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On the *h*-Additive Functions and Their Symmetry Properties

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Abstract: Our goal in this paper is to generalize the class of additive functions to a more general class of functions, h-additive functions, and give several characterizations of this class by using its symmetry. Also, we provide a definition of this class of h_n-additive functions and give several characteristics for it. The density of this class of functions in the plane is given as well. The characterization of the continuity of this class also is presented.

Keywords: additive function; Cauchy equation; h-additive function

MSC: 52A99; 26A51

1. Introduction

Throughout this paper, $\mathbb{Q}$, $\mathbb{Q}_+$, $\mathbb{R}$, $\mathbb{R}_+$, and G denote the set of rational numbers, positive rational numbers, the set of real numbers, the set of non-negative real numbers, and an additive group, respectively. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be additive if the equality

$$f(x+y) = f(x) + f(y) \qquad (x, y \in \mathbb{R}) \tag{1}$$

holds true. Additive functions have many applications in functional equations and other mathematical fields including real and complex analysis, functional analysis, geometry, fluid and dynamical systems, statistics, probability, economics, and partial differential equations. Equation (1) is also known as Cauchy's functional equation. Cauchy [1] first found the general continuous solution for (1). An investigation of (1) assumes various types of regularity of the known function $f : \mathbb{R} \to \mathbb{R}$; each of these regularities yields the existence of $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{R}$. For instance, Cauchy [1] supposed that f is continuous, Darboux [2] assumed that f is either bounded on an interval or monotone, Fréchet [3], Sierpiński [4] and Figiel [5] assumed that f is Lebesgue measurable, Mehdi [6] supposed that f is bounded above on a second Baire set, and Kormes [7] imposed that f is bounded on a measurable set of positive measures. Hamel [8] investigated Equation (1) without any regularity on f; he used the Hamel bases to find all nonlinear solutions. Maksa and Volkmann [9] characterized the additive functions and gave inequalities that imply them. Aczél and Dhombers [10] gave the proof of the density of additive functions in the plane. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a square-norm (or quadratic, parallelogram, or Jordan–von Neumann) function if it fulfills the equality

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \qquad (x, y \in \mathbb{R}).$$

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A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a monomial if

$$\Delta_y^n f(x) = n! f(y) \qquad (x, y \in \mathbb{R}),$$

where $\Delta_y^1 f(x) = \Delta_y f(x) = f(x+y) - f(x)$ for all $x, y \in \mathbb{R}$ and $\Delta_y^{n+1} f(x) = \Delta_y(\Delta_y^n f(x))$ for all $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$. The additive function is a monomial with $n = 1$ and the square-norm function is a monomial with $n = 2$.

Gilanyi [11] gave several characterizations of square-norm functions. Daroczy and Kátai [12] gave a characterization of a differentiable additive function. Daroczy and Kátai [13] gave a characterization of continuous additive functions. Daroczy and Kátai [14] characterized the additive functions in the circle group. Páles and Shihab [15] proved that the discontinuous additive functions satisfy several inequalities. For more details and characteristics of additive functions, we refer to references [10,16–19]. Ng [20] characterized the Wright convex function by the convex and additive functions. Maksa and Páles [21] generalized Ng's Theorem to a higher-order convexity. Páles and Shihab [22] gave a new proof of the decomposition theorem of Maksa and Páles. In addition, Páles and Shihab [23] gave a generalization of Ng's Theorem for the convexity with respect to the Chebyshev system.

In this paper, we generalize the additive functions to a more general class, namely h-additive functions, and give several characterizations for it in Section 2. In Section 3, we study such a class of h-additive functions, namely h_n-additive functions.

Our investigation of this class assumes that f is continuous to introduce the existence of $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{R}$. We give a solution for the problem studied by Cauchy and the authors mentioned above in our new class of h_n-additive functions. We prove the homogeneity of this class with respect to rational numbers. The density of discontinuous functions in this class is given. The continuity characterization of this class is presented as well. The motivation of this generalization is to introduce a class of functions that satisfy several symmetry properties of the standard class of additive functions.

2. h-Additive Functions

In this section, we give a definition of h-additive functions and several characterizations of this class. In what follows, we say that a function $f : G \to \mathbb{R}$ is h-additive if there exists a function $h : G \to \mathbb{R}$ such that the equality

$$f(x+y) = h(x)f(x) + h(y)f(y) \qquad (x, y \in G) \tag{2}$$

holds true. We give an example of an h-additive function.

