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Article

Soliton and Similarity Solutions of $\mathcal{N} = 2, 4$ Supersymmetric Equations

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Abstract: We produce soliton and similarity solutions of supersymmetric extensions of Burgers, Korteweg–de Vries and modified KdV equations. We give new representations of the τ-functions in Hirota bilinear formalism. Chiral superfields are used to obtain such solutions. We also introduce new solitons called virtual solitons whose nonlinear interactions produce no phase shifts.

Keywords: supersymmetric equations; solitons; Hirota bilinear formalism

Classification: MSC 35Q51, 35Q53, 81Q60

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1. Introduction

The study of $\mathcal{N} = 2$ supersymmetric (SUSY) extensions of nonlinear evolution equations has been largely studied in the past [1–8] in terms of integrability conditions and solutions. Such extensions are given as a Grassmann-valued partial differential equation with one dependent variable $A(x, t; \theta_1, \theta_2)$ which is assumed to be bosonic to get nontrivial extensions. The independent variables are given as a set of even (commuting) space x and time t variables and a set of odd (anticommuting) variables θ_1, θ_2. Since the odd variables satisfy $\theta_1^2 = \theta_2^2 = \{\theta_1, \theta_2\} = 0$, the dependent variable A admits the following finite Taylor expansion

$$A(x, t; \theta_1, \theta_2) = u(x, t) + \theta_1\xi_1(x, t) + \theta_2\xi_2(x, t) + \theta_1\theta_2 v(x, t) \tag{1}$$

where u and v are bosonic complex valued functions and ξ_1 and ξ_2 are fermionic complex valued functions. In this paper, we show that some of these extensions can be related to a linear partial differential equation (PDE) by assuming that A is a chiral superfield [9]. Proving the integrability of an equation by linearization has been largely studied in the classical case [10,11] and has found new developments in the $\mathcal{N}=1$ formalism [12]. We propose a similar development in the $\mathcal{N}=2$ formalism. In $\mathcal{N}=2$ SUSY, we consider a pair of supercovariant derivatives defined as

$$D_1 = \partial_{\theta_1} + \theta_1\partial_x, \quad D_2 = \partial_{\theta_2} + \theta_2\partial_x \tag{2}$$

which satisfy the anticommutation relations $\{D_1, D_1\} = \{D_2, D_2\} = 2\partial_x$ and $\{D_1, D_2\} = 0$. We consider also the complex supercovariant derivatives

$$D_{\pm} = \frac{1}{2}(D_1 \pm iD_2) \tag{3}$$

which satisfy $\{D_{\pm}, D_{\pm}\} = 0$ and $\{D_+, D_-\} = \partial_x$. In terms of the complex Grassmann variables $\theta_{\pm} = \frac{1}{\sqrt{2}}(\theta_1 \pm i\theta_2)$, the derivatives Equation (3) admits the following representation

$$D_{\pm} = \frac{1}{\sqrt{2}}(\partial_{\theta_{\mp}} + \theta_{\pm}\partial_x) \tag{4}$$

and the superfield A given in Equation (1) writes

$$A(x, t; \theta_+, \theta_-) = u(x, t) + \theta_+\rho_-(x, t) + \theta_-\rho_+(x, t) + i\theta_+\theta_-v(x, t) \tag{5}$$

The fermionic complex valued functions $\rho_{\pm}$ are defined as $\rho_{\pm} = \frac{1}{\sqrt{2}}(\xi_1 \pm i\xi_2)$.

Chiral superfields are superfields of type Equation (5) satisfying $D_+A = 0$. In terms of components, we get

$$A(x, t; \theta_+, \theta_-) = u(x, t) + \theta_+\rho_-(x, t) + \theta_+\theta_-u_x(x, t) \tag{6}$$

or equivalently $\xi_2 = i\xi_1$ and $v = -iu_x$.

