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New Numerical Quadrature Functional Inequalities on Hilbert Spaces in the Framework of Different Forms of Generalized Convex Mappings

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Abstract: The purpose of this article is to investigate some tensorial norm inequalities for continuous functions of self-adjoint operators in Hilbert spaces. Our first approach is to develop a gradient descent inequality and some relational properties for continuous functions involving Huber convex functions, as well as several new bounds for Simpson type inequality that is twice differentiable using different types of generalized convex mappings. It is believed that this study will provide a valuable contribution towards developing a new perspective on functional inequalities by utilizing some other types of generalized mappings.

Keywords: Simpson inequality; functional inequality; convex mapping; Hilbert space

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1. Introduction

In many areas of mathematics, including approximation theory, convex programming, and mathematical statistics, convexity is an important concept. A variety of convex functions have recently been extensively studied by scholars in applied sciences. Convex functions are highly important in formulating different inequalities. The link between convexity and inequality is a broad field of study with important applications in practical arithmetic. For example, in numerical methods, inequalities derived from convex functions are used to estimate errors and improve algorithms [1]; in information theory, specifically in estimating entropies and divergences [2]; in statistics, they aid in understanding distributions and the behavior of systems under various constraints, leading to insights [3]; and in economics, convexity in preferences or utility functions can result in inequalities that describe optimal allocations of resources [4]. For some further recent applications in various disciplines, we refer to [5–9].

In recent years, fractional calculus has made significant advances in many areas of mathematics and science. In recent years, new definitions of integrals and fractional derivatives have emerged, expanding upon the traditional definitions in some way. Furthermore, one prominent topic of mathematical analysis study has been the thorough examination of those new definitions. Many materials and processes exhibit non-local behavior, where the current state depends on the history of the system. Fractional derivatives capture this memory effect naturally, as they involve integrals over time or space. Due to the usefulness of non-integer calculus, researchers have exploit it to develop convex integral disparities that play a significant role in approximation theory. The following are a few instances of inequalities that may be used to identify the error limits of quadrature formulas: Jensen [10], Simpson's [11], Ostrowski [12], Hermite-Hadamard [13], trapezoidal [14], and several other. To build these convex integral inequalities, the researchers used a variety of convex mappings, integral operators (classical, fractional, and stochastic), order relations (cr-order, pseudo-order, left-right order, inclusion orders), and other techniques. For instance, in [15], authors used convex symmetric coordinated functions to create Hermite and Hadamard inequality; in [16], authors used a fractional Riemann-Liouville integral to create Newton type inequalities for generalized convex functions; in [17], authors created Simpson type inequalities by using various function classes; and in [18], authors created Bullen-type result using generalized integral operators. The authors of [19] improve Young's inequality with a number of intriguing bounds and applications, and in [20], they develop Holder's inequality by solving delay differential equations using mean continuity and proving its uniqueness. The authors in [21] developed an Ostrowski type inequality using differentiable s-convex mapping, whereas the authors in [22] developed trapezoid type inequalities using quantum integral operators.

Simpson's inequality holds significance as it not only provides a theoretical foundation for the accuracy of numerical integration techniques, but also aids researchers in selecting the most effective methods based on the characteristics of the functions they are studying, particularly in the context of quadrature error estimation and complex definite integrals. Thomas Simpson, a mathematician, popularized Simpson's rule in the 18th century, and it is the basis for Simpson's inequality. By approximating a function with a quadratic polynomial, the rule offers a way to estimate the integral of that function. Specifically, it states that for a function $\Im$ that is continuous on the interval $[\xi_1, \xi_2]$, the integral can be approximated as:

- Simpson's $\frac{1}{3}$ rule, often known as the quadrature formula:

$$\int_{\xi_1}^{\xi_2} \Im(\kappa)\mathrm{d}\kappa \approx \frac{\xi_2 - \xi_1}{6}\left(\Im(\xi_1) + 4\Im\left(\frac{\xi_1 + \xi_2}{2}\right) + \Im(\xi_2)\right).$$

- Simpson's $\frac{3}{8}$ rule, often known as the Simpson's 2nd formula:

$$\int_{\xi_1}^{\xi_2} \Im(\kappa)\mathrm{d}\kappa \approx \frac{\xi_2 - \xi_1}{8}\left[\Im(\xi_1) + 3\Im\left(\frac{2\xi_1 + \xi_2}{3}\right) + 3\Im\left(\frac{\xi_1 + 2\xi_2}{3}\right) + \Im(\xi_2)\right].$$

The most often used three-point Simpson-type inequality has the following definition.

Theorem 1 ([23]). *Let $\Im : [\xi_1, \xi_2] \to \mathrm{R}$ be a continuous mapping, and assume that $\left\|\Im^{(4)}\right\|_\infty = \sup_{\kappa\in(\xi_1,\xi_2)}\left|\Im^{(4)}(\kappa)\right| < \infty$. The inequality listed below is therefore true:*

$$\left|\frac{1}{6}\left[\Im(\xi_1) + 4\Im\left(\frac{\xi_1 + \xi_2}{2}\right) + \Im(\xi_2)\right] - \frac{1}{\xi_2 - \xi_1}\int_{\xi_1}^{\xi_2} \Im(\kappa)\mathrm{d}\kappa\right| \leq \frac{1}{2880}\left\|\Im^{(4)}\right\|_\infty (\xi_2 - \xi_1)^4.$$

Numerous techniques have been employed by scholars to examine Simpson's inequality. For instance, the authors of [24] demonstrated multiple new bounds using coordinated convex type mappings and q-class integral operators; in [25], authors used several fractional integral operators for differentiable mappings and discovered various improved bounds; in [26], authors demonstrated refinement and reversal using preinvex mappings and quantum calculus; in [27], authors used the idea of tempered fractional integral operators; and in [28], authors employed multiplicative calculus to determine various bounds and reversals for such inequalities. For additional information on these kinds of related outcomes, readers are directed to [29–32] and the references therein.

The use of self-adjoint operators, a basic class of operators in arithmetic and physics, allows for expansions of well-known numerical inequalities to the domain of linear operators acting on Hilbert spaces. They extend the concept of Hermitian matrices, which are square matrices that have the property of being identical to their own conjugate transpose, which ensures orthogonal eigenvectors and true eigenvalues. Numerous disciplines, such as functional analysis, matrix theory, quantum physics, and optimization, depend heavily on these inequalities. Recently, a large number of authors have studied classical inequalities in relation to operators on Hilbert spaces. For instance, in [33], authors used bounded linear operators in Hilbert spaces to generate numerical radius-type inequalities, whereas in [34], authors created multiple means type inequalities for linear operators in the setup of Hilbert spaces; in [35], authors propose Hölder-type inequalities that involve power series, which have intriguing applications in Hilbert spaces; and in [36], authors study variational problem associated with inequalities and graphs in Hilbert spaces. See the references in [37–44] for further results on a similar kind connected to developed results.

Silvestru Sever Dragomir [45] presents several new novel modifications and refinements of Young's results in tensorial framework.

Theorem 2 ([45])**.** *Let* $\mathtt{H}$ *represent a Hilbert space. If the self-adjoint operators* $\mathtt{A}$ *and* $\mathtt{B}$ *hold the following conditions* $0 < v_1 \le \mathtt{B}, \mathtt{A} \le v_2$*, for some constants* v_1, v_2 *together with associated tensorial product of self-adjoint operators* $\mathtt{A} \otimes \mathtt{B}$ *in* $\mathtt{H}$*, then*

$$\begin{aligned} 0 &\le \frac{v_1}{v_2^2}\kappa(1-\kappa)\left(\frac{\mathtt{B}^2 \otimes 1 + 1 \otimes \mathtt{A}^2}{2} - \mathtt{B} \otimes \mathtt{A}\right) \\ &\le (1-\kappa)\mathtt{B} \otimes 1 + \kappa 1 \otimes \mathtt{A} - \mathtt{B}^{1-\kappa} \otimes \mathtt{A}^{\kappa} \\ &\le \frac{v_2}{v_1^2}\kappa(1-\kappa)\left(\frac{\mathtt{B}^2 \otimes 1 + 1 \otimes \mathtt{A}^2}{2} - \mathtt{B} \otimes \mathtt{A}\right). \end{aligned}$$

Vuk Stojiljkovic [46] developed the Simpson and Ostrowski type inequality by employing classical integral operators and twice differentiable mappings to continuous functions on self-adjoint operators in Hilbert space.

Theorem 3 ([46])**.** *Assume that* $\Im$ *is continuously differentiable on* Δ*,* $\mathtt{A}$ *and* $\mathtt{B}$ *are selfadjoint operators with associated sepctrums* $\mathcal{SP}(\mathtt{B}), \mathcal{SP}(\mathtt{A}) \subset \Delta$ *together with tensorial product of self-adjoint operators* $\mathtt{A} \otimes \mathtt{B}$ *in* $\mathtt{H}$*, then*

$$\begin{aligned} &\int_0^1 \Im((1-\kappa)\mathtt{B} \otimes 1 + \kappa 1 \otimes \mathtt{A})\mathrm{d}\kappa - \Im\left(\frac{\mathtt{B} \otimes 1 + 1 \otimes \mathtt{A}}{2}\right) \\ =&\frac{(1 \otimes \mathtt{A} - \mathtt{B} \otimes 1)^2}{16}\left[\int_0^1 \kappa^2 \Im''((1-\kappa)\mathtt{B} \otimes 1 + \kappa 1 \otimes \mathtt{A})\mathrm{d}\kappa\right. \\ &\left.+ \int_0^1 (\kappa-1)^2 \Im''\left(\left(\frac{1-\kappa}{2}\right)\mathtt{B} \otimes 1 + \left(\frac{1+\kappa}{2}\right)1 \otimes \mathtt{A}\right)\mathrm{d}\kappa\right]. \end{aligned}$$

Theorem 4 ([47])**.** *Assume that* $\Im$ *is continuously differentiable on* Δ *with* $|\Im'|$ *is convex on* Δ*,* $\mathtt{A}$ *and* $\mathtt{B}$ *are selfadjoint operators with associated sepctrums* $\mathcal{SP}(\mathtt{A}), \mathcal{SP}(\mathtt{B}) \subset \Delta$ *together with tensorial product of self-adjoint operators* $\mathtt{A} \otimes \mathtt{B}$ *in* $\mathtt{H}$*, then*

$$\begin{aligned} &\left\|\frac{1}{6}\left[\Im(\mathtt{B}) \otimes 1 + 4\Im\left(\frac{\mathtt{B} \otimes 1 + 1 \otimes v}{2}\right) + 1 \otimes \Im(v)\right] - \int_0^1 \Im((1-\kappa)\mathtt{B} \otimes 1 + \kappa 1 \otimes v)\mathrm{d}\kappa\right\| \\ &\leqslant \frac{5}{72}\|1 \otimes \mathtt{A} - \mathtt{B} \otimes 1\|\left(\|\Im'(\mathtt{B})\| + \|\Im'(v)\|\right). \end{aligned}$$

Shuhei employed positive semidefinite operators on a Hilbert space to derive the following Callebaut type inequality for tensorial product

Theorem 5 ([48])**.** *Let* $\mathtt{A}$ *and* $\mathtt{B}$ *are positive as well as semidefinite operators with associated sepctrums* $\mathcal{SP}(\mathtt{B}), \mathcal{SP}(\mathtt{A}) \subset \Delta$*. Then*

$$(\mathtt{B}\#\mathtt{A}) \otimes (\mathtt{B}\#\mathtt{A}) \leqslant \frac{1}{2}\Big\{(\mathtt{B}\sigma\mathtt{A}) \otimes \Big(\mathtt{B}\sigma^{\perp}\mathtt{A}\Big) + \Big(\mathtt{B}\sigma^{\perp}\mathtt{A}\Big) \otimes (\mathtt{B}\sigma\mathtt{A})\Big\}$$
$$\leqslant \frac{1}{2}\{(\mathtt{B} \otimes \mathtt{A}) + (\mathtt{A} \otimes \mathtt{B})\},$$

where # *is the geometric mean,* $\otimes$ *is a tensorial product of self-adjoint operators,* σ *and* $\sigma^{\perp}$ *are operator means and their dual.*

Significance of the Study

The importance of tensorial functional inequalities lies in their versatility and ability to bridge abstract mathematical concepts with practical applications across disciplines. Tensorial functional inequalities extend classical scalar inequalities to multidimensional and tensor-valued contexts. For instance, tensor versions of the spectral norm inequality [49], triangle inequality [50], or determinant-related inequalities [51] expand the applicability of classical results to higher dimensions. As a consequence of its importance, we extend Budak et al. [52] result to tensor settings by using continuous self-adjoint operators in Hilbert spaces. We use convex, quasi-convex, and also check the maximum bound over the interval domian in comparison to their results. We present a novel and significant study in which mathematical inequalities are developed using Hilbert spaces in tensor frameworks, and this is the first time that a gradient inequality has been constructed using self-adjoint operators in Hilbert spaces. In a recent study, authors examined Simpson type inequalities using classical integral operators in Hilbert spaces, while in this study, we use fractional integral operators to refine earlier results under different operator orders. We also use a very interesting Huber convex function in order to show some relational properties of tensorial arithmetic operations, which opens up an entirely new avenue for inequality theory. Since we know that the theory of tensor Hilbert spaces is very well known in literature but related to developed results it's relatively new not rushed comparative to classical results of such types. As a result, we hope that this article is somehow an initiative, and we believe that researchers are working in this direction to develop more interesting results in the future by taking motivation from this.

Our motivation to create a new and enhanced version of different inequalities in tensorial Hilbert spaces comes mostly from the works of these authors [47,52–54]. The use of fresh approaches and viewpoints, which have almost ever been covered in a few papers, significantly broadens and enriches inequality theory. The work is organized into four sections. In Section 2, we will go over some fundamental principles related to Hilbert spaces, including basic definitions and various arithmetic operations on tensor Hilbert spaces. In Section 3, we developed gradient inequality and various essential lemmas and bounds for Simpson type inequalities using operator convex mappings. In Section 4, we discuss a main findings and some future possible work related to these results.

2. Preliminaries

In this part, we review some primary ideas related to extended convex mappings and arithmetic operations on tensor Hilbert spaces. For more crucial concepts and findings pertaining to this part, we direct readers to the subsequent article [44].

