrt{g}$, (Ω, Π) also holds for all the conditions of the L family. The remaining proof of the Corollary is taken as same as the proof of Theorem 5. □

4. Applications

In this part, we proved the applications of integral equations and chemical science.

4.1. An Application to Integral Equations

Let us consider the Banach space $C([0, I], \mathbb{R})$ of all continuous functions defined on a real interval $[0, I]$ (where $I > 0$) endowed with the supremum norm

$$\| g \| = \sup_{c\in[0,1]} | g(c) | \text{ for all } g \in C([0, I], \mathbb{R}),$$

with the following complete bi-polar metric:

$$\alpha(g, j) = \sup_{c\in[0,1]} | g(c) - j(c) | .$$

Consider the following integral equation:

$$g(c) = l(c) + \int_0^c D(c, k, g(k))dk, \text{ for all } c \in [0, I]. \tag{21}$$

Consider the FBMS with the product t-norm as follows:

$$\alpha(g, j, p) = \frac{p}{p + \alpha(g, j)} \text{ for all } g, j \in E \cup F, C, D \in ([0, I], \mathbb{R}) \text{ and all } p > 0. \tag{22}$$

According to George and Veeramani, the standard fuzzy metric space and the corresponding metric space are endowed by the same topology. Therefore, the fuzzy metric space defined by (22) is complete.

Theorem 6. *Suppose that the integral operator A on $C([0, I], \mathbb{R})$ is*

$$A(j(e)) = j(e) + \int_0^e D(e, m, j(e))de,$$

where $h : [0, I] \times [0, I] \rightarrow [0, \infty)$ is such that $h \in L^1([0, I], \mathbb{R})$, for each $g, j \in C([0, I], \mathbb{R})$, $c, t \in [0, I]$, and A fulfills the following conditions:

$$| D(e, m, g(m)) - D(e, m, j(m)) | \leq h(m, e) | g(e) - j(e) |,$$

for all $g, j \in C([0, I], \mathbb{R})$, $c, t \in [0, I]$, where

$$\sup_{c \in [0,I]} \int_0^c m(e, m) de \leq \lambda < 1.$$

Then, the integral Equation (21) has a unique solution.

Proof. As $g, j \in C \cup D$, $c, t \in [0, I]$, we have that

$$\begin{aligned} | A(g(c) - A(j(c))) | &\leq \int_0^c | D(c, p, g(p)) - D(c, p, j(p)) | \; dp \\ &\leq \alpha(g, j) \int_0^c m(e, m) de \\ &\leq \lambda \alpha(g, j). \end{aligned}$$

Therefore, the following holds:

$$\alpha(Ag, Aj) \leq \lambda \alpha(g, j).$$

Using (22), we can write

$$\alpha(Ag, Aj) \leq \lambda \alpha(g, j) \leq \alpha(g, j),$$

which can be interpreted as follows:

$$p + \alpha(Ag, Aj) \leq p + \alpha(g, j).$$

Hence, we have

$$\frac{p}{p + \alpha(Ag, Aj)} \geq \frac{p}{p + \alpha(g, j)},$$

which means that the following holds:

$$\alpha(Ag, Aj, p) \geq \alpha(g, j, p) \geq \alpha(g, j, p)^2.$$

If we take $\Omega(p) = p$ and $\Pi(p) = \sqrt{p}$, then we can write the above inequality as

$$\Omega(\alpha(Ag, Aj, p)) \geq \Pi(\alpha(g, j, p)).$$

Since all the conditions of Theorem 5 hold, we conclude that (21) has a unique solution. □

4.2. An Application to Chemical Science

Consider a diffusing substance placed in an absorbing medium between parallel walls such that δ_1 and δ_2 are the stipulated concentrations at walls. Furthermore, suppose $\Phi(b)$ to be the given source of density and $A(b)$ to be the known absorption coefficient. Then, the concentration $g(b)$ of the substance under the aforementioned hypothesis governs the following boundary value problem:

$$-g'' + A(b)g = \Phi(b); b \in [0, 1] = I \; g(0) = \delta_1, g(0) = \delta_2 \tag{23}$$