In the subsequent sections, we produce solutions of $\mathcal{N}=2$ SUSY extensions of the Korteweg–de Vries [1] (SKdV_a), modified Korteweg–de Vries [6] (SmKdV) and Burgers [5] (SB) equations from a chiral superfield point of view. In this instance, the equations, in terms of the complex covariant derivatives Equation (3), reads, respectively, as

$$A_t = (-A_{xx} + i(a+2)A[D_+, D_-]A + i(a-1)[D_+A, D_-A] + aA^3)_x \tag{7}$$

$$A_t = -A_{xxx} - 2A_x^3 - 6([D_+, D_-]A)^2A_x \tag{8}$$

$$A_t = (i[D_+, D_-]A + 2A^2)_x \tag{9}$$

where $[X, Y] = XY - YX$ is the commutator. In Equation (7), a is an arbitrary parameter but we will consider only the integrable cases [1] where $a = -2, 1, 4$.

In this paper, we start by presenting a general reduction procedure of these equations using chiral superfields (Section II). We thus treat SKdV_{-2} and SmKdV together and construct classical N super soliton solutions [4,7,8,13] and an infinite set of similarity solutions [7]. In Section IV, we demonstrate the existence of special N super soliton solutions, called virtual solitons [5], for the SUSY extensions of the KdV equation with $a = 1, 4$ and the Burgers equation using a related linear partial differential equation. The last section is devoted to a $\mathcal{N}=4$ extension of the KdV equation [6] in an attempt to construct a general N super virtual soliton solution.

2. General Approach and Chiral Solutions

Here, we propose a general approach for the construction of chiral solutions of SUSY extensions. This approach avoids treating SUSY extensions in terms of components of the bosonic field A given in Equation (1). Assuming $D_+A = 0$, we get the chiral property $\{D_+, D_-\}A = D_+D_-A = A_x$ and the Equations (7–9) reduce to

$$A_t + (A_{xx} - i(a+2)AA_x - aA^3)_x = 0 \tag{10}$$

$$A_t + A_{xxx} + 8A_x^3 = 0 \tag{11}$$

$$A_t - (iA_x + 2A^2)_x = 0 \tag{12}$$

Note that these equations may be evidently treated as classical [14] PDE's, but remains SUSY extensions due to the Grassmannian dependence of the bosonic field A.

The absence of the Grassmannian variables θ_+ and θ_- derivatives in Equations (10–12) indicates that the odd sectors of chiral solutions should be free from fermionic constraint. This property is in accordance with the integrability of these extensions due to arbitrary bosonization of the fermionic components [15] of the bosonic superfield A.

From the classical case, we know that the methods of resolution of all these equations are similar. The same could be said for the SUSY case. Indeed, if we assume the introduction of a potential $\tilde{A}$ such that $A = \tilde{A}_x$ in Equation (10) and after one integration with respect to x, we get

$$\tilde{A}_t + \tilde{A}_{xxx} - i(a+2)\tilde{A}_x\tilde{A}_{xx} - a\tilde{A}_x^3 = 0 \tag{13}$$

where the constant of integration is set to zero. The same is done on Equation (12) and leads to

$$\tilde{A}_t - i\tilde{A}_{xx} - 2\tilde{A}_x^2 = 0 \tag{14}$$

We thus observe that the Equations (11,13,14) are now on an equal footing, *i.e.*, the order of the equation in x is equal to the number of appearance of ∂_x in the nonlinear terms. This is standard in Hirota formalism. The choice $a = -2$ in Equation (13) gives, up to a slight change of variable, the SmKdV Equation (11). This means that the known [7] N super soliton solutions and similarity solutions of SKdV$_{-2}$ will lead to similar types of solutions for the SmKdV Equation (11).