Example 1. *Let $f(x) = x + c$ and $h(x) = \frac{x+\frac{1}{2}c}{x+c}$, where $f, h : \mathbb{R} \to \mathbb{R}$ and c is a real constant with $c \neq -x$. Therefore, the right-hand side of (2) is*

$$\frac{x+\frac{1}{2}c}{x+c}(x+c) + \frac{y+\frac{1}{2}c}{y+c}(y+c) = x + y + c,$$

which is the left-hand side of (2). This equality shows that f is an $\frac{x+\frac{1}{2}c}{x+c}$-additive function.

Clearly, the particular case of (2) when $h(x) = 1$ is an additive function. In the present paper, we consider $h : G \to \mathbb{R}$ as an arbitrary function. We consider the norm defined by $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in \mathbb{R}$.

In the next two theorems, we give characterizations of h-additive functions. The next theorem is a counterpart of a result given by Maksa and Volkmann [9].

Theorem 1. *Let $f : G \to \mathbb{R}$ be an arbitrary function. If the inequality*

$$\|f(x+y) - h(x)f(x)\| \leq \|h(y)f(y)\| \qquad (x, y \in G) \tag{3}$$

holds, then f is h-additive.

Proof. According to (3), we have that

$$\|f(x+y)\|^2 + \|h(x)f(x)\|^2 - 2\langle f(x+y), h(x)f(x)\rangle \leq \|h(y)f(y)\|^2 \qquad (x, y \in G)$$

By exchanging the role of x and y in this equation, we have that

$$\|f(x+y)\|^2 + \|h(y)f(y)\|^2 - 2\langle f(x+y), h(y)f(y)\rangle \leq \|h(x)f(x)\|^2 \qquad (x, y \in G)$$

Adding the above two inequalities, we have that

$$2\|f(x+y)\|^2 - 2\langle f(x+y), h(x)f(x) + h(y)f(y)\rangle \leq 0 \qquad (x, y \in G) \tag{4}$$

This inequality, after being divided by 2, implies that

$$\langle f(x+y), f(x+y) - h(x)f(x) - h(y)f(y)\rangle \leq 0 \qquad (x, y \in G). \tag{5}$$

According to the inequality (4), we have that

$$\langle f(x+y), -h(x)f(x) - h(y)f(y)\rangle \leq 0 \qquad (x, y \in G). \tag{6}$$

In view of the positivity of $\|h(x)f(x) + h(y)f(y)\|$, we have that

$$-\|h(x)f(x) + h(y)f(y)\|^2 \leq 0 \qquad (x, y \in G)$$

This inequality implies that

$$\langle -h(x)f(x) - h(y)f(y), -h(x)f(x) - h(y)f(y)\rangle \leq 0 \qquad (x, y \in G). \tag{7}$$

Adding (6) and (7), we have that

$$\langle f(x+y) - h(x)f(x) - h(y)f(y), -h(x)f(x) - h(y)f(y)\rangle \leq 0 \qquad (x, y \in G). \tag{8}$$

Adding (5) and (8), we have that

$$\langle f(x+y) - h(x)f(x) - h(y)f(y), f(x+y) - h(x)f(x) - h(y)f(y)\rangle \leq 0 \qquad (x, y \in G).$$

This inequality implies that

$$\|f(x+y) - h(x)f(x) - h(y)f(y)\|^2 \leq 0 \qquad (x, y \in G).$$

Therefore, the equality (2) is proven. Thus, f is h-additive. □

Corollary 1. *Let $f : G \to \mathbb{R}$ be an arbitrary function. If the inequality*

$$\|f(x+y) - h(y)f(y)\| \leq \|h(x)f(x)\| \qquad (x, y \in G) \tag{9}$$

then the equality (2) holds.

Proof. By changing the role of x and y in the inequality (9), we have that

$$\|f(x+y) - h(x)f(x)\| \leq \|h(y)f(y)\| \qquad (x, y \in G).$$

Now, applying Theorem 1 yields that Equation (2) holds for $x, y \in G$. □

The next example shows that the inequality

$$\|h(x)f(x) + h(y)f(y)\| \leq \|f(x+y)\| \qquad (x, y \in G), \tag{10}$$

in general does not imply Equation (2). If we assume that $h = 0$ and f is a non-zero constant c, then we have that this inequality holds but the equality (2) does not hold.

