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On Symmetrically Stochastic System of Fractional Differential Equations and Variational Inequalities

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Abstract: In this work, we are devoted to discussing a system of fractional stochastic differential variational inequalities with Lévy jumps (SFSDVI with Lévy jumps), that comprises both parts, that is, a system of stochastic variational inequalities (SSVI) and a system of fractional stochastic differential equations(SFSDE) with Lévy jumps. Here it is noteworthy that the SFSDVI with Lévy jumps consists of both sections that possess a mutual symmetry structure. Invoking Picard's successive iteration process and projection technique, we obtain the existence of only a solution to the SFSDVI with Lévy jumps via some appropriate restrictions. In addition, the major outcomes are invoked to deduce that there is only a solution to the spatial-price equilibria system in stochastic circumstances. The main contributions of the article are listed as follows: (a) putting forward the SFSDVI with Lévy jumps that could be applied for handling different real matters arising from varied domains; (b) deriving the unique existence of solutions to the SFSDVI with Lévy jumps under a few mild assumptions; (c) providing an applicable instance for spatial-price equilibria system in stochastic circumstances affected with Lévy jumps and memory.

Keywords: SFSDVI; SSVI; SFSDE; Lévy jumps

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1. Introduction

Let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ be the norm and inner product in $\mathbf{R}^n$ (or $\mathbf{R}^m$), respectively. We also use $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ to denote the norm and inner product in the product space $\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}$, respectively, that is,

$$\|\mathbf{y}\| = \sqrt{\|y_1\|^2+\|y_2\|^2} \text{ and } \langle\mathbf{y},\mathbf{x}\rangle = \langle y_1,x_1\rangle+\langle y_2,x_2\rangle, \ \forall\mathbf{y},\mathbf{x}\in\mathbf{R}^{n_1}\times\mathbf{R}^{n_2},$$

with $\mathbf{y}=(y_1,y_2)$ and $\mathbf{x}=(x_1,x_2)$. In the same way, the norm and inner product in $\mathbf{R}^{m_1}\times\mathbf{R}^{m_2}$ could be formulated, respectively.

Let DVI and SDE represent a differential variational inequality and a stochastic differential equation, respectively. Suppose $\{B_t\}$ is standard Brownian motion of l-dimension. Recently, stochastic differential VI was proposed and discussed in [1], outlined as follows:

$$\begin{cases} dy(s) = b(s, y(s), h(s))ds + \sigma(s, y(s), h(s))dB_s, \ \ s \in [0, T], \ y(0) = y_0, \\ \langle F(\omega, s, y(\omega, s), h(\omega, s)), x - h(\omega, s)\rangle \geq 0, \ \ \forall x \in \mathcal{K}, \ a.s. \ \omega \in \Omega, \ a.e. \ s \in [0, T], \end{cases} \tag{1}$$

where y_0 is a fixed random vector, $\mathcal{K} \subset \mathbf{R}^m$ is of both convexity and closedness, and single-valued mappings F, σ, b are measurable.

It was shown in [1] that there is only a solution to the above stochastic DVI and the solution is continue to parametric stochastic DVI. Their outcomes were also applied to treat some practical problems such as the spatial-price equilibria problems in stochastic circumstances. Meanwhile, Euler iterative approach is applied in [2] for settling stochastic DVI and the major results are exploited to handle some practical problems such as the circuits with electrical diodes in stochastic circumstances.

It is noteworthy that the above stochastic DVI is actually the classical DVI considered in [3] with stochastic circumstance effects. In accordance with [3], it is known that DVI furnishes an efficient modeling pattern to different applicable matters. So, stochastic differential VI could be exploited to address varied practical matters arising in different fields such as mechanics, finance and economy in stochastic circumstances. A range of theoretic results, iteration processes and computational instances had been acquired broadly for classical differential VI; refer to [4–17].

To the best of our awareness, in past decade, many scholars had found that there are jump and memory features for certain systems to display. Moreover, these features could not be enough explained by SDEs driven just by Brownian motions. As a result, on the basis of fractional calculus and Lévy jumps, certain academics captured memorability and instability of systems, independently; refer to [8,9,18–22]. With the help of the matters related to the stochastic systems effected with memory and jumps, Zeng et al. [23] presented and discussed a fractional stochastic DVI with Lévy jump (FSDVI with Lévy jump), formulated as follows:

$$\begin{cases} dy(s) = b(s, y(s-), h(s-))ds + \sigma_1(s, y(s-), h(s-))(ds)^\alpha + \sigma(s, y(s-), h(s-))dB_s \\ \qquad + \int_{\|y\|<c} G(s, y(s-), h(s-), y)\tilde{N}(ds, dy), \ \ \alpha \in (\frac{1}{2}, 1), \\ y(0) = y_0, \\ \langle F(\omega, s, y(\omega, s), h(\omega, s)), x - h(\omega, s)\rangle \geq 0, \ \ \forall x \in \mathcal{K}, \ a.s. \ \omega \in \Omega, \ a.e. \ s \in [0, T]. \end{cases} \tag{2}$$

To estimate the above fractional differential part, we realize that it serves as a special term for which it is not hard to reckon fractional differentiation expressed as $(ds)^\alpha$ for $\alpha \in (\frac{1}{2}, 1)$; refer to [20].

Taking into account the matters relevant to these stochastic systems effected by memory and jumps with $\alpha \in (\frac{1}{2}, 1)$ we now introduce and explore a system of fractional stochastic DVIs with Lévy jumps (SFSDVI with Lévy jumps), specified as follows:

$$\begin{cases} dx_1(s) = b_1(s, \mathbf{x}(s-), \mathbf{u}(s-))ds + \sigma_1^1(s, \mathbf{x}(s-), \mathbf{u}(s-))(ds)^\alpha + \sigma_0^1(s, \mathbf{x}(s-), \mathbf{u}(s-))dB_1(s) \\ \qquad + \int_{\|x_1\|<c_1} G_1(s, \mathbf{x}(s-), \mathbf{u}(s-), x_1)\widetilde{N}_1(ds, dx_1), \\ dx_2(s) = b_2(s, \mathbf{x}(s-), \mathbf{u}(s-))ds + \sigma_1^2(s, \mathbf{x}(s-), \mathbf{u}(s-))(ds)^\alpha + \sigma_0^2(s, \mathbf{x}(s-), \mathbf{u}(s-))dB_2(s) \\ \qquad + \int_{\|x_2\|<c_2} G_2(s, \mathbf{x}(s-), \mathbf{u}(s-), x_2)\widetilde{N}_2(ds, dx_2), \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases} \tag{3}$$

and

$$\begin{cases} \langle F_1(\omega, s, x_2(\omega, s), u_1(\omega, s)), v_1 - u_1(\omega, s)\rangle \geq 0, \ \ \forall v_1 \in K_1, \ a.e. \ s \in [0, T], \ a.s. \ \omega \in \Omega, \\ \langle F_2(\omega, s, x_1(\omega, s), u_2(\omega, s)), v_2 - u_2(\omega, s)\rangle \geq 0, \ \ \forall v_2 \in K_2, \ a.e. \ s \in [0, T], \ a.s. \ \omega \in \Omega, \end{cases} \tag{4}$$

in which $\mathbf{x}_0, \mathbf{x} \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ and $\mathbf{u} \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}$, with $\mathbf{x}_0 = (x_{1,0}, x_{2,0})$, $\mathbf{x} = (x_1, x_2)$ and $\mathbf{u} = (u_1, u_2)$. Here it is noteworthy that the SFSDVI with Lévy jumps consists of both sections that possess a mutual symmetry structure. Under making no misleading, we could employ $x_k(s)$ and $u_k(s)$ instead of $x_k(\omega, s)$ and $u_k(\omega, s)$ in the subsequent statement for $k = 1, 2$. Refer to the further description for more notations and detailed information.

A few special cases of the issue (3) and (4) are released as follows.

(i) In case $G_k = 0$ for each k, the issue (3) and (4) reduces to

$$\begin{cases} dx_1(s) = b_1(s, \mathbf{x}(s-), \mathbf{u}(s-))ds + \sigma_1^1(s, \mathbf{x}(s-), \mathbf{u}(s-))(ds)^\alpha + \sigma_0^1(s, \mathbf{x}(s-), \mathbf{u}(s-))dB_1(s), \\ dx_2(s) = b_2(s, \mathbf{x}(s-), \mathbf{u}(s-))ds + \sigma_1^2(s, \mathbf{x}(s-), \mathbf{u}(s-))(ds)^\alpha + \sigma_0^2(s, \mathbf{x}(s-), \mathbf{u}(s-))dB_2(s), \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases} \tag{5}$$

and

$$\begin{cases} \langle F_1(\omega, s, x_2(\omega, s), u_1(\omega, s)), v_1 - u_1(\omega, s)\rangle \geq 0, \ \ \forall v_1 \in K_1, \ a.e. \ s \in [0, T], \ a.s. \ \omega \in \Omega, \\ \langle F_2(\omega, s, x_1(\omega, s), u_2(\omega, s)), v_2 - u_2(\omega, s)\rangle \geq 0, \ \ \forall v_2 \in K_2, \ a.e. \ s \in [0, T], \ a.s. \ \omega \in \Omega. \end{cases} \tag{6}$$

Issue (5) and (6) serves as a new matter.

(ii) In case $\sigma_1^k = 0$ and $G_k = 0$ for each k, the issue (3) and (4) reduces to a generalization of SDVI (1) studied in [1,2,23].

(iii) In case $\sigma_0^k = 0$ and $G_k = 0$ for each k, the issue (3) and (4) reduces to a generalization of a special situation of FDVI presented and considered by varied academics; refer to [8,9,19,24–26].

Precisely speaking, through suitable selections of the measurability mappings $\sigma_0^k, \sigma_1^k, b_k, G_k$ and the set K_k for $k = 1, 2$, one could derive a range of prominent (stochastic) DVIs and their systems as special examples in terms of SFSDVI with Lévy jumps (3) and (4). Resembling FSDVI with Lévy jump (2), employed for treating numerous matters in stochastic circumstances, SFSDVI with Lévy jumps (3) and (4) could be exploited for expressing different systems of realistic stochastic problems, with memory and jumps. We shall in Section 4 provide an application of (3) and (4) to the spatial-price equilibria systems in stochastic circumstances influenced with memory and Lévy jumps.

As well as we know, there is no research work for one to study the symmetrical SFSDVI with Lévy jumps like (3) and (4). So, it will be interesting and valuable to investigate (3) and (4).

The main contributions of the article over other ones (see e.g., [11,23]) are listed as follows: (a) putting forward the SFSDVI with Lévy jumps (3) and (4) that could be applied for handling different real matters arising from varied domains; (b) deriving the unique existence of solutions to (3) and (4) under a few mild assumptions; (c) providing an applicable instance for spatial-price equilibria system in stochastic circumstances affected with Lévy jumps and memory.

2. Basic Concepts and Formulations

To deal with the symmetrical SFSDVI with Lévy jumps (3) and (4), one first releases some preliminaries, including some notions and basic tools.

