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Four Classes of Symmetric Sums over Cyclically Binomial Products

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Abstract: Four classes of multiple symmetric sums over cyclic products of binomial coefficients are examined. By incorporating the generating function approach and recursive construction method, they are expressed analytically as coefficients of rational functions. Several recurrence relations and generating functions are explicitly determined when the dimension of the multiple sums does not exceed five.

Keywords: binomial coefficient; Fibonacci number; circular product; multiple sum; generating function; recurrence relation

MSC: Primary 11B39; 05A10; Secondary 11B65

1. Introduction and Outline

There exist thousands of interesting binomial identities in the literature (cf. [1–5]) that play important roles in mathematics and applied science. In 1965, Carlitz [6] discovered, by examining the characteristic polynomial of a certain binomial matrix, the following beautiful formula for the multiple sums of circular binomial products defined by

$$\sum_{0\leq k_1,k_2,\cdots,k_m\leq n}\binom{n-k_1}{k_m}\prod_{\imath=1}^{m-1}\binom{n-k_{\imath+1}}{k_\imath}=\frac{F_{mn+m}}{F_m},$$

where the empty product is equal to one by convention, and the Fibonacci numbers (cf. [7]) are defined recursively by

$$F_n=F_{n-1}+F_{n-2}:\ F_0=0 \text{ and } F_1=1.$$

These numbers also admit the generating function

$$\sum_{n=0}^{\infty}F_nx^n=\frac{x}{1-x-x^2},$$

and the explicit Binet formula

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}:\ \alpha,\ \beta=\frac{1\pm\sqrt{5}}{2}.$$

Among the known multiple binomial sums (cf. [8–11]), Carlitz' formula seems to be a unique deep result characterized by the cyclic property, despite the fact that since its publication, it has passed over a half century. This formula was recorded by Benjianmin and Quinn in their monograph [12] (Identity 142). There exist two further proofs of it. One

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is due to Benjianmin and Rouse [13] who found a combinatorial proof by counting domino tilings. Recently, Mikic [14] provided an analytic proof by the principle of induction. The second author [15,16] introduced “the recursive construction" method, by which he not only provided a more transparent proof for Carlitz’ formula, but also found the analogous identities for the alternating multiple sums.

Let $\mathbb{N}$ be the set of natural numbers with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For $m \in \mathbb{N}$, denote by $\mathbf{k} \in \mathbb{N}_0^m$ a vector of an m-tuple of nonnegative integers. Define further three sums of its coordinates by

$$\lfloor \mathbf{k} | = \sum_{i=1}^{\lceil \frac{m}{2} \rceil} k_{2i-1}, \quad |\mathbf{k} \rceil = \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} k_{2i} \quad \text{and} \quad |\mathbf{k}| = \sum_{i=1}^{m} k_i = \lfloor \mathbf{k}| + |\mathbf{k}\rceil.$$

Encouraged by the recent works (cf. [15–17]), we shall investigate, in this paper, the following symmetric sums associated with cyclic products of the binomial coefficients

$$\sum_{\mathbf{k}\in[0,n]^m} \nabla(\mathbf{k}) \binom{n+k_m}{2k_1} \prod_{\imath=1}^{m-1} \binom{n+k_\imath}{2k_{\imath+1}},$$

specified by the sign factors

$$\nabla(\mathbf{k}) \in \left\{1,\ (-1)^{|\mathbf{k}|},\ (-1)^{\lfloor \mathbf{k}|},\ (-1)^{|\mathbf{k}\rceil}\right\}.$$

The corresponding four classes of symmetric sums will be denoted by $A_m(n)$, $B_m(n)$, $C_m(n)$, and $D_m(n)$, respectively. They will be examined individually in the next four sections by incorporating the generating function approach and the “recursive construction” method. For each class of the multiple sums, it will be expressed analytically as a coefficient of a certain rational function (involving deliberately introduced lambda polynomials). Then, for small numbers m (from 1 to 5), the corresponding ordinary generating functions will be explicitly derived. Finally, the paper will end with a brief comment about remaining problems concerning topics related to generating functions.

Throughout the paper, numerical computations were carried out by “Mathematica” (Wolfram, Version 11). By employing the same program, we verified experimentally all the displayed expressions in order to assure accuracy.

2. The First Class of Multiple Sums $A_m(n)$

In this section, we are going to examine the positive multiple circular sums

$$A_m(n) := \sum_{\mathbf{k}\in[0,n]^m} \binom{n+k_m}{2k_1} \prod_{\imath=1}^{m-1} \binom{n+k_\imath}{2k_{\imath+1}}.$$

They will be determined by extracting coefficients from rational functions involving the λ_m-polynomials defined recursively by

$$\lambda_{m+1}(x) = \lambda_m^2(x) - x^{2^m} \ \text{ with } \ \lambda_0(x) = 1.$$

The first few terms are recorded as follows:

$$\begin{aligned}
&\lambda_1 = 1 - x,\\
&\lambda_2 = 1 - 2x,\\
&\lambda_3 = (1-x)^2(1-2x-x^2) = 1 - 4x + 4x^2 - x^4,\\
&\lambda_4 = (1-2x)^2(1-4x+4x^2-2x^4) = 1 - 8x + 24x^2 - 32x^3 + 14x^4 + 8x^5 - 8x^6,\\
&\lambda_5 = (1-x)^4(1-2x-x^2)^2(1-8x+24x^2-32x^3+14x^4+8x^5-8x^6-x^8)\\
&\quad = 1 - 16x + 112x^2 - 448x^3 + 1116x^4 - 1744x^5 + 1552x^6 - 384x^7\\
&\quad - 700x^8 + 736x^9 - 160x^{10} - 128x^{11} + 64x^{12} - x^{16}.
\end{aligned}$$

These polynomials are reversals of those introduced by Kimberling [18]

$$\lambda_n(x) = x^{2^{n-1}} \mathrm{p}_n(x^{-1}),$$

where $\mathrm{p}_n(x)$ are defined by the simpler recurrence relation

$$\mathrm{p}_{n+1}(x) = \mathrm{p}_n^2(x) - 1 \ \text{ with } \ \mathrm{p}_1(x) = x - 1.$$

We record the following properties of $\lambda_n(x)$ transcribed from those of $\mathrm{p}_n(x)$:

- Reciprocal relations ($n > 1$)

$$\begin{aligned}
\lambda_n(x) &= (2x-1)^{2^{n-1}} \times \lambda_n\Big(\frac{x}{2x-1}\Big),\\
\lambda_{n+1}(x) &= (1-x)^{2^n} \times \lambda_n\Big(\frac{x^2}{(1-x)^2}\Big),\\
\lambda_{n+2}(x) &= (1-2x)^{2^n} \times \lambda_n\Big(\frac{x^4}{(1-2x)^2}\Big);
\end{aligned}$$

  where the last recurrence is derived by iterating the middle one.
- The roots of $\lambda_n(x)$ are the nonzero ones of $\mathrm{p}_n(\frac{1}{x})$. If γ is a zero of $\lambda_n(x)$, then $\big(1 \pm \frac{1}{\sqrt{\gamma}}\big)^{-1}$ are zeros of $\lambda_{n+1}(x)$.
- The increasing positive and decreasing negative sequences of zeros of these polynomials are given explicitly by

$$\frac{1}{1 \pm \sqrt{2}}, \frac{1}{1 \pm \sqrt{1+\sqrt{2}}}, \frac{1}{1 \pm \sqrt{1+\sqrt{1+\sqrt{2}}}}, \cdots$$

  and have the respective limits $\frac{1}{1\pm\alpha}$, where α denotes the golden ratio as in the Binet formula for the Fibonacci numbers.

The main result concerning $A_m(n)$ is given in the following theorem.

Theorem 1. *Let Λ_m be the polynomials defined by $\Lambda_m(x) = \lambda_m(x^2)$. Then, the circular sums $A_m(n)$ can be evaluated by extracting the coefficient:*

$$A_m(n) = [x^{2^m n - n}] \frac{\Lambda_{m-1}(1+x)^n}{\Lambda_m - x^{2^m - 1}} \prod_{j=1}^{m-2} \Lambda_j^{2n+1}.$$

The initial numbers $A_m(n)$ are illustrated in the following Table 1.

Table 1. $A_m(n)$ values.

$m \backslash n$	0	1	2	3	4	5	6	7
1	1	2	5	13	34	89	233	610
2	1	2	13	89	610	4181	28657	196418
3	1	2	41	745	13354	239633	4300049	77161234
4	1	2	137	6561	307170	14433225	678056065	31854160578
5	1	2	465	58593	7168474	881979489	108476292993	13341671147890

It is difficult to determine generating functions for $A_m(n)$ when m is considered as an indeterminate. However, when m is specified by small integers, we have the following concrete results, which show consequently that for $1 \le m \le 5$, the rational generating functions below allow us to express $A_m(n)$ as a linear sum of the n-th powers of the reciprocal zeros from the denominator polynomial in the corresponding generating function.