We look for a condition on the function h such that the inequality (10) implies (2).

Theorem 2. *Let $f : G \to \mathbb{R}$ be an arbitrary function. If* (10) *and*

$$\langle h(x)f(x) + h(y)f(y) - f(x+y), -f(x+y)\rangle \leq 0 \qquad (x, y \in G) \tag{11}$$

hold, then the equality (2) *holds.*

Proof. Equation (10) is equivalent to

$$\|h(x)f(x) + h(y)f(y) - f(x+y) + f(x+y)\|^2 \leq \|f(x+y)\|^2 \qquad (x, y \in G).$$

This inequality implies that

$$\|h(x)f(x) + h(y)f(y) - f(x+y)\|^2 + \|f(x+y)\|^2 + 2\langle h(x)f(x) + h(y)f(y) - f(x+y), f(x+y)\rangle \\ \leq \|f(x+y)\|^2 \qquad (x, y \in G).$$

This inequality implies that

$$\|h(x)f(x) + h(y)f(y) - f(x+y)\|^2 + 2\langle h(x)f(x) + h(y)f(y) - f(x+y), f(x+y)\rangle \\ \leq 0 \qquad (x, y \in G).$$

Adding this inequality to (11), we have that

$$\|h(x)f(x) + h(y)f(y) - f(x+y)\|^2 + \langle h(x)f(x) + h(y)f(y) - f(x+y), f(x+y)\rangle \\ \leq 0 \qquad (x, y \in G).$$

From this inequality, we infer that

$$\langle h(x)f(x) + h(y)f(y) - f(x+y), h(x)f(x) + h(y)f(y)\rangle \leq 0 \qquad (x, y \in G).$$

Now, combining this inequality with (11), we obtain that

$$\langle h(x)f(x) + h(y)f(y) - f(x+y), h(x)f(x) + h(y)f(y) - f(x+y)\rangle \leq 0 \qquad (x, y \in G).$$

Therefore, this inequality yields that

$$\|h(x)f(x) + h(y)f(y) - f(x+y)\|^2 \leq 0 \qquad (x, y \in G),$$

which implies Equation (2). □

3. A New Class of h-Additive Functions

In this section, we define a subclass of additive functions called h_n-additive functions. For all $n \in \mathbb{N}$, a function $f : G \to \mathbb{R}$ is said to be h_n-additive if there exists a function $h : G \to \mathbb{R}$ such that for $x_1, \ldots, x_n \in G$, the function f satisfies the equation

$$f(x_1 + \cdots + x_n) = h(x_1)f(x_1) + \cdots + h(x_n)f(x_n) \qquad (x_1, \ldots, x_n \in G). \tag{12}$$

One can easily see that any h_n-additive function is h additive, but the converse implication is not always true. The first result of this section shows the homogeneity of the multiplication of h and h_n-additive functions with rational numbers. The next theorem is a counterpart of a result given by Cauchy [1].

Theorem 3. *If a function $f : \mathbb{R} \to \mathbb{R}$ is h_n-additive and satisfies*

$$h(0) = 0, \tag{13}$$

then

$$h(rx)f(rx) = rh(x)f(x) \qquad (r \in \mathbb{Q}, x \in \mathbb{R}). \tag{14}$$

Furthermore, if h and f are continuous, then $h(x)f(x) = cx$, where $x \in \mathbb{R}$ and c is a constant.

Proof. Putting $x = y = 0$ in (2), we have that

$$f(0) = h(0)f(0) + h(0)f(0).$$

Using (13), this equality shows that

$$f(0) = 0. \tag{15}$$

By definition, we have that

$$f(x_1 + \cdots + x_n) = h(x_1)f(x_1) + \cdots + h(x_n)f(x_n) \qquad (x_1, \ldots, x_n \in \mathbb{R}).$$