Definition 1 ([55])**.** *A inner product on a complex linear space $\mathcal{K}$ is a map*

$$(\cdot,\cdot):\mathcal{K}\times\mathcal{K}\to\mathbb{C}$$

such that, for all $\kappa_1,\kappa_2,\kappa_3\in\mathcal{K}$ and $\lambda\in\mathbb{C}$, we have

$$\begin{aligned}\langle\kappa_1+\kappa_2,\kappa_3\rangle&=\langle\kappa_1,\kappa_3\rangle+\langle\kappa_2+\kappa_3\rangle\\ \langle\lambda\kappa_1,\kappa_2\rangle&=\lambda\langle\kappa_1,\kappa_2\rangle\\ \langle\kappa_1,\kappa_2\rangle&=\overline{\langle\kappa_2,\kappa_1\rangle}\\ \langle\kappa_1,\kappa_1\rangle&\geq 0,\quad\langle\kappa_1,\kappa_1\rangle=0\Longleftrightarrow\kappa_1=0.\end{aligned}$$

Definition 2 ([55])**.** *A bilinear mapping $\Im:\mathtt{B}\times\mathtt{A}\to\mathcal{K}$ and a tensor product of $\mathtt{B}$ with $\mathtt{A}$ provide a Hilbert space $\mathcal{K}$ such that*

- *The collection of all vectors $\Im(\xi_1,\xi_2)(\xi_1\in\mathtt{B},\xi_2\in\mathtt{A})$ is a total subset of $\mathcal{K}$ its closed linear span is equal to $\mathcal{K}$;*
- *$(\Im(\xi_1,\xi_2)\mid\Im(\xi_3,\xi_4))=(\xi_1\mid\xi_2)(\xi_3\mid\xi_4)$ for $\xi_1,\xi_2\in\mathtt{B},\xi_3,\xi_4\in\mathtt{A}$. If $(\mathcal{K},\Im)$ is multiplication of $\mathtt{B}$ and $\mathtt{A}$, it is common to write $\xi_1\otimes\xi_2$ in place of $\Im(\xi_1,\xi_2)$. A tensor product of $\mathtt{B}\otimes\mathtt{A}$ and a mapping $(\xi_1,\xi_2)\mapsto\xi_1\otimes\xi_2$ of $\mathtt{B}\times\mathtt{A}$ into $\mathtt{B}\otimes\mathtt{A}$, holds*

$$\begin{aligned}(\xi_1+\xi_2)\otimes\xi_2&=\xi_1\otimes\xi_2+\xi_2\otimes\xi_2\\ (\lambda\xi_1)\otimes\xi_2&=\lambda(\xi_1\otimes\xi_2)\\ \xi_1\otimes(\xi_3+\xi_4)&=\xi_1\otimes\xi_3+\xi_1\otimes\xi_3\\ \xi_1\otimes(\lambda\xi_2)&=\lambda(\xi_1\otimes\xi_2),\end{aligned}$$

where $\lambda\in\mathcal{K}$.

Let $\Im:\Delta_1\times\ldots\times\Delta_\mathtt{p}\to\mathrm{R}$ be a bounded real-valued mapping defined on the Cartesian product of the intervals. Let $\mathtt{M}=(\mathtt{M}_1,\ldots,\mathtt{M}_\mathtt{p})$ be an p-tuple of adjoint operators on Hilbert spaces $\mathtt{E}_1,\ldots,\mathtt{E}_\mathtt{p}$. Then

$$\mathtt{M}_\mathtt{i}=\int_{\Delta_\mathtt{i}}\kappa_\mathtt{i}\mathtt{dE}_\mathtt{i}(\kappa_\mathtt{i})$$

is the spectrum of operators for $\mathtt{i}=1,\ldots,\mathtt{p}$; following [48], we define $\mathtt{M}_\mathtt{i}$ as follows:

$$\Im(\mathtt{M}_1,\ldots,\mathtt{M}_\mathtt{p}):=\int_{\Delta_1}\ldots\int_{\Delta_\mathtt{p}}\Im(\kappa_1,\ldots,\kappa_\mathtt{p})\mathtt{dE}_1(\kappa_1)\otimes\ldots\otimes\mathtt{dE}_\mathtt{p}(\kappa_\mathtt{p}).$$

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. The author expands the construction [48] in [53] and defines it as:

$$\Im(\mathtt{M}_1,\ldots,\mathtt{M}_\mathtt{p})=\Im_1(\mathtt{M}_1)\otimes\ldots\otimes\Im_\mathtt{p}(\mathtt{M}_\mathtt{p}),$$

whenever $\Im$ can be separated as a product $\Im(\mathtt{a}_1,\ldots,\mathtt{a}_\mathtt{p})=\Im_1(\mathtt{a}_1)\ldots\Im_\mathtt{p}(\mathtt{a}_\mathtt{p})$ of p functions each depending on only one variable.

It is known that, if $\Im$ is super-multiplicative (sub-multiplicative) on $[0,\infty)$, namely

$$\Im(\xi_1\xi_2)\geq(\leq)\Im(\xi_1)\Im(\xi_2)\text{ for all }\xi_1\xi_2\in[0,\infty)$$

and if $\Im$ is continuous on $[0,\infty)$, then

$$\Im(\mathtt{A}\otimes\mathtt{B})\geq(\leq)\Im(\mathtt{A})\otimes\Im(\mathtt{B})\text{ for all }\mathtt{A},\mathtt{B}\geq 0,$$

this follows by observing that, if

$$\mathtt{A} = \int_{[0,\infty)} \xi_1 \mathtt{dE}(\xi_1) \text{ and } \mathtt{B} = \int_{[0,\infty)} \xi_2 \mathtt{dF}(\xi_2),$$

are the spectral resolutions of $\mathtt{A}$ and $\mathtt{B}$, for the continuous function $\Im$ on $[0,\infty)$, then

$$\Im(\mathtt{A} \otimes \mathtt{B}) = \int_{[0,\infty)} \int_{[0,\infty)} \Im(\xi_1\xi_2) \mathtt{dE}(\xi_1) \otimes \mathtt{dF}(\xi_2).$$

Recall the geometric operator mean for the positive operators $\mathtt{B}, \mathtt{A} > 0$, that is

$$\mathtt{B}\#_{\mathtt{p}}\mathtt{A} := \mathtt{B}^{1/2}\left(\mathtt{B}^{-1/2}\mathtt{A}\mathtt{B}^{-1/2}\right)^{\mathtt{p}}\mathtt{B}^{1/2},$$

where $\mathtt{p} \in [0,1]$ and

$$\mathtt{B}\#\mathtt{A} := \mathtt{B}^{1/2}\left(\mathtt{B}^{-1/2}\mathtt{A}\mathtt{B}^{-1/2}\right)^{1/2}\mathtt{B}^{1/2}.$$

By the definitions of # and $\otimes$, we have

$$\mathtt{B}\#\mathtt{A} = \mathtt{A}\#\mathtt{B} \text{ and } (\mathtt{B}\#\mathtt{A}) \otimes (\mathtt{A}\#\mathtt{B}) = (\mathtt{B} \otimes \mathtt{A})\#(\mathtt{A} \otimes \mathtt{B}).$$

Recall the following property of the tensorial product

$$(\mathtt{AC}) \otimes (\mathtt{BD}) = (\mathtt{A} \otimes \mathtt{B})(\mathtt{C} \otimes \mathtt{D}), \tag{1}$$

that holds for any $\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D} \in \mathtt{B}(\mathtt{H})$, the Banach algebra of all bounded linear operators on Hilbert space $\mathtt{H}$. If we take $\mathtt{C} = \mathtt{A}$ and $\mathtt{D} = \mathtt{B}$, then we get

$$\mathtt{A}^2 \otimes \mathtt{B}^2 = (\mathtt{A} \otimes \mathtt{B})^2$$

By induction and using (1), we derive that

$$\mathtt{A}^{\mathtt{n}} \otimes \mathtt{B}^{\mathtt{n}} = (\mathtt{A} \otimes \mathtt{B})^{\mathtt{n}} \text{ for natural } \mathtt{n} \geq 0.$$

In particular

$$\mathtt{B}^{\sigma} \otimes 1 = (\mathtt{B} \otimes 1)^{\sigma} \text{ and } 1 \otimes \mathtt{A}^{\sigma} = (1 \otimes \mathtt{A})^{\sigma},$$

for all $\sigma \geq 0$.

We also observe that, by (1), the operators $\mathtt{A} \otimes 1$ and $1 \otimes \mathtt{B}$ are commutative and

$$(\mathtt{B} \otimes 1)(1 \otimes \mathtt{A}) = (1 \otimes \mathtt{A})(\mathtt{B} \otimes 1) = \mathtt{B} \otimes \mathtt{A}.$$

Moreover, for two natural numbers $\mathtt{m}, \mathtt{n}$, we have

$$(\mathtt{B} \otimes 1)^{\mathtt{m}}(1 \otimes \mathtt{A})^{\mathtt{n}} = (1 \otimes \mathtt{A})^{\mathtt{m}}(\mathtt{B} \otimes 1)^{\mathtt{n}} = \mathtt{B}^{\mathtt{n}} \otimes \mathtt{A}^{\mathtt{m}}.$$

Definition 3 ([56]). *A mapping $\Im : \Delta \subseteq \mathsf{R} \to \mathsf{R}$ is said to be convex (concave) on Δ, if*

$$\Im(\kappa\xi_1 + (1-\kappa)\xi_2) \leq (\geq) \kappa\Im(\xi_1) + (1-\kappa)\Im(\xi_2)$$

valid for all $\xi_1, \xi_2 \in \Delta$ and $\kappa \in [0,1]$.

Definition 4 ([56])**.** *A mapping* $\Im : \Delta \to \mathrm{R}$ *is said to be quasi-convex, if*

$$\Im((1-\kappa)\xi_1+\kappa\xi_2) \le \max\{\Im(\xi_2),\Im(\xi_1)\} = \frac{1}{2}(\Im(\xi_2)+\Im(\xi_1)+|\Im(\xi_2)-\mathtt{A}(\xi_1)|)$$

for all $\xi_1, \xi_2 \in \Delta$ *and* $\kappa \in [0,1]$.

Lemma 1 ([52])**.** *Let* $\Im : [\xi_1, \xi_2] \to \mathrm{R}$ *be a twice differentiable mapping on* (ξ_1, ξ_2) *such that* $\Im'' \in \mathcal{L}([\xi_1, \xi_2])$*. Then, the following equality holds:*

$$\begin{aligned}
&\frac{1}{6}\left[\Im(\xi_1)+4\Im\left(\frac{\xi_1+\xi_2}{2}\right)+\Im(\xi_2)\right]-\frac{2^{v-1}\Gamma(v+1)}{(\xi_2-\xi_1)^v}\left[\mathcal{J}^v_{\frac{\xi_1+\xi_2}{2}-}\Im(\xi_1)+\mathcal{J}^v_{\frac{\xi_1+\xi_2}{2}+}\Im(\xi_2)\right]\\
=&\frac{(\xi_2-\xi_1)^2}{6}\left[\int_0^{\frac{1}{2}}\left(\kappa-\frac{3.2^v\kappa^{v+1}}{v+1}\right)\left[\Im''(\xi_2\kappa+(1-\kappa)\xi_1)\right]\mathrm{d}\kappa\right.\\
&\left.+\int_{\frac{1}{2}}^1\left((1-\kappa)-\frac{3.2^v(1-\kappa)^{v+1}}{v+1}\right)\left[\Im''(\xi_2\kappa+(1-\kappa)\xi_1)\right]\mathrm{d}\kappa\right].
\end{aligned} \tag{2}$$

Proof. With the help of the integration by parts, it follows

$$\begin{aligned}
\mathcal{K}_1=&\int_0^{\frac{1}{2}}\kappa\left(1-\frac{3\cdot2^v}{v+1}\kappa^v\right)\Im''(\kappa\xi_2+(1-\kappa)\xi_1)\mathrm{d}\kappa\\
=&\kappa\left(\kappa-\frac{3\cdot2^v}{v+1}\kappa^v\right)\frac{\Im'(\kappa\xi_2+(1-\kappa)\xi_1)}{\xi_2-\xi_1}\Bigg|_0^{\frac{1}{2}}\\
&+\frac{1}{\xi_2-\xi_1}\int_0^{\frac{1}{2}}(1-3\cdot2^v\kappa^v)\Im'(\kappa\xi_2+(1-\kappa)\xi_1)\mathrm{d}\kappa\\
=&\frac{1}{\xi_2-\xi_1}\left[\frac{1}{2}-\frac{3}{2(v+1)}\right]\Im'\left(\frac{\xi_1+\xi_2}{2}\right)\\
&-\frac{1}{\xi_2-\xi_1}\left[\frac{1-3\cdot2^v\kappa^v}{\xi_2-\xi_1}\Im(\kappa\xi_2+(1-\kappa)\xi_1)\right]\Bigg|_0^{\frac{1}{2}}\\
&+\frac{3\cdot2^v v}{\xi_2-\xi_1}\int_0^{\frac{1}{2}}\kappa^{v-1}\Im'(\kappa\xi_2+(1-\kappa)\xi_1)\mathrm{d}\kappa\\
=&\frac{1}{\xi_2-\xi_1}\left[\frac{1}{2}-\frac{3}{2(v+1)}\right]\Im'\left(\frac{\xi_1+\xi_2}{2}\right)+\frac{2}{(\xi_2-\xi_1)^2}\Im\left(\frac{\xi_1+\xi_2}{2}\right)\\
&+\frac{1}{(\xi_2-\xi_1)^2}\Im(\xi_1)-\frac{2^v3\Gamma(v+1)}{(\xi_2-\xi_1)^{v+2}}\mathcal{J}^v_{\left(\frac{\xi_1+\xi_2}{2}\right)-}\Im(\xi_1).
\end{aligned} \tag{3}$$

Similarly, we obtain

$$\begin{aligned}
\mathcal{K}_2=&\int_{\frac{1}{2}}^1(1-\kappa)\left(1-\frac{3\cdot2^v}{v+1}(1-\kappa)^v\right)\Im''(\kappa\xi_2+(1-\kappa)\xi_1)\mathrm{d}\kappa\\
=&-\frac{1}{\xi_2-\xi_1}\left[\frac{1}{2}-\frac{3}{2(v+1)}\right]\Im'\left(\frac{\xi_1+\xi_2}{2}\right)+\frac{2}{(\xi_2-\xi_1)^2}\Im\left(\frac{\xi_1+\xi_2}{2}\right)\\
&+\frac{1}{(\xi_2-\xi_1)^2}\Im(\xi_2)-\frac{2^v3\Gamma(v+1)}{(\xi_2-\xi_1)^{v+2}}\mathcal{J}^v_{\left(\frac{\xi_1+\xi_2}{2}\right)+}\Im(\xi_2)
\end{aligned} \tag{4}$$

Equations (3) and (4) yield the following equality:

$$\begin{aligned}
\frac{(\xi_2-\xi_1)^2}{6}(\mathcal{K}_1+\mathcal{K}_2)=&\frac{1}{6}\left[\Im(\xi_1)+4\Im\left(\frac{\xi_1+\xi_2}{2}\right)+\Im(\xi_2)\right]\\
&-\frac{2^{v-1}\Gamma(v+1)}{(\xi_2-\xi_1)^v}\left[\mathcal{J}^v_{\left(\frac{\xi_1+\xi_2}{2}\right)+}\Im(\xi_2)+\mathcal{J}^v_{\left(\frac{\xi_1+\xi_2}{2}\right)-}\Im(\xi_1)\right].
\end{aligned}$$

This is the end of the proof of Lemma 1. □

3. The Major Results

Theorem 6. *Assume* A *and* B *are self-adjoint operators with* $\mathcal{SP}(\mathtt{B}) \subset \Delta$ *and* $\mathcal{SP}(\mathtt{A}) \subset \Delta$*. Let* $\Im$ *be convex and differentiable mapping on* Δ*, then the inequality stated below holds true:*

$$\begin{aligned}\big(\Im'(\mathtt{B}) \otimes 1\big)(\mathtt{B} \otimes 1 - 1 \otimes \mathtt{A}) &\geq \Im(\mathtt{B}) \otimes 1 - 1 \otimes \Im(\mathtt{A}) \\ &\geq (\mathtt{B} \otimes 1 - 1 \otimes \mathtt{A})\big(1 \otimes \Im'(\mathtt{A})\big). \end{aligned} \tag{5}$$