- $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\geq 0}, \mathbf{P})$ denotes a complete probability space with filtration $\{\mathcal{F}_s\}$.
- $B(s)$ denotes Brownian motion that is l-dimensional and $\mathcal{F}_s$-adapted.
- $N : \mathbf{R}^+ \times (\mathbf{R}^n \setminus \{0\})$ and $B(s)$ are independent of each other, with N being jump $\mathcal{F}_s$-adapted measure; and the associated compensation is martingale measure, specified as follows:

$$\widetilde{N}(ds, dx) := N(ds, dx) - \nu(dx)ds,$$

whose intensity measure $\nu(\cdot)$ meets

$$\int_{\mathbf{R}^n\setminus\{0\}} \frac{x^2}{1+x^2}\nu(dx) < \infty.$$

- $\mathcal{L}^2(\Omega, \mathbf{R}^n)$ denotes the Hilbert space of $\mathbf{R}^n$-valued squared-integrable random variables, equipped with norm $\|\cdot\|_{\mathcal{L}^2} = (\mathbf{E}\|\cdot\|^2)^{1/2}$.
- $H_m[b,c] = \mathcal{L}^2_{ad}(\Omega \times [b,c], \mathbf{R}^m)$ denotes the Hilbert space of $\mathbf{R}^m$-valued $\mathcal{F}_s$-adapted stochastic processes, fulfilling $\int_b^c \mathbf{E}\|f(\omega,\tau)\|^2 d\tau < \infty, \forall f \in H_m[b,c]$, whose inner product is endowed by

$$\langle h, v\rangle_{H_m[b,c]} = \int_b^c \mathbf{E}(\langle h(\omega,s), v(\omega,s)\rangle)ds, \quad \forall h, v \in H_m[b,c],$$

  with $[b,c] \subset [0,T]$.
- $U_m[b,c] = \{h(\omega,\tau) \in \mathcal{L}^2_{ad}(\Omega \times [b,c], \mathbf{R}^m) : h(\omega,\tau) \in \mathcal{K},\ a.s.\ \omega \in \Omega,\ a.e.\ \tau \in [b,c]\}$, with $\mathcal{K} \subset \mathbf{R}^m$ being both convex and closed.

  Let $K_k \subset \mathbf{R}^{m_k}$ be convex and closed. We specify the spatial-products $\mathbf{X} = \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ and $\mathbf{V} = \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}$. In what follows, we furnish the specific details for (3) and (4). For $i = 1,2$, assume the following conditions hold throughout.
- $\int_0^s \int_{\|x_i\|<c_i} G_i(\iota, \mathbf{x}(\iota-), \mathbf{u}(\iota-), x_i)\tilde{N}_i(d\iota, dx_i)$ is an $\mathbf{R}^{n_i}$-valued martingale of square integrability, satisfying

$$\mathbf{P}(\int_{\|x_i\|<c_i} \|G_i(\iota, \mathbf{x}(\iota-), \mathbf{u}(\iota-), x_i)\|\nu_i(dx_i)d\iota < \infty) = 1$$

  in which the constant $c_i > 0$ denotes the jump size of allowable maximality.
- $\mathbf{x}_0$ is the starting datum fulfilling $\mathbf{E}\|\mathbf{x}_0\|^2 < \infty$.
- $b_i : [0,T] \times \mathbf{X} \times \mathbf{V} \to \mathbf{R}^{n_i}$.
- $\sigma_0^i : [0,T] \times \mathbf{X} \times \mathbf{V} \to \mathbf{R}^{n_i \times l_i}$.
- $G_i : [0,T] \times \mathbf{X} \times \mathbf{V} \times \mathbf{R}^{n_i} \to \mathbf{R}^{n_i}$.
- $F_1 : \Omega \times [0,T] \times \mathbf{R}^{n_2} \times K_1 \to \mathbf{R}^{m_1}$ and $F_2 : \Omega \times [0,T] \times \mathbf{R}^{n_1} \times K_2 \to \mathbf{R}^{m_2}$.
- $\sigma_1^i : [0,T] \times \mathbf{X} \times \mathbf{V} \to \mathbf{R}^{n_i}$ is of continuity with respect to s.

Next, we provide vital notion involving solutions of (3) and (4), and other concepts that will be useful to demonstrate the major results.

Definition 1. *Let $\frac{1}{2} < \alpha < 1$. The pair $(\mathbf{x}(s), \mathbf{u}(s))$ is said to be a solution of (3) and (4) if $\mathbf{x}(s) \in \mathcal{L}^2_{ad}(\Omega \times [0,T], \mathbf{X})$ satisfying*

$$\begin{cases} dx_1(s) = b_1(s,\mathbf{x}(s-),\mathbf{u}(s-))ds + \sigma_1^1(s,\mathbf{x}(s-),\mathbf{u}(s-))(ds)^\alpha + \sigma_0^1(s,\mathbf{x}(s-),\mathbf{u}(s-))dB_1(s) \\ \qquad + \int_{\|x_1\|<c_1} G_1(s,\mathbf{x}(s-),\mathbf{u}(s-),x_1)\widetilde{N}_1(ds,dx_1), \\ dx_2(s) = b_2(s,\mathbf{x}(s-),\mathbf{u}(s-))ds + \sigma_1^2(s,\mathbf{x}(s-),\mathbf{u}(s-))(ds)^\alpha + \sigma_0^2(s,\mathbf{x}(s-),\mathbf{u}(s-))dB_2(s) \\ \qquad + \int_{\|x_2\|<c_2} G_2(s,\mathbf{x}(s-),\mathbf{u}(s-),x_2)\widetilde{N}_2(ds,dx_2), \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases} \tag{7}$$

and

$$\begin{cases} u_1(s) \in \mathrm{SOL}(U_{m_1}[0,T], F_1(\omega,s,x_2(\omega,s),u_1(\omega,s))), \\ u_2(s) \in \mathrm{SOL}(U_{m_2}[0,T], F_2(\omega,s,x_1(\omega,s),u_2(\omega,s))), \end{cases} \tag{8}$$

in which $\mathrm{SOL}(U_{m_i}[0,T], F_i(\omega,s,x_j(\omega,s),u_i(\omega,s)))$ is the solution set of the SVI: seek $u_i \in U_{m_i}[0,T]$ s.t.

$$\langle F_i(\omega,s,x_j(\omega,s),u_i(\omega,s)), v_i - u_i(\omega,s)\rangle \geq 0, \ \ \forall v_i \in K_i,\ a.s.\ \omega \in \Omega,\ a.e.\ s \in [0,T].$$

If the pair $(\mathbf{x}(s), \mathbf{u}(s))$ is unique in the almost everywhere sense, we say that there holds the unique existence of solutions to (3) and (4).

Take a fixed $f \in \mathcal{L}^1([b,c];\mathbf{R}^d)$ arbitrarily. In terms of [20], we recall the left Riemann-Liouville α-order fractional integral, specified below

$$(I^{\alpha}_{a+}f)(t) = \frac{1}{\Gamma(\alpha)}\int_b^t (t-s)^{\alpha-1}f(s)ds, \quad t > b,$$

with $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$. In addition, from [27] we known that if f is also absolutely continuous, then left Riemann-Liouville α-order fractional derivative is specified below

$$(D^{\alpha}_{b+}f)(s) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{ds}\int_b^s (s-t)^{-\alpha}f(t)dt, \quad s > b.$$

Next, it is noteworthy to mention that we are concentrated on the situation of $b = 0$ in the formulation above, that is,

$$(D^{\alpha}_{0+}f)(s) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{ds}\int_0^s (s-t)^{-\alpha}f(t)dt.$$

According to [28], one has $D^{\alpha}_{0+} = \frac{d^\alpha}{(ds)^\alpha}$ and $(d^\alpha f)(s) = \Gamma(1+\alpha)(df)(s) = (D^{\alpha}_{0+}f)(s)(ds)^\alpha$. Setting $g(s) = (D^{\alpha}_{0+}f)(s)$, one obtains

$$\begin{aligned}\int_0^t g(s)(ds)^\alpha &= \Gamma(1+\alpha)f(t) = \Gamma(1+\alpha)D^{-\alpha}_{0+}g(t)\\ &= \frac{\Gamma(1+\alpha)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds = \alpha\int_0^t (t-s)^{\alpha-1}g(s)ds.\end{aligned}$$

Therefore, the system of fractional stochastic differential equations (3) could be rephrased as

$$\begin{cases} x_1(t) = x_{1,0} + \int_0^t b_1(s,\mathbf{x}(s-),\mathbf{u}(s-))ds + \alpha\int_0^t (t-s)^{\alpha-1}\sigma_1^1(s,\mathbf{x}(s-),\mathbf{u}(s-))ds \\ \qquad + \int_0^t \sigma_0^1(s,\mathbf{x}(s-),\mathbf{u}(s-))dB_1(s) + \int_0^t\int_{\|x_1\|<c_1} G_1(s,\mathbf{x}(s-),\mathbf{u}(s-),x_1)\widetilde{N}_1(ds,dx_1), \\ x_2(t) = x_{2,0} + \int_0^t b_2(s,\mathbf{x}(s-),\mathbf{u}(s-))ds + \alpha\int_0^t (t-s)^{\alpha-1}\sigma_1^2(s,\mathbf{x}(s-),\mathbf{u}(s-))ds \\ \qquad + \int_0^t \sigma_0^2(s,\mathbf{x}(s-),\mathbf{u}(s-))dB_2(s) + \int_0^t\int_{\|x_2\|<c_2} G_2(s,\mathbf{x}(s-),\mathbf{u}(s-),x_2)\widetilde{N}_2(ds,dx_2). \end{cases}$$

For detailed information, refer to [18,20,28].

In what follows one releases the following lemmas for the subsequent usage.

Lemma 1 ([1]). *Let $[b,c] \subset [0,T]$ and $\emptyset \neq \mathcal{K} \subset \mathbf{R}^m$ where $\mathcal{K}$ is closed and convex. Then $\emptyset \neq U_m[b,c] \subset \mathcal{L}^2_{ad}(\Omega \times [b,c], \mathbf{R}^m)$ where $U_m[b,c]$ is closed and convex.*

Let K be a nonempty closed convex subset of a real Hilbert space H. We then know from [23] that for each x in H, there exists the unique y in K, denoted by $P_K(x)$, that is $y = P_K(x)$, s.t. $\text{dist}(x,K) = \min_{v\in K}\|x-v\|_H = \|x-y\|_H$. Moreover, for a point $y \in K$, it holds that: $y = P_K(x) \Leftrightarrow \langle x-y, v-y\rangle_H \le 0, \ \forall v \in K$. In addition, let $A : H \to H$ be a mapping. It then follows from [23] that there holds the equivalence of the relations below:

(a) $v \in K$ is a solution to the VI: $\langle Av, y-v\rangle_H \ge 0$ for all $y \in K$;
(b) $v = P_K(v - \mu Av)$ with coefficient $\mu > 0$.

Lemma 2 ([1]). *Take an element $x \in \mathcal{L}^2_{ad}(\Omega \times [b,c], \mathbf{R}^n)$ arbitrarily. One then has that, for $v \in U_m[b,c]$, the following relations are equivalent:*

(i) $v(\omega,s) \in \mathcal{K}$ solves the SVI:

$$\langle F(\omega,s,x(\omega,s),v(\omega,s)), y - v(\omega,s)\rangle \ge 0, \quad \forall y \in \mathcal{K},\ a.s.\ \omega \in \Omega,\ a.e.\ s \in [b,c];$$

(ii) $v \in U_m[b,c]$ *solves the VI:*

$$\langle \widetilde{F}(x,v), w-v\rangle_{H_m[b,c]} \geq 0, \quad \forall w \in U_m[b,c],$$

in which $\widetilde{F}(x,v)(\omega,s) := F(\omega,s,x(\omega,s),v(\omega,s))$ *for all* $(x,v) \in \mathcal{L}^2_{ad}(\Omega\times[b,c],\mathbf{R}^n)\times U_m[b,c]$ *and* $(\omega,s)\in\Omega\times[b,c]$.