Now setting

$$\tilde{A}(x, t; \theta_+, \theta_-) = \beta_a \log H_a(x, t; \theta_+, \theta_-) \tag{15}$$

in Equation (13), we obtain

$$H_a^2(H_{a,t} + H_{a,xxx}) - (3 + i(2+a)\beta_a)H_aH_{a,x}H_{a,xx} - (\beta_a - i)(a\beta_a - 2i)H_{a,x}^3 = 0 \tag{16}$$

The above equation reduces to the linear dispersive equation [14]

$$H_{a,t} + H_{a,xxx} = 0 \tag{17}$$

for the special and only values $a = 1$ with $\beta_1 = i$ and $a = 4$ with $\beta_4 = \frac{i}{2}$. For $a = -2$, Equation (16) writes

$$H_{-2}^2(H_{-2,t} + H_{-2,xxx}) - 3H_{-2}H_{-2,x}H_{-2,xx} + 2(\beta_{-2}^2 + 1)H_{-2,x}^3 = 0 \tag{18}$$

which does not linearize but can be bilinearized taking $\beta_{-2} = i$. It is discussed in the next Section.

A similar change of variable as in Equation (15) but with $\tilde{A} = \beta_B \log H_B$ and $\beta_B = \frac{i}{2}$ in Equation (14) is assumed and leads to the linear Schrödinger Equation

$$H_{B,t} - iH_{B,xx} = 0 \tag{19}$$

3. SKdV$_{-2}$ and SmKdV Equations

It is well known [13,14,16–19] that we can generate via the Hirota bilinear formalism N soliton and similarity solutions in the classical case and in SUSY $\mathcal{N} = 1$ extensions. Recently, the formalism was adapted to $\mathcal{N} = 2$ extensions [4,7,8] by splitting the equation into two $\mathcal{N} = 1$ equations, one fermionic and one bosonic. Our approach consists of treating the equation as a $\mathcal{N} = 2$ extension without splitting it, but imposing chirality conditions.

Equation (11) can be bilinearized using the Hirota derivative defined as

$$\mathcal{D}_x^n(f \cdot g) = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)|_{x=x_1=x_2} \tag{20}$$

Indeed, we take $\tilde{A}$ as in Equation (15) with $\beta_{-2} = i$ and $H = \frac{\tau_1}{\tau_2}$, where $\tau_i = \tau_i(x, t; \theta_+, \theta_-)$ are bosonic chiral superfields for $i = 1, 2$. Equation (11) leads to the set of bilinear equations

$$(\mathcal{D}_t + \mathcal{D}_x^3)(\tau_1 \cdot \tau_2) = 0 \tag{21}$$

$$\mathcal{D}_x^2(\tau_1 \cdot \tau_2) = 0 \tag{22}$$

This set is analogous to the corresponding bilinear equations in the classical mKdV equation [14] but we deal with superfields τ_1 and τ_2.

In order to get chiral solutions, we have to solve the set of bilinear equations with the additional chiral property $D_+\tau_i = 0$ for $i = 1, 2$. It will lead to new solutions of the SmKdV equation which are related to our recent contribution [7].

3.1. *N Super Soliton Solutions*

The one soliton solution is easily retrieved. Indeed, we cast

$$\tau_1 = 1 + a_1 e^{\Psi_1}, \quad \tau_2 = 1 - a_1 e^{\Psi_1} \tag{23}$$

where a_1 is an even parameter. Ψ_1 is a $\mathcal{N} = 2$ chiral bosonic superfield defined as

$$\Psi_1 = \kappa_1 x - \kappa_1^3 t + \theta_+\zeta_1 + \theta_+\theta_-\kappa_1 \tag{24}$$

and never appears on this form in other approaches of $\mathcal{N} = 2$ SUSY. The parameters κ_1 and ζ_1 are, respectively, even and odd. The τ-functions Equation (23) together with Equation (24) solve the set of bilinear Equations (21,22) and give rise to a one super soliton solution. Since $D_+\Psi_1 = 0$, the resulting traveling wave solution is chiral.