Proof. Using the gradient inequality for the differentiable convex $\Im$ on Δ, we obtain

$$\Im'(\eta)(\eta - \zeta) \geq \Im(\eta) - \Im(\zeta) \geq \Im'(\zeta)(\eta - \zeta),$$

for all $\eta, \zeta \in \Delta$. Assume that the spectral resolutions of B and A

$$\mathtt{B} = \int_\Delta \eta \mathtt{dE}(\eta) \text{ and } \mathtt{A} = \int_\Delta \zeta \mathtt{dF}(\zeta).$$

These imply that

$$\begin{aligned}\int_\Delta\int_\Delta \Im'(\eta)(\eta - \zeta)\mathtt{dE}_\eta \otimes \mathtt{dE}_\zeta &\geq \int_\Delta\int_\Delta (\Im(\eta) - \Im(\zeta))\mathtt{dE}_\eta \otimes \mathtt{dE}_\zeta \\ &\geq \int_\Delta\int_\Delta \Im'(\zeta)(\eta - \zeta)\mathtt{dE}_\eta \otimes \mathtt{dE}_\zeta. \end{aligned} \tag{6}$$

Observe that

$$\begin{aligned}&\int_\Delta\int_\Delta \Im'(\eta)(\eta - \zeta)\mathtt{dE}_\eta \otimes \mathtt{dE}_\zeta \\ &= \int_\Delta\int_\Delta \big(\Im'(\eta)\eta - \Im'(\eta)\zeta\big)\mathtt{dE}_\eta \otimes \mathtt{dE}_\zeta \\ &= \int_\Delta\int_\Delta \Im'(\eta)\eta\mathtt{dE}_\eta \otimes \mathtt{dE}_\zeta - \int_\Delta\int_\Delta \Im'(\eta)\zeta\mathtt{dE}_\eta \otimes \mathtt{dE}_\zeta \\ &= \Im'(\mathtt{B})\mathtt{B} \otimes 1 - \Im'(\mathtt{B}) \otimes \mathtt{A} \otimes 1, \end{aligned}$$

this implies that

$$\int_\Delta\int_\Delta (\Im(\eta) - \Im(\zeta))\mathtt{dE}_\eta \otimes \mathtt{dE}_\zeta = \Im'(\mathtt{B})\mathtt{B} \otimes 1 - \Im'(\mathtt{B}) \otimes \mathtt{A} \otimes 1 \tag{7}$$

and

$$\begin{aligned}&\int_\Delta\int_\Delta \Im'(\zeta)(\eta - \zeta)\mathtt{dE}_\eta \otimes \mathtt{dE}_\zeta \\ &= \int_\Delta\int_\Delta \big(\eta\Im'(\zeta) - \Im'(\zeta)\zeta\big)\mathtt{dE}_\eta \otimes \mathtt{dE}_\zeta \\ &= \int_\Delta\int_\Delta \eta\Im'(\zeta)\mathtt{dE}_\eta \otimes \mathtt{dE}_\zeta - \int_\Delta\int_\Delta \Im'(\zeta)\zeta\mathtt{dE}_\eta \otimes \mathtt{dE}_\zeta \\ &= \mathtt{B} \otimes \Im'(\mathtt{A}) - 1 \otimes \big(\Im'(\mathtt{A})\mathtt{A}\big) \end{aligned}$$

and by (7) we derive the inequality of interest:

$$\begin{aligned}\big(\Im'(\mathtt{B})\mathtt{B}\big) \otimes 1 - \Im'(\mathtt{B}) \otimes \mathtt{A} &\geq \Im(\mathtt{B}) \otimes 1 - 1 \otimes \Im(\mathtt{A}) \\ &\geq \mathtt{B} \otimes \Im'(\mathtt{A}) - 1 \otimes \big(\Im'(\mathtt{A})\mathtt{A}\big). \end{aligned} \tag{8}$$

Now, by applying the tensorial property

$$(\mathtt{mn}) \otimes (\mathtt{pq}) = (\mathtt{m} \otimes \mathtt{p})(\mathtt{n} \otimes \mathtt{q}),$$

for any $\mathtt{m}, \mathtt{n}, \mathtt{p}, \mathtt{q} \in \Delta$, we have

$$\begin{aligned}\big(\Im'(\mathtt{B})\mathtt{B}\big) \otimes 1 &= \big(\Im'(\mathtt{B}) \otimes 1\big)(\mathtt{B} \otimes 1)\\ \Im'(\mathtt{B}) \otimes \mathtt{A} &= \big(\Im'(\mathtt{B}) \otimes 1\big)(1 \otimes \mathtt{A})\\ \mathtt{B} \otimes \Im'(\mathtt{A}) &= (\mathtt{B} \otimes 1)\big(1 \otimes \Im'(\mathtt{A})\big)\end{aligned}$$

and

$$1 \otimes \big(\Im'(\mathtt{A})\mathtt{A}\big) = 1 \otimes \big(\mathtt{A}\Im'(\mathtt{A})\big) = (1 \otimes \mathtt{A})\big(1 \otimes \Im'(\mathtt{A})\big).$$

Therefore

$$\begin{aligned}\big(\Im'(\mathtt{B})\mathtt{B}\big) \otimes 1 - \Im'(\mathtt{B}) \otimes \mathtt{A} &= \big(\Im'(\mathtt{B}) \otimes 1\big)(\mathtt{B} \otimes 1) - \big(\Im'(\mathtt{B}) \otimes 1\big)(1 \otimes \mathtt{A})\\ &= \big(\Im'(\mathtt{B}) \otimes 1\big)(\mathtt{B} \otimes 1 - 1 \otimes \mathtt{A})\end{aligned}$$

and

$$\begin{aligned}\mathtt{B} \otimes \Im'(\mathtt{A}) - 1 \otimes \big(\Im'(\mathtt{A})\mathtt{A}\big) &= (\mathtt{B} \otimes 1)\big(1 \otimes \Im'(\mathtt{A})\big) - (1 \otimes \mathtt{A})\big(1 \otimes \Im'(\mathtt{A})\big)\\ &= (\mathtt{B} \otimes 1 - 1 \otimes \mathtt{A})\big(1 \otimes \Im'(\mathtt{A})\big)\end{aligned}$$

and by (8) we derive (5). □

Corollary 1. *Let self-adjoint operators* $\mathtt{B}$ *and* $\mathtt{A}$ *with* $\mathcal{SP}(\mathtt{B}) \subset \Delta$ *and* $\mathcal{SP}(\mathtt{A}) \subset \Delta$. *If* $\mathtt{B}_\mathtt{j} \in \mathtt{B}(\mathtt{H})$ *with spectra* $\mathcal{SP}(\mathtt{B}_\mathtt{j}) \subset \Delta, \mathtt{p}_\mathtt{j} \geq 0$ *for* $\mathtt{j} \in \{1, \ldots, \mathtt{n}\}$ *with* $\sum_{\mathtt{j}=1}^{\mathtt{n}} \mathtt{p}_\mathtt{j} = 1$, *then by Theorem 6 we have*

$$\begin{aligned}&\left(\sum_{\mathtt{j}=1}^{\mathtt{n}} \mathtt{p}_\mathtt{j}\Im'(\mathtt{B}_\mathtt{j})\mathtt{B}_\mathtt{j}\right) \otimes 1 - \left(\sum_{\mathtt{j}=1}^{\mathtt{n}} \mathtt{p}_\mathtt{j}\Im'(\mathtt{B}_\mathtt{j})\right) \otimes B\\ &\geq \left(\sum_{\mathtt{j}=1}^{\mathtt{n}} \mathtt{p}_\mathtt{j}\Im(\mathtt{B}_\mathtt{j})\right) \otimes 1 - 1 \otimes \Im(\mathtt{A})\\ &\geq \left(\left(\sum_{\mathtt{j}=1}^{\mathtt{n}} \mathtt{p}_\mathtt{j}\mathtt{B}_\mathtt{j}\right) \otimes 1 - 1 \otimes \mathtt{A}\right)\big(1 \otimes \Im'(\mathtt{A})\big).\end{aligned}$$

In particular, we have

$$\begin{aligned}&\left(\sum_{\mathtt{j}=1}^{\mathtt{n}} \mathtt{p}_\mathtt{j}\Im'(\mathtt{B}_\mathtt{j})\mathtt{B}_\mathtt{j}\right) \otimes 1 - \left(\sum_{\mathtt{j}=1}^{\mathtt{n}} \mathtt{p}_\mathtt{j}\Im'(\mathtt{B}_\mathtt{j})\right) \otimes \left(\sum_{\mathtt{j}=1}^{\mathtt{n}} \mathtt{p}_\mathtt{j}\mathtt{B}_\mathtt{j}\right)\\ &\geq \left(\sum_{\mathtt{j}=1}^{\mathtt{n}} \mathtt{p}_\mathtt{j}\Im(\mathtt{B}_\mathtt{j})\right) \otimes 1 - 1 \otimes \Im\left(\sum_{\mathtt{j}=1}^{\mathtt{n}} \mathtt{p}_\mathtt{j}\mathtt{B}_\mathtt{j}\right)\\ &\geq \left(\left(\sum_{\mathtt{j}=1}^{\mathtt{n}} \mathtt{p}_\mathtt{j}\mathtt{B}_\mathtt{j}\right) \otimes 1 - 1 \otimes \left(\sum_{\mathtt{j}=1}^{\mathtt{n}} \mathtt{p}_\mathtt{j}\mathtt{B}_\mathtt{j}\right)\right)\left(1 \otimes \Im'\left(\sum_{\mathtt{j}=1}^{\mathtt{n}} \mathtt{p}_\mathtt{j}\mathtt{B}_\mathtt{j}\right)\right).\end{aligned}$$

We have the following representation results for continuous functions:

Lemma 2. *Let* $\mathtt{A}$ *and* $\mathtt{B}$ *be self-adjoint operators whose spectra are contained in* Δ_1 *and* Δ_2 *respectively. Suppose that* $\Im, \vartheta$ *are continuous on* Δ_1, ζ, χ *are continuous on* Δ_2, *and* φ *is convex on* Δ, *then sum of intervals* $\vartheta(\Delta_1) + \Im(\Delta_2)$ *has the following equality:*

$$\begin{aligned}&(\Im(\mathtt{B}) \otimes 1 + 1 \otimes \zeta(\mathtt{A}))\varphi(\vartheta(\mathtt{B}) \otimes 1 + 1 \otimes \chi(\mathtt{A}))\\ &= \int_{\Delta_1}\int_{\Delta_2} (\Im(\xi_2) + \zeta(\xi_1))\varphi(\vartheta(\xi_2) + \chi(\xi_1))\mathtt{dE}_{\xi_2} \otimes \mathtt{dF}_{\xi_1},\end{aligned} \tag{9}$$

where $\mathtt{B}$ *and* $\mathtt{A}$ *have the spectral resolutions*

$$\mathtt{B} = \int_{\Delta_1} \xi_2 \mathtt{dE}(\xi_2) \text{ and } \mathtt{A} = \int_{\Delta_2} \xi_1 \mathtt{dF}(\xi_1).$$

Proof. By Stone-Weierstrass theorem [57], any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for continous function. Consider the Huber convex function, which is defined as

$$\varphi(\mu) = \begin{cases} \frac{1}{2}\mu^2 \mathtt{m}, & |\mu| \le \delta \\ \delta\left(|\mu|^{\mathtt{m}} - \frac{\delta}{2}\right), & |\mu| > \delta. \end{cases}$$

If $\mathtt{m}, \mathtt{n}$ are natural numbers and $|\mu| \le \delta$, then we have

$$\begin{aligned}
\Im &:= \int_{\Delta_1}\int_{\Delta_2} (\Im(\xi_2) + \zeta(\xi_1))\frac{1}{2}(\vartheta(\xi_2) + \chi(\xi_1))^{2\mathtt{m}} \mathtt{dE}_{\xi_2} \otimes \mathtt{dF}_{\xi_1} \\
&= \int_{\Delta_1}\int_{\Delta_2} (\Im(\xi_2) + \zeta(\xi_1)) \sum_{\mathtt{m}=0}^{\mathtt{n}} C_{\mathtt{n}}^{\mathtt{m}} \frac{1}{2}[\vartheta(\xi_2)]^{2\mathtt{m}}[\chi(\xi_1)]^{2\mathtt{n}-2\mathtt{m}} \mathtt{dE}_{\xi_2} \otimes \mathtt{dF}_{\xi_1} \\
&= \sum_{\mathtt{m}=0}^{\mathtt{n}} C_{\mathtt{n}}^{\mathtt{m}} \int_{\Delta_1}\int_{\Delta_2} (\Im(\xi_2) + \zeta(\xi_1))\frac{1}{2}[\vartheta(\xi_2)]^{2\mathtt{m}}[\chi(\xi_1)]^{2\mathtt{n}-2\mathtt{m}} \mathtt{dE}_{\xi_2} \otimes \mathtt{dF}_{\xi_1} \\
&= \sum_{\mathtt{m}=0}^{\mathtt{n}} C_{\mathtt{n}}^{\mathtt{m}} \left[\int_{\Delta_1}\int_{\Delta_2} \Im(\xi_2)\frac{1}{2}[\vartheta(\xi_2)]^{2\mathtt{m}}[\chi(\xi_1)]^{2\mathtt{n}-2\mathtt{m}} \mathtt{dE}_{\xi_2} \otimes \mathtt{dF}_{\xi_1}\right. \\
&\quad \left. + \int_{\Delta_1}\int_{\Delta_2} \zeta(\xi_1)\frac{1}{2}[\vartheta(\xi_2)]^{2\mathtt{m}}[\chi(\xi_1)]^{2\mathtt{n}-2\mathtt{m}} \mathtt{dE}_{\xi_2} \otimes \mathtt{dF}_{\xi_1}\right].
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_{\Delta_1}\int_{\Delta_2} \Im(\xi_2)\frac{1}{2}[\vartheta(\xi_2)]^{2\mathtt{m}}[\chi(\xi_1)]^{2\mathtt{n}-2\mathtt{m}} \mathtt{dE}_{\xi_2} \otimes \mathtt{dF}_{\xi_1} \\
&= \Im(\mathtt{B})\frac{1}{2}[\vartheta(\mathtt{B})]^{2\mathtt{m}} \otimes [\chi(\mathtt{A})]^{2\mathtt{n}-2\mathtt{m}} = (\Im(\mathtt{B}) \otimes 1)\frac{1}{2}\left([\vartheta(\mathtt{B})]^{2\mathtt{n}} \otimes [\chi(\mathtt{A})]^{2\mathtt{n}-2\mathtt{m}}\right) \\
&= (\Im(\mathtt{B}) \otimes 1)\frac{1}{2}\left([\vartheta(\mathtt{B})]^{2\mathtt{n}} \otimes 1\right)\left(1 \otimes [\chi(\mathtt{A})]^{2\mathtt{n}-2\mathtt{m}}\right) \\
&= (\Im(\mathtt{B}) \otimes 1)\frac{1}{2}(\vartheta(\mathtt{B}) \otimes 1)^{2\mathtt{m}}(1 \otimes \chi(\mathtt{A}))^{2\mathtt{n}-2\mathtt{m}}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Delta_1}\int_{\Delta_2} \frac{1}{2}[\vartheta(\xi_2)]^{2\mathtt{m}} \mathtt{A}(\xi_1)[\chi(\xi_1)]^{2\mathtt{n}-2\mathtt{m}} \mathtt{dE}_{\xi_2} \otimes \mathtt{dF}_{\xi_1} \\
&= \frac{1}{2}[\vartheta(\mathtt{B})]^{2\mathtt{m}} \otimes \left(\mathtt{A}(\mathtt{A})[\chi(\mathtt{A})]^{2\mathtt{n}-2\mathtt{m}}\right) = (1 \otimes \zeta(\mathtt{A}))\frac{1}{2}\left([\vartheta(\mathtt{B})]^{2\mathtt{n}} \otimes [\chi(\mathtt{A})]^{2\mathtt{n}-2\mathtt{m}}\right) \\
&= (1 \otimes \zeta(\mathtt{A}))\frac{1}{2}\left([\vartheta(\mathtt{B})]^{2\mathtt{n}} \otimes 1\right)\left(1 \otimes [\chi(\mathtt{A})]^{2\mathtt{n}-2\mathtt{m}}\right) \\
&= (1 \otimes \zeta(\mathtt{A}))\frac{1}{2}(\vartheta(\mathtt{B}) \otimes 1)^{2\mathtt{m}}(1 \otimes \chi(\mathtt{A}))^{2\mathtt{n}-2\mathtt{m}},
\end{aligned}$$