Lemma 3 ([29], Doob-type Inequality). *Suppose that* $q\in[1,\infty)$ *and the martingale* $\ell(s)$ *is right-continuous s.t.* $\mathbf{E}\|\ell(s)\|^q<\infty$ *for all* $s\geq 0$. *Then*

$$\mathbf{P}(\sup_{s\in[0,T]}\|\ell(s)\|>\epsilon)\leq \frac{\mathbf{E}\|\ell(T)\|^q}{\epsilon^q} \quad \text{for all } T>0$$

and for $q>1$,

$$\mathbf{E}(\sup_{s\in[0,T]}\|\ell(s)\|^q)\leq (\frac{q}{q-1})^q\mathbf{E}\|\ell(T)\|^q \quad \text{for all } T>0.$$

Lemma 4 ([30], Itô-type Isometry). *Take a positive number* T *arbitrarily. Then*

$$\mathbf{E}[(\int_0^T h(\omega,s)dB_s)^2]=\mathbf{E}[\int_0^T h^2(\omega,s)ds], \quad \text{for all } h\in\mathcal{V}(0,T),$$

in which $\mathcal{V}(0,T)$ *denotes the family of functions* $h:\Omega\times[0,T]\to\mathbf{R}$ *satisfying:*

(a) h *is* $\mathcal{B}\times\mathcal{F}$ *measurable, in which* $\mathcal{B}$ *denotes Borel-*σ*-algebra on* $[0,T]$;

(b) h *is* $\mathcal{F}_s$*-adapted;*

(c) $\mathbf{E}[\int_0^T h^2(\omega,s)ds]<\infty$.

3. Solvability of Problem (3) and (4)

We are now ready to present and demonstrate that there holds the unique existence of solutions of the symmetrical SFSDVI with Lévy jumps (3) and (4). For $i=1,2$, assume the following conditions hold throughout.

Assumption 1. *Take* $s,\iota\in[0,T]$ *arbitrarily, with constant* $T>0$, $\mathbf{x}_2,\mathbf{x}_1,\mathbf{x}\in\mathbf{X}$, $\mathbf{u}_2,\mathbf{u}_1,\mathbf{u}\in\mathbf{V}$, $\tilde{x}_{i,2},\tilde{x}_{i,1},\tilde{x}_i\in\mathcal{L}^2_{ad}(\Omega\times[0,T],\mathbf{R}^{n_i})$ *and* $\tilde{u}_{i,2},\tilde{u}_{i,1}\in U_{m_i}[0,T]$. *Suppose throughout that there exist positive constants* $\overline{C}_i, L_{b_i}, L_{\sigma_1^i}, L_{\sigma_0^i}, L_{G_i}, K_{b_i}, K_{\sigma_1^i}, K_{\sigma_0^i}, K_{G_i}$, *and* L_{F_i} *with* $L_{F_i}>\overline{C}_i$ *s.t.*

(i)
$$\|b_i(s,\mathbf{x},\mathbf{u})\|^2\leq K_{b_i}(1+\|\mathbf{x}\|^2+\|\mathbf{u}\|^2);$$
$$\|\sigma_0^i(s,\mathbf{x},\mathbf{u})\|^2_{\mathbf{R}^{n_i\times l_i}}\leq K_{\sigma_0^i}(1+\|\mathbf{x}\|^2+\|\mathbf{u}\|^2);$$
$$\|\sigma_1^i(s,\mathbf{x},\mathbf{u})\|^2\leq K_{\sigma_1^i}(1+\|\mathbf{x}\|^2+\|\mathbf{u}\|^2);$$
$$\int_{\|\tilde{x}_i\|<c_i}\|G_i(s,\mathbf{x},\mathbf{u},\tilde{x}_i)\|^2\nu_i(d\tilde{x}_i)\leq K_{G_i}(1+\|\mathbf{x}\|^2+\|\mathbf{u}\|^2);$$

(ii)
$$\|b_i(\iota,\mathbf{x}_2,\mathbf{u}_2)-b_i(\iota,\mathbf{x}_1,\mathbf{u}_1)\|^2\leq L_{b_i}(\|\mathbf{u}_1-\mathbf{u}_2\|^2+\|\mathbf{x}_1-\mathbf{x}_2\|^2);$$
$$\|\sigma_0^i(\iota,\mathbf{x}_2,\mathbf{u}_2)-\sigma_0^i(\iota,\mathbf{x}_1,\mathbf{u}_1)\|^2_{\mathbf{R}^{n_i\times l_i}}\leq L_{\sigma_0^i}(\|\mathbf{u}_1-\mathbf{u}_2\|^2+\|\mathbf{x}_1-\mathbf{x}_2\|^2);$$
$$\|\sigma_1^i(\iota,\mathbf{x}_2,\mathbf{u}_2)-\sigma_1^i(\iota,\mathbf{x}_1,\mathbf{u}_1)\|^2\leq L_{\sigma_1^i}(\|\mathbf{u}_1-\mathbf{u}_2\|^2+\|\mathbf{x}_1-\mathbf{x}_2\|^2);$$
$$\int_{\|\tilde{x}_i\|<c_i}\|G_i(\iota,\mathbf{x}_2,\mathbf{u}_2,\tilde{x}_i)-G_i(\iota,\mathbf{x}_1,\mathbf{u}_1,\tilde{x}_i)\|^2\nu_i(d\tilde{x}_i)\leq L_{G_i}(\|\mathbf{u}_1-\mathbf{u}_2\|^2+\|\mathbf{x}_1-\mathbf{x}_2\|^2);$$

(iii) *for* $j=1,2$ *and* $j\neq i$
$$L_{F_i}(\|\tilde{u}_{i,1}-\tilde{u}_{i,2}\|_{H_{m_i}[0,T]}+\|\tilde{x}_{j,1}-\tilde{x}_{j,2}\|_{H_{n_j}[0,T]})\geq\|\widetilde{F}_i(\tilde{x}_{j,1},\tilde{u}_{i,1})-\widetilde{F}_i(\tilde{x}_{j,2},\tilde{u}_{i,2})\|_{H_{m_i}[0,T]};$$
$$\overline{C}_i\|\tilde{u}_{i,1}-\tilde{u}_{i,2}\|_{H_{m_i}[0,T]}\leq\langle\widetilde{F}_i(\tilde{x}_j,\tilde{u}_{i,1})-\widetilde{F}_i(\tilde{x}_j,\tilde{u}_{i,2}),\tilde{u}_{i,1}-\tilde{u}_{i,2}\rangle_{H_{m_i}[0,T]}.$$

Because of the associated inferences with [2], one hence obtains two consequences below.

Lemma 5 ([2]). *If condition (iii) of Assumption 1 holds, then,* $\forall x_j \in \mathcal{L}^2_{ad}(\Omega \times [0,T], \mathbf{R}^{n_j})$, $\exists | u_i \in U_{m_i}[0,T]$ *s.t.*

$$\langle F_i(\omega, s, x_j(\omega, s), u_i(\omega, s)), v_i - u_i(\omega, s)\rangle \geq 0, \quad \forall v_i \in K_i,\ a.e.\ s \in [0,T],\ a.s.\ \omega \in \Omega.$$

Lemma 6 ([2]). *If condition (iii) in Assumption 1 holds, then,* $\forall x_{j,1} \in \mathcal{L}^2_{ad}(\Omega \times [0,T], \mathbf{R}^{n_j})$ *(resp.,* $x_{j,2} \in \mathcal{L}^2_{ad}(\Omega \times [0,T], \mathbf{R}^{n_j})$*),* $\exists | u_{i,1} \in U_{m_i}[0,T]$ *(resp.,* $u_{i,2} \in U_{m_i}[0,T]$*) s.t.*

$$\begin{cases} \langle F_i(\omega, s, x_{j,1}(\omega, s), u_{i,1}(\omega, s)), v_i - u_{i,1}(\omega, s)\rangle \geq 0, & \forall v_i \in K_i,\ a.e.\ s \in [0,T],\ a.s.\ \omega \in \Omega, \\ \langle F_i(\omega, s, x_{j,2}(\omega, s), u_{i,2}(\omega, s)), v_i - u_{i,2}(\omega, s)\rangle \geq 0, & \forall v_i \in K_i,\ a.e.\ s \in [0,T],\ a.s.\ \omega \in \Omega. \end{cases}$$

In addition, one has that $\exists M_i > 0$ s.t.

$$\mathbf{E}\int_0^t \|u_{i,1}(\omega, s) - u_{i,2}(\omega, s)\|^2 ds \leq M_i \mathbf{E}\int_0^t \|x_{j,1}(\omega, s) - x_{j,2}(\omega, s)\|^2 ds, \quad \forall t \in [0,T].$$

In order to achieve the main results, we now analyze the convergent behavior of $\{(\mathbf{x}_k(s), \mathbf{u}_k(s)\}$, with $\mathbf{x}_k(s) = (x_{1,k}, x_{2,k})$ and $\mathbf{u}_k(s) = (u_{1,k}, u_{2,k})$, constructed below:

$$\begin{cases} \mathbf{x}_1(\tau) = \mathbf{x}_0, \\ u_{i,k}(\tau) = P_{U_{m_i}[0,T]}(u_{i,k}(\tau) - \rho_i \widetilde{F}_i(x_{j,k}(\tau), u_{i,k}(\tau))), \\ x_{i,k+1} = x_{i,0} + \int_0^\tau b_i(\iota, \mathbf{x}_k(\iota-), \mathbf{u}_k(\iota-))d\iota + \alpha \int_0^\tau (\tau-\iota)^{\alpha-1}\sigma_1^i(\iota, \mathbf{x}_k(\iota-), \mathbf{u}_k(\iota-))d\iota \\ \qquad + \int_0^\tau \sigma_0^i(\iota, \mathbf{x}_k(\iota-), \mathbf{u}_k(\iota-))dB_i(\iota) + \int_0^\tau \int_{\|x_i\|<c_i} G_i(\iota, \mathbf{x}_k(\iota-), \mathbf{u}_k(\iota-), x_i)\widetilde{N}_i(d\iota, dx_i). \end{cases} \tag{9}$$

Meanwhile, we will establish a few natures of $\{\mathbf{x}_k\}$.

Lemma 7. *For* $i = 1,2$, *if* $(x_{i,k-1}, u_{i,k-1}) \in \mathcal{L}^2_{ad}(\Omega \times [0,T], \mathbf{R}^{n_i}) \times U_{m_i}[0,T]$, *then* $\mathbf{E}(\sup_{s\in[0,T]} \|x_{i,k}(s)\|^2) < \infty$.

Proof. For convenience, one puts $\wp_k = (\iota, \mathbf{x}_k(\iota-), \mathbf{u}_k(\iota-))$ and $\Lambda_k = \|\mathbf{u}_k(\iota-)\|^2 + \|\mathbf{x}_k(\iota-)\|^2 + 1$. Using (9) and the relation below,

$$(\|\sum_{j=1}^r \ell_j\|)^2 \leq r\sum_{j=1}^r \|\ell_j\|^2, \tag{10}$$

one has

$$\begin{aligned} &\mathbf{E}(\sup_{\tau\in[0,T]} \|x_{i,k}(\tau)\|^2) \\ &\leq 5\mathbf{E}(\|x_{i,0}\|^2) + 5\mathbf{E}(\sup_{\tau\in[0,T]} \|\int_0^\tau b_i(\wp_{k-1})d\iota\|^2) \\ &\quad + 5\mathbf{E}(\sup_{\tau\in[0,T]} \|\int_0^\tau \sigma_0^i(\wp_{k-1})dB_i(\iota)\|^2) \\ &\quad + 5\mathbf{E}(\sup_{\tau\in[0,T]} \|\int_0^\tau \int_{\|x_i\|<c_i} G_i(\wp_{k-1}, x_i)\widetilde{N}_i(d\iota, dx_i)\|^2) \\ &\quad + 5\alpha^2\mathbf{E}(\sup_{\tau\in[0,T]} \|\int_0^\tau (\tau-s)^{\alpha-1}\sigma_1^i(\wp_{k-1})d\iota\|^2) \\ &= 5\mathbf{E}(\|x_{i,0}\|^2) + 5I_{i,1} + 5I_{i,2} + 5I_{i,3} + 5\alpha_i^2 I_{i,4}. \end{aligned} \tag{11}$$