Proposition 1. *For $1 \le m \le 5$, the following recurrence relations and generating functions hold:*

(a) $A_1(n) = \sum_{k=0}^{n} \binom{n+k}{2k} = 3A_1(n-1) - A_1(n-2) = [y^n]\dfrac{1-y}{1-3y+y^2}.$

(b) $A_2(n) = 7A_2(n-1) - A_2(n-2) = [y^n]\dfrac{1-5y}{1-7y+y^2}.$

(c)
$$\begin{aligned} A_3(n) &= 17A_3(n-1) + 16A_3(n-2) + 17A_3(n-3) - A_3(n-4) \\ &= [y^n]\frac{1-15y-9y^2-y^3}{(1+y+y^2)(1-18y+y^2)}. \end{aligned}$$

(d)
$$\begin{aligned} A_4(n) &= 43A_4(n-1) + 175A_4(n-2) + 561A_4(n-3) - 59A_4(n-4) + A_4(n-5) \\ &= [y^n]\frac{1-41y-124y^2-241y^3+9y^4}{(1-47y+y^2)(1+4y+12y^2-y^3)}. \end{aligned}$$

(e)
$$\begin{aligned} A_5(n) &= 111A_5(n-1) + 1375A_5(n-2) + 12274A_5(n-3) + 1586A_5(n-4) \\ &\quad + 4402A_5(n-5) + 1439A_5(n-6) + 111A_5(n-7) - A_5(n-8) \\ &= [y^n]\frac{1-109y-1132y^2-8046y^3-858y^4-1484y^5-251y^6-y^7}{(1-123y+y^2)(1+12y+100y^2+14y^3+36y^4+12y^5+y^6)}. \end{aligned}$$

In particular, three initial sums can be expressed in terms of the Fibonacci numbers, where the former two can be found in [19] (A001519 and A172968):

$$A_1(n) = F_{2n+1}, \tag{1}$$

$$A_2(n) = F_{4n-1}, \tag{2}$$

$$A_3(n) = \frac{4F_{6n+1} - F_{6n}}{19} + \frac{3}{19} \times \begin{cases} 5, & n \equiv_3 0; \\ -2, & n \equiv_3 1; \\ -3, & n \equiv_3 2. \end{cases} \tag{3}$$

We illustrate proofs only for the first two formulae. The third one can be validated similarly. By making use of the binomial theorem and then the Binet formula for the Fibonacci numbers, we can confirm the two binomial sums as follows:

$$[x^n]\frac{(1+x)^n}{1-x-x^2}=\sum_{k=0}^{n}\binom{n}{k}F_{1+n-k}=\sum_{k=0}^{n}\binom{n}{k}\frac{\alpha^{1+n-k}-\beta^{1+n-k}}{\alpha-\beta}=\frac{\alpha(1+\alpha)^n-\beta(1+\beta)^n}{\alpha-\beta}=\frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta}=F_{2n+1},$$

$$[x^{3n}]\frac{(1-x)(1+x)^n}{1-x-x^2}=\sum_{k=0}^{n}\binom{n}{k}F_{3n-k-1}=\sum_{k=0}^{n}\binom{n}{k}\frac{\alpha^{3n-k-1}-\beta^{3n-k-1}}{\alpha-\beta}=\frac{\alpha^{3n-1}(1-\beta)^n-\beta^{3n-1}(1-\alpha)^n}{\alpha-\beta}=\frac{\alpha^{4n-1}-\beta^{4n-1}}{\alpha-\beta}=F_{4n-1}.$$

2.1. *Proof of Theorem 1*

This will be realized by the recursive construction. According to the binomial relations

$$\binom{n+k_1}{2k_2}=[x^{2n+2k_1}]\frac{x^{4k_2}}{(1-x^2)^{1+2k_2}}, \tag{4}$$

$$\binom{n+k_m}{2k_1}=[x^{n+k_m-2k_1}](1+x)^{n+k_m}; \tag{5}$$

we can express the sum with respect to k_1 in $A_m(n)$ as

$$A_m^{\langle 1\rangle}=\sum_{k_1=0}^{n}\binom{n+k_m}{2k_1}\binom{n+k_1}{2k_2}=[x^{3n+k_m}]\frac{(1+x)^{n+k_m}}{(1-x^2)^{1+2k_2}}x^{4k_2}.$$

For the next sum with respect to k_2, we can reformulate it as

$$A_m^{\langle 2\rangle}=\sum_{k_2=0}^{n}A_m^{\langle 1\rangle}\binom{n+k_2}{2k_3}\qquad \boxed{y=\frac{x^4}{\Lambda_1^2}}$$
$$=[x^{3n+k_m}]\frac{(1+x)^{n+k_m}}{\Lambda_1}\sum_{k_2=0}^{\infty}\binom{n+k_2}{2k_3}y^{k_2},$$

where the upper limit of the sum is released to ∞, since the corresponding coefficient of $[x^{3n+k_m}]$ vanishes when $k_2>n$.

For the last series, making the replacement $k_2\to j-n$ on the summation index, we can evaluate it in the following manner:

$$\sum_{k_2=0}^{\infty}\binom{n+k_2}{2k_3}y^{k_2}=\sum_{j=n}^{\infty}\binom{j}{2k_3}y^{j-n}=\sum_{j=2k_3}^{\infty}\binom{j}{2k_3}y^{j-n}-\sum_{j=2k_3}^{n-1}\binom{j}{2k_3}y^{j-n}=\frac{y^{2k_3-n}}{(1-y)^{2k_3+1}}-\sum_{j=2k_3}^{n-1}\binom{j}{2k_3}y^{j-n}. \tag{6}$$

From this, we can reduce $A_m^{\langle 2\rangle}$ to the following closed expression:

$$A_m^{\langle 2\rangle}=[x^{3n+k_m}]\frac{(1+x)^{n+k_m}}{\Lambda_1}\frac{y^{2k_3-n}}{(1-y)^{2k_3+1}}=[x^{7n+k_m}](1+x)^{n+k_m}\frac{x^{8k_3}}{\Lambda_2^{1+2k_3}}\Lambda_1^{2n+1},$$

where the sum in (6) is dropped off, because the corresponding functions

$$(1+x)^{n+k_m}\frac{y^{j-n}}{\Lambda_1} = x^{4j-4n}(1+x)^{n+k_m}\Lambda_1^{2n-2j-1} \quad \text{for} \quad j<n$$

result substantially in Laurent polynomials with positive powers of exponents at most $n+k_m-2 < 3n+k_m$, whose contributions to $A_m^{\langle 2\rangle}$ are annihilated by $[x^{3n+k_m}]$.

Analogously, we can deal with the sum with respect to k_3 in $A_m(n)$

$$\begin{aligned}
A_m^{\langle 3\rangle} &= \sum_{k_3=0}^{n} A_m^{\langle 2\rangle}\binom{n+k_3}{2k_4} \qquad \boxed{y=\frac{x^8}{\Lambda_2^2}}\\
&= [x^{7n+k_m}](1+x)^{n+k_m}\frac{\Lambda_1^{2n+1}}{\Lambda_2}\sum_{k_3=0}^{\infty}\binom{n+k_3}{2k_4}y^{k_3}\\
&= [x^{7n+k_m}](1+x)^{n+k_m}\frac{\Lambda_1^{2n+1}}{\Lambda_2}\frac{y^{2k_4-n}}{(1-y)^{1+2k_4}}\\
&= [x^{15n+k_m}](1+x)^{n+k_m}\frac{x^{16k_4}}{\Lambda_3^{1+2k_4}}\big(\Lambda_1\Lambda_2\big)^{2n+1},
\end{aligned}$$

and the next sum with respect to k_4 in $A_m(n)$:

$$\begin{aligned}
A_m^{\langle 4\rangle} &= \sum_{k_4=0}^{n} A_m^{\langle 3\rangle}\binom{n+k_4}{2k_5} \qquad \boxed{y=\frac{x^{16}}{\Lambda_3^2}}\\
&= [x^{15n+k_m}](1+x)^{n+k_m}\frac{(\Lambda_1\Lambda_2)^{2n+1}}{\Lambda_3}\sum_{k_4=0}^{n}\binom{n+k_4}{2k_5}y^{k_4}\\
&= [x^{15n+k_m}](1+x)^{n+k_m}\frac{(\Lambda_1\Lambda_2)^{2n+1}}{\Lambda_3}\frac{y^{2k_5-n}}{(1-y)^{1+2k_5}}\\
&= [x^{31n+k_m}](1+x)^{n+k_m}\frac{x^{32k_5}}{\Lambda_4^{1+2k_5}}\big(\Lambda_1\Lambda_2\Lambda_3\big)^{2n+1}.
\end{aligned}$$

Iterating this process $(\ell-1)$-times (where $1<\ell\le m$), we can show, by induction, that the sum with respect to $k_{\ell-1}$ in $A_m(n)$ results in the following expression:

$$\begin{aligned}
A_m^{\langle \ell-1\rangle} &= \sum_{k_{\ell-1}=0}^{n} A_m^{\langle \ell-2\rangle}\binom{n+k_{\ell-1}}{2k_\ell}\\
&= [x^{2^\ell n-n+k_m}](1+x)^{n+k_m}\frac{x^{2^\ell k_\ell}}{\Lambda_{\ell-1}^{1+2k_\ell}}\prod_{j=1}^{\ell-2}\Lambda_j^{2n+1}.
\end{aligned}$$

Finally, by summing over k_m, we arrive at the formula for $A_m(n)$:

$$\begin{aligned}
A_m(n) = A_m^{\langle m\rangle} &= \sum_{k_m=0}^{n} A_m^{\langle m-1\rangle}\\
&= \sum_{k_m=0}^{n}[x^{2^m n-n+k_m}](1+x)^{n+k_m}\frac{x^{2^m k_m}}{\Lambda_{m-1}^{1+2k_m}}\prod_{j=1}^{m-2}\Lambda_j^{2n+1}\\
&= [x^{2^m n-n}]\frac{(1+x)^n}{\Lambda_{m-1}}\prod_{j=1}^{m-2}\Lambda_j^{2n+1}\sum_{k_m=0}^{\infty}\frac{(1+x)^{k_m}}{\Lambda_{m-1}^{2k_m}}x^{(2^m-1)k_m}\\
&= [x^{2^m n-n}]\frac{(1+x)^n}{\Lambda_{m-1}}\prod_{j=1}^{m-2}\Lambda_j^{2n+1}\Big/\left\{1-\frac{(1+x)x^{2^m-1}}{\Lambda_{m-1}^2}\right\}\\
&= [x^{2^m n-n}]\frac{\Lambda_{m-1}(1+x)^n}{\Lambda_m-x^{2^m-1}}\prod_{j=1}^{m-2}\Lambda_j^{2n+1},
\end{aligned}$$

which is exactly the expression displayed in Theorem 1. □

2.2. Proof of Proposition 1

We demonstrate (e) in detail as an example. The others can be demonstrated analogously. Consider a linear combination of $m+1$ terms:

$$\Delta_m(n) = \sum_{k=0}^{m} \beta_k A_5(n-k).$$

To find a linear recurrence relation, we have to figure out a minimum $m \in \mathbb{N}$ and $\beta_k \in \mathbb{Z}$ such that $\Delta_m(n) = 0$. According to Theorem 1, we can reformulate

$$\Delta_m(n) = \sum_{k=0}^{m} \beta_k [x^{31(n-k)}] \frac{\Lambda_4(1+x)^{n-k}}{\Lambda_5 - x^{31}} \prod_{j=1}^{3} \Lambda_j^{1+2n-2k} = [x^{31n}]\Phi_m(x),$$

where $\Phi_m(x)$ is a rational function associated with a polynomial $P_m(x)$:

$$\Phi_m(x) = (1+x)^{n-m}(\Lambda_1\Lambda_2\Lambda_3)^{1+2n-2m} \frac{\Lambda_4 P_m(x)}{\Lambda_5 - x^{31}},$$
$$P_m(x) = \sum_{k=0}^{m} \beta_k x^{31k}(1+x)^{m-k}(\Lambda_1\Lambda_2\Lambda_3)^{2m-2k}.$$

Denote by $R_m(x)$ the remainder polynomial of $P_m(x)$ divided by $\Lambda_5 - x^{31}$. In order to guarantee $\Delta_m(n) = 0$, it is reasonable to assume that $P_m(x)$ is divisible by $\Lambda_5 - x^{31}$. Therefore, we have to find the minimum m such that $R_m(x) = 0$.