If $x_1 = \cdots = x_n = x$, we have that

$$f(nx) = nh(x)f(x) \qquad (x \in \mathbb{R}). \tag{16}$$

Assume $x = \frac{n}{m}t$, where $m \in \mathbb{N}$ and $t \in \mathbb{R}$. This implies that

$$mx = nt.$$

This equation yields that

$$f(mx) = f(nt).$$

In view of (16), we conclude that

$$mh(x)f(x) = nh(t)f(t).$$

This equation yields that

$$mh(\tfrac{n}{m}t)f(\tfrac{n}{m}t) = nh(t)f(t).$$

This equation implies that

$$h(\tfrac{n}{m}t)f(\tfrac{n}{m}t) = \frac{n}{m}h(t)f(t). \tag{17}$$

If $t = 1$, we obtain

$$h(\tfrac{n}{m})f(\tfrac{n}{m}) = \tfrac{n}{m}h(1)f(1).$$

Assume $h(1)f(1) := c$ and $r := \frac{n}{m}$. Therefore, this equality implies that

$$h(r)f(r) = cr \qquad (r \in \mathbb{Q}_+). \tag{18}$$

If $r = 0$, then by (15), we have that (18) holds. Since h and f are continuous, by taking limit on both sides of (18), we obtain

$$h(x)f(x) = cx \qquad (x \in \mathbb{R}_+).$$

We show that (18) also holds for negative rational r. Since f is h-additive, (2) holds. Putting $y = -x$ in (2), we have that

$$f(0) = h(x)f(x) + h(-x)f(-x) \qquad (x \in \mathbb{R}). \tag{19}$$

In view of our assumption $f(0) = 0$, this equation yields that

$$h(x)f(x) = -h(-x)f(-x) \qquad (x \in \mathbb{R}).$$

Now, this equation and (17) imply that

$$h(-\tfrac{n}{m}t)f(-\tfrac{n}{m}t) = -\tfrac{n}{m}h(t)f(t).$$

Therefore, we have that (18) also holds for negative rational r. This proves (14).

Putting $x = 1$ in (14), we have that

$$h(r)f(r) = rh(1)f(1) \qquad (r \in \mathbb{Q}).$$

Assume that $h(1)f(1) := c$; therefore, we have that

$$h(r)f(r) = cr \qquad (r \in \mathbb{Q}).$$

Furthermore, the continuity of h and f allows us to take a limit on both sides of this equation; thus, we have that

$$h(x)f(x) = cx \qquad (x \in \mathbb{R}).$$

□

Our next result, as a corollary of Theorem 3, shows the homogeneity of the h_n-additive function.

Corollary 2. *Assume that a function h satisfies the equality*

$$h(rx) = h(x) \qquad (r \in \mathbb{Q}_+, x \in \mathbb{R}).$$

If a function $f : \mathbb{R} \to \mathbb{R}$ is h_n-additive and satisfies $f(0) = 0$, then

$$f(rx) = rf(x) \qquad (r \in \mathbb{Q}, x \in \mathbb{R}). \tag{20}$$

Furthermore, if f is continuous, then $f(x) = cx$, where $x \in \mathbb{R}$ and c is a constant.

Our next result shows that the h_n-additive function is dense in the plane. To prove this result, we need this lemma in the sequel.

Lemma 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be an h_n-additive function and*

$$f(0) = 0.$$

Assume that $n \in \mathbb{N}$ for $r_1, \ldots, r_n \in \mathbb{Q}$ and $x_1, \ldots, x_n \in \mathbb{R}$; we have that

$$f(r_1x_1 + \cdots + r_nx_n) = r_1h(x_1)f(x_1) + \cdots + r_nh(x_n)f(x_n).$$

Proof. By definition of h_n-additive functions and Theorem 3, we have that

$$\begin{aligned} f(r_1x_1 + \cdots + r_nx_n) =& h(r_1x_1)f(r_1x_1) + \cdots + h(r_nx_n)f(r_nx_n) \\ &= r_1h(x_1)f(x_1) + \cdots + r_nh(x_n)f(x_n). \end{aligned}$$

So, we obtain what we needed. □

Theorem 4. *If the function $f : \mathbb{R} \to \mathbb{R}$ is a discontinuous h_n-additive and*

$$h(0) = 0,$$

then everywhere is dense in $\mathbb{R}^2$.