Noticing Lemmas 3 and 4 and Hölder-type inequality, by condition (i) of Assumption 1, one obtains

$$\begin{aligned} I_{i,1} &\leq \mathbf{E}(\sup_{\tau\in[0,T]} \tau\int_0^\tau \|b_i(\wp_{k-1})\|^2 d\iota) \\ &\leq T\mathbf{E}\int_0^T K_{b_i}\Lambda_{k-1}d\iota < \infty, \end{aligned} \tag{12}$$

$$\begin{aligned} I_{i,2} &\leq (\tfrac{2}{2-1})^2\mathbf{E}(\int_0^T \|\sigma_0^i(\wp_{k-1})\|^2 d\iota) \\ &\leq 4\mathbf{E}\int_0^T K_{\sigma_0^i}\Lambda_{k-1}d\iota < \infty, \end{aligned} \tag{13}$$

$$\begin{aligned} I_{i,3} &\leq (\tfrac{2}{2-1})^2\mathbf{E}(\int_0^T \int_{\|x_i\|<c_i} \|G_i(\wp_{k-1}, x_i)\|^2 \nu_i(dx_i)d\iota) \\ &\leq 4\mathbf{E}\int_0^T K_{G_i}\Lambda_{k-1}d\iota < \infty, \end{aligned} \tag{14}$$

$$\begin{aligned} I_{i,4} &\leq \mathbf{E}(\sup_{t\in[0,T]}[\int_0^t (t-\iota)^{2\alpha-2}d\iota \int_0^t \|\sigma_1^i(\wp_{k-1})\|^2 d\iota]) \\ &\leq \mathbf{E}(\tfrac{T^{2\alpha-1}}{2\alpha-1}\int_0^T K_{G_i}\Lambda_{k-1}d\iota) < \infty. \end{aligned} \tag{15}$$

So, we conclude from the above inequalities that

$$\mathbf{E}(\sup_{\tau\in[0,T]}\|x_{i,k}(\tau)\|^2) \leq 5\mathbf{E}(\|x_{i,0}\|^2) + 5I_{i,1} + 5I_{i,2} + 5I_{i,3} + 5\alpha^2 I_{i,4} < \infty.$$

Obviously, the above lemma ensures that: if $(x_{i,k-1}, u_{i,k-1}) \in \mathcal{L}^2_{ad}(\Omega\times[0,T],\mathbf{R}^{n_i}) \times U_{m_i}[0,T]$, then $\mathbf{E}(\sup_{s\in[0,T]}\|x_{i,k}(s)\|^2) < \infty$. This arrives at $x_{i,k} \in \mathcal{L}^2_{ad}(\Omega\times[0,T],\mathbf{R}^{n_i})$. □

We are now in a position to state and demonstrate that there holds the unique existence of solutions of issue (3) and (4).

Theorem 1. *There holds the unique existence of solutions of issue (3) and (4) provided that that Assumption 1 is satisfied.*

Proof. For $i = 1, 2$, one defines $x_{i,1}(\tau) := x_{i,0}\ \forall \tau \in [0,T]$. Hence it is easily known from Lemma 5 and Lemmas 1 and 2 that $\exists| u_{i,1} \in U_{m_i}[0,T]$ such that

$$u_{i,1}(\tau) = P_{U_{m_i}[0,T]}(u_{i,1}(\tau) - \rho_i \widetilde{F}_i(x_{j,1}(\tau), u_{i,1}(\tau))).$$

Also, we set

$$\begin{aligned} x_{i,k+1} = x_{i,0} &+ \int_0^\tau b_i(\iota, \mathbf{x}_k(\iota-), \mathbf{u}_k(\iota-))d\iota + \alpha\int_0^\tau (\tau-\iota)^{\alpha-1}\sigma_1^i(\iota, \mathbf{x}_k(\iota-), \mathbf{u}_k(\iota-))d\iota \\ &+ \int_0^\tau \sigma_0^i(\iota, \mathbf{x}_k(\iota-), \mathbf{u}_k(\iota-))dB_i(\iota) + \int_0^\tau\int_{\|x_i\|<c_i} G_i(\iota, \mathbf{x}_k(\iota-), \mathbf{u}_k(\iota-), x_i)\widetilde{N}_i(d\iota, dx_i). \end{aligned}$$

Then for any given $(x_{j,1}, u_{i,1}) \in \mathcal{L}^2_{ad}(\Omega\times[0,T],\mathbf{R}^{n_j}) \times U_{m_i}[0,T]$, it follows that $x_{j,2} \in \mathcal{L}^2_{ad}(\Omega\times[0,T],\mathbf{R}^{n_j})$ (because of Lemma 7). By Lemma 5 and Lemmas 1 and 2, we deduce that $\exists| u_{i,2} \in U_{m_i}[0,T]$ such that

$$u_{i,2}(\tau) = P_{U_{m_i}[0,T]}(u_{i,2}(\tau) - \rho_i \widetilde{F}_i(x_{j,2}(\tau), u_{i,2}(\tau))).$$

Conducting such process persistently, we could fabricate $\{(\mathbf{x}_k(\tau), \mathbf{u}_k(\tau)\}$, in which $\mathbf{x}_k(\tau) = (x_{1,k}, x_{2,k})$ and $\mathbf{u}_k(\tau) = (u_{1,k}, u_{2,k})$, satisfying the following:

$$\begin{cases} u_{i,k}(\tau) = P_{U_{m_i}[0,T]}(u_{i,k}(\tau) - \rho_i \widetilde{F}_i(x_{j,k}(\tau), u_{i,k}(\tau))), \\ x_{i,k+1} = x_{i,0} + \int_0^\tau b_i(\iota, \mathbf{x}_k(\iota-), \mathbf{u}_k(\iota-))d\iota + \alpha\int_0^\tau (\tau-\iota)^{\alpha-1}\sigma_1^i(\iota, \mathbf{x}_k(\iota-), \mathbf{u}_k(\iota-))d\iota \\ \qquad + \int_0^\tau \sigma_0^i(\iota, \mathbf{x}_k(\iota-), \mathbf{u}_k(\iota-))dB_i(\iota) + \int_0^\tau\int_{\|x_i\|<c_i} G_i(\iota, \mathbf{x}_k(\iota-), \mathbf{u}_k(\iota-), x_i)\widetilde{N}_i(d\iota, dx_i), \end{cases}$$

where $0 < \rho_i < \frac{2\overline{C}_i}{L^2_{F_i}}$.

Now let us show the convergence of $\{(\mathbf{x}_k, \mathbf{u}_k)\}_{k\geq 1}$ in $\mathcal{L}^2_{ad}(\Omega\times[0,T],\mathbf{X}) \times U_{m_1+m_2}[0,T]$, where

$$U_{m_1+m_2}[0,T] := \{u(\omega,s) \in \mathcal{L}^2_{ad}(\Omega\times[0,T],\mathbf{V}) : u(\omega,s) \in K_1\times K_2,\ a.s.\ \omega\in\Omega,\ a.e.\ s\in[0,T]\}.$$

Indeed, for convenience, one puts $\wp_k = (\iota, \mathbf{x}_k(\iota-), \mathbf{u}_k(\iota-))$, $\xi_k = \|\mathbf{x}_{k-1}(\iota-) - \mathbf{x}_k(\iota-)\|^2$, $\zeta_k = \|\mathbf{u}_{k-1}(\iota-) - \mathbf{u}_k(\iota-)\|^2$, $\xi_{i,k} = \|x_{i,k-1}(\iota-) - x_{i,k}(\iota-)\|^2$ and $\zeta_{i,k} =$

$\|u_{i,k-1}(\iota-) - u_{i,k}(\iota-)\|^2$. Noticing condition (ii) of Assumption 1, from Lemma 3 and Hölder-type inequality we get

$$\begin{aligned}
&\mathbf{E}\sup_{s\in[0,\tau]} \|x_{i,k}(s) - x_{i,k+1}(s)\|^2 \\
&\leq 4\mathbf{E}T\int_0^\tau \|b_i(\wp_{k-1}) - b_i(\wp_k)\|^2 d\iota + 4\mathbf{E}\sup_{s\in[0,\tau]}(\int_0^s \|\sigma_0^i(\wp_{k-1}) - \sigma_0^i(\wp_k)\|dB_i(\iota))^2 \\
&\quad + 4\mathbf{E}\sup_{s\in[0,\tau]}(\int_0^s \int_{\|x_i\|<c_i} \|G_i(\wp_{k-1}, x_i) - G_i(\wp_k, x_i)\|\widetilde{N}_i(d\iota, dx_i))^2 \\
&\quad + 4\alpha^2\mathbf{E}\sup_{t\in[0,\tau]}(\int_0^t (t-\iota)^{\alpha-1}\|\sigma_1^i(\wp_{k-1}) - \sigma_1^i(\wp_k)\|d\iota)^2 \\
&= \bar{I}_{i,1} + \bar{I}_{i,2} + \bar{I}_{i,3} + \bar{I}_{i,4}.
\end{aligned} \tag{16}$$

According to Hölder-type inequality and Lemma 3, one gets

$$\bar{I}_{i,1} \leq 4TL_{b_i}\mathbf{E}\int_0^\tau \zeta_k + \zeta_k d\iota, \tag{17}$$

$$\begin{aligned}
\bar{I}_{i,2} &\leq 4\times 4\mathbf{E}\int_0^\tau \|\sigma_0^i(\wp_{k-1}) - \sigma_0^i(\wp_k)\|^2 d\iota \\
&\leq 16L_{\sigma_0^i}\mathbf{E}\int_0^\tau \zeta_k + \zeta_k d\iota,
\end{aligned} \tag{18}$$

$$\begin{aligned}
\bar{I}_{i,3} &\leq 4\times 4\mathbf{E}\int_0^\tau \int_{\|x_i\|<c_i} \|G_i(\wp_{k-1}, x_i) - G_i(\wp_k, x_i)\|^2 \nu_i(dx_i)d\iota \\
&\leq 16L_{G_i}\mathbf{E}\int_0^\tau \zeta_k + \zeta_k d\iota,
\end{aligned} \tag{19}$$

$$\begin{aligned}
\bar{I}_{i,4} &\leq 4\alpha^2\tfrac{T^{2\alpha-1}}{2\alpha-1}\mathbf{E}\int_0^\tau \|\sigma_1^i(\wp_{k-1}) - \sigma_1^i(\wp_k)\|^2 d\iota \\
&\leq 4\alpha^2\tfrac{T^{2\alpha-1}}{2\alpha-1}L_{\sigma_1^i}\mathbf{E}\int_0^\tau \zeta_k + \zeta_k d\iota.
\end{aligned} \tag{20}$$