With the help of *Mathematica*, we find that $m = 8$ is the minimum integer such that the linear system constructed by annihilating the coefficient list (equal to zero) of x^k $(0 \le k \le 31)$ in $R_8(x)$ admits a non-trivial solution

$$\{\beta_k\}_{k=0}^{8} = \{1, -111, -1375, -12274, -1586, -4402, -1439, -111, 1\}.$$

In view of $0 \le k \le m = 8 \le n$, the corresponding polynomial $\Phi_8(x)$ has the degree

$$24 + 31k + 25(n-m) + 25(m-k) - 32 = 25n + 6k - 8 \le 25n + 40.$$

Consequently, for $n \ge 8$, the related coefficient $\Delta_8(n) = [x^{31n}]\Phi_8(x) = 0$, which confirms the recurrence relation stated in Proposition 1(e).

To determine the generating function for the sequence $A_5(n)$, we appeal to a theorem due to Stanley [20] (Theorem 4.1.1, p. 202), which claims in this case that there is a polynomial $Q(y)$ of degree < 8 such that

$$\sum_{n=0}^{\infty} A_5(n) y^n = \frac{Q(y)}{1 - 111y - 1375y^2 - 12274y^3 - 1586y^4 - 4402y^5 - 1439y^6 - 111y^7 + y^8}.$$

Taking into account the initial values in Table 1

$$\{A_5(n)\}_{n=0}^{7} = \{1, 2, 465, 58593, 7168474, 881979489, 108476292993, 13341671147890\},$$

and then comparing the coefficients of $y^k (0 \le k \le 7)$, we find the explicit expression

$$Q(y) = 1 - 109y - 1132y^2 - 8046y^3 - 858y^4 - 1484y^5 - 251y^6 - y^7.$$

This completes the proof of Proposition 1(e).

3. The Second Class of Multiple Sums $B_m(n)$

Now, we turn to investigate the alternating circular sums

$$B_m(n) := \sum_{\mathbf{k}\in[0,n]^m} (-1)^{|\mathbf{k}|} \binom{n+k_m}{2k_1} \prod_{\imath=1}^{m-1} \binom{n+k_\imath}{2k_{\imath+1}}.$$

These multiple sums will be expressed by the coefficients extracted from rational functions in terms of $\overline{\lambda}_m$-polynomials defined recursively by

$$\overline{\lambda}_{m+1}(x) = \overline{\lambda}_m^2(x) + x^{2^m} \quad \text{with} \quad \overline{\lambda}_1(x) = 1 - x.$$

The first five terms are given explicitly as follows:

$$\begin{aligned}
&\overline{\lambda}_1 = 1 - x,\\
&\overline{\lambda}_2 = 1 - 2x + 2x^2,\\
&\overline{\lambda}_3 = 1 - 4x + 8x^2 - 8x^3 + 5x^4,\\
&\overline{\lambda}_4 = 1 - 8x + 32x^2 - 80x^3 + 138x^4 - 168x^5 + 144x^6 - 80x^7 + 26x^8,\\
&\overline{\lambda}_5 = 1 - 16x + 128x^2 - 672x^3 + 2580x^4 - 7664x^5 + 18208x^6 - 35296x^7 + 56472x^8\\
&\quad - 74944x^9 + 82432x^{10} - 74624x^{11} + 54792x^{12} - 31776x^{13} + 13888x^{14} - 4160x^{15} + 677x^{16}.
\end{aligned}$$

It is not difficult to show that these polynomials admit the reciprocal relations

$$\begin{aligned}
\overline{\lambda}_n(x) &= (2x-1)^{2^{n-1}} \times \overline{\lambda}_n\Big(\frac{x}{2x-1}\Big),\\
\overline{\lambda}_{n+1}(x) &= (1-x)^{2^n} \times \overline{\lambda}_n\Big(\frac{-x^2}{(1-x)^2}\Big),\\
\overline{\lambda}_{n+2}(x) &= (1-2x+2x^2)^{2^n} \times \overline{\lambda}_n\Big(\frac{-x^4}{(1-2x+2x^2)^2}\Big);
\end{aligned}$$

where the last recurrence relation of the second order is obtained by iterating the middle one. Their reversal polynomials

$$\overline{\lambda}_n(x) = x^{2^{n-1}} \overline{\mathrm{p}}_n(x^{-1})$$

satisfy the simpler recurrence relation

$$\overline{\mathrm{p}}_{n+1}(x) = \overline{\mathrm{p}}_n^2(x) + 1 \;\text{ with }\; \overline{\mathrm{p}}_1(x) = x - 1.$$

The main result is given in the following theorem.

Theorem 2 ($m \neq 1$)**.** *Let $\overline{\Lambda}_m$ be the polynomials defined by $\overline{\Lambda}_m(x) = \overline{\lambda}_m(x^2)$. Then, the circular sums $B_m(n)$ can be evaluated by extracting the coefficient:*

$$B_m(n) = (-1)^{mn} [x^{2^m n - n}] \frac{\overline{\Lambda}_{m-1}(\mathbf{i}+x)^n}{\overline{\Lambda}_m + \mathbf{i}x^{2^m-1}} \prod_{\jmath=1}^{m-2} \overline{\Lambda}_\jmath^{2n+1}.$$

The initial numbers $B_m(n)$ are illustrated in the following Table 2.

Table 2. $B_m(n)$ values.

$m \backslash n$	0	1	2	3	4	5	6	7
1	1	0	−1	−1	0	1	1	0
2	1	2	9	37	146	585	2341	9362
3	1	0	−19	−25	456	1075	−10319	−37200
4	1	2	57	649	9506	143481	2209033	34782914
5	1	0	−151	−721	55800	459511	−22082399	−264884400

For small natural numbers m, it is possible to determine explicitly recurrence relations and generating functions for $B_m(n)$. Some partial results are given in the proposition below.

Proposition 2. *For $1 \leq m \leq 5$, the following recurrence relations and generating functions hold:*

$$\text{(a) } B_1(n) = \sum_{k=0}^{n} (-1)^k \binom{n+k}{2k} = [x^n] \frac{(\mathbf{i}+x)^n}{1-\mathbf{i}x-x^2}$$
$$= B_1(n-1) - B_1(n-2) = [y^n] \frac{1-y}{1-y+y^2},$$
$$\text{(b) } B_2(n) = 3B_2(n-1) + 3B_2(n-2) + 4B_2(n-3) = [y^n] \frac{1-y}{(1-4y)(1+y+y^2)},$$
$$\text{(c) } B_3(n) = -24B_3(n-2) - 25B_3(n-3) = [y^n] \frac{1+5y^2}{(1+y)(1-y+25y^2)},$$
$$\text{(d) } B_4(n) = 15B_4(n-1) + 63B_4(n-2) - 535B_4(n-3)$$
$$- 3255B_4(n-4) - 3303B_4(n-5) - 2704B_4(n-6)$$
$$= [y^n] \frac{1-13y-36y^2+203y^3+505y^4+312y^5}{(1-16y)(1+y+y^2)(1-48y^2-169y^3)},$$
$$\text{(e) } B_5(n) = B_5(n-1) - 481B_5(n-2) - 1082B_5(n-3) - 25414B_5(n-4)$$
$$- 132218B_5(n-5) - 327673B_5(n-6) + 301177B_5(n-7) - 458329B_5(n-8)$$
$$= [y^n] \frac{1-y+330y^2+512y^3+9304y^4+25746y^5+7927y^6+8801y^7}{(1-y+y^2)(1+480y^2+1562y^3+26496y^4+157152y^5+458329y^6)}.$$

Among them, the first sum can be located in [19] (A010892), which admits the closed form expression as follows:

$$B_1(n) = (-1)^{\lfloor \frac{n+2}{3} \rfloor} \Big\{ (n+2)^2 (\text{mod } 3) \Big\}$$
$$= \frac{1}{2} \Big\{ (-1)^{\lfloor \frac{n+1}{3} \rfloor} + (-1)^{\lfloor \frac{n+2}{3} \rfloor} \Big\}.$$

3.1. Proof of Theorem 2

By combining (4) with the binomial coefficient

$$(-1)^{k_1} \binom{n+k_m}{2k_1} = [x^{n+k_m-2k_1}] (\mathbf{i}+x)^{n+k_m}, \tag{7}$$

we can express the sum with respect to k_1 in $B_m(n)$ as

$$B_m^{\langle 1 \rangle} = \sum_{k_1=0}^{n} (-1)^{k_1} \binom{n+k_m}{2k_1} \binom{n+k_1}{2k_2} = [x^{3n+k_m}] \frac{(\mathbf{i}+x)^{n+k_m}}{(1-x^2)^{1+2k_2}} x^{4k_2}.$$