Proof. Let $x_1 \neq 0$ in $\mathbb{R}$. Assume that f is a discontinuous h-additive, h_n-additive function with $f(0) = 0$; therefore, by Theorem 3, for some $x \in \mathbb{R}$, the function f does not satisfy the equation

$$h(x)f(x) = cx \qquad (c \quad \text{is constant}).$$

Thus, we can find a non-zero point $x_2 \in \mathbb{R}$ such that

$$\frac{h(x_1)f(x_1)}{x_1} \neq \frac{h(x_2)f(x_2)}{x_2}. \tag{21}$$

In other words, we have that

$$\begin{vmatrix} x_1 & h(x_1)f(x_1) \\ x_2 & h(x_2)f(x_2) \end{vmatrix} \neq 0.$$

Therefore, the two rows $p_1 = (x_1, h(x_1)f(x_1))$ and $p_2 = (x_2, h(x_2)f(x_2))$ are linearly independent. Therefore, they span the plane $\mathbb{R}^2$. So, any vector in $\mathbb{R}^2$ can be written as a linear combination of these two vectors. Let r_1 and r_2 be two rational numbers. By performing simple calculations and applying Lemma 1, we have that

$$\begin{aligned} r_1p_1 + r_2p_2 =& r_1(x_1, h(x_1)f(x_1)) + r_2(x_2, h(x_2)f(x_2)) \\ &= (r_1x_1 + r_2x_2, r_1h(x_1)f(x_1) + r_2h(x_2)f(x_2)) \\ &= (r_1x_1 + r_2x_2, f(r_1x_1 + r_2x_2)). \end{aligned}$$

For $x_1 \neq 0$ and $x_2 \neq 0$ satisfying (21), we have that the set

$$G := \{(x, y) : x = r_1x_1 + r_2x_2, y = f(x); \quad (r_1, r_2) \in \mathbb{Q}^2\}$$

is dense everywhere in $\mathbb{R}^2$. This shows that f is dense in $\mathbb{R}^2$. □

It is known that additive functions are dense everywhere in $\mathbb{R}^2$; therefore, this theorem generalizes the density of h_n-additive functions.

Note that the h-additive function is a special case of h_n-additive functions for $n = 2$. Therefore, in view of the above theorem in the case where $n = 2$, we conclude that the h-additive function is dense everywhere in $\mathbb{R}^2$.

Darboux [24] proved that if an additive function is continuous at a single point x_0, then it is continuous throughout its domain. Therefore, Theorem 4 is a counterpart of Darboux's Theorem. The motivation of Theorem 4 is to introduce a class of functions larger than additive functions satisfying Darboux's Theorem.

In the following, we generalize this result to the h-additive function.

Theorem 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be an h_n-additive function. If f and h are continuous at a single point x_0 and h is idempotent, then f is continuous on $\mathbb{R}$.*

Proof. If $h = 0$, then $f = 0$, which is continuous on all of the real line. Assume that h is non-zero, and f and h are continuous at x_0; therefore, for $x \in \mathbb{R}$, we have that

$$\begin{aligned}\lim_{u\to x} f(u) &= \lim_{u-x+x_0\to x_0} f(u - x + x_0 + x - x_0) = \lim_{t\to x_0} f(t + x - x_0)\\ &= \lim_{t\to x_0} h(t)f(t) + \lim_{t\to x_0} h(x)f(x) - \lim_{t\to x_0} h(x_0)f(x_0)\\ &= h(x_0)f(x_0) + h(x)f(x) - h(x_0)f(x_0) = h(x)f(x).\end{aligned}$$

Now, we have that

$$\lim_{u\to x} f(u) = \frac{h^2(x)}{h(x)} f(x).$$

Since h is idempotent, we have that

$$\lim_{u\to x} f(u) = f(x).$$

This equality shows that f is continuous on $\mathbb{R}$. □

It is known that additive functions satisfy the above theorem; therefore, this theorem is a generalization of additive functions. The motivation of Theorem 5 is to introduce a class of functions larger than additive functions that characterize the continuity of the h_n-additive functions.

According to the above theorem, when $n = 2$, we have the following corollary:

Corollary 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be an h-additive function. If f and h are continuous at a single point x_0 and h is idempotent, then f is continuous on $\mathbb{R}$.*

4. Conclusions

In this paper, we define such a class of functions called "h-additive functions". This class is more general than the additive functions. The additive functions have many applications. In this work, we show that h-additive functions have several of these. We generalize Cauchy's Theorem to the h-additive function setting. We prove the homogeneity of this class with respect to rational numbers. The density of discontinuous functions of this class is given. The continuity characterization of this class is presented as well. We leave several investigations of additive functions that can be generalized to our new class for future research.

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