where $\frac{1}{2}(\vartheta(\mathtt{B}) \otimes 1)$ and $\frac{1}{2}(1 \otimes \chi(\mathtt{A}))$ are commute with each other. Therefore

$$\begin{aligned}
\Im &= (\Im(\mathtt{B}) \otimes 1 + 1 \otimes \zeta(\mathtt{A})) \sum_{\mathtt{m}=0}^{\mathtt{n}} C_{\mathtt{n}}^{\mathtt{m}} \frac{1}{2}(\vartheta(\mathtt{B}) \otimes 1)^{2\mathtt{m}}(1 \otimes \chi(\mathtt{A}))^{2\mathtt{n}-2\mathtt{m}} \\
&= (\Im(\mathtt{B}) \otimes 1 + 1 \otimes \zeta(\mathtt{A}))\frac{1}{2}(\vartheta(\mathtt{B}) \otimes 1 + 1 \otimes \chi(\mathtt{A}))^{2\mathtt{n}}.
\end{aligned}$$

We now analyze a second case: if $|\mu| > \delta$, then

$$\begin{aligned}\Im &:= \int_{\Delta_1}\int_{\Delta_2}(\Im(\xi_2)+\zeta(\xi_1))\delta\left(|(\vartheta(\xi_2)+\chi(\xi_1))|^{\mathtt{m}}-\frac{\delta}{2}\right)\mathtt{dE}_{\xi_2}\otimes\mathtt{dF}_{\xi_1}\\
&= \int_{\Delta_1}\int_{\Delta_2}(\Im(\xi_2)+\zeta(\xi_1))\sum_{\mathtt{m}=0}^{\mathtt{n}}C_{\mathtt{n}}^{\mathtt{m}}\delta\left(|(\vartheta(\xi_2))|^{\mathtt{m}}-\frac{\delta}{2}\right)\delta\left(|(\chi(\xi_1))|^{\mathtt{n-m}}-\frac{\delta}{2}\right)\mathtt{dE}_{\xi_2}\otimes\mathtt{dF}_{\xi_1}\\
&= \sum_{\mathtt{m}=0}^{\mathtt{n}}C_{\mathtt{n}}^{\mathtt{m}}\int_{\Delta_1}\int_{\Delta_2}(\Im(\xi_2)+\zeta(\xi_1))\delta\left(|(\vartheta(\xi_2))|^{\mathtt{m}}-\frac{\delta}{2}\right)\delta\left(|(\chi(\xi_1))|^{\mathtt{n-m}}-\frac{\delta}{2}\right)\mathtt{dE}_{\xi_2}\otimes\mathtt{dF}_{\xi_1}\\
&= \sum_{\mathtt{m}=0}^{\mathtt{n}}C_{\mathtt{n}}^{\mathtt{m}}\left[\int_{\Delta_1}\int_{\Delta_2}\Im(\xi_2)\delta\left(|(\vartheta(\xi_2))|^{\mathtt{m}}-\frac{\delta}{2}\right)\delta\left(|(\chi(\xi_1))|^{\mathtt{n-m}}-\frac{\delta}{2}\right)\mathtt{dE}_{\xi_2}\otimes\mathtt{dF}_{\xi_1}\right.\\
&\left.+\int_{\Delta_1}\int_{\Delta_2}\zeta(\xi_1)\delta\left(|(\vartheta(\xi_2))|^{\mathtt{m}}-\frac{\delta}{2}\right)\delta\left(|(\chi(\xi_1))|^{\mathtt{n-m}}-\frac{\delta}{2}\right)\mathtt{dE}_{\xi_2}\otimes\mathtt{dF}_{\xi_1}\right].\end{aligned}$$

Observe that

$$\begin{aligned}&\int_{\Delta_1}\int_{\Delta_2}\Im(\xi_2)\delta\left(|(\vartheta(\xi_2))|^{\mathtt{m}}-\frac{\delta}{2}\right)\delta\left(|(\chi(\xi_1))|^{\mathtt{n-m}}-\frac{\delta}{2}\right)\mathtt{dE}_{\xi_2}\otimes\mathtt{dF}_{\xi_1}\\
&= \Im(\mathtt{B})\delta\left(|(\vartheta(\mathtt{B}))|^{\mathtt{m}}-\frac{\delta}{2}\right)\otimes\delta\left(|(\chi(\mathtt{A}))|^{\mathtt{n-m}}-\frac{\delta}{2}\right)\\
&= (\Im(\mathtt{B})\otimes 1)\left[\delta\left(|(\vartheta(\mathtt{B}))|^{\mathtt{m}}-\frac{\delta}{2}\right)\otimes\delta\left(|(\chi(\mathtt{A}))|^{\mathtt{n-m}}-\frac{\delta}{2}\right)\right]\\
&= (\Im(\mathtt{B})\otimes 1)\left[\delta\left(|(\vartheta(\mathtt{B}))|^{\mathtt{m}}\otimes 1-\frac{\delta}{2}\right)\otimes\delta\left(1\otimes|(\chi(\mathtt{A}))|^{\mathtt{n-m}}-\frac{\delta}{2}\right)\right]\\
&= (\Im(\mathtt{B})\otimes 1)\left[\delta\left((|(\vartheta(\mathtt{B}))|\otimes 1)^{\mathtt{m}}-\frac{\delta}{2}\right)\otimes\delta\left((1\otimes|(\chi(\mathtt{A}))|)^{\mathtt{n-m}}-\frac{\delta}{2}\right)\right]\end{aligned}$$

and

$$\begin{aligned}&\int_{\Delta_1}\int_{\Delta_2}\zeta(\xi_1)\delta\left(|(\vartheta(\xi_2))|^{\mathtt{m}}-\frac{\delta}{2}\right)\delta\left(|(\chi(\xi_1))|^{\mathtt{n-m}}-\frac{\delta}{2}\right)\mathtt{dE}_{\xi_2}\otimes\mathtt{dF}_{\xi_1}\\
&= \zeta(\xi_1)\delta\left(|(\vartheta(\mathtt{B}))|^{\mathtt{m}}-\frac{\delta}{2}\right)\otimes\delta\left(|(\chi(\mathtt{A}))|^{\mathtt{n-m}}-\frac{\delta}{2}\right)\\
&= (1\otimes\zeta(\xi_1))\left[\delta\left(|(\vartheta(\mathtt{B}))|^{\mathtt{m}}-\frac{\delta}{2}\right)\otimes\delta\left(|(\chi(\mathtt{A}))|^{\mathtt{n-m}}-\frac{\delta}{2}\right)\right]\\
&= (1\otimes\zeta(\xi_1))\left[\delta\left(|(\vartheta(\mathtt{B}))|^{\mathtt{m}}\otimes 1-\frac{\delta}{2}\right)\otimes\delta\left(1\otimes|(\chi(\mathtt{A}))|^{\mathtt{n-m}}-\frac{\delta}{2}\right)\right]\\
&= (1\otimes\zeta(\xi_1))\left[\delta\left((|(\vartheta(\mathtt{B}))|\otimes 1)^{\mathtt{m}}-\frac{\delta}{2}\right)\otimes\delta\left((1\otimes|(\chi(\mathtt{A}))|)^{\mathtt{n-m}}-\frac{\delta}{2}\right)\right],\end{aligned}$$

where $\delta\left(|(\vartheta(\mathtt{B}))|\otimes 1)-\frac{\delta}{2}\right)$ and $\delta\left(1\otimes|(\chi(\mathtt{A}))|)-\frac{\delta}{2}\right)$ are commute with each other. Therefore, we have

$$\begin{aligned}\Im &= (\Im(\mathtt{B})\otimes 1+1\otimes\zeta(\mathtt{A}))\sum_{\mathtt{m}=0}^{\mathtt{n}}C_{\mathtt{n}}^{\mathtt{m}}\delta\left((|(\vartheta(\mathtt{B}))|\otimes 1)^{\mathtt{m}}-\frac{\delta}{2}\right)\delta\left((1\otimes|(\chi(\mathtt{A}))|)^{\mathtt{n-m}}-\frac{\delta}{2}\right)\\
&= (\Im(\mathtt{B})\otimes 1+1\otimes\zeta(\mathtt{A}))\left[\delta\left((|(\vartheta(\mathtt{B}))|\otimes 1)-\frac{\delta}{2}\right)+\delta\left((1\otimes|(\chi(\mathtt{A}))|)-\frac{\delta}{2}\right)\right]^{\mathtt{n}}.\end{aligned}$$

□

Several Novel Bounds for Simpson Type Inequalities Using Operator Convex Mappings in Hilbert Spaces

In developing upper bounds for Simpson type inequalities, we utilized the generalized fractional integral operator and its associated identities, which we pre-owned in our major conclusions.

Definition 5 ([52]). *Let $\Im : [\xi_1, \xi_2] \to \mathsf{R}$ be a real-valued mapping on $[\xi_1, \xi_2]$. For $v > 0$ the associated Riemann-Liouville integrals are represented as:*

$$\mathcal{J}^{v}_{\xi_1+}\Im(\wp) = \frac{1}{\Gamma(v)} \int_{\xi_1}^{\wp} (\wp - \varepsilon)^{v-1}\Im(\varepsilon)\mathrm{d}\varepsilon,$$

for $\xi_1 < \wp \leqslant \xi_2$ and

$$\mathcal{J}^{v}_{\xi_2-}\Im(\wp) = \frac{1}{\Gamma(v)} \int_{\wp}^{\xi_2} (\varepsilon - \wp)^{v-1}\Im(\varepsilon)\mathrm{d}\varepsilon,$$

for $\xi_1 \leqslant \wp < \xi_2$, where Γ is the gamma function.

Lemma 3. *Let $\Im : [\xi_1, \xi_2] \to \mathsf{R}$ be a real-valued mapping on $[\xi_1, \xi_2]$. For any $\wp \in (\xi_1, \xi_2)$ we have*

$$\begin{aligned}
&\mathcal{J}^{v}_{\xi_1+}\Im(\wp) + \mathcal{J}^{v}_{\xi_2-}\Im(\wp) \\
&= \frac{1}{\Gamma(v+1)}[(\wp - \xi_1)^{v}\Im(\mathtt{B}) + (\xi_2 - \wp)^{v}\Im(\mathtt{A})] \\
&\quad + \frac{1}{\Gamma(v+1)}\left[\int_{\xi_1}^{\wp} (\wp - \varepsilon)^{v}\Im'(\varepsilon)\mathrm{d}\varepsilon - \int_{\wp}^{\xi_2} (\varepsilon - \wp)^{v}\Im'(\varepsilon)\mathrm{d}\varepsilon\right].
\end{aligned} \tag{10}$$

Proof. Since $\Im : [\xi_1, \xi_2] \to \mathsf{R}$ be a continuous mapping, then the symmetry of integrals become as:

$$\int_{\xi_1}^{\wp} (\wp - \varepsilon)^{v}\Im'(\varepsilon)\mathrm{d}\varepsilon \text{ and } \int_{\wp}^{\xi_2} (\varepsilon - \wp)^{v}\Im'(\varepsilon)\mathrm{d}\varepsilon.$$

It follows that

$$\begin{aligned}
&\frac{1}{\Gamma(v+1)} \int_{\xi_1}^{\wp} (\wp - \varepsilon)^{v}\Im'(\varepsilon)\mathrm{d}\varepsilon \\
&= \frac{1}{\Gamma(v)} \int_{\xi_1}^{\wp} (\wp - \varepsilon)^{v-1}\Im(\varepsilon)\mathrm{d}\varepsilon - \frac{1}{\Gamma(v+1)}(\wp - \xi_1)^{v}\Im(\xi_1) \\
&= \mathcal{J}^{v}_{\xi_1+}\Im(\wp) - \frac{1}{\Gamma(v+1)}(\wp - \xi_1)^{v}\Im(\xi_1),
\end{aligned} \tag{11}$$

for $\xi_1 < \wp \leqslant \xi_2$ and

$$\begin{aligned}
&\frac{1}{\Gamma(v+1)} \int_{\wp}^{\xi_2} (\varepsilon - \wp)^{v}\Im'(\varepsilon)\mathrm{d}\varepsilon \\
&= \frac{1}{\Gamma(v+1)}(\xi_2 - \wp)^{v}\Im(\xi_2) - \frac{1}{\Gamma(v)} \int_{\wp}^{\xi_2} (\varepsilon - \wp)^{v-1}\Im(\varepsilon)\mathrm{d}\varepsilon \\
&= \frac{1}{\Gamma(v+1)}(\xi_2 - \wp)^{v}\Im(\xi_2) - \mathcal{J}^{v}_{\xi_2-}\Im(\wp),
\end{aligned} \tag{12}$$

for $\xi_1 \leqslant \wp < \xi_2$. From (11), one has

$$\mathcal{J}^{v}_{\xi_1+}\Im(\wp) = \frac{1}{\Gamma(v+1)}(\wp - \xi_1)^{v}\Im(\xi_1) + \frac{1}{\Gamma(v+1)} \int_{\xi_1}^{\wp} (\wp - \varepsilon)^{v}\Im'(\varepsilon)\mathrm{d}\varepsilon,$$

for $\xi_1 < \wp \leqslant \xi_2$ and from (12), one has

$$\mathcal{J}^{v}_{\xi_2-}\Im(\wp) = \frac{1}{\Gamma(v+1)}(\xi_2 - \wp)^{v}\Im(\xi_2) - \frac{1}{\Gamma(v+1)} \int_{\wp}^{\xi_2} (\varepsilon - \wp)^{v}\Im'(\varepsilon)\mathrm{d}\varepsilon.$$