For the next sum with respect to k_2, by following the same procedure as that for "$A_m^{\langle 2\rangle}$" (in the proof of Theorem 1), we can evaluate $B_m^{\langle 2\rangle}$ in the closed form as follows:

$$\begin{aligned}B_m^{\langle 2\rangle} &= \sum_{k_2=0}^{n}(-1)^{k_2}B_m^{\langle 1\rangle}\binom{n+k_2}{2k_3} \qquad \boxed{y=\frac{\mathbf{i}x^2}{\overline{\Lambda}_1}} \\ &= [x^{3n+k_m}]\frac{(\mathbf{i}+x)^{n+k_m}}{\overline{\Lambda}_1}\sum_{k_2=0}^{\infty}\binom{n+k_2}{2k_3}y^{2k_2} \\ &= [x^{3n+k_m}]\frac{(\mathbf{i}+x)^{n+k_m}}{\overline{\Lambda}_1}\frac{y^{4k_3-2n}}{(1-y^2)^{1+2k_3}} \\ &= (-1)^n[x^{7n+k_m}](\mathbf{i}+x)^{n+k_m}\frac{x^{8k_3}}{\overline{\Lambda}_2^{1+2k_3}}\overline{\Lambda}_1^{2n+1}.\end{aligned}$$

Analogously, we can deal with the sum with respect to k_3 in $B_m(n)$

$$\begin{aligned}B_m^{\langle 3\rangle} &= \sum_{k_3=0}^{n}(-1)^{k_3}B_m^{\langle 2\rangle}\binom{n+k_3}{2k_4} \qquad \boxed{y=\frac{\mathbf{i}x^4}{\overline{\Lambda}_2}} \\ &= (-1)^n[x^{7n+k_m}](\mathbf{i}+x)^{n+k_m}\frac{\overline{\Lambda}_1^{2n+1}}{\overline{\Lambda}_2}\sum_{k_3=0}^{\infty}\binom{n+k_3}{2k_4}y^{2k_3} \\ &= (-1)^n[x^{7n+k_m}](\mathbf{i}+x)^{n+k_m}\frac{\overline{\Lambda}_1^{2n+1}}{\overline{\Lambda}_2}\frac{y^{4k_4-2n}}{(1-y^2)^{1+2k_4}} \\ &= [x^{15n+k_m}](\mathbf{i}+x)^{n+k_m}\frac{x^{16k_4}}{\overline{\Lambda}_3^{1+2k_4}}\big(\overline{\Lambda}_1\overline{\Lambda}_2\big)^{2n+1},\end{aligned}$$

and the next sum with respect to k_4 in $B_m(n)$:

$$\begin{aligned}B_m^{\langle 4\rangle} &= \sum_{k_4=0}^{n}(-1)^{k_4}B_m^{\langle 3\rangle}\binom{n+k_4}{2k_5} \qquad \boxed{y=\frac{\mathbf{i}x^8}{\overline{\Lambda}_3}} \\ &= [x^{15n+k_m}](\mathbf{i}+x)^{n+k_m}\frac{(\overline{\Lambda}_1\overline{\Lambda}_2)^{2n+1}}{\overline{\Lambda}_3}\sum_{k_4=0}^{\infty}\binom{n+k_4}{2k_5}y^{2k_4} \\ &= [x^{15n+k_m}](\mathbf{i}+x)^{n+k_m}\frac{(\overline{\Lambda}_1\overline{\Lambda}_2)^{2n+1}}{\overline{\Lambda}_3}\frac{y^{4k_5-2n}}{(1-y^2)^{1+2k_5}} \\ &= (-1)^n[x^{31n+k_m}](\mathbf{i}+x)^{n+k_m}\frac{x^{32k_5}}{\overline{\Lambda}_4^{1+2k_5}}\big(\overline{\Lambda}_1\overline{\Lambda}_2\overline{\Lambda}_3\big)^{2n+1}.\end{aligned}$$

Iterating this process $(\ell-1)$-times (where $1<\ell\le m$), we can show, by induction, that the sum with respect to $k_{\ell-1}$ in $B_m(n)$ results in the following expression:

$$\begin{aligned}B_m^{\langle \ell-1\rangle} &= \sum_{k_{\ell-1}=0}^{n}(-1)^{k_{\ell-1}}B_m^{\langle \ell-2\rangle}\binom{n+k_{\ell-1}}{2k_\ell} \\ &= (-1)^{\ell n}[x^{2^\ell n-n+k_m}](\mathbf{i}+x)^{n+k_m}\frac{x^{2^\ell k_\ell}}{\overline{\Lambda}_{\ell-1}^{1+2k_\ell}}\prod_{j=1}^{\ell-2}\overline{\Lambda}_j^{2n+1}.\end{aligned}$$

Finally, by summing over k_m, we confirm the formula displayed in Theorem 2:

$$\begin{aligned}
B_m(n) = B_m^{\langle m\rangle} &= \sum_{k_m=0}^{n} (-1)^{k_m} B_m^{\langle m-1\rangle} \\
&= (-1)^{mn} \sum_{k_m=0}^{\infty} (-1)^{k_m} [x^{2^m n-n+k_m}](\mathbf{i}+x)^{n+k_m} \frac{x^{2^m k_m}}{\overline{\Lambda}_{m-1}^{1+2k_m}} \prod_{j=1}^{m-2} \overline{\Lambda}_j^{2n+1} \\
&= (-1)^{mn} [x^{2^m n-n}] \frac{(\mathbf{i}+x)^n}{\overline{\Lambda}_{m-1}} \prod_{j=1}^{m-2} \overline{\Lambda}_j^{2n+1} \sum_{k_m=0}^{\infty} \frac{(\mathbf{i}+x)^{k_m}}{(-1)^{k_m} \overline{\Lambda}_{m-1}^{2k_m}} x^{(2^m-1)k_m} \\
&= (-1)^{mn} [x^{2^m n-n}] \frac{(\mathbf{i}+x)^n}{\overline{\Lambda}_{m-1}} \prod_{j=1}^{m-2} \overline{\Lambda}_j^{2n+1} \Big/ \left\{1 + \frac{(\mathbf{i}+x)x^{2^m-1}}{\Lambda_{m-1}^2}\right\} \\
&= (-1)^{mn} [x^{2^m n-n}] \frac{\overline{\Lambda}_{m-1}(\mathbf{i}+x)^n}{\overline{\Lambda}_m + \mathbf{i}x^{2^m-1}} \prod_{j=1}^{m-2} \overline{\Lambda}_j^{2n+1}. \qquad \square
\end{aligned}$$

3.2. Proof of Proposition 2

Since Theorem 2 is not valid for $m = 1$, we have to show the first equality in (a) separately. According to the binomial relation (7), it can be easily confirmed as follows:

$$\begin{aligned}
B_1(n) = \sum_{k=0}^{n} (-1)^k \binom{n+k}{2k} &= \sum_{k=0}^{n} \left[x^{n-k}\right] (\mathbf{i}+x)^{n+k} \\
&= \left[x^n\right] \sum_{k=0}^{\infty} x^k (\mathbf{i}+x)^{n+k} = \left[x^n\right] \frac{(\mathbf{i}+x)^n}{1-\mathbf{i}x-x^2}.
\end{aligned}$$

Now, as an example, we present a detailed demonstration for (d). The others can be verified analogously. Consider a linear combination of $m+1$ terms

$$\Delta_m(n) = \sum_{k=0}^{m} \beta_k B_4(n-k).$$

To find a linear recurrence relation, we have to figure out a minimum $m \in \mathbb{N}$ and $\beta_k \in \mathbb{Z}$ such that $\Delta_m(n)$=0. According to Theorem 2, we can reformulate

$$\Delta_m(n) = \sum_{k=0}^{m} \beta_k [x^{15(n-k)}] \frac{\overline{\Lambda}_3(\mathbf{i}+x)^{n-k}}{\overline{\Lambda}_4 + \mathbf{i}x^{15}} (\overline{\Lambda}_1\overline{\Lambda}_2)^{2n-2k+1} = [x^{15n}]\Phi_m(x),$$

where $\Phi_m(x)$ is a rational function associated with a polynomial $P_m(x)$:

$$\begin{aligned}
\Phi_m(x) &= (\mathbf{i}+x)^{n-m} (\overline{\Lambda}_1\overline{\Lambda}_2)^{2n-2m+1} \frac{\overline{\Lambda}_3 P_m(x)}{\overline{\Lambda}_4 + \mathbf{i}x^{15}}, \\
P_m(x) &= \sum_{k=0}^{m} \beta_k x^{15k} (\mathbf{i}+x)^{m-k} (\overline{\Lambda}_1\overline{\Lambda}_2)^{2m-2k}.
\end{aligned}$$

Denote by $R_m(x)$ the remainder polynomial of $P_m(x)$ divided by $\overline{\Lambda}_4 + \mathbf{i}x^{15}$. In order to guarantee $\Delta_m(n) = 0$, it is reasonable to assume that $P_m(x)$ is divisible by $\overline{\Lambda}_4 + \mathbf{i}x^{15}$. Therefore, we have to find the minimum m such that $R_m(x) = 0$.

With the help of *Mathematica*, we find that $m = 6$ is the minimum integer such that the linear system constructed by annihilating the coefficient list (equal to zero) of x^k $(0 \le k \le 15)$ in $R_6(x)$ admits a non-trivial solution

$$\{\beta_k\}_{k=0}^{6} = \{1, -15, -63,\ 535,\ 3255,\ 3303,\ 2704\}.$$

In view of $0 \leq k \leq m = 6 \leq n$, the corresponding polynomial $\Phi_6(x)$ has the degree

$$14 + 15k + 13(n-m) + 13(m-k) - 16 = 13n + 2k - 2 \leq 13n + 10.$$

Consequently, for $n \geq 6$, the related coefficient $\Delta_6(n) = [x^{15n}]\Phi_6(x) = 0$, which confirms the recurrence relation stated in Proposition 2(d).