Using the above inequalities, we obtain

$$\begin{aligned}
&\mathbf{E}\sup_{s\in[0,\tau]} \|x_{i,k}(s) - x_{i,k+1}(s)\|^2 \\
&\leq (4TL_{b_i} + 16L_{\sigma_0^i} + 16L_{G_i} + 4\alpha^2\tfrac{T^{2\alpha-1}}{2\alpha-1}L_{\sigma_1^i})\mathbf{E}\int_0^\tau \zeta_k + \zeta_k d\iota,
\end{aligned}$$

that along with Lemma 6, leads to

$$\begin{aligned}
&\mathbf{E}\sup_{s\in[0,\tau]} \|x_{i,k}(s) - x_{i,k+1}(s)\|^2 \\
&\leq (4TL_{b_i} + 16L_{\sigma_0^i} + 16L_{G_i} + 4\alpha^2\tfrac{T^{2\alpha-1}}{2\alpha-1}L_{\sigma_1^i}) \\
&\quad \times \mathbf{E}\int_0^\tau \zeta_k + \zeta_{1,k} + \zeta_{2,k} d\iota \\
&\leq (4TL_{b_i} + 16L_{\sigma_0^i} + 16L_{G_i} + 4\alpha^2\tfrac{T^{2\alpha-1}}{2\alpha-1}L_{\sigma_1^i}) \\
&\quad \times \mathbf{E}\int_0^\tau \zeta_k + M_1\zeta_{2,k} + M_2\zeta_{1,k} d\iota \\
&\leq (4TL_{b_i} + 16L_{\sigma_0^i} + 16L_{G_i} + 4\alpha^2\tfrac{T^{2\alpha-1}}{2\alpha-1}L_{\sigma_1^i}) \\
&\quad \times (1 + M_1 + M_2)\mathbf{E}\int_0^\tau \zeta_k d\iota \\
&= \beta_i\mathbf{E}\int_0^\tau \zeta_k d\iota,
\end{aligned}$$

where $\beta_i = (4TL_{b_i} + 16L_{\sigma_0^i} + 16L_{G_i} + 4\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}L_{\sigma_1^i})(1 + M_1 + M_2)$. Therefore,

$$\begin{aligned}
&\mathbf{E}\sup_{\iota\in[0,\tau]} \|\mathbf{x}_k(\iota) - \mathbf{x}_{k+1}(\iota)\|^2 \\
&\leq \mathbf{E}\sup_{\iota\in[0,\tau]} \|x_{1,k}(\iota) - x_{1,k+1}(\iota)\|^2 \\
&\quad + \mathbf{E}\sup_{\iota\in[0,\tau]} \|x_{2,k}(\iota) - x_{2,k+1}(\iota)\|^2 \\
&\leq \beta_1\mathbf{E}\int_0^\tau \zeta_k d\iota + \beta_2\mathbf{E}\int_0^\tau \zeta_k d\iota \\
&= (\beta_1 + \beta_2)\mathbf{E}\int_0^\tau \zeta_k d\iota \\
&\leq \beta\mathbf{E}\int_0^\tau \sup_{\iota\in[0,s]} \|\mathbf{x}_{k-1}(\iota) - \mathbf{x}_k(\iota)\|^2 ds,
\end{aligned} \tag{21}$$

where $\beta = \beta_1 + \beta_2$.

When $k=1$, we obtain from $\mathbf{x}_1(\tau)=\mathbf{x}_0$

$$\begin{aligned}\mathbf{E}\sup_{t\in[0,\tau]}\|x_{i,2}(t)-x_{i,1}(t)\|^2 &\le \mathbf{E}\sup_{t\in[0,T]}\|x_{i,2}(t)-x_{i,0}(t)\|^2\\ &\le 4T\mathbf{E}\int_0^T K_{b_i}(1+\|\mathbf{x}_0\|^2+\|\mathbf{u}_1(s-)\|^2)ds\\ &\quad+4\times 4\mathbf{E}\int_0^T K_{\sigma_0^i}(1+\|\mathbf{x}_0\|^2+\|\mathbf{u}_1(s-)\|^2)ds\\ &\quad+4\times 4\mathbf{E}\int_0^T K_{G_i}(1+\|\mathbf{x}_0\|^2+\|\mathbf{u}_1(s-)\|^2)ds\\ &\quad+4\alpha^2\tfrac{T^{2\alpha-1}}{2\alpha-1}\mathbf{E}\int_0^T K_{\sigma_1^i}(1+\|\mathbf{x}_0\|^2+\|\mathbf{u}_1(s-)\|^2)ds\\ &=\eta_i.\end{aligned}$$

This hence arrives at

$$\begin{aligned}&\mathbf{E}\sup_{\iota\in[0,\tau]}\|\mathbf{x}_2(\iota)-\mathbf{x}_1(\iota)\|^2\\ &\le \mathbf{E}\sup_{\iota\in[0,T]}\|x_{1,2}(\iota)-x_{1,0}\|^2+\mathbf{E}\sup_{\iota\in[0,T]}\|x_{2,2}(\iota)-x_{2,0}\|^2\\ &\le \eta_1+\eta_2=\eta,\end{aligned}$$

where

$$\eta=\sum_{i=1}^{2}4(TK_{b_i}+4K_{\sigma_0^i}+4K_{G_i}+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}K_{\sigma_1^i})\mathbf{E}\int_0^T(1+\|\mathbf{x}_0\|^2+\|\mathbf{u}_1(s-)\|^2)ds.$$

Thus, if $k=2$, then by (21) one gets

$$\begin{aligned}\mathbf{E}\sup_{\iota\in[0,\tau]}\|\mathbf{x}_3(\iota)-\mathbf{x}_2(\iota)\|^2 &\le \beta\mathbf{E}\int_0^\tau\sup_{\mu\in[0,s]}\|\mathbf{x}_2(\mu)-\mathbf{x}_1(\mu)\|^2ds\\ &\le \beta\eta\tau.\end{aligned}$$

Similarly, if $k=3$, we get

$$\begin{aligned}\mathbf{E}\sup_{t\in[0,\tau]}\|\mathbf{x}_4(t)-\mathbf{x}_3(t)\|^2 &\le \beta\mathbf{E}\int_0^\tau\sup_{\mu\in[0,s]}\|\mathbf{x}_3(\mu)-\mathbf{x}_2(\mu)\|^2ds\\ &\le \beta\int_0^\tau\beta\eta s ds\\ &=\beta^2\eta\tfrac{\tau^2}{2!}.\end{aligned}$$

Conducting such process persistently, we could infer that

$$\mathbf{E}\sup_{t\in[0,\tau]}\|\mathbf{x}_{k+1}(t)-\mathbf{x}_k(t)\|^2\le\eta\frac{\beta^{k-1}\tau^{k-1}}{(k-1)!}\to 0\quad\text{for each }\tau.$$

Utilizing the similar reasoning to that of the proof in [30], (Theorem 5.2.1), one has that for $1\le k<m$,

$$\begin{aligned}(\int_0^T\mathbf{E}\|\mathbf{x}_m(s)-\mathbf{x}_k(s)\|^2ds)^{1/2} &\le \sum_{l=k}^{m-1}(\int_0^T\mathbf{E}\|\mathbf{x}_{l+1}(s)-\mathbf{x}_l(s)\|^2ds)^{1/2}\\ &\le \sum_{l=k}^{m-1}(\int_0^T\mathbf{E}\sup_{t\in[0,s]}\|\mathbf{x}_{l+1}(t)-\mathbf{x}_l(t)\|^2ds)^{1/2}\\ &\le \sum_{l=k}^{m-1}(\tfrac{\eta}{\beta}\cdot\tfrac{\beta^lT^l}{l!})^{1/2}\to 0\quad(k\to\infty).\end{aligned}$$

This ensures that $\{\mathbf{x}_k\}$ is a Cauchy sequence in $\mathcal{L}^2_{ad}(\Omega\times[0,T],\mathbf{X})$. So it follows that $\{x_{j,k}\}$ is Cauchy sequence in $\mathcal{L}^2_{ad}(\Omega\times[0,T],\mathbf{R}^{n_j})$, and so is $\{u_{i,k}\}$ in $U_{m_i}[0,T]$ (by Lemma 6). Therefore, one infers that $\{(x_{j,k},u_{i,k})\}$ is Cauchy sequence in $\mathcal{L}^2_{ad}(\Omega\times[0,T],\mathbf{R}^{n_j})\times U_{m_i}[0,T]$. As a result, $\exists(x_j^*,u_i^*)\in\mathcal{L}^2_{ad}(\Omega\times[0,T],\mathbf{R}^{n_j})\times U_{m_i}[0,T]$ such that $(x_{j,k},u_{i,k})\to(x_j^*,u_i^*)$ as $n\to\infty$. Whereby, we obtain that $\mathbf{x}_k\to\mathbf{x}^*$ and $\mathbf{u}_k\to\mathbf{u}^*$, with $\mathbf{x}^*=(x_1^*,x_2^*)$ and $\mathbf{u}^*=(u_1^*,u_2^*)$. Because $P_{U_{m_i}[0,T]}$ is of continuity, it could be readily seen that

$$u_i^*(\iota) = P_{U_{m_i}[0,T]}(u_i^*(\iota) - \rho_i \widetilde{F}_i(x_j^*(\iota), u_i^*(\iota))). \tag{22}$$

We now define $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ as follows

$$\begin{aligned} \bar{x}_i(\tau) \quad &:= x_{i,0} + \int_0^\tau b_i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-))d\iota + \alpha \int_0^\tau (\tau-\iota)^{\alpha-1}\sigma_1^i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-))d\iota \\ &+ \int_0^\tau \sigma_0^i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-))dB_i(\iota) + \int_0^\tau \int_{\|x_i\|<c_i} G_i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-), x_i)\widetilde{N}_i(d\iota, dx_i), \end{aligned} \tag{23}$$

that along with (21), arrives at

$$\begin{aligned} &\mathbf{E}\sup_{\iota\in[0,\tau]} \|\bar{\mathbf{x}}(\iota) - \mathbf{x}_k(\iota)\|^2 \\ &\le \sum_{i=1}^2 (4TL_{b_i} + 16L_{\sigma_0^i} + 16L_{G_i} + 4\alpha^2 \tfrac{T^{2\alpha-1}}{2\alpha-1} L_{\sigma_1^i})(1 + M_1 + M_2) \\ &\quad \times \mathbf{E}\int_0^\tau \|\mathbf{x}^*(s-) - \mathbf{x}_{k-1}(s-)\|^2 ds \\ &= \beta \mathbf{E}\int_0^\tau \|\mathbf{x}^*(s-) - \mathbf{x}_{k-1}(s-)\|^2 ds, \end{aligned} \tag{24}$$

where

$$\beta = \sum_{j=1}^2 (4TL_{b_j} + 16L_{\sigma_0^j} + 16L_{G_j} + 4\alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1} L_{\sigma_1^j})(1 + M_1 + M_2).$$

Thanks to $\mathbf{x}_k \to \mathbf{x}^*$, it is easily known that $\mathbf{x}_k \to \bar{\mathbf{x}}$. Thus, one deduces that $\bar{\mathbf{x}}$ is equal to $\mathbf{x}^*$. Consequently, from (22) and (23) we obtain

$$\left\{ \begin{aligned} &u_i^*(\iota) = P_{U_{m_i}[0,T]}(u_i^*(\iota) - \rho_i \widetilde{F}_i(x_j^*(\iota), u_i^*(\iota))), \\ &x_i^*(\iota) = x_{i,0} + \int_0^\iota b_i(\tau, \mathbf{x}^*(\tau-), \mathbf{u}^*(\tau-))d\tau + \alpha \int_0^\iota (\iota-\tau)^{\alpha-1}\sigma_1^i(\tau, \mathbf{x}^*(\tau-), \mathbf{u}^*(\tau-))d\tau \\ &\quad + \int_0^\iota \sigma_0^i(\tau, \mathbf{x}^*(\tau-), \mathbf{u}^*(\tau-))dB_i(\tau) + \int_0^\iota \int_{\|x_i\|<c_i} G_i(\tau, \mathbf{x}^*(\tau-), \mathbf{u}^*(\tau-), x_i)\widetilde{N}_i(d\tau, dx_i). \end{aligned} \right.$$

Next, let us show the uniqueness of solutions to issue (3) and (4). Indeed, assume that $(\mathbf{x}_1(t), \mathrm{u}_1(t))$ and $(\mathbf{x}_2(t), \mathrm{u}_2(t))$ are both solutions of issue (3) and (4), with $\mathbf{x}_i = (x_{1,i}, x_{2,i})$ and $\mathbf{u}_i = (u_{1,i}, u_{2,i})$. Utilizing the similar reasoning to that of the above proof, one gets

$$\mathbf{E}\sup_{s\in[0,T]} \|\mathbf{x}_1(s) - \mathbf{x}_2(s)\|^2 \le \beta \mathbf{E}\int_0^T \sup_{\iota\in[0,s]} \|\mathbf{x}_1(\iota) - \mathbf{x}_2(\iota)\|^2 ds,$$

where

$$\beta = \sum_{i=1}^2 (4TL_{b_i} + 16L_{\sigma_0^i} + 16L_{G_i} + 4\alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1} L_{\sigma_1^i})(1 + M_1 + M_2).$$

Putting $f(\tau) = \mathbf{E}\sup_{s\in[0,\tau]} \|\mathbf{x}_1(s) - \mathbf{x}_2(s)\|^2 \ \forall \tau \in [0,T]$, one has

$$f(T) \le \int_0^T \beta f(s) ds,$$

that along with Gronwall-type inequality, arrives at $f(\tau) = 0 \ \forall \tau \in [0,T]$. For $j = 1,2$, it then follows that

$$\mathbf{E}\sup_{s\in[0,T]} \|x_{j,1}(s) - x_{j,2}(s)\|^2 \le \mathbf{E}\sup_{s\in[0,T]} \|\mathbf{x}_1(s) - \mathbf{x}_2(s)\|^2 = f(T) = 0.$$

Thus, by Lemma 6 we get

$$\begin{aligned} &\mathbf{E}\sup_{s\in[0,T]} \|x_{j,1}(s) - x_{j,2}(s)\|^2 = 0 \ \Rightarrow \\ &\quad \|x_{j,1}(s) - x_{j,2}(s)\|^2_{H_{n_j}[0,T]} = 0 \text{ and } \|u_{i,1}(s) - u_{i,2}(s)\|^2_{H_{m_i}[0,T]} = 0. \end{aligned}$$

□

It is noteworthy that, setting $G_i = 0$ and $\sigma_1^i = 0$ in the above theorem, we can derive an extension of ([2], Theorem 3.1) since it incorporates the SSDE and SSVI. In what follows, for achieving the valuable property of solutions, we now furnish a basic tool.