Theorem 5 is equivalent to the following integral equation:

$$g(b) = \delta_1 + (\delta_2 - \delta_1)b + \int_0^1 S(b,l)(\Phi(l) - A(l)g(l))dl, b \in [0,1], \tag{24}$$

where $S(b,l) : [0,1] \times \mathbb{R} \to \mathbb{R}$ is Green's function, which is continuous and given by

$$S(b,l) = b(1-l)\ 0 \le b \le l \le 1, l(1-b)\ 0 \le l \le b \le 1. \tag{25}$$

Suppose that $C(I,\mathbb{R}) = E \cup F$ is the space of all real-valued continuous functions defined on I, and let $E \cup F$ be endowed with the FBMS α defined by

$$\alpha(g,s) = \| g - s \|,$$

where $\| g \| = \sup\{| g(b) : b \in I\}$. Obviously, $(E,F,\alpha,\circ)$ is a complete FBMS with the product t-norm that is defined as

$$\alpha(g,j,p) = \frac{p}{p + \alpha(g,j)} \text{ for all } g,j \in C([0,I],\mathbb{R}) \text{ and all } p > 0. \tag{26}$$

Let the mapping $A : E \cup F \to E \cup F$ be defined as

$$Ag(b) = \delta_1 + (\delta_2 - \delta_1)b + \int_0^1 S(b,l)(\Phi(l) - A(l)g(l))dl,$$

where

$$\sup_{b\in[0,1]} \int_0^1 S(b,l)dl \le \lambda < 1.$$

Then, $s \in E \cap F$ is the unique solution of (24) if and only if it is a fixed point of A. The following theorem is provided for proving the existence of a fixed point of A.

Theorem 7. *Consider Theorem (5) and suppose that there exist $\kappa > 0$ and a continuous function $A(l) : I \to \mathbb{R}$ such that the following assertion holds:*

$$0 \le | A(l)g(l) - A(l)s(l) | \le s(l) - g(l).$$

Then, consider the integral Equation (24). Consequently, the boundary value problem (23) governing the concentration of diffusing subsequence has a unique solution in $E \cup F$.

Proof. Clearly, considering $g \in E \cup F$ and $b \in I$, the mapping $A : E \cup F \to E \cup F$ is well defined. Also, A is an FBMS:

$$\begin{aligned} &| Ag(b) - As(b) | \\ &= \left| \int_0^1 S(b,l)(\Phi(l) - A(l)g(l))dl - \int_0^1 S(b,l)(\Phi(l) - A(l)s(l))dl \right| \\ &\le \int_0^1 S(b,l) \mid (\Phi(l) - A(l)g(l)) - (\Phi(l) - A(l)s(l)) \mid dl \\ &\le \| g - s \| \sup_{b\in[0,1]} \int_0^1 S(b,l)dl \\ &\le \lambda \| g - s \| . \end{aligned}$$

Therefore, the following holds:

$$\alpha(Ag, Aj) \leq \lambda\alpha(g, j).$$

Using (26), we can write

$$\alpha(Ag, Aj) \leq \lambda\alpha(g, j) < \alpha(g, j),$$

which can be interpreted as the following:

$$p + \alpha(Ag, Aj) < p + \alpha(g, j).$$

Hence, we have

$$\frac{p}{p + \alpha(Ag, Aj)} > \frac{p}{p + \alpha(g, j)},$$

which means that the following holds:

$$\alpha(Ag, Aj, p) > \alpha(g, j, p) > \alpha(g, j, p)^2.$$

If we take $\Omega(p) = p$ and $\Pi(p) = \sqrt{p}$, then we can write the above inequality as

$$\Omega(\alpha(Ag, Aj, p)) > \Pi(\alpha(g, j, p)).$$

Since all the conditions of Theorem 5 hold, we conclude that the integral Equation (24) has a unique solution. Consequently, the boundary value problem (23) has a unique solution. □

5. Conclusions

In this work, we proved some fixed point theorems for bi-polar metric spaces and fuzzy bi-polar metric spaces. We provided some lemmas, corollaries, remarks and non-trivial examples. We solved the integral equation by applying our main result. Furthermore, we solved a boundary value problem that occurred in chemical science by applying the main result. Thus, researchers can enhance the results in the setting of fuzzy bi-polar multiplicative metric spaces, fuzzy bi-polar p-metric spaces, and many other structures.

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