Since we exhibit the three super soliton solution of the SmKdV equation in Figures 1 and 2, we give the general expressions of τ_1 and τ_2:

$$\tau_1(x,t;\theta_+,\theta_-) = 1 + \sum_{i=1}^{3} a_i e^{\Psi_i} + \sum_{i<j} a_i a_j A_{ij} e^{\Psi_i+\Psi_j} + a_1 a_2 a_3 A_{12} A_{13} A_{23} e^{\Psi_1+\Psi_2+\Psi_3} \tag{25}$$

$$\tau_2(x,t;\theta_+,\theta_-) = 1 - \sum_{i=1}^{3} a_i e^{\Psi_i} + \sum_{i<j} a_i a_j A_{ij} e^{\Psi_i+\Psi_j} - a_1 a_2 a_3 A_{12} A_{13} A_{23} e^{\Psi_1+\Psi_2+\Psi_3} \tag{26}$$

where $A_{ij} = \left(\frac{\kappa_i-\kappa_j}{\kappa_i+\kappa_j}\right)^2$ and the Ψ_i's are defined as in Equation (24). The functions τ_1 and τ_2 solves the bilinear Equations (21) and (22) and are such that $D_+\tau_i = 0$ for $i = 1,2$. The generalization to a N super soliton solution is direct using the τ-functions expressed above. The forms of the τ-functions given above are new representations of super soliton solutions and have never been introduced before.

Figure 1. The function Im(v) of the three soliton solution of the SmKdV equation where $\kappa_1 = \frac{4}{3}\kappa_2 = 2\kappa_3 = \frac{4}{3}$ and $t = -20, 0, 20$.

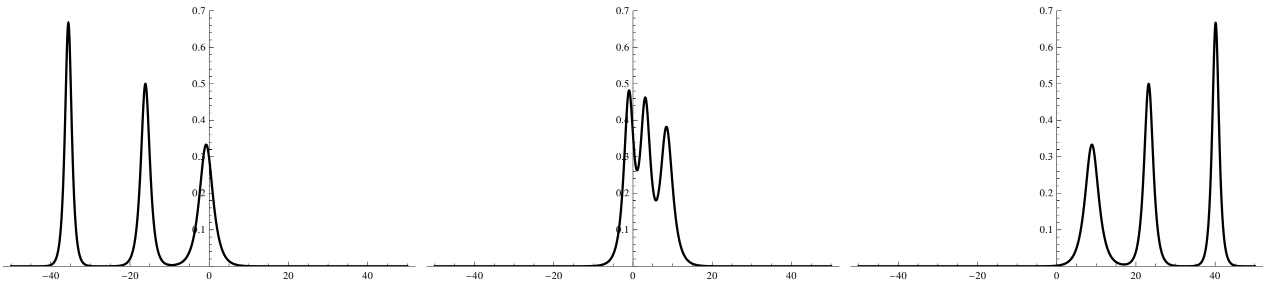
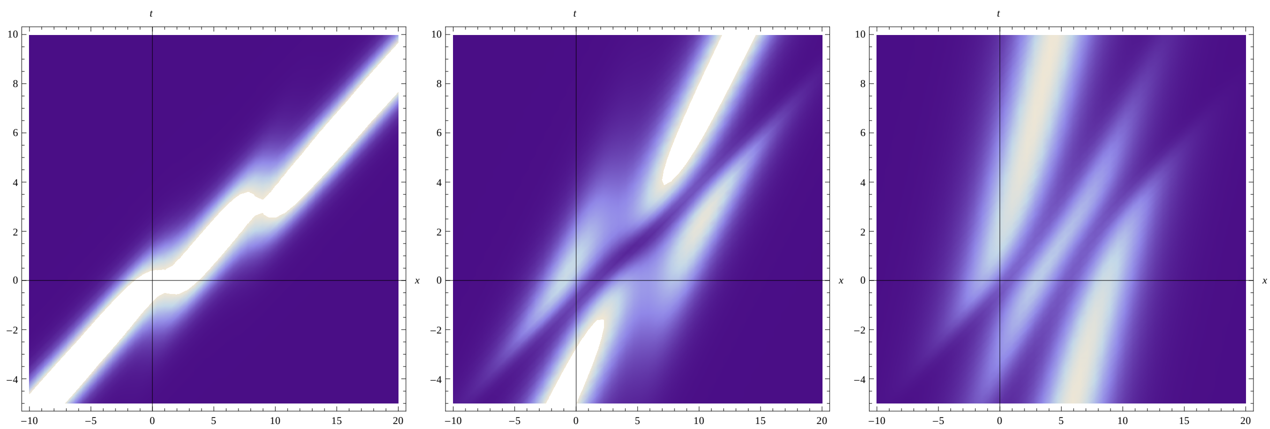