Now, consider the generating function for the sequence $B_4(n)$:

$$\sum_{n=0}^{\infty} B_4(n) y^n = \frac{Q(y)}{1 - 15y - 63y^2 + 535y^3 + 3255y^4 + 3303y^5 + 2704y^6},$$

where $Q(y)$ is a polynomial of degree < 6. Recalling the initial values in Table 2

$$\{B_4(n)\}_{n=0}^{5} = \{1,\ 2,\ 57,\ 649,\ 9506,\ 143481\},$$

and then comparing the coefficients of $y^k (0 \leq k \leq 5)$, we find the explicit expression

$$Q(y) = 1 - 13y - 36y^2 + 203y^3 + 505y^4 + 312y^5.$$

This completes the proof of Proposition 2(d). □

4. The Third Class of Multiple Sums $C_m(n)$

In this section, we shall consider the partially alternating circular sums

$$C_m(n) := \sum_{\mathbf{k} \in [0,n]^m} (-1)^{\lfloor \mathbf{k} \rfloor} \binom{n+k_m}{2k_1} \prod_{\imath=1}^{m-1} \binom{n+k_\imath}{2k_{\imath+1}}.$$

These multiple sums will be shown to be the coefficients of certain rational functions expressed in terms of the $\widehat{\lambda}_m$-polynomials defined recursively by

$$\widehat{\lambda}_{m+1}(x) = \widehat{\lambda}_m^2(x) + (-1)^m x^{2^m} \quad \text{with} \quad \widehat{\lambda}_1(x) = 1 - x.$$

The first terms are recorded as follows:

$$\begin{aligned}
\widehat{\lambda}_1 &= 1 - x, \\
\widehat{\lambda}_2 &= 1 - 2x, \\
\widehat{\lambda}_3 &= 1 - 4x + 4x^2 + x^4, \\
\widehat{\lambda}_4 &= (1-2x)^2(1 - 4x + 4x^2 + 2x^4) = 1 - 8x + 24x^2 - 32x^3 + 18x^4 - 8x^5 + 8x^6, \\
\widehat{\lambda}_5 &= 1 - 16x + 112x^2 - 448x^3 + 1124x^4 - 1840x^5 + 2032x^6 \\
&\quad - 1664x^7 + 1220x^8 - 800x^9 + 352x^{10} - 128x^{11} + 64x^{12} + x^{16}.
\end{aligned}$$

It is not difficult to show that these polynomials admit the reciprocity

$$\widehat{\lambda}_n(x) = (2x-1)^{2^{n-1}} \times \widehat{\lambda}_n\Big(\frac{x}{2x-1}\Big).$$

Unlike polynomials $\lambda_n(x)$, there exists no recurrence of the first order for $\widehat{\lambda}_n(x)$. However, we do have the following similar relation of the second order:

$$\widehat{\lambda}_{n+2}(x) = (1-2x)^{2^n} \times \widehat{\lambda}_n\Big(\frac{-x^4}{(1-2x)^2}\Big).$$

Their reversal polynomials

$$\widehat{\lambda}_n(x) = x^{2^{n-1}}\widehat{\mathrm{p}}_n(x^{-1})$$

satisfy the simpler recurrence relation

$$\widehat{\mathrm{p}}_{n+1}(x) = \widehat{\mathrm{p}}_n^2(x) + (-1)^n \text{ with } \widehat{\mathrm{p}}_1(x) = x - 1.$$

The main result is displayed in the following theorem.

Theorem 3 ($m \neq 1$)**.** *Let $\widehat{\Lambda}_m$ be the polynomials defined by $\widehat{\Lambda}_m(x) = \widehat{\lambda}_m(x^2)$. Then, the circular sums $C_m(n)$ can be evaluated by extracting the coefficient:*

$$C_m(n) = (-1)^{\lfloor \frac{m-2}{2} \rfloor n}[x^{2^m n - n}]\frac{\widehat{\Lambda}_{m-1}(\mathbf{i} + x)^n}{\widehat{\Lambda}_m - (-1)^m \mathbf{i} x^{2^m - 1}} \prod_{j=1}^{m-2} \widehat{\Lambda}_j^{2n+1}.$$

The initial numbers $C_m(n)$ are illustrated in the following Table 3.

Table 3. $C_m(n)$ values.

$m\backslash n$	0	1	2	3	4	5	6	7
1	1	0	−1	−1	0	1	1	0
2	1	0	−7	−11	80	251	−745	−4328
3	1	2	25	209	1906	17569	156641	1402914
4	1	2	73	917	22034	444201	8419653	185957346
5	1	0	−199	−2305	196296	4609975	−194038143	−7113842448

When m takes small integer values, we have the following specific results.

Proposition 3. *For $1 \leq m \leq 5$, the following recurrence relations and generating functions hold:*

(a) $C_1(n) = B_1(n)$, (as in Proposition 2(a))

(b) $$C_2(n) = C_2(n-1) - 13C_2(n-2) - 4C_2(n-3) = [y^n]\frac{1 - y + 6y^2}{1 - y + 13y^2 + 4y^3},$$

(c) $$\begin{aligned} C_3(n) &= 15C_3(n-1) - 17C_3(n-2) - 334C_3(n-3) - 318C_3(n-4) \\ &+ 2370C_3(n-5) - 1889C_3(n-6) - 449C_3(n-7) - 225C_3(n-8) \\ &= [y^n]\frac{1 - 13y + 12y^2 + 202y^3 + 182y^4 - 852y^5 + 413y^6 + 15y^7}{1 - 15y + 17y^2 + 334y^3 + 318y^4 - 2370y^5 + 1889y^6 + 449y^7 + 225y^8}, \end{aligned}$$

(d) $$\begin{aligned} C_4(n) &= 3C_4(n-1) + 351C_4(n-2) + 970C_4(n-3) - 23198C_4(n-4) + 62042C_4(n-5) \\ &+ 49495C_4(n-6) + 740594C_4(n-7) - 11529C_4(n-8) + 400C_4(n-9) \\ &= [y^n]\frac{1-y-284y^2-974y^3+14918y^4-30224y^5-16499y^6-18693y^7+1580y^8}{(1+y+y^2)(1+25y+177y^2-16y^3)(1-29y+200y^2-469y^3+25y^4)}, \end{aligned}$$

(e) $C_5(n) = [y^n]\dfrac{\mathcal{P}(y)}{\mathcal{Q}(y)}$ *with the recurrence relation and generating function below:*

$$
\begin{aligned}
C_5(n) &= C_5(n-1) - 3169C_5(n-2) - 21586C_5(n-3) - 3435374C_5(n-4) - 45426994C_5(n-5) - 1641656533C_5(n-6) - 22796690795C_5(n-7) - 404710417493C_5(n-8) \\
&- 3268156293016C_5(n-9) - 33804380372328C_5(n-10) - 168964607932440C_5(n-11) - 1551352907563762C_5(n-12) - 5662669029968910C_5(n-13) \\
&- 51311836314455858C_5(n-14) - 73534936060644716C_5(n-15) - 832439305397569812C_5(n-16) - 853560785996751020C_5(n-17) + 135477961341669886C_5(n-18) \\
&- 8078215424165184382C_5(n-19) - 10611777228552903298C_5(n-20) + 13180286747017953512C_5(n-21) - 15069371330241310184C_5(n-22) - 13705816558156601026 4C_5(n-23) \\
&- 20033526716984442849 3C_5(n-24) - 12546607822743341675 5C_5(n-25) - 36094996417462873389C_5(n-26) - 4989491094531447186C_5(n-27) - 3006547445177974 22C_5(n-28) \\
&+ 282494402142292878C_5(n-29) + 6231014641182359C_5(n-30) - 2058508750923991C_5(n-31) - 299576175776361C_5(n-32).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}(y) &:= 1 - y + 2970y^2 + 19480y^3 + 3003344y^4 + 38240514y^5 + 1331675283y^6 + 17764633845y^7 + 302849024360y^8 + 2339919260208y^9 + 23154697178736y^{10} \\
&+ 110191221451944y^{11} + 964828564000202y^{12} + 3335028990722262y^{13} + 28705156431721740y^{14} + 38519258204084768y^{15} + 413235225927011856y^{16} \\
&+ 396135612564869084y^{17} - 748951061540538 50y^{18} + 3239280736313773386y^{19} + 3951798773900542664y^{20} - 4593418192387788976y^{21} \\
&+ 4367695021266031824y^{22} + 3788003183629348884 0y^{23} + 4939406022178487565 3y^{24} + 2700067331887941135 5y^{25} + 6586465980610113834y^{26} \\
&+ 7387917420886067 92y^{27} + 352653606724728 00y^{28} - 257575864127345 42y^{29} - 603688976514057y^{30} + 48682293193809y^{31},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}(y) &:= 1 - y + 3169y^2 + 21586y^3 + 3435374y^4 + 45426994y^5 + 1641656533y^6 + 22796690795y^7 + 404710417493y^8 + 3268156293016y^9 + 33804380372328y^{10} \\
&+ 168964607932440y^{11} + 1551352907563762y^{12} + 5662669029968910y^{13} + 51311836314455858y^{14} + 73534936060644716y^{15} + 832439305397569812y^{16} \\
&+ 853560785996751020y^{17} - 135477961341669886y^{18} + 8078215424165184382y^{19} + 10611777228552903298y^{20} - 13180286747017953512y^{21} \\
&+ 15069371330241310184y^{22} + 137058165581566010264y^{23} + 200335267169844428493y^{24} + 125466078227433416755y^{25} + 36094996417462873389y^{26} \\
&+ 4989491094531447186y^{27} + 300654744517797422y^{28} - 282494402142292878y^{29} - 6231014641182359y^{30} + 2058508750923991y^{31} + 299576175776361y^{32}.
\end{aligned}
$$