□

Lemma 4. *Let* $\Im : [\xi_1, \xi_2] \to \mathsf{R}$ *be a real-valued mapping on* $[\xi_1, \xi_2]$*. For any* $\wp \in (\xi_1, \xi_2)$*, we have*

$$\begin{aligned}
&\mathcal{J}^{v}_{\wp-}\Im(\xi_1) + \mathcal{J}^{v}_{\wp+}\Im(\xi_2) \\
&= \frac{1}{\Gamma(v+1)}[(\wp-\xi_1)^v + (\xi_2-\wp)^v]\Im(\wp) \\
&\quad + \frac{1}{\Gamma(v+1)}\left[\int_{\wp}^{\xi_2}(\xi_2-\varepsilon)^v\Im'(\varepsilon)\mathrm{d}\varepsilon - \int_{\xi_1}^{\wp}(\varepsilon-\xi_1)^v\Im'(\varepsilon)\mathrm{d}\varepsilon\right].
\end{aligned}$$

Proof. Since we have

$$\mathcal{J}^{v}_{\wp+}\Im(\xi_2) = \frac{1}{\Gamma(v)}\int_{\wp}^{\xi_2}(\xi_2-\varepsilon)^{v-1}\Im(\varepsilon)\mathrm{d}\varepsilon,$$

for $\xi_1 \leqslant \wp < \xi_2$ and

$$\mathcal{J}^{v}_{\wp-}\Im(\xi_1) = \frac{1}{\Gamma(v)}\int_{\xi_1}^{\wp}(\varepsilon-\xi_1)^{v-1}\Im(\varepsilon)\mathrm{d}\varepsilon,$$

for $\xi_1 < \wp \leqslant \xi_2$. Since $\Im : [\xi_1, \xi_2] \to \mathsf{R}$ be an continuous function $[\xi_1, \xi_2]$, then the integrals

$$\int_{\xi_1}^{\wp}(\varepsilon-\xi_1)^v\Im'(\varepsilon)\mathrm{d}\varepsilon \text{ and } \int_{\wp}^{\xi_2}(\xi_2-\varepsilon)^v\Im'(\varepsilon)\mathrm{d}\varepsilon,$$

holds and with integrating, we have

$$\begin{aligned}
&\frac{1}{\Gamma(v+1)}\int_{\xi_1}^{\wp}(\varepsilon-\xi_1)^v\Im'(\varepsilon)\mathrm{d}\varepsilon \\
&= \frac{1}{\Gamma(v+1)}(\wp-\xi_1)^v\Im(\wp) - \frac{1}{\Gamma(v)}\int_{\xi_1}^{\wp}(\varepsilon-\xi_1)^{v-1}\Im(\varepsilon)\mathrm{d}\varepsilon \\
&= \frac{1}{\Gamma(v+1)}(\wp-\xi_1)^v\Im(\wp) - \mathcal{J}^{v}_{\wp-}\Im(\xi_1),
\end{aligned} \tag{13}$$

for $\xi_1 < \wp \leqslant \xi_2$ and

$$\begin{aligned}
&\frac{1}{\Gamma(v+1)}\int_{\wp}^{\xi_2}(\xi_2-\varepsilon)^v\Im'(\varepsilon)\mathrm{d}\varepsilon \\
&= \frac{1}{\Gamma(v)}\int_{\wp}^{\xi_2}(\xi_2-\varepsilon)^{v-1}\Im(\varepsilon)\mathrm{d}\varepsilon - \frac{1}{\Gamma(v+1)}(\xi_2-\wp)^v\Im(\wp) \\
&= \mathcal{J}^{v}_{\wp+}\Im(\xi_2) - \frac{1}{\Gamma(v+1)}(\xi_2-\wp)^v\Im(\wp),
\end{aligned} \tag{14}$$

for $\xi_1 \leqslant \wp < \xi_2$. From (13) we have

$$\mathcal{J}^{v}_{\wp-}\Im(\xi_1) = \frac{1}{\Gamma(v+1)}(\wp-\xi_1)^v\Im(\wp) - \frac{1}{\Gamma(v+1)}\int_{\xi_1}^{\wp}(\varepsilon-\xi_1)^v\Im'(\varepsilon)\mathrm{d}\varepsilon,$$

for $\xi_1 < \wp \leqslant \xi_2$ and from (14)

$$\mathcal{J}^{v}_{\wp+}\Im(\xi_2) = \frac{1}{\Gamma(v+1)}(\xi_2-\wp)^v\Im(\wp) + \frac{1}{\Gamma(v+1)}\int_{\wp}^{\xi_2}(\xi_2-\varepsilon)^v\Im'(\varepsilon)\mathrm{d}\varepsilon.$$

□

Corollary 2. *Let* $\Im : [\xi_1, \xi_2] \to \mathrm{R}$ *be an absolutely continuous function on* $[\xi_1, \xi_2]$*. We have the following midpoint equalities*

$$\begin{aligned}
&\mathcal{J}^{v}_{\xi_1+}\Im\left(\frac{\xi_1+\xi_2}{2}\right)+\mathcal{J}^{v}_{\xi_2-}\Im\left(\frac{\xi_1+\xi_2}{2}\right)\\
&=\frac{1}{2^{v-1}\Gamma(v+1)}\frac{\Im(\xi_1)+\Im(\xi_2)}{2}\\
&+\frac{1}{\Gamma(v+1)}\left[\int_{\xi_1}^{\frac{\xi_1+\xi_2}{2}}\left(\frac{\xi_1+\xi_2}{2}-\varepsilon\right)^{v}\Im'(\varepsilon)\mathrm{d}\varepsilon-\int_{\frac{\xi_1+\xi_2}{2}}^{\xi_2}\left(\varepsilon-\frac{\xi_1+\xi_2}{2}\right)^{v}\Im'(\varepsilon)\mathrm{d}\varepsilon\right]
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{J}^{v}_{\frac{\xi_1+\xi_2}{2}-}\Im(\xi_1)+\mathcal{J}^{v}_{\frac{\xi_1+\xi_2}{2}+}\Im(\xi_2)\\
&=\frac{1}{2^{v-1}\Gamma(v+1)}\Im\left(\frac{\xi_1+\xi_2}{2}\right)(\xi_2-\xi_1)^{v}\\
&+\frac{1}{\Gamma(v+1)}\left[\int_{\frac{\xi_1+\xi_2}{2}}^{\xi_2}(\varepsilon-\xi_2)^{v}\Im'(\varepsilon)\mathrm{d}\varepsilon-\int_{\xi_1}^{\frac{\xi_1+\xi_2}{2}}(\varepsilon-\xi_1)^{v}\Im'(\varepsilon)\mathrm{d}\varepsilon\right],
\end{aligned} \tag{15}$$

for $\xi_1 \leqslant \frac{\xi_1+\xi_2}{2} < \xi_2$*. From (15) we have*

$$\begin{aligned}
\mathcal{J}^{v}_{\frac{\xi_1+\xi_2}{2}-}\Im(\xi_1)&=\frac{1}{2^{v-1}\Gamma(v+1)}\Im\left(\frac{\xi_1+\xi_2}{2}\right)(\xi_2-\xi_1)^{v}-\frac{1}{\Gamma(v+1)}\left[\int_{\xi_1}^{\frac{\xi_1+\xi_2}{2}}(\varepsilon-\xi_1)^{v}\Im'(\varepsilon)\mathrm{d}\varepsilon\right]\\
&=\frac{1}{2^{v-1}\Gamma(v+1)}\Im\left(\frac{\xi_1+\xi_2}{2}\right)(\xi_2-\xi_1)^{v}-\frac{\kappa^{v}(\xi_2-\xi_1)^{v+1}}{2^{v+1}\Gamma(v+1)}\left[\int_0^1\Im'\left((1-\kappa)\xi_1+\left(\frac{\xi_1+\xi_2}{2}\right)\kappa\right)\mathrm{d}\kappa\right],
\end{aligned} \tag{16}$$

for $\xi_1 < \frac{\xi_1+\xi_2}{2} \leqslant \xi_2$ *and from (15) we have*

$$\begin{aligned}
\mathcal{J}^{v}_{\frac{\xi_1+\xi_2}{2}+}\Im(\xi_2)&=\frac{1}{2^{v-1}\Gamma(v+1)}\Im\left(\frac{\xi_1+\xi_2}{2}\right)(\xi_2-\xi_1)^{v}+\frac{1}{\Gamma(v+1)}\left[\int_{\frac{\xi_1+\xi_2}{2}}^{\xi_2}(\xi_2-\varepsilon)^{v}\Im'(\varepsilon)\mathrm{d}\varepsilon\right]\\
&=\frac{1}{2^{v-1}\Gamma(v+1)}\Im\left(\frac{\xi_1+\xi_2}{2}\right)(\xi_2-\xi_1)^{v}-\frac{(1-\kappa)^{v}(\xi_2-\xi_1)^{v+1}}{2^{v+1}\Gamma(v+1)}\left[\int_0^1\Im'\left((1-\kappa)\left(\frac{\xi_1+\xi_2}{2}\right)+\xi_2\kappa\right)\mathrm{d}\kappa\right].
\end{aligned} \tag{17}$$

Lemma 5. *Assume that* $\Im$ *is continuously differentiable on* Δ*,* $\mathtt{A}$ *and* $\mathtt{B}$ *are selfadjoint operators with* $\mathcal{SP}(\mathtt{B}), \mathcal{SP}(\mathtt{A}) \subset \Delta$*, then*

$$\begin{aligned}
&\left[\frac{1}{6}(\Im(\mathtt{B})\otimes 1)+\frac{2}{3}\Im\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)+\frac{1}{6}(1\otimes\Im(\mathtt{A}))\right]\\
&\quad-\left[\Im\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)-\frac{\kappa^{v}(\xi_2-\xi_1)}{4}\left[\int_0^1\Im'\left((1-\kappa)\mathtt{B}\otimes 1+\left(\frac{\kappa 1\otimes\mathtt{A}}{2}\right)\right)\mathrm{d}\kappa\right]\right.\\
&\quad\left.+\Im\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)-\frac{(1-\kappa)^{v}(\xi_2-\xi_1)}{4}\left[\int_0^1\Im'\left(\left(\frac{1-\kappa}{2}\right)\mathtt{B}\otimes 1+\left(\frac{1+\kappa}{2}\right)1\otimes\mathtt{A}\right)\mathrm{d}\kappa\right]\right.\\
&=\frac{(1\otimes\mathtt{A}-\mathtt{B}\otimes 1)^2}{6}\left[\int_0^{\frac{1}{2}}\left(\kappa-\frac{3.2^{v}\kappa^{v+1}}{v+1}\right)\left[\Im''(\mathtt{B}\otimes 1\kappa+1\otimes\mathtt{A}(1-\kappa))\mathrm{d}\kappa\right]\right.\\
&\quad\left.+\int_{\frac{1}{2}}^{1}\left((1-\kappa)-\frac{3.2^{v}(1-\kappa)^{v+1}}{v+1}\right)\Im''(\mathtt{B}\otimes 1\kappa+1\otimes\mathtt{A}(1-\kappa))\mathrm{d}\kappa\right].
\end{aligned} \tag{18}$$

Proof. Taking into account the following result [52], which refines Simpson type inequality in the fractional framework via differentiable convex mappings.

Let mapping $\Im : [\xi_1, \xi_2] \to \mathrm{R}$ be defined over interval (ξ_1, ξ_2) such that $\Im'' \in \mathcal{L}([\xi_1, \xi_2])$. Then, we have

$$\frac{1}{6}\left[\Im(\xi_1) + 4\Im\left(\frac{\xi_1+\xi_2}{2}\right) + \Im(\xi_2)\right] - \frac{2^{v-1}\Gamma(v+1)}{(\xi_2-\xi_1)^v}\left[\mathcal{J}^v_{\frac{\xi_1+\xi_2}{2}-}\Im(\xi_1) + \mathcal{J}^v_{\frac{\xi_1+\xi_2}{2}+}\Im(\xi_2)\right]$$
$$= \frac{(\xi_2-\xi_1)^2}{6}\left[\int_0^{\frac{1}{2}}\left(\kappa - \frac{3.2^v\kappa^{v+1}}{v+1}\right)\left[\Im''(\xi_2\kappa + (1-\kappa)\xi_1)\right]\mathrm{d}\kappa\right.$$
$$\left. + \int_{\frac{1}{2}}^1\left((1-\kappa) - \frac{3.2^v(1-\kappa)^{v+1}}{v+1}\right)\left[\Im''(\xi_2\kappa + (1-\kappa)\xi_1)\right]\mathrm{d}\kappa\right]. \tag{19}$$

By using substitution from Equations (16) and (17), we have

$$\frac{1}{6}\left[\Im(\xi_1) + 4\Im\left(\frac{\xi_1+\xi_2}{2}\right) + \Im(\xi_2)\right] - \frac{2^{v-1}\Gamma(v+1)}{(\xi_2-\xi_1)^v}\left[\frac{1}{2^{v-1}\Gamma(v+1)}\Im\left(\frac{\xi_1+\xi_2}{2}\right)(\xi_2-\xi_1)^v\right.$$
$$- \frac{\kappa^v(\xi_2-\xi_1)^{v+1}}{2^{v+1}\Gamma(v+1)}\left[\int_0^1 \Im'\left((1-\kappa)\xi_1 + \left(\frac{\xi_1+\xi_2}{2}\right)\kappa\right)\mathrm{d}\kappa\right] + \frac{1}{2^{v-1}\Gamma(v+1)}\Im\left(\frac{\xi_1+\xi_2}{2}\right)(\xi_2-\xi_1)^v$$
$$\left. - \frac{(1-\kappa)^v(\xi_2-\xi_1)^{v+1}}{2^{v+1}\Gamma(v+1)}\left[\int_0^1 \Im'\left((1-\kappa)\left(\frac{\xi_1+\xi_2}{2}\right) + \xi_2\kappa\right)\mathrm{d}\kappa\right]\right]$$
$$= \frac{(\xi_2-\xi_1)^2}{6}\left[\int_0^{\frac{1}{2}}\left(\kappa - \frac{3.2^v\kappa^{v+1}}{v+1}\right)\left[\Im''(\xi_2\kappa + (1-\kappa)\xi_1)\right]\mathrm{d}\kappa\right.$$
$$\left. + \int_{\frac{1}{2}}^1\left((1-\kappa) - \frac{3.2^v(1-\kappa)^{v+1}}{v+1}\right)\left[\Im''(\xi_2\kappa + (1-\kappa)\xi_1)\right]\mathrm{d}\kappa\right]. \tag{20}$$