Lemma 8. *For $q \in (1, \infty)$, the following holds*

$$t^q - \tau^q \geq (t-\tau)^q, \quad t \geq \tau \geq 0.$$

For $q \in (0,1)$, the following holds

$$t^q - \tau^q \leq (t-\tau)^q, \quad t \geq \tau \geq 0.$$

Whereby, we are ready to show the result below.

Theorem 2. *For $\mathbf{u}^* \in U_{m_1+m_2}[0,T]$, $\mathbf{x}^* \in \mathcal{L}^2_{ad}(\Omega \times [0,T], \mathbf{X})$ and $\alpha \in (\frac{1}{2}, 1)$, one has that $\exists$ (nonnegative constants) C_1, C_2 and C_3, s.t.*

$$C_3 + C_2(t-\iota)^{2\alpha-1} + C_1(t-\iota) \geq \mathbf{E}\|\mathbf{x}^*(t) - \mathbf{x}^*(\iota)\|^2, \quad 0 \leq \iota \leq t \leq T. \tag{25}$$

Proof. For $s, t \in [0,T]$ with $t \geq s$, one obtains from (9),

$$\begin{aligned} x_i^*(t) - x_i^*(s) = &\int_s^t b_i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-))d\iota + \int_s^t \sigma_0^i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-))dB_i(\iota) \\ &+ \int_s^t \int_{\|x_i\|<c_i} G_i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-), x_i)\widetilde{N}_i(d\iota, dx_i) \\ &+ \alpha \int_0^s [(t-\iota)^{\alpha-1}\sigma_1^i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-)) - (s-\iota)^{\alpha-1}\sigma_1^i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-))]d\iota \\ &+ \alpha \int_s^t (t-\iota)^{\alpha-1}\sigma_1^i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-))d\iota, \end{aligned} \tag{26}$$

which together with (10), leads to

$$\begin{aligned} &\|x_i^*(t) - x_i^*(s)\|^2 \\ &\leq 4\| \int_s^t b_i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-))d\iota\|^2 + 4\| \int_s^t \sigma_0^i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-))dB_i(\iota)\|^2 \\ &\quad + 4\| \int_s^t \int_{\|x_i\|<c_i} G_i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-), x_i)\widetilde{N}_i(d\iota, dx_i)\|^2 \\ &\quad + 4\alpha^2\| \int_0^s [\frac{\sigma_1^i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-))}{(t-\iota)^{1-\alpha}} - \frac{\sigma_1^i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-))}{(s-\iota)^{1-\alpha}}]d\iota \\ &\quad + \int_s^t \frac{\sigma_1^i(\iota, \mathbf{x}^*(\iota-), \mathbf{u}^*(\iota-))}{(t-\iota)^{1-\alpha}}d\iota\|^2 \\ &= 4J_{i,1} + 4J_{i,2} + 4J_{i,3} + 4\alpha^2 J_{i,4}. \end{aligned} \tag{27}$$

Utilizing condition (i) in Assumption 1, from Hölder-type inequality and Lemmas 3 and 4 one gets

$$\begin{aligned} &\mathbf{E}4(J_{i,1} + J_{i,2} + J_{i,3}) \leq 4(T-s)\mathbf{E} \int_s^t K_{b_i}(\|\mathbf{u}^*(\iota-)\|^2 + \|\mathbf{x}^*(\iota-)\|^2 + 1)d\iota \\ &\quad + 4\mathbf{E} \int_s^t K_{\sigma_0^i}(\|\mathbf{u}^*(\iota-)\|^2 + \|\mathbf{x}^*(\iota-)\|^2 + 1)d\iota \\ &\quad + 4\mathbf{E} \int_s^t K_{G_i}(\|\mathbf{u}^*(\iota-)\|^2 + \|\mathbf{x}^*(\iota-)\|^2 + 1)d\iota \\ &\leq [4(T-s)K_{b_i} + K_{\sigma_0^i} + K_{G_i}][(t-s) + \mathbf{E} \int_s^t \|\mathbf{x}^*(\tau-)\|^2 + \|\mathbf{u}^*(\tau-)\|^2 d\tau] \\ &\leq [4(T-s)K_{b_i} + K_{\sigma_0^i} + K_{G_i}][(t-s) + \mathbf{E} \int_s^t \|\mathbf{x}^*(\tau-)\|^2 d\tau \\ &\quad + 2\mathbf{E} \int_s^t \|\mathbf{u}^*(\tau-) - \mathbf{u}_1(\tau-)\|^2 + \|\mathbf{u}_1(\tau-)\|^2 d\tau] \\ &\leq [4(T-s)K_{b_i} + K_{\sigma_0^i} + K_{G_i}][(t-s) + \mathbf{E} \int_s^t \|\mathbf{x}^*(\tau-)\|^2 d\tau \\ &\quad + 2(M_1+M_2)\mathbf{E} \int_s^t \|\mathbf{x}^*(\tau-) - \mathbf{x}_0\|^2 d\tau + 2\mathbf{E} \int_s^t \|\mathbf{u}_1(\tau-)\|^2 d\tau] \\ &\leq [4(T-s)K_{b_i} + K_{\sigma_0^i} + K_{G_i}][(t-s) + \mathbf{E} \int_s^t \|\mathbf{x}^*(\tau-)\|^2 d\tau \\ &\quad + 4(M_1+M_2)\mathbf{E} \int_s^t \|\mathbf{x}_0\|^2 + \|\mathbf{x}^*(\tau-)\|^2 d\tau + 2\mathbf{E} \int_s^t \|\mathbf{u}_1(\tau-)\|^2 d\tau] \\ &\leq 4[(T-s)K_{b_i} + K_{\sigma_0^i} + K_{G_i}][(t-s) \\ &\quad + (1 + 4(M_1+M_2))\mathbf{E} \sup_{\iota \in [0,T]} \|\mathbf{x}^*(\iota)\|^2 (t-s) \\ &\quad + (M_1+M_2)4\mathbf{E}\|\mathbf{x}_0\|^2(t-s) + 2\mathbf{E} \int_s^t \|\mathbf{u}_1(\iota-)\|^2 d\iota] \\ &\leq C_{i,1}(t-s) + C_{i,3}, \end{aligned} \tag{28}$$

where

$$C_{i,1} = 4[TK_{b_i} + K_{\sigma_0^i} + K_{G_i}][1 + (1 + 4(M_1 + M_2))\mathbf{E} \sup_{\iota \in [0,T]} \|\mathbf{x}^*(\iota)\|^2 + (M_1 + M_2)4\mathbf{E}\|\mathbf{x}_0\|^2],$$

and

$$C_{i,3} = 8[TK_{b_i} + K_{\sigma_0^i} + K_{G_i}]\mathbf{E}\int_0^T \|\mathbf{u}_1(\iota-)\|^2 d\iota.$$

Utilizing condition (i) in Assumption 1, from Hölder-type inequality and Lemma 8 one has

$$\begin{aligned} \mathbf{E}J_{i,4} &\le 2\mathbf{E}\| \int_0^s [\tfrac{\sigma_1^i(\iota,\mathbf{x}^*(\iota-),\mathbf{u}^*(\iota-))}{(t-\iota)^{1-\alpha}} - \tfrac{\sigma_1^i(\iota,\mathbf{x}^*(\iota-),\mathbf{u}^*(\iota-))}{(s-\iota)^{1-\alpha}}]d\iota\|^2 \\ &\quad + 2\mathbf{E}\| \int_s^t \tfrac{\sigma_1^i(\iota,\mathbf{x}^*(\iota-),\mathbf{u}^*(\iota-))}{(t-\iota)^{1-\alpha}} d\iota\|^2 \\ &\le 2\mathbf{E}\int_0^s \|\sigma_1^i(\iota,\mathbf{x}^*(\iota-),\mathbf{u}^*(\iota-))\|^2 d\iota \int_0^s [(t-\iota)^{\alpha-1} - (s-\iota)^{\alpha-1}]^2 d\iota \\ &\quad + 2\mathbf{E}\int_s^t \|\sigma_1^i(\iota,\mathbf{x}^*(\iota-),\mathbf{u}^*(\iota-))\|^2 d\iota \int_s^t (t-\iota)^{2\alpha-2} d\iota \\ &\le 2\mathbf{E}\int_0^T K_{\sigma_1^i}(\|\mathbf{u}^*(\iota-)\|^2 + \|\mathbf{x}^*(\iota-)\|^2 + 1)d\iota \\ &\quad \times [\int_0^s (s-\iota)^{2\alpha-2} - (t-\iota)^{2\alpha-2} d\iota + \int_s^t (t-\iota)^{2\alpha-2} d\iota] \\ &\le 2\mathbf{E}\int_0^T K_{\sigma_1^i}(\|\mathbf{u}^*(\iota-)\|^2 + \|\mathbf{x}^*(\iota-)\|^2 + 1)d\iota \\ &\quad \times [\tfrac{2}{2\alpha-1}(t-s)^{2\alpha-1} + \tfrac{\|s^{2\alpha-1} - t^{2\alpha-1}\|^2}{2\alpha-1}] \\ &\le 2\mathbf{E}\int_0^T K_{\sigma_1^i}(\|\mathbf{u}^*(\iota-)\|^2 + \|\mathbf{x}^*(\iota-)\|^2 + 1)d\iota \times \tfrac{3}{2\alpha-1}(t-s)^{2\alpha-1}. \end{aligned} \tag{29}$$

Let

$$C_{i,2} = \frac{24\alpha^2}{2\alpha-1}\mathbf{E}\int_0^T K_{\sigma_1^i}(\|\mathbf{u}^*(\iota-)\|^2 + \|\mathbf{x}^*(\iota-)\|^2 + 1)d\iota.$$

Then

$$4\alpha^2 \mathbf{E}J_{i,4} \le C_{i,2}(t-s)^{2\alpha-1}. \tag{30}$$

From the inequalities above, it follows that for $s, \iota \in [0,T]$ with $\iota \ge s$,

$$C_{i,3} + C_{i,2}(\iota - s)^{2\alpha-1} + C_{i,1}(\iota - s) \ge \mathbf{E}\|x_i^*(\iota) - x_i^*(s)\|^2.$$

Therefore,

$$\begin{aligned} \mathbf{E}\|\mathbf{x}^*(t) - x^*(s)\|^2 &= \mathbf{E}\|x_1^*(t) - x_1^*(s)\|^2 + \mathbf{E}\|x_2^*(t) - x_2^*(s)\|^2 \\ &\le (C_{1,1} + C_{2,1})(t-s) + (C_{1,2} + (C_{2,2})(t-s)^{2\alpha-1} + (C_{1,3} + C_{2,3}) \\ &= C_1(t-s) + C_2(t-s)^{2\alpha-1} + C_3, \end{aligned}$$

where $C_1 = C_{1,1} + C_{2,1}$, $C_2 = C_{1,2} + C_{2,2}$ and $C_3 = C_{1,3} + C_{2,3}$. □

It is noteworthy that, if $\mathbf{u}_1(t)$ is bounded, then for $s, \iota \in [0,T]$ with $\iota \ge s$, (3.22) can be changed into

$$\mathbf{E}\|\mathbf{x}^*(\iota) - \mathbf{x}^*(s)\|^2 \le \overline{C}_1(\iota - s) + \overline{C}_2(\iota - s)^{2\alpha-1}.$$

4. Applications to Stochastic SPE Systems

In the rest of this paper, we denote by the FSDE, SPE, APP, AP and BM the fractional stochastic differential equation, spatial price equilibria, asset price process, asset price and Brownian motion, respectively. Also, let the SC, DM, SM, PC and SS represent the stochastic circumstance, demand market, supply market, price of commodity and stochastic system, respectively.