Figure 2. The density plots of the functions f_1, f_2 and f_3, respectively from left to right, of the three soliton solution of the SmKdV equation where $\kappa_1 = \frac{4}{3}\kappa_2 = 2\kappa_3 = \frac{4}{3}$.



In Figure 1, we may enjoy the three soliton solution Im(v) of the SmKdV equation given by

$$v(x,t) = \frac{1}{2}\partial_x \log\left(\frac{\tau_1(x,t;0,0)}{\tau_2(x,t;0,0)}\right) \tag{27}$$

as a function of x, for the special values $\kappa_1 = \frac{4}{3}\kappa_2 = 2\kappa_3 = \frac{4}{3}$, $a_i = i$ in Equations (25) and (26) and $t = -20, 0, 20$. In Figure 2, we explore the behavior of the fermionic component ρ_- of the superfield A for the same special values. To achieve this, we write ρ_- as

$$\rho_-(x,t) = \zeta_1\, f_1(x,t) + \zeta_2\, f_2(x,t) + \zeta_3\, f_3(x,t) \tag{28}$$

and trace out the bosonic functions f_1, f_2 and f_3.

3.2. Similarity Solutions

In a recent paper [7], we have proven the existence of an infinite set of rational similarity solutions of the SKdV$_{-2}$ using a SUSY version of the Yablonskii–Vorob'ev polynomials [16–18]. We propose in this subsection to retrieve those solutions and find an infinite set of similarity solution for the SmKdV equation. To give us a hint into what change of variables we have to cast, we have used the symmetry reduction method associated to a dilatation invariance [2].

Let us define the following τ-functions [7]

$$\tau_{1,n}(\tilde{z},t) = t^{\frac{n(n+1)}{6}} Q_n(\tilde{z}) \tag{29}$$

where $\tilde{z} = t^{-\frac{1}{3}}(x + \theta_+\zeta + \theta_+\theta_-)$ and the functions $Q_n(\tilde{z})$ are the Yablonskii–Vorob'ev polynomials defined by the recurrence relation

$$3^{\frac{1}{3}} Q_{n+1}Q_{n-1} = \tilde{z}Q_n^2 - 12(Q_nQ_{n,\tilde{z}\tilde{z}} - Q_{n,\tilde{z}}^2) \tag{30}$$

with $Q_0(\tilde{z}) = 3^{-\frac{1}{3}}$ and $Q_1(\tilde{z}) = \tilde{z}$. We would like to insist that $\tilde{z}$ is a $\mathcal{N} = 2$ bosonic superfield (as it is the case for the Ψ_i in the preceding subsection). Using the fact that the Yablonskii–Vorob'ev polynomials satisfy the following bilinear equations [17]

$$\left(\mathcal{D}_{\tilde{z}}^3 - \frac{1}{3}\tilde{z}\mathcal{D}_{\tilde{z}} - \frac{n+1}{3}\right)(Q_n \cdot Q_{n+1}) = 0 \tag{31}$$