By making several simplifications, we may have

$$\frac{1}{6}\left[\Im(\xi_1) + 4\Im\left(\frac{\xi_1+\xi_2}{2}\right) + \Im(\xi_2)\right] - \left[\Im\left(\frac{\xi_1+\xi_2}{2}\right) - \frac{\kappa^v(\xi_2-\xi_1)}{4}\left[\int_0^1 \Im'\left((1-\kappa)\xi_1 + \left(\frac{\xi_2}{2}\right)\kappa\right)\mathrm{d}\kappa\right]\right.$$
$$\left. + \Im\left(\frac{\xi_1+\xi_2}{2}\right) - \frac{(1-\kappa)^v(\xi_2-\xi_1)}{4}\left[\int_0^1 \Im'\left(\left(\frac{1-\kappa}{2}\right)\xi_1 + \left(\left(\frac{1+\kappa}{2}\right)\xi_2\right)\mathrm{d}\kappa\right)\right]\right]$$
$$= \frac{(\xi_2-\xi_1)^2}{6}\left[\int_0^{\frac{1}{2}}\left(\kappa - \frac{3.2^v\kappa^{v+1}}{v+1}\right)\left[\Im''(\xi_2\kappa + (1-\kappa)\xi_1)\right]\mathrm{d}\kappa\right.$$
$$\left. + \int_{\frac{1}{2}}^1\left((1-\kappa) - \frac{3.2^v(1-\kappa)^{v+1}}{v+1}\right)\left[\Im''(\xi_2\kappa + (1-\kappa)\xi_1)\right]\mathrm{d}\kappa\right]. \tag{21}$$

The spectral resolutions of self-adjoint operators $\mathtt{A}$ and $\mathtt{B}$ are represented as follows:

$$\mathtt{A} = \int_\Delta \xi_1 \mathtt{dE}(\xi_1) \text{ and } \mathtt{B} = \int_\Delta \xi_2 \mathtt{dF}(\xi_2).$$

$\int_\Delta \int_\Delta$ over $\mathtt{dE}_{\xi_1} \otimes \mathtt{dF}_{\xi_2}$ in (21), then we get

$$\int_\Delta\int_\Delta \frac{1}{6}\left[\Im(\xi_1) + 4\Im\left(\frac{\xi_1+\xi_2}{2}\right) + \Im(\xi_2)\right]\mathtt{dE}_{\xi_1}\otimes\mathtt{dF}_{\xi_2}$$
$$- \left[\int_\Delta\int_\Delta\left(\Im\left(\frac{\xi_1+\xi_2}{2}\right) - \frac{\kappa^v(\xi_2-\xi_1)}{4}\left[\int_0^1 \Im'\left((1-\kappa)\xi_1 + \left(\frac{\xi_2}{2}\right)\kappa\right)\mathrm{d}\kappa\right]\right)\mathtt{dE}_{\xi_1}\otimes\mathtt{dF}_{\xi_2}\right.$$
$$+ \int_\Delta\int_\Delta\left(\Im\left(\frac{\xi_1+\xi_2}{2}\right) - \frac{(1-\kappa)^v(\xi_2-\xi_1)}{4}\left[\int_0^1 \Im'\left(\left(\frac{1-\kappa}{2}\right)\xi_1 + \left(\frac{1+\kappa}{2}\right)\xi_2\right)\mathrm{d}\kappa\right]\mathtt{dE}_{\xi_1}\otimes\mathtt{dF}_{\xi_2}\right)$$

$$=\frac{(\xi_2-\xi_1)^2}{6}\int_\Delta\int_\Delta\left[\int_0^{\frac{1}{2}}\left(\kappa-\frac{3.2^{\upsilon}\kappa^{\upsilon+1}}{\upsilon+1}\right)\left[\Im''(\xi_2\kappa+(1-\kappa)\xi_1)\right]\mathrm{d}\kappa\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}\left((1-\kappa)-\frac{3.2^{\upsilon}(1-\kappa)^{\upsilon+1}}{\upsilon+1}\right)\left[\Im''(\xi_2\kappa+(1-\kappa)\xi_1)\right]\mathrm{d}\kappa\right]\mathrm{d}\mathtt{E}_{\xi_1}\otimes\mathrm{d}\mathtt{F}_{\xi_2}. \tag{22}$$

Considering Lemma 2 and Fubini's theorem [58], we have

$$\int_\Delta\int_\Delta \Im(\xi_2)\mathrm{d}\mathtt{E}_{\xi_1}\otimes\mathrm{d}\mathtt{F}_{\xi_2}=(\Im(\mathtt{B})\otimes 1),$$
$$\int_\Delta\int_\Delta \Im\left(\frac{\xi_1+\xi_2}{2}\right)\mathrm{d}\mathtt{E}_{\xi_1}\otimes\mathrm{d}\mathtt{F}_{\xi_2}=\Im\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right),$$
$$\int_\Delta\int_\Delta \Im(\xi_1)\mathrm{d}\mathtt{E}_{\xi_1}\otimes\mathrm{d}\mathtt{F}_{\xi_2}=(1\otimes\Im(\mathtt{A})),$$
$$\int_\Delta\int_\Delta\int_0^1 \Im'\left((1-\kappa)\xi_1+\left(\frac{\xi_2}{2}\right)\kappa\right)\mathrm{d}\kappa\mathrm{d}\mathtt{E}_{\xi_1}\otimes\mathrm{d}\mathtt{F}_{\xi_2}$$
$$=\int_0^1\int_\Delta\int_\Delta \Im'\left((1-\kappa)\xi_1+\left(\frac{\xi_2}{2}\right)\kappa\right)\mathrm{d}\mathtt{E}_{\xi_1}\otimes\mathrm{d}\mathtt{F}_{\xi_2}\mathrm{d}\kappa$$
$$=\int_0^1 \Im'\left((1-\kappa)\mathtt{B}\otimes 1+\left(\frac{\kappa 1\otimes\mathtt{A}}{2}\right)\right)\mathrm{d}\kappa,$$
$$\int_\Delta\int_\Delta\int_0^1 \Im'\left(\left(\frac{1-\kappa}{2}\right)\xi_1+\left(\frac{1+\kappa}{2}\right)\xi_2\right)\mathrm{d}\kappa\mathrm{d}\mathtt{E}_{\xi_1}\otimes\mathrm{d}\mathtt{F}_{\xi_2}$$
$$=\int_0^1\int_\Delta\int_\Delta \Im'\left(\left(\frac{1-\kappa}{2}\right)\xi_1+\left(\frac{1+\kappa}{2}\right)\xi_2\right)\mathrm{d}\mathtt{E}_{\xi_1}\otimes\mathrm{d}\mathtt{F}_{\xi_2}\mathrm{d}\kappa$$
$$=\int_0^1\int_\Delta\int_\Delta \Im'\left(\left(\frac{1-\kappa}{2}\right)\mathtt{B}\otimes 1+\left(\frac{1+\kappa}{2}\right)1\otimes\mathtt{A}\right)\mathrm{d}\kappa,$$
$$\Im''\Big(\xi_2\kappa+\xi_1(1-\kappa)\Big)\mathrm{d}\kappa\mathrm{d}\mathtt{E}_{\xi_1}\otimes\mathrm{d}\mathtt{F}_{\xi_2}=\Im''(\mathtt{B}\otimes 1\kappa+1\otimes\mathtt{A}(1-\kappa))\mathrm{d}\kappa. \tag{23}$$

A same technique has been taking into consideration we have

$$\int_\Delta\int_\Delta\frac{(\xi_2-\xi_1)^2}{6}\left[\int_0^{\frac{1}{2}}\left(\kappa-\frac{3.2^{\upsilon}\kappa^{\upsilon+1}}{\upsilon+1}\right)\left[\Im''(\xi_2\kappa+(1-\kappa)\xi_1)\right]\mathrm{d}\kappa\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}\left((1-\kappa)-\frac{3.2^{\upsilon}(1-\kappa)^{\upsilon+1}}{\upsilon+1}\right)\left[\Im''(\xi_2\kappa+(1-\kappa)\xi_1)\right]\mathrm{d}\kappa\right]\mathrm{d}\mathtt{E}_{\xi_1}\otimes\mathrm{d}\mathtt{F}_{\xi_2}$$
$$=\frac{(1\otimes\mathtt{A}-\mathtt{B}\otimes 1)^2}{6}\left[\int_0^{\frac{1}{2}}\left(\kappa-\frac{3.2^{\upsilon}\kappa^{\upsilon+1}}{\upsilon+1}\right)\left[\Im''(\mathtt{B}\otimes 1\kappa+1\otimes\mathtt{A}(1-\kappa))\mathrm{d}\kappa\right]\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}\left((1-\kappa)-\frac{3.2^{\upsilon}(1-\kappa)^{\upsilon+1}}{\upsilon+1}\right)\Im''(\mathtt{B}\otimes 1\kappa+1\otimes\mathtt{A}(1-\kappa))\mathrm{d}\kappa\right]. \tag{24}$$

We obtain the required result by accounting for (23) and (24) in (22).
□

Remark 1.

- *If we choose $\upsilon=1$ in Lemma 5, then it refines Lemma 2.1 as presented by the authors using classical integral operator in [54].*
- *If we choose $\upsilon=1$ in Lemma 5, then it refines Lemma 2.3 as presented by the authors using classical integral operator in [47].*
- *If we choose $\upsilon=1$ in Lemma 5, then it refines Lemma 3 as presented by the authors using classical integral operator in [59].*

Theorem 7. *Assume that $\Im$ is continuously differentiable on Δ with $|\Im''|$ is convex on Δ, $\mathtt{A}$ and $\mathtt{B}$ are selfadjoint operators with $\mathcal{SP}(\mathtt{A}), \mathcal{SP}(\mathtt{B}) \subset \Delta$, then*

$$\begin{aligned}&\Bigg\| \left[\frac{1}{6}(\Im(\mathtt{B})\otimes 1) + \frac{2}{3}\Im\left(\frac{\mathtt{B}\otimes 1 + 1\otimes \mathtt{A}}{2}\right) + \frac{1}{6}(1\otimes \Im(\mathtt{A}))\right] \\ &- \left[\Im\left(\frac{\mathtt{B}\otimes 1 + 1\otimes \mathtt{A}}{2}\right) - \frac{\kappa^{v}(\xi_2-\xi_1)}{4}\left[\int_0^1 \Im'\left((1-\kappa)\mathtt{B}\otimes 1 + \left(\frac{\kappa 1\otimes \mathtt{A}}{2}\right)\right)\mathtt{d}\kappa\right]\right. \\ &+ \Im\left(\frac{\mathtt{B}\otimes 1 + 1\otimes \mathtt{A}}{2}\right) - \frac{(1-\kappa)^{v}(\xi_2-\xi_1)}{4}\left[\int_0^1 \Im'\left(\left(\frac{1-\kappa}{2}\right)\mathtt{B}\otimes 1 + \left(\frac{1+\kappa}{2}\right)1\otimes \mathtt{A}\right)\mathtt{d}\kappa\right]\Bigg\| \\ &\leq \frac{\|(1\otimes \mathtt{A} - \mathtt{B}\otimes 1)^2\|}{6}\left[\left(\frac{v^2\left(\left(\frac{(v+1)^2}{9}\right)^{\frac{1}{v}} + v\left(\frac{(v+1)^2}{9}\right)^{\frac{1}{v}} + 3\right.}{4(v+1)(v+2)} - \frac{1}{8}\right)\Big\| |\Im''(\mathtt{B})| + |\Im''(\mathtt{A})| \Big\|\right].\end{aligned}$$

Proof. By assumption that $|\Im''|$ is convex on Δ, we have

$$|\Im''(\xi_2\kappa + \xi_1(1-\kappa))| \leq \kappa|\Im''(\xi_2)| + (1-\kappa)|\Im''(\xi_1)|$$

Similarly, we get

$$|\Im''(\xi_1\kappa + \xi_2(1-\kappa))| \leq \kappa|\Im''(\xi_1)| + ((1-\kappa))|\Im''(\xi_2)|$$

for all for $\tau \in [0,1]$ and $\xi_1, \xi_2 \in \Delta$.

Applying $\int_\Delta \int_\Delta$ over $\mathtt{dE}_{\xi_1} \otimes \mathtt{dF}_{\xi_2}$, then we get

$$\begin{aligned}|\Im''(1\otimes \mathtt{B}\kappa + 1\otimes \mathtt{A}(1-\kappa))| &= \int_\Delta\int_\Delta |\Im''(\xi_2\kappa + \xi_1(1-\kappa))|\mathtt{dE}_{\xi_1}\otimes \mathtt{dF}_{\xi_2} \\ &\leq \int_\Delta\int_\Delta \kappa|\Im''(\xi_2)| + (1-\kappa)|\Im''(\xi_1)|\mathtt{dE}_{\xi_1}\otimes \mathtt{dF}_{\xi_2} \\ &\leq \kappa 1\otimes |\Im''(\mathtt{B})| + (1-\kappa)|\Im''(\mathtt{A})|\otimes 1.\end{aligned} \tag{25}$$

If we apply norm in (25), then we have

$$\begin{aligned}&\|\Im''(1\otimes \mathtt{B}\kappa + 1\otimes \mathtt{A}(1-\kappa))\| \\ &\leq \|\kappa 1\otimes |\Im''(\mathtt{B})| + (1-\kappa)|\Im''(\mathtt{A})|\otimes 1\| \leq \kappa\|\Im''(\mathtt{B})\| + (1-\kappa)\|\Im''(\mathtt{A})\|.\end{aligned}$$

Similarly, we get

$$\begin{aligned}&\|\Im''(1\otimes \mathtt{A}\kappa + 1\otimes \mathtt{B}(1-\kappa))\| \\ &\leq \|\kappa 1\otimes |\Im''(\mathtt{A})| + ((1-\kappa))|\Im''(\mathtt{B})|\otimes 1\| \leq \kappa\|\Im''(\mathtt{A})\| + ((1-\kappa))\|\Im''(\mathtt{B})\|.\end{aligned}$$