It is well known that, the spatial-price equilibria models have played an important role in solving some practical problems arising from energy markets, agriculture, economics, and finance; see e.g., [1,5,23,31]. In 2024, Zeng et al. [23] exploited a FSDE driven by BM to indicate APP and modeled SPE in SC using FSDVI (2). Note that, for FSDVI (2), they had explained that the APP reveals the jumps [21] and memory [32] features, and BM is not strong enough to acquire the dynamics of AP changes. Accordingly, they had utilized the FSDVI (2) possessing Lévy jump to express stochastic SPE possessing jumps and memory.

Inspired by the study [23], we introduce and discuss a system of stochastic spatial-price equilibria, where each stochastic spatial-price equilibrium involves a commodity possessing jumps and memory in the time term of $[0, T]$. In what follows, we release certain symbols. Let $n_1 = n_2 = n$, $m_1 = m_2 = n$ and $l_1 = l_2 = n$ in the above section. Then for each $l = 1, 2$,

- $S_{l,i}$: the ith-SM, $\forall i$.
- $D_{l,j}$: the jth-DM, $\forall j$.
- $a^l_{ij}(\omega, \iota)$: the number of commodities transported from the SM $S_{l,i}$ to the DM $D_{l,j}$ at ι-time, and $\mathbf{a}_l(\omega, \iota) = (a^l_{ij}(\omega, \iota)) \in \mathbf{R}^{n\times n}$.
- $\bar{S}_{l,i}(\omega, \iota) = \sum_{j=1}^n a^l_{ij}(\omega, \iota)$: the number of commodities supplied by SM $S_{l,i}$ at ι-time, and $\bar{\mathbf{S}}_l(\omega, \iota) = (\bar{S}_{l,1}(\omega, \iota), ..., \bar{S}_{l,n}(\omega, \iota)) \in \mathbf{R}^n$.
- $\bar{D}_{l,j}(\omega, \iota) = \sum_{i=1}^n a^l_{ij}(\omega, \iota)$: the demand for commodities in DM $D_{l,j}$ at ι-time, and $\bar{\mathbf{D}}_l(\omega, \iota) = (\bar{D}_{l,1}(\omega, \iota), ..., \bar{D}_{l,n}(\omega, \iota)) \in \mathbf{R}^n$.
- $p_{l,i}(\omega, \iota)$: the supply PC related to SM $S_{l,i}$ at ι-time, and $\mathbf{p}_l(\omega, \iota) = (p_{l,1}(\omega, \iota), ..., p_{l,n}(\omega, \iota)) \in \mathcal{L}^2_{ad}(\Omega \times [0, T], \mathbf{R}^n)$.
- $q_{l,j}(\omega, \iota)$: the demand PC related to DM $D_{l,j}$ at ι-time, and $\mathbf{q}_l(\omega, \iota) = (q_{l,1}(\omega, \iota), ..., q_{l,n}(\omega, \iota)) \in \mathcal{L}^2_{ad}(\Omega \times [0, T], \mathbf{R}^n)$.
- $c^l_{ij}(\omega, \iota) = c^l_{ij}(a^l_{ij}(\omega, \iota))$: a unit transported cost from $S_{l,i}$ to $D_{l,j}$ at ι-time, and $\mathbf{c}_l(\omega, \iota) = (c^l_{ij}(\omega, \iota)) \in \mathbf{R}^{n\times n}$.
- $\mathcal{L}^2_{ad} = \mathcal{L}^2_{ad}(\Omega \times [0, T], \mathbf{R}^n) \times \mathcal{L}^2_{ad}(\Omega \times [0, T], \mathbf{R}^n) \times \mathcal{L}^2_{ad}(\Omega \times [0, T], \mathbf{R}^{n\times n})$ and
$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathcal{L}^2_{ad}} = \mathbf{E}\int_0^T \langle \mathbf{a}(\omega, \iota), \mathbf{b}(\omega, \iota)\rangle d\iota, \quad \mathbf{a}, \mathbf{b} \in \mathcal{L}^2_{ad}.$$
- $\mathbf{u}_l(\omega, \iota) = (\bar{\mathbf{S}}_l(\omega, \iota), \bar{\mathbf{D}}_l(\omega, \iota), \mathbf{a}_l(\omega, \iota)) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^{n\times n}$.
- $K_l = \{(A_l, B_l, C_l) : A_l = (A_{l,1}, A_{l,2}, ..., A_{l,n}) \in \mathbf{R}^n,\ B_l = (B_{l,1}, B_{l,2}, ..., B_{l,n}) \in \mathbf{R}^n,\ C_l = (C^l_{ij}) \in \mathbf{R}^{n\times n},\ C^l_{ij} \geq 0,\ A_{l,i} = \sum_{j=1}^n C^l_{ij},\ B_{l,j} = \sum_{i=1}^n C^l_{ij}\}$.
- $U_{K_l}[0, T] = \{\mathbf{u}_l \in \mathcal{L}^2_{ad} : \mathbf{u}_l(\omega, \iota) \in K_l,\ a.e.\ \iota \in [0, T],\ a.s.\ \omega \in \Omega\}$.

Thanks to the impact of jump and memory statuses on the APP, we presume always that for $l = 1, 2$, APPs $\mathbf{p}_l(\omega, t), \mathbf{q}_l(\omega, t)$ solve the FSDES possessing jumps:

$$\begin{cases} d\mathbf{p}_l(\iota) = b^l_1(\iota, \mathbf{p}_l(\iota-), \bar{\mathbf{S}}_l(\iota-))d\iota + \sigma^l_1(\iota, \mathbf{p}_l(\iota-), \bar{\mathbf{S}}_l(\iota-))(d\iota)^\alpha + f^l_1(\iota, \mathbf{p}_l(\iota-), \bar{\mathbf{S}}_l(\iota-))dB^l_1(\iota) \\ \qquad\qquad + \int_{\|x\|<c_l} G^l_1(\iota, \mathbf{p}_l(\iota-), \bar{\mathbf{S}}_l(\iota-), x)\widetilde{N}^l_1(d\iota, dx), \\ \mathbf{p}_l(0) = \mathbf{p}_{l,0}, \\ d\mathbf{q}_l(\iota) = b^l_2(\iota, \mathbf{q}_l(\iota-), \bar{\mathbf{D}}_l(\iota-))d\iota + \sigma^l_2(\iota, \mathbf{q}_l(\iota-), \bar{\mathbf{D}}_l(\iota-))(d\iota)^\alpha + f^l_2(\iota, \mathbf{q}_l(\iota-), \bar{\mathbf{D}}_l(\iota-))dB^l_2(\iota) \\ \qquad\qquad + \int_{\|x\|<c_l} G^l_2(\iota, \mathbf{q}_l(\iota-), \bar{\mathbf{D}}_l(\iota-), x)\widetilde{N}^l_2(d\iota, dx), \\ \mathbf{q}_l(0) = \mathbf{q}_{l,0}, \end{cases} \tag{31}$$

where $b^l_i, \sigma^l_i, f^l_i, G^l_i$ are of suitable measurability, $\sigma^l_1(\iota, \mathbf{p}_l(\omega, \iota), \bar{\mathbf{S}}_l(\omega, \iota))$ and $\sigma^l_2(\iota, \mathbf{q}_l(\omega, \iota), \bar{\mathbf{D}}_l(\omega, \iota))$ are of continuity w.r.t. ι, $B^l_1(\iota)$ and $B^l_2(\iota)$ are two $\mathcal{F}_\iota$-adapted BMs, N^l_1, N^l_2 are both $\mathcal{F}_\iota$-adapted Poisson measure, and their martingale measures of associated compensation are formulated as $\widetilde{N}^l_i(d\iota, dx) := N^l_i(d\iota, dx) - \nu^l_i(dx)d\iota$ for $i = 1, 2$. Moreover, we assume that $N^l_1, N^l_2, B^l_1, B^l_2$ are independent mutually.

Resembling the concept given in [23], we could put forward the following concept of spatial-price equilibria system point in a stochastic circumstance affected with Lévy jumps and memory.

Definition 2. *Given* $\mathbf{u}^* = (u_1^*, u_2^*)$, *where* $u_l^*(\omega,\iota) = (\bar{\mathbf{S}}_l^*(\omega,\iota), \bar{\mathbf{D}}_l^*(\omega,\iota), \mathbf{a}_l^*(\omega,\iota)), l = 1,2,$ *s.t.* $u_l^* \in U_{K_l}[0,T]$. $\mathbf{u}^*$ *is termed as a SPE system point in SC iff there hold the relations below: for* $l, m = 1,2$ *and* $l \neq m$

$$p_{l,i}^*(\omega,\iota) + c_{ij}^m(a_{ij}^{m*}(\omega,\iota)) \begin{cases} = q_{l,j}^*(\omega,\iota) & \text{if } a_{ij}^{m*} \geq 0 \\ \geq q_{l,j}^*(\omega,\iota) & \text{if } a_{ij}^{m*} = 0 \end{cases} \quad a.e.\ \iota \in [0,T],\ a.s.\ \omega \in \Omega, \tag{32}$$

with $\mathbf{p}_l^*(\omega,\iota)$ *and* $\mathbf{q}_l^*(\omega,\iota)$ *satisfying (31).*

Lemma 9. $K_l \neq \emptyset \neq U_{K_l}[0,T]$, *and they are of both convexity and closedness for* $l = 1,2$.

Proof. First, it is easy to check that K_l is nonempty convex closed and hence $U_{K_l}[0,T]$ is nonempty.

Let us show that $U_{K_l}[0,T] \subset \mathcal{L}_{ad}^2$ is convex. Indeed, for each $u_{l,1}, u_{l,2} \in U_{K_l}[0,T]$ and each $\mu \in [0,1]$, we know that $u_{l,1}, u_{l,2} \in \mathcal{L}_{ad}^2$ and $u_{l,1}, u_{l,2} \in K_l$. Because $K_l \neq \emptyset$, which is of both convexity and closedness, the following relation is valid:

$$\mu u_{l,1}(\omega,\iota) + (1-\mu)u_{l,2}(\omega,\iota) \in K_l, \quad a.e.\ \iota \in [0,T],\ a.s.\ \omega \in \Omega,$$

and hence $U_{K_l}[0,T]$ is of convexity.