$$\mathcal{D}_{\tilde{z}}^2(Q_n \cdot Q_{n+1}) = 0 \tag{32}$$

we have that the pair of bilinear Equations (21) and (22) are such that [7,16–18]

$$(\mathcal{D}_t + \mathcal{D}_x^3)(\tau_{1,n} \cdot \tau_{1,n+1}) = 0 \tag{33}$$

$$\mathcal{D}_x^2(\tau_{1,n} \cdot \tau_{1,n+1}) = 0 \tag{34}$$

From the choice of the variable $\tilde{z}$, we also have $D_+\tau_{i,n} = 0$ for all integers n. Taking $\tau_{2,n} = \tau_{1,n+1}$, we have an infinite set of similarity solutions of the SmKdV Equation given by

$$\tilde{A}_n(\tilde{z},t) = \frac{i}{2}\log\left(\frac{\tau_{1,n}(\tilde{z},t)}{\tau_{1,n+1}(\tilde{z},t)}\right) \tag{35}$$

for all integers $n \geq 0$ and $\tau_{1,n}$ defined as in Equation (29). To get similarity solutions A_n of the SKdV$_{-2}$, we use the above solution with $A_n = 2t^{-\frac{1}{3}}\partial_{\tilde{z}}\tilde{A}_n$. Plots of some similarity solutions are given in our recent contribution [7].

4. SKdV$_1$, SKdV$_4$ and SB Equations and Virtual Solitons

In this section, we exhibit N super soliton solutions, called N super virtual solitons, for the three equations SKdV$_1$, SKdV$_4$ and SB. Virtual solitons are soliton-like solutions which exhibit no phase shifts in nonlinear interactions. In terms of classical N soliton solutions [3–5,7,14,16,19], this is equivalent to say that the interaction coefficients A_{ij} between soliton i and soliton j are zero, $\forall i \neq j$. They manifest as traveling wave solutions for negative time $t \ll 0$ and decrease spontaneously at time $t = 0$ to split into a N soliton profile which exhibit no phase shifts. It is often said that the traveling wave solution was charged with $N - 1$ soliton, called virtual solitons [5].

Using the change of variable Equation (15) for the unknown bosonic field $\tilde{A}$, we have seen that the bosonic field H_a must be a chiral superfield and solve the linear dispersive Equation (17) when $a = 1$ and $a = 4$. For the Burgers equation, the bosonic field H_B had to be chiral and solves Equation (19).

It is easy to show that they admit the following solution

$$H(x,t;\theta_+,\theta_-) = 1 + \sum_{i=1}^{N} a_i e^{\Psi_i} \tag{36}$$

where the bosonic superfields Ψ_i are given as

$$\Psi_i = \kappa_i x + \omega(\kappa_i)t + \theta_+\zeta_i + \theta_+\theta_-\kappa_i \tag{37}$$

The frequencies $\omega(\kappa_i)$ are such that $\omega(\kappa_i) = -\kappa_i^3$ for SKdV$_a$ and $\omega(\kappa_i) = i\,\kappa_i^2$ for SB. It looks like a typical KdV type soliton solution where all the interaction coefficients A_{ij} are set to zero.

We see that the virtual soliton solutions of the SKdV$_1$ and SKdV$_4$ equations are completely similar due to the form of $\tilde{A}$ which differs only by the constant value of β_a. The expression of the original bosonic field is obtained from

$$A = \beta\,\frac{H_x}{H} \tag{38}$$

where $\beta = \beta_a$ for the SKdV$_a$ equation and $\beta = \beta_B$ for the SB equation. Thus, we can give the explicit forms of the superfield components u and ρ_-. Indeed, we have

$$u(x,t) = \beta\,\frac{\sum_{i=1}^{N} a_i\kappa_i e^{\eta_i}}{1+\sum_{i=1}^{N} a_i e^{\eta_i}}, \quad \rho_-(x,t) = \beta\sum_{i=1}^{N}\zeta_i\, f_i(x,t) \tag{39}$$

where $\eta_i = \kappa_i x + \omega(\kappa_i)t$ and the bosonic functions $f_i(x,t)$ are defined as

$$f_i(x,t) = \frac{a_i\kappa_i e^{\eta_i} + \sum_{j=1}^{N} a_i a_j(\kappa_i - \kappa_j)e^{\eta_i+\eta_j}}{\left(1+\sum_{j=1}^{N} a_j e^{\eta_j}\right)^2} \tag{40}$$