Using the norm in (21) and considering triangle inequality, we have

$$\begin{aligned}&\Bigg\| \left[\frac{1}{6}(\Im(\mathtt{B})\otimes 1) + \frac{2}{3}\Im\left(\frac{\mathtt{B}\otimes 1 + 1\otimes \mathtt{A}}{2}\right) + \frac{1}{6}(1\otimes \Im(\mathtt{A}))\right] \\ &- \left[\Im\left(\frac{\mathtt{B}\otimes 1 + 1\otimes \mathtt{A}}{2}\right) - \frac{\kappa^{v}(\xi_2-\xi_1)}{4}\left[\int_0^1 \Im'\left((1-\kappa)\mathtt{B}\otimes 1 + \left(\frac{\kappa 1\otimes \mathtt{A}}{2}\right)\right)\mathtt{d}\kappa\right]\right.\end{aligned}$$

$$+\Im\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)-\frac{(1-\kappa)^{\upsilon}(\xi_2-\xi_1)}{4}\left[\int_0^1\Im'\left(\left(\frac{1-\kappa}{2}\right)\mathtt{B}\otimes 1+\left(\frac{1+\kappa}{2}\right)1\otimes\mathtt{A}\right)\mathrm{d}\kappa\right]\Bigg\|$$

$$\le\frac{\|(1\otimes\mathtt{A}-\mathtt{B}\otimes 1)^2\|}{6}\left(\left\|\int_0^{\frac{1}{2}}\left(\kappa-\frac{3.2^{\upsilon}\kappa^{\upsilon+1}}{\upsilon+1}\right)[\Im''(\mathtt{B}\otimes 1\kappa+1\otimes\mathtt{A}(1-\kappa))\mathrm{d}\kappa]\right.\right.$$

$$\left.\left.+\int_{\frac{1}{2}}^1\left((1-\kappa)-\frac{3.2^{\upsilon}(1-\kappa)^{\upsilon+1}}{\upsilon+1}\right)\Im''(\mathtt{B}\otimes 1\kappa+1\otimes\mathtt{A}(1-\kappa))\right\|\right)$$

$$\le\frac{\|(1\otimes\mathtt{A}-\mathtt{B}\otimes 1)^2\|}{6}\left(\left\|\int_0^{\frac{1}{2}}\left(\kappa-\frac{3.2^{\upsilon}\kappa^{\upsilon+1}}{\upsilon+1}\right)\left[\kappa 1\otimes|\Im''(\mathtt{B})|+(1-\kappa)|\Im''(\mathtt{B})|\otimes 1\right]\mathrm{d}\kappa\right.\right.$$

$$\left.\left.+\int_{\frac{1}{2}}^1\left((1-\kappa)-\frac{3.2^{\upsilon}(1-\kappa)^{\upsilon+1}}{\upsilon+1}\right)\kappa 1\otimes|\Im''(\mathtt{A})|+(1-\kappa)|\Im''(\mathtt{A})|\otimes 1\right\|\right)$$

$$\le\frac{\|(1\otimes\mathtt{A}-\mathtt{B}\otimes 1)^2\|}{6}\left(\left\|\int_0^{\frac{1}{2}}\left(\kappa^2-\frac{3.2^{\upsilon}\kappa^{\upsilon+2}}{\upsilon+1}\right)\otimes|\Im''(\mathtt{B})|+\int_{\frac{1}{2}}^1\left((\kappa-\kappa^2)-\frac{3.2^{\upsilon}\kappa(1-\kappa)^{\upsilon+1}}{\upsilon+1}\right)\otimes|\Im''(\mathtt{A})|\right\|\right)\mathrm{d}\kappa$$

$$\le\frac{\|(1\otimes\mathtt{A}-\mathtt{B}\otimes 1)^2\|}{6}\left[\left(\frac{\upsilon^2\left(\frac{(\upsilon+1)^2}{9}\right)^{\frac{1}{\upsilon}}+\upsilon\left(\frac{(\upsilon+1)^2}{9}\right)^{\frac{1}{\upsilon}}+3}{4(\upsilon+1)(\upsilon+2)}-\frac{1}{8}\right)\left\||\Im''(\mathtt{B})|+|\Im''(\mathtt{A})|\right\|\right]. \tag{26}$$

□

Remark 2.

- *If we choose* $\upsilon=1$ *and tensorial arithmetic operations in Theorem 7 are degenerated, then Theorem 7 simplifies to Theorem 2.2 provided by the authors in [60].*
- *If tensorial arithmetic operations in Theorem 7 are degenerated, then Theorem 7 simplifies to Theorem 2.3 provided by the authors in [61].*
- *If we choose* $\upsilon=1$ *in Theorem 7, then it refines Theorem 2.3 as presented by the authors using classical integral operator in Ref. [54].*
- *If we choose* $\upsilon=1$ *in Theorem 7, then it refines Theorem 2.3 as presented by the authors using classical integral operator in Ref. [47].*
- *If we choose* $\upsilon=1$ *in Theorem 7, then it refines Theorem 9 as presented by the authors using classical integral operator in Ref. [59].*

Remark 3. *If two self-adjoint operators* $\mathtt{A}$ *and* $\mathtt{B}$ *on a Hilbert space commute, meaning their commutator satisfies* $[\mathtt{A},\mathtt{B}]=\mathtt{AB}-\mathtt{BA}=0$*, they exhibit several important properties. Commutativity ensures that* $\mathtt{A}$ *and* $\mathtt{B}$ *can be simultaneously diagonalized, implying the existence of a common eigenbasis where both operators act as scalar multipliers. This is a significant feature in quantum mechanics and functional analysis, as it allows for the simultaneous measurement of the observables associated with* $\mathtt{A}$ *and* $\mathtt{B}$ *without interference. Moreover, the commutativity extends to functions of these operators, such as their exponentials, ensuring that* $e^{\mathtt{B}}e^{\mathtt{A}}=e^{\mathtt{A}}e^{\mathtt{B}}$*. This property is particularly useful in applications involving operator exponentiation, such as time evolution and transformations in quantum mechanics. Thus, the commutativity of self-adjoint operators is a cornerstone in the study of their spectral and functional behavior. It is known that if* $\mathtt{A}$ *and* $\mathtt{B}$ *are commuting, i.e.,* $\mathtt{AB}=\mathtt{BA}$*, then the exponential function satisfies the property*

$$e^{\mathtt{B}}e^{\mathtt{A}}=e^{\mathtt{A}}e^{\mathtt{B}}=e^{(\mathtt{B}+\mathtt{A})}.$$

Also, if $\mathtt{B}$ *is invertible and* $\xi_1,\xi_2\in\mathrm{R}$ *with* $\xi_1<\xi_2$*, then*

$$\int_{\xi_1}^{\xi_2}e^{\kappa\mathtt{B}}\mathrm{d}\kappa=\frac{[e^{\xi_2\mathtt{B}}-e^{\xi_1\mathtt{B}}]}{\mathtt{B}}.$$

Moreover, if $\mathtt{A}$ *and* $\mathtt{B}$ *are commuting and* $\mathtt{A}-\mathtt{B}$ *is invertible, then*

$$\int_0^1 e^{((1-v)\mathtt{B}+v\mathtt{A})}\mathrm{d}v = \int_0^1 e^{(v(\mathtt{A}-\mathtt{B}))}e^{\mathtt{B}}\mathrm{d}v = \left(\int_0^1 e^{(v(\mathtt{A}-\mathtt{B}))}\mathrm{d}v\right)e^{\mathtt{B}}$$
$$= \frac{[e^{(\mathtt{A}-\mathtt{B})}-\mathtt{I}]e^{\mathtt{B}}}{\mathtt{A}-\mathtt{B}} = \frac{[e^{\mathtt{A}}-e^{\mathtt{B}}]}{\mathtt{A}-\mathtt{B}}.$$

Since the operators $\mathtt{A} = \mathtt{U}\otimes 1$ *and* $\mathtt{B} = 1\otimes \mathtt{V}$ *are commutative and if* $1\otimes\mathtt{V}-\mathtt{U}\otimes 1$ *is invertible, then*

$$\int_0^1 \exp((1-\alpha)\mathtt{U}\otimes 1+\alpha 1\otimes \mathtt{V})\mathrm{d}\alpha$$
$$= (1\otimes\mathtt{V}-\mathtt{U}\otimes 1)^{-1}[\exp(1\otimes\mathtt{V})-\exp(\mathtt{U}\otimes 1)].$$

Corollary 3. *Under the assumptions of Theorem 7 with* $\Im(\mu)=\exp(\mu)$ *is continuously differentiable on* Δ*. Let* $v=\frac{1}{4}$*, then one has*

$$\Bigg\| \left[\frac{1}{6}(\exp(\mathtt{B})\otimes 1)+\frac{2}{3}\exp\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)+\frac{1}{6}(1\otimes\exp(\mathtt{A}))\right]$$
$$-\left[\exp\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)-\frac{\kappa^{v}(\xi_2-\xi_1)}{4}\left[\int_0^1 \exp'\left((1-\kappa)\mathtt{B}\otimes 1+\left(\frac{\kappa 1\otimes\mathtt{A}}{2}\right)\right)\mathrm{d}\kappa\right]\right.$$
$$\left.+\exp\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)-\frac{(1-\kappa)^{v}(\xi_2-\xi_1)}{4}\left[\int_0^1 \exp'\left(\left(\frac{1-\kappa}{2}\right)\mathtt{B}\otimes 1+\left(\frac{1+\kappa}{2}\right)1\otimes\mathtt{A}\right)\mathrm{d}\kappa\right]\right]\Bigg\|$$
$$\le \frac{\|(1\otimes\mathtt{A}-\mathtt{B}\otimes 1)^2\|}{6}\left[\frac{1}{4\left(\frac{1}{4}+2\right)}\left(\frac{1}{4}\left(\frac{\frac{1}{4}+1}{3}\right)\frac{2}{\frac{2}{4}}+\frac{3}{\frac{1}{4}+1}\right)-\frac{1}{8}\right]\Big\| |\exp''(\mathtt{B})|+|\exp''(\mathtt{A})|\Big\|.$$

Theorem 8. *Assume that* $\Im$ *is continuously differentiable on* Δ *with* $\|\Im''\|_{\Delta,\infty} := \sup_{v\in\Delta}|\Im''(v)| < \infty$ *and* $\mathtt{A}$ *and* $\mathtt{B}$ *are selfadjoint operators with* $\mathcal{SP}(\mathtt{A}),\mathcal{SP}(\mathtt{B})\subset\Delta$*, then*

$$\Bigg\| \left[\frac{1}{6}(\Im(\mathtt{B})\otimes 1)+\frac{2}{3}\Im\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)+\frac{1}{6}(1\otimes\Im(\mathtt{A}))\right]$$
$$-\left[\Im\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)-\frac{\kappa^{v}(\xi_2-\xi_1)}{4}\left[\int_0^1 \Im'\left((1-\kappa)\mathtt{B}\otimes 1+\left(\frac{\kappa 1\otimes\mathtt{A}}{2}\right)\right)\mathrm{d}\kappa\right]\right.$$
$$\left.+\Im\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)-\frac{(1-\kappa)^{v}(\xi_2-\xi_1)}{4}\left[\int_0^1 \Im'\left(\left(\frac{1-\kappa}{2}\right)\mathtt{B}\otimes 1+\left(\frac{1+\kappa}{2}\right)1\otimes\mathtt{A}\right)\mathrm{d}\kappa\right]\right]\Bigg\|$$
$$\le \frac{\|(1\otimes\mathtt{A}-\mathtt{B}\otimes 1)^2\|}{6}\left[\left(\frac{\kappa^2}{2}+\frac{6\ln^{-v-1}(2)\Gamma(v+1,\ln(2)(1-\kappa))}{v+1}\right.\right.$$
$$\left.\left.+\frac{\kappa^2}{2}+\kappa+\frac{6\ln^{-v-1}(2)\Gamma(v+1,\ln(2)(1-\kappa))}{v+1}\right)\|\Im'\|_{\Delta,+\infty}\right].$$

Proof. Considering Lemma 5 and applying the triangle inequality, we arrive at

$$\Bigg\| \left[\frac{1}{6}(\Im(\mathtt{B})\otimes 1)+\frac{2}{3}\Im\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)+\frac{1}{6}(1\otimes\Im(\mathtt{A}))\right]$$
$$-\left[\Im\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)-\frac{\kappa^{v}(\xi_2-\xi_1)}{4}\left[\int_0^1 \Im'\left((1-\kappa)\mathtt{B}\otimes 1+\left(\frac{\kappa 1\otimes\mathtt{A}}{2}\right)\right)\mathrm{d}\kappa\right]\right.$$
$$\left.+\Im\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)-\frac{(1-\kappa)^{v}(\xi_2-\xi_1)}{4}\left[\int_0^1 \Im'\left(\left(\frac{1-\kappa}{2}\right)\mathtt{B}\otimes 1+\left(\frac{1+\kappa}{2}\right)1\otimes\mathtt{A}\right)\mathrm{d}\kappa\right]\right]\Bigg\|$$

$$\leq \frac{\|(1\otimes \mathtt{A}-\mathtt{B}\otimes 1)^2\|}{6}\left\|\int_0^{\frac{1}{2}}\left(\kappa-\frac{3.2^{\upsilon}\kappa^{\upsilon+1}}{\upsilon+1}\right)[\Im''(\mathtt{B}\otimes 1\kappa+1\otimes \mathtt{A}(1-\kappa))d\kappa]\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left((1-\kappa)-\frac{3.2^{\upsilon}(1-\kappa)^{\upsilon+1}}{\upsilon+1}\right)\Im''(\mathtt{B}\otimes 1\kappa+1\otimes \mathtt{A}(1-\kappa))\right\|$$
$$\leq \frac{\|(1\otimes \mathtt{A}-\mathtt{B}\otimes 1)^2\|}{6}\left(\left\|\int_0^{\frac{1}{2}}\left(\kappa-\frac{3.2^{\upsilon}\kappa^{\upsilon+1}}{\upsilon+1}\right)[\Im''(\mathtt{B}\otimes 1\kappa+1\otimes \mathtt{A}(1-\kappa))d\kappa]\right.\right.$$
$$\left.\left.+\int_{\frac{1}{2}}^1\left((1-\kappa)-\frac{3.2^{\upsilon}(1-\kappa)^{\upsilon+1}}{\upsilon+1}\right)\Im''(\mathtt{B}\otimes 1\kappa+1\otimes \mathtt{A}(1-\kappa))\right\|\right). \tag{27}$$

Observe that, by Lemma 2

$$\left|\Im''(\mathtt{B}1\otimes\kappa+1\otimes \mathtt{A}(1-\kappa))\right| = \int_\Delta\int_\Delta\left|\Im''(\xi_2\kappa+\xi_1(1-\kappa))\right|d\mathtt{E}_{\xi_1}\otimes d\mathtt{F}_{\xi_2}.$$

Since

$$\left|\Im''(\xi_2\kappa+\xi_1(1-\kappa))\right|\leqslant\|\Im'\|_{\Delta,+\infty},$$

for all $\tau\in[0,1]$ and $\xi_1,\xi_2\in\Delta$.