4.1. Proof of Theorem 3

First, the sum with respect to k_1 in $C_m(n)$ is exactly the same as $B_m^{\langle 1\rangle}$ in the proof of Theorem 2:

$$
C_m^{\langle 1\rangle} = \sum_{k_1=0}^{n} (-1)^{k_1} \binom{n+k_m}{2k_1} \binom{n+k_1}{2k_2} = [x^{3n+k_m}] \frac{(\mathbf{i}+x)^{n+k_m}}{(1-x^2)^{1+2k_2}} x^{4k_2}.
$$

Then, by carrying out the same analysis as that conducted for "$A_m^{\langle 2\rangle}$" (in the proof of Theorem 1), we can compute the next sum with respect to k_2 as below:

$$
\begin{aligned}
C_m^{\langle 2\rangle} &= \sum_{k_2=0}^{n} C_m^{\langle 1\rangle} \binom{n+k_2}{2k_3} \qquad \boxed{y = \frac{x^2}{\widehat{\Lambda}_1}} \\
&= [x^{3n+k_m}] \frac{(\mathbf{i}+x)^{n+k_m}}{\widehat{\Lambda}_1} \sum_{k_2=0}^{\infty} \binom{n+k_2}{2k_3} y^{2k_2} \\
&= [x^{3n+k_m}] \frac{(\mathbf{i}+x)^{n+k_m}}{\widehat{\Lambda}_1} \frac{y^{4k_3-2n}}{(1-y^2)^{1+2k_3}} \\
&= [x^{7n+k_m}] (\mathbf{i}+x)^{n+k_m} \frac{x^{8k_3}}{\widehat{\Lambda}_2^{1+2k_3}} \widehat{\Lambda}_1^{2n+1}.
\end{aligned}
$$

Analogously, we can deal with the sum with respect to k_3 in $C_m(n)$

$$
\begin{aligned}
C_m^{\langle 3\rangle} &= \sum_{k_3=0}^{n} (-1)^{k_3} C_m^{\langle 2\rangle} \binom{n+k_3}{2k_4} \qquad \boxed{y = \frac{\mathbf{i}x^4}{\widehat{\Lambda}_2}} \\
&= [x^{7n+k_m}] (\mathbf{i}+x)^{n+k_m} \frac{\widehat{\Lambda}_1^{2n+1}}{\widehat{\Lambda}_2} \sum_{k_3=0}^{\infty} \binom{n+k_3}{2k_4} y^{2k_3} \\
&= [x^{7n+k_m}] (\mathbf{i}+x)^{n+k_m} \frac{\widehat{\Lambda}_1^{2n+1}}{\widehat{\Lambda}_2} \frac{y^{4k_4-2n}}{(1-y^2)^{1+2k_4}} \\
&= (-1)^n [x^{15n+k_m}] (\mathbf{i}+x)^{n+k_m} \frac{x^{16k_4}}{\widehat{\Lambda}_3^{1+2k_4}} \big(\widehat{\Lambda}_1 \widehat{\Lambda}_2\big)^{2n+1},
\end{aligned}
$$

and the next sum with respect to k_4 in $C_m(n)$:

$$\begin{aligned} C_m^{\langle 4\rangle} &= \sum_{k_4=0}^{n} C_m^{\langle 3\rangle} \binom{n+k_4}{2k_5} \qquad \boxed{y = \frac{x^8}{\widehat{\Lambda}_3}} \\ &= (-1)^n [x^{15n+k_m}](\mathbf{i}+x)^{n+k_m} \frac{(\widehat{\Lambda}_1\widehat{\Lambda}_2)^{2n+1}}{\widehat{\Lambda}_3} \sum_{k_4=0}^{\infty} \binom{n+k_4}{2k_5} y^{2k_4} \\ &= (-1)^n [x^{15n+k_m}](\mathbf{i}+x)^{n+k_m} \frac{(\widehat{\Lambda}_1\widehat{\Lambda}_2)^{2n+1}}{\widehat{\Lambda}_3} \frac{y^{4k_5-2n}}{(1-y^2)^{1+2k_5}} \\ &= (-1)^n [x^{31n+k_m}](\mathbf{i}+x)^{n+k_m} \frac{x^{32k_5}}{\widehat{\Lambda}_4^{1+2k_5}} \big(\widehat{\Lambda}_1\widehat{\Lambda}_2\widehat{\Lambda}_3\big)^{2n+1}. \end{aligned}$$

Iterating this process $(\ell-1)$-times (where $1 < \ell \le m$), we can show, by induction, that the sum with respect to $k_{\ell-1}$ in $C_m(n)$ results in the following expression:

$$C_m^{\langle \ell-1\rangle} = (-1)^{\lfloor \frac{\ell-2}{2} \rfloor n} [x^{2^\ell n - n + k_m}](\mathbf{i}+x)^{n+k_m} \frac{x^{2^\ell k_\ell}}{\widehat{\Lambda}_{\ell-1}^{1+2k_\ell}} \prod_{\imath=1}^{\ell-2} \widehat{\Lambda}_\imath^{2n+1}.$$

Finally, keeping in mind the parity of m and then summing over k_m, we can proceed with the following calculations:

$$\begin{aligned} C_m(n) = C_m^{\langle m\rangle} &= \sum_{k_m=0}^{n} (\pm 1)^{k_m} C_m^{\langle m-1\rangle} \qquad \boxed{\text{“}\pm 1\text{” corresponds to even/odd } m} \\ &= (-1)^{\lfloor \frac{m-2}{2} \rfloor n} \sum_{k_m=0}^{n} (\pm 1)^{k_m} [x^{2^m n - n + k_m}](\mathbf{i}+x)^{n+k_m} \frac{x^{2^m k_m}}{\widehat{\Lambda}_{m-1}^{1+2k_m}} \prod_{\imath=1}^{m-2} \widehat{\Lambda}_\imath^{2n+1} \\ &= (-1)^{\lfloor \frac{m-2}{2} \rfloor n} [x^{2^m n - n}] \frac{(\mathbf{i}+x)^n}{\widehat{\Lambda}_{m-1}} \prod_{\imath=1}^{m-2} \widehat{\Lambda}_\imath^{2n+1} \sum_{k_m=0}^{n} \frac{(\mathbf{i}+x)^{k_m}}{(\pm 1)^{k_m} \widehat{\Lambda}_{m-1}^{2k_m}} x^{(2^m-1)k_m} \\ &= (-1)^{\lfloor \frac{m-2}{2} \rfloor n} [x^{2^m n - n}] \frac{(\mathbf{i}+x)^n}{\widehat{\Lambda}_{m-1}} \prod_{\imath=1}^{m-2} \widehat{\Lambda}_\imath^{2n+1} \Big/ \left\{ 1 \mp \frac{(\mathbf{i}+x)x^{2^m-1}}{\widehat{\Lambda}_{m-1}^2} \right\} \\ &= (-1)^{\lfloor \frac{m-2}{2} \rfloor n} [x^{2^m n - n}] \frac{\widehat{\Lambda}_{m-1}(\mathbf{i}+x)^n}{\widehat{\Lambda}_m - (-1)^m \mathbf{i} x^{2^m-1}} \prod_{\imath=1}^{m-2} \widehat{\Lambda}_\imath^{2n+1}, \end{aligned}$$

which confirms the algebraic expression stated in Theorem 3. □

4.2. Proof of Proposition 3

We demonstrate (b) in detail as an example. The others can be confirmed analogously. Consider a linear combination of $m+1$ terms:

$$\Delta_m(n) = \sum_{k=0}^{m} \beta_k C_2(n-k).$$

To find a linear recurrence relation, we have to figure out a minimum $m \in \mathbb{N}$ and $\beta_k \in \mathbb{Z}$ such that $\Delta_m(n)$=0. According to Theorem 3, we can reformulate

$$\Delta_m(n) = \sum_{k=0}^{m} \beta_k [x^{3(n-k)}] \frac{\widehat{\Lambda}_1(\mathbf{i}+x)^{n-k}}{\widehat{\Lambda}_2 - \mathbf{i}x^3} = [x^{3n}]\Phi_m(x),$$

where $\Phi_m(x)$ is a rational function associated with a polynomial $P_m(x)$:

$$\Phi_m(x) = (\mathbf{i}+x)^{n-m}\frac{\widehat{\Lambda}_1 P_m(x)}{\widehat{\Lambda}_2 - \mathbf{i}x^3},$$
$$P_m(x) = \sum_{k=0}^{m} \beta_k x^{3k}(\mathbf{i}+x)^{m-k}.$$

Denote by $R_m(x)$ the remainder polynomial of $P_m(x)$ divided by $\widehat{\Lambda}_2 - \mathbf{i}x^3$. In order to guarantee $\Delta_m(n) = 0$, it is reasonable to assume that $P_m(x)$ is divisible by $\widehat{\Lambda}_2 - \mathbf{i}x^3$. Therefore, we have to find the minimum m such that $R_m(x) = 0$.

With the help of *Mathematica*, we find that $m = 3$ is the minimum integer such that the linear system constructed by annihilating the coefficient list (equal to zero) of x^k $(0 \le k \le 3)$ in $R_3(x)$ admits a non-trivial solution

$$\{\beta_k\}_{k=0}^{3} = \{1, -1,\ 13,\ 4\}.$$

In view of $0 \le k \le m = 3 \le n$, the corresponding polynomial $\Phi_3(x)$ has the degree

$$2 + 3k + (n-m) + (m-k) - 4 \le n + 2k - 2 \le n + 4.$$

Consequently, for $n \ge 3$, the related coefficient $\Delta_3(n) = [x^{3n}]\Phi_3(x) = 0$, which confirms the recurrence relation stated in Proposition 3(b).