In Figure 3, we may enjoy the three virtual soliton solution Im(u) of the SKdV$_1$ Equation for $\kappa_1 = \frac{4}{3}\kappa_2 = 2\kappa_3 = \frac{4}{3}$ and $a_i = 1$ in Equation (36) and $t = 0, 10, 20$. In Figure 4, we observe the behavior of the function v where $v = -iu_x$, $\kappa_1 = \frac{4}{3}\kappa_2 = 2\kappa_3 = \frac{4}{3}$ and $a_i = 1$ in Equation (36) and $t = -20, 0, 20$. For the same special values, Figure 5 gives the density plots of the bosonic functions f_1, f_2 and f_3 as given in Equation (40).

Figure 3. The function $\mathrm{Im}(u)$ of the three virtual soliton solution of the SKdV_1 equation where $\kappa_1 = \frac{4}{3}\kappa_2 = 2\kappa_3 = \frac{4}{3}$ and $t = 0, 10, 20$.

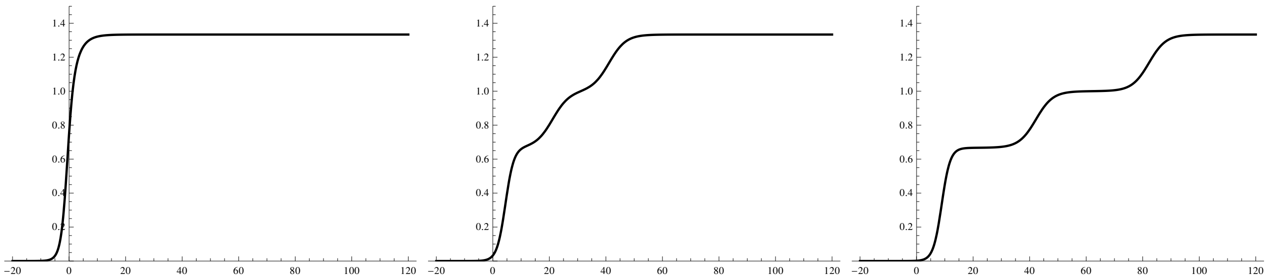


Figure 4. The function v of the three virtual soliton solution of the SKdV_1 equation where $\kappa_1 = \frac{4}{3}\kappa_2 = 2\kappa_3 = \frac{4}{3}$ and $t = -20, 0, 20$.