Taking $\int_\Delta\int_\Delta$ over $d\mathtt{E}_{\xi_1}\otimes d\mathtt{F}_{\xi_2}$, then we get

$$\begin{aligned}\left|\Im''(1\otimes \mathtt{B}\kappa+1\otimes \mathtt{A}(1-\kappa))\right| &= \int_\Delta\int_\Delta\left|\Im''(\xi_2\kappa+\xi_1(1-\kappa))\right|d\mathtt{E}_{\xi_1}\otimes d\mathtt{F}_{\xi_2}\\ &\leqslant\|\Im'\|_{\Delta,+\infty}\int_\Delta\int_\Delta d\mathtt{E}_{\xi_1}\otimes d\mathtt{F}_{\xi_2}=\|\Im'\|_{\Delta,+\infty}.\end{aligned} \tag{28}$$

Similarly, we get

$$\begin{aligned}\left|\Im''(1\otimes \mathtt{A}\kappa+1\otimes \mathtt{B}(1-\kappa))\right| &= \int_\Delta\int_\Delta\left|\Im''(\xi_1\kappa+\xi_2(1-\kappa))\right|d\mathtt{E}_{\xi_1}\otimes d\mathtt{F}_{\xi_2}\\ &\leqslant\|\Im'\|_{\Delta,+\infty}\int_\Delta\int_\Delta d\mathtt{E}_{\xi_1}\otimes d\mathtt{F}_{\xi_2}=\|\Im'\|_{\Delta,+\infty}.\end{aligned} \tag{29}$$

Considering right-hand side of Equation (27), it now follows that

$$\frac{\|(1\otimes \mathtt{A}-\mathtt{B}\otimes 1)^2\|}{6}\left(\left\|\int_0^{\frac{1}{2}}\left(\kappa-\frac{3.2^{\upsilon}\kappa^{\upsilon+1}}{\upsilon+1}\right)[\Im''(\mathtt{B}\otimes 1\kappa+1\otimes \mathtt{A}(1-\kappa))d\kappa]\right.\right.$$
$$\left.\left.+\int_{\frac{1}{2}}^1\left((1-\kappa)-\frac{3.2^{\upsilon}(1-\kappa)^{\upsilon+1}}{\upsilon+1}\right)\Im''(\mathtt{B}\otimes 1\kappa+1\otimes \mathtt{A}(1-\kappa))\right\|\right)$$
$$\leq \frac{\|(1\otimes \mathtt{A}-\mathtt{B}\otimes 1)^2\|}{6}\left(\left\|\int_0^{\frac{1}{2}}\left(\kappa-\frac{3.2^{\upsilon}\kappa^{\upsilon+1}}{\upsilon+1}\right)\right\|\left\|\Im'''(1\otimes \mathtt{B}\kappa+1\otimes \mathtt{B}(1-\kappa))\right\|\right.$$
$$\left.+\left\|\int_{\frac{1}{2}}^1\left((1-\kappa)-\frac{3.2^{\upsilon}(1-\kappa)^{\upsilon+1}}{\upsilon+1}\right)\right\|\left\|\Im''(1\otimes \mathtt{B}\kappa+1\otimes \mathtt{B}(1-\kappa))d\kappa\right\|\right)$$
$$\leq \frac{\|(1\otimes \mathtt{A}-\mathtt{B}\otimes 1)^2\|}{6}\left(\left\|\frac{\kappa^2}{2}-\frac{6\ln^{-\upsilon-1}(2)\Gamma(\upsilon+1,\ln(2)(1-\kappa))}{\theta+1}\right\|\left\|\Im'''(1\otimes \mathtt{B}\kappa+1\otimes \mathtt{B}(1-\kappa))\right\|\right.$$
$$\left.+\left\|-\frac{\kappa^2}{2}+\kappa-\frac{6\ln^{-\upsilon-1}(2)\Gamma(\upsilon+1,\ln(2)(1-\kappa))}{\upsilon+1}\right\|\left\|\Im''(1\otimes \mathtt{B}\kappa+1\otimes \mathtt{B}(1-\kappa))d\kappa\right\|\right)$$
$$\leq \frac{\|(1\otimes \mathtt{A}-\mathtt{B}\otimes 1)^2\|}{6}\left(\left\|\frac{\kappa^2}{2}+\frac{6\ln^{-\upsilon-1}(2)\Gamma(\upsilon+1,\ln(2)(1-\kappa))}{\upsilon+1}\right\|\|\Im'\|_{\Delta,+\infty}\right.$$

$$
\begin{aligned}
&+\left\|\frac{\kappa^2}{2}+\kappa+\frac{6\ln^{-v-1}(2)\Gamma(v+1,\ln(2)(1-\kappa))}{v+1}\right\|\|\Im'\|_{\Delta,+\infty}\Bigg)\\
&\le\frac{\|(1\otimes\mathtt{A}-\mathtt{B}\otimes 1)^2\|}{6}\left[\left(\frac{\kappa^2}{2}+\frac{6\ln^{-v-1}(2)\Gamma(v+1,\ln(2)(1-\kappa))}{v+1}\right.\right.\\
&\left.\left.+\frac{\kappa^2}{2}+\kappa+\frac{6\ln^{-v-1}(2)\Gamma(v+1,\ln(2)(1-\kappa))}{v+1}\right)\|\Im'\|_{\Delta,+\infty}\right)\Bigg]. \qquad (30)
\end{aligned}
$$

Using equation (30) in (27), we get required result. □

Theorem 9. *Assume that $\Im$ is continuously differentiable on Δ with $|\Im''|$ is quasi-convex on Δ, $\mathtt{A}$ and $\mathtt{B}$ are selfadjoint operators with $\mathcal{SP}(\mathtt{A}), \mathcal{SP}(\mathtt{B})\subset\Delta$, then*

$$
\begin{aligned}
&\Bigg\|\left[\frac{1}{6}(\Im(\mathtt{B})\otimes 1)+\frac{2}{3}\Im\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)+\frac{1}{6}(1\otimes\Im(\mathtt{A}))\right]\\
&-\left[\Im\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)-\frac{\kappa^v(\xi_2-\xi_1)}{4}\left[\int_0^1\Im'\left((1-\kappa)\mathtt{B}\otimes 1+\left(\frac{\kappa 1\otimes\mathtt{A}}{2}\right)\right)d\kappa\right]\right.\\
&\left.+\Im\left(\frac{\mathtt{B}\otimes 1+1\otimes\mathtt{A}}{2}\right)-\frac{(1-\kappa)^v(\xi_2-\xi_1)}{4}\left[\int_0^1\Im'\left(\left(\frac{1-\kappa}{2}\right)\mathtt{B}\otimes 1+\left(\frac{1+\kappa}{2}\right)1\otimes\mathtt{A}\right)d\kappa\right]\right\|\\
&\le\frac{\|(1\otimes\mathtt{A}-\mathtt{B}\otimes 1)^2\|}{6}\left(\frac{1}{4(v+2)}\left(v\left(\frac{v+1}{3}\right)^{\frac{2}{v}}+\frac{3}{v+1}\right)-\frac{1}{8}\right)\\
&\times\left\|\frac{1}{2}\big(|\Im''(\mathtt{B})|\otimes 1+1\otimes|\Im''(\mathtt{A})|+||\Im''(\mathtt{B})|\otimes 1-1\otimes|\Im''(\mathtt{A})||\big)\right\|.
\end{aligned}
$$

Proof. By assumption $|\Im''|$ is quasi convex on Δ, then we have

$$
\begin{aligned}
&\big|(\Im''(\xi_2\kappa+\xi_1(1-\kappa))-\Im''(\xi_1\kappa+\xi_2(1-\kappa)))\big|\\
&\le\big|(\Im''(\xi_2\kappa+\xi_1(1-\kappa))+\Im''(\xi_1\kappa+\xi_2(1-\kappa)))\big|\\
&\le\frac{1}{2}\big(|\Im''(\xi_2)|+|\Im''(\xi_1)|+||\Im''(\xi_2)|-|\Im''(\xi_1)||\big)
\end{aligned}
$$

$\forall\,\tau\in[0,1]$ and $\xi_1,\xi_2\in\Delta$.
Taking $\int_\Delta\int_\Delta$ over $d\mathrm{E}_{\xi_1}\otimes d\mathrm{F}_{\xi_2}$ yields:

$$
\begin{aligned}
&\big|(\Im''(1\otimes\mathtt{B}\kappa+1\otimes\mathtt{A}(1-\kappa))-\Im''(1\otimes\mathtt{A}\kappa+1\otimes\mathtt{B}(1-\kappa)))\big|\\
&=\int_\Delta\int_\Delta\Big|(\Im''(\xi_2\kappa+\xi_1(1-\kappa))-\Im''(\xi_1\kappa+\xi_2(1-\kappa)))\Big|d\mathrm{E}_{\xi_1}\otimes d\mathrm{F}_{\xi_2}\\
&\le\frac{1}{2}\int_\Delta\int_\Delta\big(|\Im''(\xi_2)|+|\Im''(\xi_1)|+||\Im''(\xi_2)|-|\Im''(\xi_1)||\big)d\mathrm{E}_{\xi_1}\otimes d\mathrm{F}_{\xi_2}\\
&=\frac{1}{2}\big(|\Im''(\mathtt{B})|\otimes 1+1\otimes|\Im''(\mathtt{A})|+||\Im''(\mathtt{B})|\otimes 1-1\otimes|\Im''(\mathtt{A})||\big).
\end{aligned}
$$

Applying the norm in above inequality result it follow as:

$$
\begin{aligned}
&\big\|(\Im''(1\otimes\mathtt{B}\kappa+1\otimes\mathtt{A}(1-\kappa))-\Im''(1\otimes\mathtt{A}\kappa+1\otimes\mathtt{B}(1-\kappa)))\big\|\\
&\le\left\|\frac{1}{2}\big(|\Im''(\mathtt{B})|\otimes 1+1\otimes|\Im''(\mathtt{A})|+||\Im''(\mathtt{A})|\otimes 1-1\otimes|\Im''(\mathtt{B})||\big)\right\|
\end{aligned}
$$

$$\leqslant \frac{1}{2}(\||\Im''(\mathtt{B})| \otimes 1 + 1 \otimes |\Im''(\mathtt{A})|\| + \||\Im''(\mathtt{B})| \otimes 1 - 1 \otimes |\Im''(\mathtt{A})|\|).$$

Using the norm in (21) and considering triangular inequality, we have

$$\begin{aligned}
&\left\| \left[\frac{1}{6}(\Im(\mathtt{B}) \otimes 1) + \frac{2}{3}\Im\left(\frac{\mathtt{B} \otimes 1 + 1 \otimes \mathtt{A}}{2}\right) + \frac{1}{6}(1 \otimes \Im(\mathtt{A}))\right]\right.\\
&- \left[\Im\left(\frac{\mathtt{B} \otimes 1 + 1 \otimes \mathtt{A}}{2}\right) - \frac{\kappa^{\upsilon}(\xi_2 - \xi_1)}{4}\left[\int_0^1 \Im'\left((1-\kappa)\mathtt{B} \otimes 1 + \left(\frac{\kappa 1 \otimes \mathtt{A}}{2}\right)\right) d\kappa\right]\right.\\
&\left.\left. + \Im\left(\frac{\mathtt{B} \otimes 1 + 1 \otimes \mathtt{A}}{2}\right) - \frac{(1-\kappa)^{\upsilon}(\xi_2 - \xi_1)}{4}\left[\int_0^1 \Im'\left(\left(\frac{1-\kappa}{2}\right)\mathtt{B} \otimes 1 + \left(\frac{1+\kappa}{2}\right)1 \otimes \mathtt{A}\right) d\kappa\right]\right\|\\
&\leq \frac{\|(1 \otimes \mathtt{A} - \mathtt{B} \otimes 1)^2\|}{6}\left(\left\| \int_0^{\frac{1}{2}}\left(\kappa - \frac{3.2^{\upsilon}\kappa^{\upsilon+1}}{\upsilon+1}\right)[\Im''(\mathtt{B} \otimes 1\kappa + 1 \otimes \mathtt{A}(1-\kappa)) d\kappa]\right.\right.\\
&\left.\left. + \int_{\frac{1}{2}}^1\left((1-\kappa) - \frac{3.2^{\upsilon}(1-\kappa)^{\upsilon+1}}{\upsilon+1}\right)\Im''(\mathtt{B} \otimes 1\kappa + 1 \otimes \mathtt{A}(1-\kappa))\right\|\right)\\
&\leq \frac{\|(1 \otimes \mathtt{A} - \mathtt{B} \otimes 1)^2\|}{6}\left(\left\| \int_0^{\frac{1}{2}}\left(\kappa - \frac{3.2^{\upsilon}\kappa^{\upsilon+1}}{\upsilon+1}\right) d\kappa \frac{1r}{2}(|\Im''(\mathtt{B})| \otimes 1 + 1 \otimes |\Im''(\mathtt{A})| + ||\Im''(\mathtt{B})| \otimes 1 - 1 \otimes |\Im''(\mathtt{A})||)\right.\right.\\
&\left.\left. + \int_{\frac{1}{2}}^1\left((1-\kappa) - \frac{3.2^{\upsilon}(1-\kappa)^{\upsilon+1}}{\upsilon+1}\right) d\kappa \frac{1}{2}(|\Im''(\mathtt{B})| \otimes 1 + 1 \otimes |\Im''(\mathtt{A})| + ||\Im''(\mathtt{B})| \otimes 1 - 1 \otimes |\Im''(\mathtt{A})||)\right\|\right)\\
&\leq \frac{\|(1 \otimes \mathtt{A} - \mathtt{B} \otimes 1)\|}{6}\left\|\left(\frac{1}{(4\upsilon+4)}\left(\upsilon\left(\frac{\upsilon+4}{2}\right)^{\frac{3}{2\upsilon}} + \frac{3}{2\upsilon+2}\right) - \frac{1}{8}\right)\right\|\\
&\times \left\|\frac{1}{2}(|\Im''(\mathtt{B})| \otimes 1 + 1 \otimes |\Im''(\mathtt{A})| + ||\Im''(\mathtt{B})| \otimes 1 - 1 \otimes |\Im''(\mathtt{A})||)\right\|\\
&\leq \frac{\|(1 \otimes \mathtt{A} - \mathtt{B} \otimes 1)^2\|}{12}\left\|\left(\frac{1}{4(\upsilon+2)}\left(\upsilon\left(\frac{\upsilon+1}{3}\right)^{\frac{2}{\upsilon}} + \frac{3}{\upsilon+1}\right) + \frac{1}{8}\right)\right\|\\
&\times \|(|\Im''(\mathtt{B})| \otimes 1 + 1 \otimes |\Im''(\mathtt{A})| + ||\Im''(\mathtt{B})| \otimes 1 + 1 \otimes |\Im''(\mathtt{A})||)\|\\
&= \frac{\|(1 \otimes \mathtt{A} - \mathtt{B} \otimes 1)^2\|}{12}\left(\frac{\upsilon^2\left(\frac{(\upsilon+1)^2}{9}\right)^{\frac{1}{\upsilon}} + \upsilon\left(\frac{(\upsilon+1)^2}{9}\right)^{\frac{1}{\upsilon}} + 3}{(4\upsilon+4)(4\upsilon+8)} + \frac{1}{8}\right)\\
&\times \|(|\Im''(\mathtt{B})| \otimes 1 + 1 \otimes |\Im''(\mathtt{A})| + ||\Im''(\mathtt{B})| \otimes 1 + 1 \otimes |\Im''(\mathtt{A})||)\|.
\end{aligned} \tag{31}$$

□

Remark 4.
- *If we choose $\upsilon = 1$ in Theorem 9, then it refines Theorem 2.4 as presented by the authors using classical integral operator in Ref. [54].*
- *If we choose $\upsilon = 1$ in Theorem 9, then it refines Theorem 2.4 as presented by the authors using classical integral operator in Ref. [47].*
- *If we choose $\upsilon = 1$ in Theorem 9, then it refines Theorem 10 as presented by the authors using classical integral operator in Ref. [59].*

4. Conclusions and Future Remarks

The tensor Hilbert spaces and its inequalities are an important topic in mathematical physics, functional analysis, and quantum mechanics. In this paper, we extend the gradient descent inequality from the classical sense to the setup of function spaces by using tensor arithmetic operations for continuous differentiable mappings. Furthermore, we refine and generalize the following results [47,54] developed by using classical integral operators by using fractional operators. We also show some non-trivial consequences and remarks that recapitulate earlier findings when our tensorial operations are degenerated.

This paper contributes to mathematical inequality theory by exploring inequalities supporting tensor Hilbert spaces, which is a rare topic in the literature. Following these results, we will advise readers to attempt to develop Simpson type inequalities involving coordinate convex [62] mappings in tensor Hilbert spaces and other types of quantum, fractional, and stochastic integral operators.

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