In what follows, it is enough to only show the closedness of $U_{K_l}[0,T]$ in $\mathcal{L}_{ad}^2$. Let the sequence $\{u_{l,n}\}$ lie in $U_{K_l}[0,T]$ s.t. $\|u_{l,n} - u_l^*\|_{\mathcal{L}_{ad}^2} \to 0$. Whereby, we know that $u_l^* \in \mathcal{L}_{ad}^2$ and

$$\int_0^T \mathbf{E}\|u_{l,n}(\omega,\iota) - u_l^*(\omega,\iota)\|^2 d\iota \to 0,$$

that hence yields

$$\|u_{l,n}(\omega,\iota) - u_l^*(\omega,\iota)\| \to 0, \quad a.e.\ \iota \in [0,T],\ a.s.\ \omega \in \Omega.$$

Because K_l is of closedness, one gets $u_l^* \in U_{K_l}[0,T]$, that is, $U_{K_l}[0,T]$ is of closedness. □

For achieving the major outcome in this section, we release the symbols below. Let $l, m = 1,2$ and $l \neq m$. For each $(\mathbf{p}_l, \mathbf{q}_l) \in \mathcal{L}_{ad}^2(\Omega \times [0,T], \mathbf{R}^n) \times \mathcal{L}_{ad}^2(\Omega \times [0,T], \mathbf{R}^n)$ and each $u_m = (\bar{\mathbf{S}}_m, \bar{\mathbf{D}}_m, \mathbf{a}_m) \in U_{K_m}[0,T]$, let

$$\widetilde{F}_m(\mathbf{p}_l, \mathbf{q}_l, u_m)(\omega,\iota) = F_m(\omega, \iota, \mathbf{p}_l(\omega,\iota), \mathbf{q}_l(\omega,\iota), u_m(\omega,\iota)), \quad \forall \omega \in \Omega, \forall \iota \in [0,T]$$

and

$$\begin{aligned} &\langle \widetilde{F}_m(\mathbf{p}_l, \mathbf{q}_l, u_m), u_m \rangle_{\mathcal{L}_{ad}^2} \\ &= \mathbf{E}\int_0^T \langle \mathbf{p}_l(\omega,\iota), \bar{\mathbf{S}}_m(\omega,\iota) \rangle - \langle \mathbf{q}_l(\omega,\iota), \bar{\mathbf{D}}_m(\omega,\iota) \rangle + \langle \mathbf{c}_m(\mathbf{a}_m(\omega,\iota)), \mathbf{a}_m(\omega,\iota) \rangle d\iota. \end{aligned}$$

Theorem 3. *Given* $\mathbf{u}^* = (u_1^*, u_2^*)$, *where for* $m = 1,2$, $u_m^*(\omega,\iota) = (\bar{\mathbf{S}}_m^*(\omega,\iota), \bar{\mathbf{D}}_m^*(\omega,\iota), \mathbf{a}_m^*(\omega,\iota))$ *such that* $u_m^* \in U_{K_m}[0,T]$. *Then the following relations are equivalent:*

(i) $\mathbf{u}^*$ *is a dynamic stochastic market equilibria system point;*
(ii) $\mathbf{u}^*$ *solves the SVIS: for* $l, m = 1,2$ *and* $l \neq m$,

$$\begin{aligned} &\langle \widetilde{F}_m(\mathbf{p}_l^*, \mathbf{q}_l^*, u_m^*), u_m - u_m^* \rangle_{\mathcal{L}_{ad}^2} \\ &= \mathbf{E}\int_0^T \{\langle \mathbf{p}_l^*(\omega,\iota), \bar{\mathbf{S}}_m(\omega,\iota) - \bar{\mathbf{S}}_m^*(\omega,\iota) \rangle - \langle \mathbf{q}_l^*(\omega,\iota), \bar{\mathbf{D}}_m(\omega,\iota) - \bar{\mathbf{D}}_m^*(\omega,\iota) \rangle \\ &\quad + \langle \mathbf{c}_m(\mathbf{a}_m^*(\omega,\iota)), \mathbf{a}_m(\omega,\iota) - \mathbf{a}_m^*(\omega,\iota) \rangle\} d\iota \geq 0, \quad \forall u_m \in U_{K_m}[0,T]. \end{aligned}$$

Proof. Such demonstration is analogous to that of Theorem 4.1 of [23]. □

Given $\alpha \in (\frac{1}{2}, 1)$. From Theorem 3 and (31), it could be readily seen that the SPE system in SC is equivalent to the SS below: for $l, m = 1, 2$ and $l \neq m$,

$$\begin{cases} d\mathbf{p}_l^*(\iota) = b_1^l(\iota, \mathbf{p}_l^*(\iota-), \bar{\mathbf{S}}_l^*(\iota-))d\iota + \sigma_1^l(\iota, \mathbf{p}_l^*(\iota-), \bar{\mathbf{S}}_l^*(\iota-))(d\iota)^\alpha + f_1^l(\iota, \mathbf{p}_l^*(\iota-), \bar{\mathbf{S}}_l^*(\iota-))dB_1^l(\iota) \\ \qquad + \int_{\|x\|<c_l} G_1^l(\iota, \mathbf{p}_l^*(\iota-), \bar{\mathbf{S}}_l^*(\iota-), x)\widetilde{N}_1^l(d\iota, dx), \;\; \mathbf{p}_l^*(0) = \mathbf{p}_{l,0}^*, \\ d\mathbf{q}_l^*(\iota) = b_2^l(\iota, \mathbf{q}_l^*(\iota-), \bar{\mathbf{D}}_l^*(\iota-))d\iota + \sigma_2^l(\iota, \mathbf{q}_l^*(\iota-), \bar{\mathbf{D}}_l^*(\iota-))(d\iota)^\alpha + f_2^l(\iota, \mathbf{q}_l^*(\iota-), \bar{\mathbf{D}}_l^*(\iota-))dB_2^l(\iota) \\ \qquad + \int_{\|x\|<c_l} G_2^l(\iota, \mathbf{q}_l^*(\iota-), \bar{\mathbf{D}}_l^*(\iota-), x)\widetilde{N}_2^l(d\iota, dx), \;\; \mathbf{q}_l^*(0) = \mathbf{q}_{l,0}^*, \\ \langle \widetilde{F}_m(\mathbf{p}_l^*, \mathbf{q}_l^*, u_m^*), u_m - u_m^* \rangle_{\mathcal{L}_{ad}^2} \geq 0, \;\; \forall u_m \in U_{K_m}[0, T]. \end{cases} \tag{33}$$

which could be rewritten as the SFDVIS possessing Lévy jumps (due to Lemma 4):

$$\begin{cases} dy_1(\iota) = b_1(\iota, y_1(\iota-), u_1(\iota-))d\iota + \sigma_1(\iota, y_1(\iota-), u_1(\iota-))(d\iota)^\alpha + f_1(\iota, y_1(\iota-), u_1(\iota-))dB_1(\iota) \\ \qquad + \int_{\|x\|<c_1} G_1(\iota, y_1(\iota-), u_1(\iota-), x)\widetilde{N}_1(d\iota, dx), \\ dy_2(\iota) = b_2(\iota, y_2(\iota-), u_2(\iota-))d\iota + \sigma_2(\iota, y_2(\iota-), u_2(\iota-))(d\iota)^\alpha + f_2(\iota, y_2(\iota-), u_2(\iota-))dB_2(\iota) \\ \qquad + \int_{\|x\|<c_2} G_2(\iota, y_2(\iota-), u_2(\iota-), x)\widetilde{N}_2(d\iota, dx), \\ y_1(0) = y_{1,0} \text{ and } y_2(0) = y_{2,0}, \end{cases} \tag{34}$$

and

$$\begin{cases} \langle \bar{F}_1(\omega, \iota, y_2(\omega, \iota), u_1(\omega, \iota)), v_1 - u_1(\omega, \iota) \rangle \geq 0, \quad \forall v_1 \in K_1, \; a.e. \; \iota \in [0, T], \; a.s. \; \omega \in \Omega, \\ \langle \bar{F}_2(\omega, \iota, y_1(\omega, \iota), u_2(\omega, \iota)), v_2 - u_2(\omega, \iota) \rangle \geq 0, \quad \forall v_2 \in K_2, \; a.e. \; \iota \in [0, T], \; a.s. \; \omega \in \Omega, \end{cases} \tag{35}$$

where for $l = 1, 2$,

$$\begin{cases} y_l(\iota) = (\mathbf{p}_l^*(\iota), \mathbf{q}_l^*(\iota))^T, \; y_l(0) = (\mathbf{p}_{l,0}^*, \mathbf{q}_{l,0}^*)^T, \; u_l(\omega, \iota) = u_l^*(\omega, \iota), \\ b_l(\iota, y_l(\iota), u_l(\iota)) = (b_1^l(\iota, \mathbf{p}_l^*(\iota), \bar{\mathbf{S}}_l^*(\iota)), b_2^l(\iota, \mathbf{q}_l^*(\iota), \bar{\mathbf{D}}_l^*(\iota)))^T, \\ \sigma_l(\iota, y_l(\iota), u_l(\iota)) = (\sigma_1^l(\iota, \mathbf{p}_l^*(\iota), \bar{\mathbf{S}}_l^*(\iota)), \sigma_2^l(\iota, \mathbf{q}_l^*(\iota), \bar{\mathbf{D}}_l^*(\iota)))^T, \\ f_l(\iota, y_l(\iota), u_l(\iota)) = \begin{pmatrix} f_1^l(\iota, \mathbf{p}_l^*(\iota), \bar{\mathbf{S}}_l^*(\iota)) & 0 \\ 0 & f_2^l(\iota, \mathbf{q}_l^*(\iota), \bar{\mathbf{D}}_l^*(\iota)) \end{pmatrix}, \\ G_l(\iota, y_l(\iota), u_l(\iota), x) = \begin{pmatrix} G_1^l(\iota, \mathbf{p}_l^*(\iota), \bar{\mathbf{S}}_l^*(\iota), x) & 0 \\ 0 & G_2^l(\iota, \mathbf{q}_l^*(\iota), \bar{\mathbf{D}}_l^*(\iota), x) \end{pmatrix}, \\ B_l(\iota) = (B_1^l(\iota), B_2^l(\iota))^T, \;\; \widetilde{N}_l(\iota, x) = (\widetilde{N}_1^l(\iota, x), \widetilde{N}_2^l(\iota, x))^T, \\ \bar{F}_1(\omega, \iota, y_2(\omega, \iota), u_1(\omega, \iota)) = F_1(\omega, \iota, \mathbf{p}_2^*(\omega, \iota), \mathbf{q}_2^*(\omega, \iota), u_1(\omega, \iota)), \\ \bar{F}_2(\omega, \iota, y_1(\omega, \iota), u_2(\omega, \iota)) = F_2(\omega, \iota, \mathbf{p}_1^*(\omega, \iota), \mathbf{q}_1^*(\omega, \iota), u_2(\omega, \iota)). \end{cases} \tag{36}$$

Therefore, under the assumptions of Theorem 1, we deduce that there is only a SPE system point in SC affected with Lévy jumps and memory, provided the APPs fulfill (31).

5. Conclusions

This paper have introduced and analyzed a new symmetrical SFSDVI with Lévy jumps (3) and (4) which can be applied for acquiring the systems' instability and memorability. By aid of Picard's successive iteration method and the equivalent relationship of solutions to (3) and (4), along with Hölder-type inequality, Itô-type isometry and Doob-type inequality, we have shown that there holds the unique existence of solutions to issue (3) and (4) via a few mild assumptions. In addition, we have presented an illustrative instance of our theoretical outcomes to the SPE system in the SCs affected with Lévy jumps and memory. It is noteworthy that the fractional Brownian Motion (FBM) has captured extensive attention in SSs [27,33,34]. As well as we know, there has been no research work for one to explore the symmetrical SFSDVI driven by FBM. Whereby, it is naturally interesting and meaningful to delve into the symmetrical SFSDVI driven by FBM. So, there is no doubt for us to aim at studying such matters in the future.

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