For the generating function of sequence $C_2(n)$, there is a polynomial $Q(y)$ of degree < 3 such that

$$\sum_{n=0}^{\infty} C_2(n) y^n = \frac{Q(y)}{1 - y + 13y^2 + 4y^3}.$$

Keeping in mind the initial values in Table 3

$$\{C_2(n)\}_{n=0}^{2} = \{1,\ 0,\ -7\},$$

and then comparing the coefficients of $y^k (0 \le k \le 2)$, we find the explicit expression

$$Q(y) = 1 - y + 6y^2.$$

This completes the proof of Proposition 3(b). □

5. The Fourth Class of Multiple Sums $D_m(n)$

Finally, in this section, we handle another class of partially alternating sums

$$D_m(n) := \sum_{\mathbf{k}\in[0,n]^m} (-1)^{|\mathbf{k}|}\binom{n+k_m}{2k_1}\prod_{\imath=1}^{m-1}\binom{n+k_\imath}{2k_{\imath+1}}.$$

These circular sums will be represented by the coefficients of certain rational functions expressed in terms of the $\widetilde{\lambda}_m$-polynomials defined recursively by

$$\widetilde{\lambda}_{m+1}(x) = \widetilde{\lambda}_m^2(x) - (-1)^m x^{2^m} \quad \text{with} \quad \widetilde{\lambda}_0(x) = 1.$$

The first terms are explicitly given as follows:

$$\begin{aligned}
\widetilde{\lambda}_1 &= 1 - x, \\
\widetilde{\lambda}_2 &= 1 - 2x + 2x^2, \\
\widetilde{\lambda}_3 &= (1-x)^2(1-2x+3x^2) = 1 - 4x + 8x^2 - 8x^3 + 3x^4, \\
\widetilde{\lambda}_4 &= 1 - 8x + 32x^2 - 80x^3 + 134x^4 - 152x^5 + 112x^6 - 48x^7 + 10x^8, \\
\widetilde{\lambda}_5 &= (1-x)^4(1-2x+3x^2)^2(1 - 8x + 32x^2 - 80x^3 + 134x^4 - 152x^5 + 112x^6 - 48x^7 + 11x^8) \\
&= 1 - 16x + 128x^2 - 672x^3 + 2572x^4 - 7568x^5 + 17632x^6 - 33056x^7 + 50232x^8 \\
&- 61888x^9 + 61440x^{10} - 48512x^{11} + 29816x^{12} - 13792x^{13} + 4544x^{14} - 960x^{15} + 99x^{16}.
\end{aligned}$$

It is not difficult to show that these polynomials admit the reciprocity

$$\widetilde{\lambda}_n(x) = (2x-1)^{2^{n-1}} \times \widetilde{\lambda}_n\Big(\frac{x}{2x-1}\Big).$$

Analogous to polynomials $\overline{\lambda}_n(x)$, the following recurrence of the second order holds

$$\widetilde{\lambda}_{n+2}(x) = (1-2x+2x^2)^{2^n} \times \widetilde{\lambda}_n\Big(\frac{x^4}{(1-2x+2x^2)^2}\Big).$$

Instead, the corresponding relation of the first order does not exist. Their reversal polynomials

$$\widetilde{\lambda}_n(x) = x^{2^{n-1}}\widetilde{\mathrm{p}}_n(x^{-1})$$

satisfy the simpler recurrence relation

$$\widetilde{\mathrm{p}}_{n+1}(x) = \widetilde{\mathrm{p}}_n^2(x) - (-1)^n \quad \text{with} \quad \widetilde{\mathrm{p}}_1(x) = x - 1.$$

The main result is reported in the following theorem.

Theorem 4. *Let $\widetilde{\Lambda}_m$ be the polynomials defined by $\widetilde{\Lambda}_m(x) = \widetilde{\lambda}_m(x^2)$. Then, the circular sums $D_m(n)$ can be evaluated by extracting the coefficient:*

$$D_m(n) = (-1)^{\lfloor \frac{m-1}{2} \rfloor n}[x^{2^m n - n}]\frac{\widetilde{\Lambda}_{m-1}(1+x)^n}{\widetilde{\Lambda}_m + (-1)^m x^{2^m-1}} \prod_{j=1}^{m-2} \widetilde{\Lambda}_j^{2n+1}.$$

The initial numbers $D_m(n)$ are illustrated in the following Table 4.

Table 4. $D_m(n)$ values.

$m \backslash n$	0	1	2	3	4	5	6	7
1	1	2	5	13	34	89	233	610
2	1	0	−7	−11	80	251	−745	−4328
3	1	0	−27	−113	2624	19915	−227039	−2839488
4	1	2	73	917	22034	444201	8419653	185957346
5	1	2	241	5873	441674	19789777	936229153	53838548626

By reversing the order of the summation indices, we can see that $C_{2m}(n) \equiv D_{2m}(n)$. When m takes small integer values, the initial concrete results are given below.

Proposition 4. *For $1 \leq m \leq 5$, the following recurrence relations and generating functions hold:*

$$\begin{aligned}
&(\text{a})\ D_1(n) = A_1(n), \qquad (\text{as in Proposition 1(a)}) \\
&(\text{b})\ D_2(n) = C_2(n), \qquad (\text{as in Proposition 3(b)}) \\
&(\text{c})\ D_3(n) = D_3(n-1) - 105D_3(n-2) - 210D_3(n-3) - 174D_3(n-4) \\
&\qquad - 370D_3(n-5) + 175D_3(n-6) + 17D_3(n-7) - 9D_3(n-8) \\
&\qquad = [y^n]\frac{1 - y + 78y^2 + 124y^3 + 76y^4 + 126y^5 - 37y^6 - 3y^7}{1 - y + 105y^2 + 210y^3 + 174y^4 + 370y^5 - 175y^6 - 17y^7 + 9y^8}, \\
&(\text{d})\ D_4(n) = C_4(n), \qquad (\text{as in Proposition 3(d)}) \\
&(\text{e})\ D_5(n) = [y^n]\frac{\mathcal{P}(y)}{\mathcal{Q}(y)} \text{ with the recurrence relation and generating function below:}
\end{aligned}$$

$$\begin{aligned}
D_5(n) &:= 63D_5(n-1) + 1855D_5(n-2) - 137814D_5(n-3) - 824470D_5(n-4) + 101717386D_5(n-5) - 1723234973D_5(n-6) + 12644207459D_5(n-7) - 30879614269D_5(n-8) \\
&- 384222576520D_5(n-9) + 3981745661560D_5(n-10) - 1320079695624D_5(n-11) - 94510665861650D_5(n-12) + 281261713370222D_5(n-13) - 610410331872274D_5(n-14) \\
&+ 1884234655150140D_5(n-15) - 12798230848456772D_5(n-16) + 7809481780556156D_5(n-17) - 27102565571628146D_5(n-18) + 954230513221006D_5(n-19) \\
&- 10702469387882802D_5(n-20) + 23834228797898616D_5(n-21) + 4591352984602488D_5(n-22) + 3231756386087928D_5(n-23) - 1988195916295021D_5(n-24) + 366659155862483D_5(n-25) \\
&+ 150382653362771D_5(n-26) + 94122716222506D_5(n-27) + 5935712164074D_5(n-28) + 274979548682D_5(n-29) - 162772053361D_5(n-30) - 651461489D_5(n-31) - 372683025D_5(n-32),
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}(y) &:= 1 - 61y - 1740y^2 + 124794y^3 + 724718y^4 - 85785372y^5 + 1398047025y^6 - 9854578941y^7 + 23094821456y^8 + 275124214296y^9 - 2725015691704y^{10} \\
&+ 835489872720y^{11} + 58882693526434y^{12} - 166361612706522y^{13} + 344348299821080y^{14} - 1005291076561284y^{15} + 6378173128148340y^{16} \\
&- 3656703006174472y^{17} + 11827309877395850y^{18} - 329680868665922y^{19} + 4075012514425104y^{20} - 8112864286505000y^{21} \\
&- 1480541554805368y^{22} - 905624131759216y^{23} + 486152413537597y^{24} - 75272465448393y^{25} - 27973222124716y^{26} \\
&- 14023574188518y^{27} - 681349551666y^{28} - 11452137148y^{29} + 9172776549y^{30} + 1370655y^{31},
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}(y) &:= 1 - 63y - 1855y^2 + 137814y^3 + 824470y^4 - 101717386y^5 + 1723234973y^6 - 12644207459y^7 + 30879614269y^8 + 384222576520y^9 - 3981745661560y^{10} \\
&+ 1320079695624y^{11} + 94510665861650y^{12} - 281261713370222y^{13} + 610410331872274y^{14} - 1884234655150140y^{15} + 12798230848456772y^{16} \\
&- 7809481780556156y^{17} + 27102565571628146y^{18} - 954230513221006y^{19} + 10702469387882802y^{20} - 23834228797898616y^{21} \\
&- 4591352984602488y^{22} - 3231756386087928y^{23} + 1988195916295021y^{24} - 366659155862483y^{25} - 150382653362771y^{26} \\
&- 94122716222506y^{27} - 5935712164074y^{28} - 274979548682y^{29} + 162772053361y^{30} + 651461489y^{31} + 372683025y^{32}.
\end{aligned}$$

5.1. Proof of Theorem 4

Just as $A_m^{\langle 1 \rangle}$ in the proof of Theorem 1, we can write down directly the sum with respect to k_1 in $D_m(n)$:

$$D_m^{\langle 1 \rangle} = \sum_{k_1=0}^{n} \binom{n+k_m}{2k_1}\binom{n+k_1}{2k_2} = [x^{3n+k_m}]\frac{(1+x)^{n+k_m}}{(1-x^2)^{1+2k_2}}x^{4k_2}.$$

Then, by replicating the same approach as "$A_m^{\langle 2 \rangle}$" in the proof of Theorem 1, we can manipulate the next sum with respect to k_2 in $D_m(n)$ as follows:

$$\begin{aligned}
D_m^{\langle 2 \rangle} &= \sum_{k_2=0}^{n} (-1)^{k_2} D_m^{\langle 1 \rangle}\binom{n+k_2}{2k_3} \qquad \boxed{y = \frac{\mathbf{i}x^2}{\widetilde{\Lambda}_1}} \\
&= [x^{3n+k_m}]\frac{(1+x)^{n+k_m}}{\widetilde{\Lambda}_1}\sum_{k_2=0}^{\infty}\binom{n+k_2}{2k_3}y^{2k_2} \\
&= [x^{3n+k_m}]\frac{(1+x)^{n+k_m}}{\widetilde{\Lambda}_1}\frac{y^{4k_3-2n}}{(1-y^2)^{1+2k_3}} \\
&= (-1)^n[x^{7n+k_m}](1+x)^{n+k_m}\frac{x^{8k_3}}{\widetilde{\Lambda}_2^{1+2k_3}}\widetilde{\Lambda}_1^{2n+1}.
\end{aligned}$$

Analogously, we can deal with the sum with respect to k_3 in $D_m(n)$

$$\begin{aligned} D_m^{\langle 3\rangle} &= \sum_{k_3=0}^{n} D_m^{\langle 2\rangle}\binom{n+k_3}{2k_4} \qquad \boxed{y=\frac{x^4}{\widetilde{\Lambda}_2}} \\ &= (-1)^n[x^{7n+k_m}](1+x)^{n+k_m}\frac{\widetilde{\Lambda}_1^{2n+1}}{\widetilde{\Lambda}_2}\sum_{k_3=0}^{\infty}\binom{n+k_3}{2k_4}y^{2k_3} \\ &= (-1)^n[x^{7n+k_m}](1+x)^{n+k_m}\frac{\widetilde{\Lambda}_1^{2n+1}}{\widetilde{\Lambda}_2}\frac{y^{4k_4-2n}}{(1-y^2)^{1+2k_4}} \\ &= (-1)^n[x^{15n+k_m}](1+x)^{n+k_m}\frac{x^{16k_4}}{\widetilde{\Lambda}_3^{1+2k_4}}\big(\widetilde{\Lambda}_1\widetilde{\Lambda}_2\big)^{2n+1}, \end{aligned}$$

and the next sum with respect to k_4 in $D_m(n)$:

$$\begin{aligned} D_m^{\langle 4\rangle} &= \sum_{k_4=0}^{n} D_m^{\langle 3\rangle}\binom{n+k_4}{2k_5} \qquad \boxed{y=\frac{\mathbf{i}x^8}{\widetilde{\Lambda}_3}} \\ &= (-1)^n[x^{15n+k_m}](1+x)^{n+k_m}\frac{(\widetilde{\Lambda}_1\widetilde{\Lambda}_2)^{2n+1}}{\widetilde{\Lambda}_3}\sum_{k_4=0}^{\infty}\binom{n+k_4}{2k_5}y^{2k_4} \\ &= (-1)^n[x^{15n+k_m}](1+x)^{n+k_m}\frac{(\widetilde{\Lambda}_1\widetilde{\Lambda}_2)^{2n+1}}{\widetilde{\Lambda}_3}\frac{y^{4k_5-2n}}{(1-y^2)^{1+2k_5}} \\ &= [x^{31n+k_m}](1+x)^{n+k_m}\frac{x^{32k_5}}{\widetilde{\Lambda}_4^{1+2k_5}}\big(\widetilde{\Lambda}_1\widetilde{\Lambda}_2\widetilde{\Lambda}_3\big)^{2n+1}. \end{aligned}$$

Iterating this process $(\ell-1)$-times (where $1<\ell\le m$), we can show, by induction, that the sum with respect to $k_{\ell-1}$ in $D_m(n)$ results in the following expression:

$$D_m^{\langle \ell-1\rangle} = (-1)^{\lfloor\frac{\ell-1}{2}\rfloor n}[x^{2^\ell n-n+k_m}](1+x)^{n+k_m}\frac{x^{2^\ell k_\ell}}{\widetilde{\Lambda}_{\ell-1}^{1+2k_\ell}}\prod_{\imath=1}^{\ell-2}\widetilde{\Lambda}_\imath^{2n+1}.$$

Finally, by summing over k_m, we can evaluate $D_m(n)$ in the following closed form:

$$\begin{aligned} D_m(n) = D_m^{\langle m\rangle} &= \sum_{k_m=0}^{n}(\mp 1)^{k_m}D_m^{\langle m-1\rangle} \qquad \boxed{\text{“}\mp 1\text{” corresponds to even/odd } m} \\ &= (-1)^{\lfloor\frac{m-1}{2}\rfloor n}\sum_{k_m=0}^{n}(\mp 1)^{k_m}[x^{2^m n-n+k_m}](1+x)^{n+k_m}\frac{x^{2^m k_m}}{\widetilde{\Lambda}_{m-1}^{1+2k_m}}\prod_{\imath=1}^{m-2}\widetilde{\Lambda}_\imath^{2n+1} \\ &= (-1)^{\lfloor\frac{m-1}{2}\rfloor n}[x^{2^m n-n}]\frac{(1+x)^n}{\widetilde{\Lambda}_{m-1}}\prod_{\imath=1}^{m-2}\widetilde{\Lambda}_\imath^{2n+1}\sum_{k_m=0}^{n}\frac{(1+x)^{k_m}}{(\mp 1)^{k_m}\widetilde{\Lambda}_{m-1}^{2k_m}}x^{(2^m-1)k_m} \\ &= (-1)^{\lfloor\frac{m-1}{2}\rfloor n}[x^{2^m n-n}]\frac{(1+x)^n}{\widetilde{\Lambda}_{m-1}}\prod_{\imath=1}^{m-2}\widetilde{\Lambda}_\imath^{2n+1}\Big/\left\{1\pm\frac{(1+x)x^{2^m-1}}{\widetilde{\Lambda}_{m-1}^2}\right\} \\ &= (-1)^{\lfloor\frac{m-1}{2}\rfloor n}[x^{2^m n-n}]\frac{\widetilde{\Lambda}_{m-1}(1+x)^n}{\widetilde{\Lambda}_m+(-1)^m x^{2^m-1}}\prod_{\imath=1}^{m-2}\widetilde{\Lambda}_\imath^{2n+1}. \end{aligned}$$

This concludes the proof of Theorem 4. □

5.2. Proof of Proposition 4

We demonstrate (c) in detail as an example. The others can be demonstrated analogously. Consider a linear combination of $m+1$ terms

$$\Delta_m(n) = \sum_{k=0}^{m} \beta_k D_3(n-k).$$

To find a linear recurrence relation, we have to figure out a minimum $m \in \mathbb{N}$ and $\beta_k \in \mathbb{Z}$ such that $\Delta_m(n)$=0. According to Theorem 4, we can reformulate

$$\Delta_m(n) = \sum_{k=0}^{m} (-1)^{n-k} \beta_k [x^{7(n-k)}] \frac{\widetilde{\Lambda}_2 (1+x)^{n-k}}{\widetilde{\Lambda}_3 - x^7} \widetilde{\Lambda}_1^{2n-2k+1} = [x^{7n}] \Phi_m(x),$$

where $\Phi_m(x)$ is a rational function associated with a polynomial $P_m(x)$:

$$\Phi_m(x) = (-1)^n (1+x)^{n-m} \widetilde{\Lambda}_1^{2n-2m+1} \frac{\widetilde{\Lambda}_2 P_m(x)}{\widetilde{\Lambda}_3 - x^7},$$
$$P_m(x) = \sum_{k=0}^{m} (-1)^k \beta_k x^{7k} (1+x)^{m-k} \widetilde{\Lambda}_1^{2m-2k}.$$

Denote by $R_m(x)$ the remainder polynomial of $P_m(x)$ divided by $\widetilde{\Lambda}_3 - x^7$. In order to guarantee $\Delta_m(n) = 0$, it is reasonable to assume that $P_m(x)$ is divisible by $\widetilde{\Lambda}_3 - x^7$. Therefore, we have to find the minimum m such that $R_m(x) = 0$.

With the help of *Mathematica*, we find that $m = 8$ is the minimum integer such that the linear system constructed by annihilating the coefficient list (equal to zero) of x^k $(0 \le k \le 7)$ in $R_8(x)$ admits a non-trivial solution

$$\{\beta_k\}_{k=0}^{8} = \{1, -1,\ 105,\ 210,\ 174,\ 370, -175, -17,\ 9\}.$$

In view of $0 \le k \le m = 8 \le n$, the corresponding polynomial $\Phi_8(x)$ has the degree

$$6 + 7k + 5(n-m) + 5(m-k) - 8 = 5n + 2k - 2 \le 5n + 14.$$

Consequently, for $n \ge 8$, the related coefficient $\Delta_8(n) = [x^{7n}]\Phi_8(x) = 0$, which confirms the recurrence relation stated in Proposition 4(c).

For the generating function for the sequence $D_3(n)$, there is a polynomial $Q(y)$ of degree < 8 such that

$$\sum_{n=0}^{\infty} D_3(n) y^n = \frac{Q(y)}{1 - y + 105y^2 + 210y^3 + 174y^4 + 370y^5 - 175y^6 - 17y^7 + 9y^8}.$$

By making use of the initial values in Table 4

$$\{D_3(n)\}_{n=0}^{7} = \{1,\ 0, -27, -113,\ 2624,\ 19915, -227039, -2839488\},$$

and then comparing the coefficients of $y^k (0 \le k \le 7)$, we find the explicit expression

$$Q(y) = 1 - y + 78y^2 + 124y^3 + 76y^4 + 126y^5 - 37y^6 - 3y^7.$$

This completes the proof of Proposition 4(c). □

6. Conclusions and Further Problems

Just like Carlitz' multiple sum, evaluating the four classes of circular sums

$$\{A_m(n),\ B_m(n),\ C_m(n),\ D_m(n)\}$$

has been shown to be challenging! By combining the generating function approach with the recursive construction method, we have successfully derived analytic expressions in terms of x-coefficients of rational functions for them, where introducing the four corresponding classes of lambda polynomials has been crucial. However, these rational functions are not generating functions in the ordinary sense (cf. [21,22]). Actually, only for small "m" from 1 to 5, we succeeded in figuring out the related generating functions. They suggest that the ordinary generating functions for the five sequences $\{A_m(n),\ B_m(n),\ C_m(n),\ D_m(n)\}$ are rational ones in general. Due to the complexity (see Propositions 3(e) and 4(e)), it seems quite a difficult task to confirm the above fact, and to determine, in general, the ordinary generating functions for these sequences when m is considered as an integer parameter. The interested reader is encouraged to make further attempts.

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