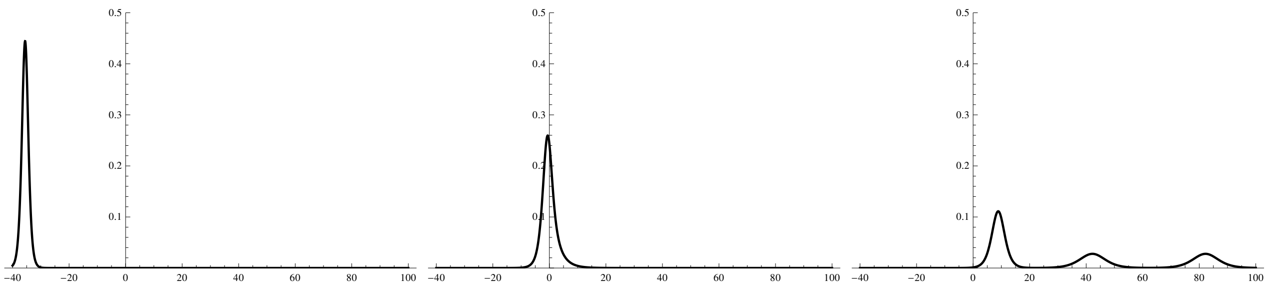
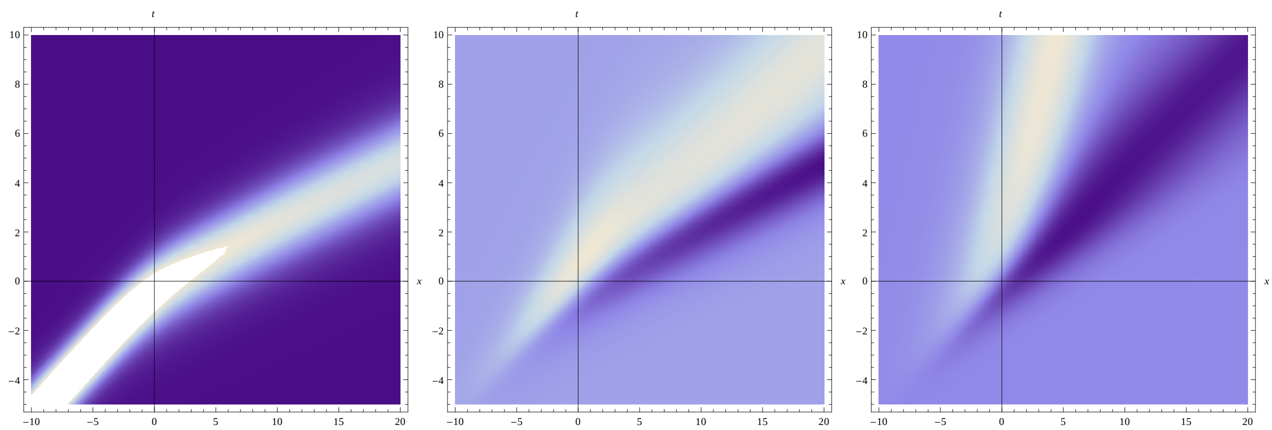


Figure 5. The density plots of the functions f_1, f_2 and f_3, respectively from left to right, of the three virtual soliton solution of the SKdV_1 equation where $\kappa_1 = \frac{4}{3}\kappa_2 = 2\kappa_3 = \frac{4}{3}$.



5. SUSY $\mathcal{N} = 4$ KdV Equation and Virtual Solitons

The SUSY $\mathcal{N} = 4$ KdV equation, as proposed by Popowicz in [6], reads

$$\Gamma_t + \Gamma_{xxx} + 4\Gamma_x^3 + 6[\check{D}_+, \check{D}_-](\Gamma_x[\hat{D}_+, \hat{D}_-]\Gamma) + 12([\hat{D}_+, \hat{D}_-]\Gamma)^2\Gamma_x = 0 \tag{41}$$

where Γ is a bosonic superfield and the complex supercovariant derivatives are defined as

$$\hat{D}_\pm = \frac{1}{2}(D_1 \pm iD_2), \quad \check{D}_\pm = \frac{1}{2}(D_3 \pm iD_4) \tag{42}$$

where $D_i = \partial_{\theta_i} + \theta_i \partial_x$ for $i = 1, 2, 3, 4$. Using the relations $\{D_i, D_j\} = 2\,\delta_{ij}\,\partial_x$, where δ_{ij} is the Kronecker delta, we have that the supercovariant derivatives Equation (42) satisfy the anticommutation rules

$$\{\hat{D}_\mu, \check{D}_\nu\} = 0, \quad \{\hat{D}_\mu, \hat{D}_\nu\} = \{\check{D}_\mu, \check{D}_\nu\} = (1 - \delta_{\mu\nu})\partial_x \tag{43}$$

where $\mu, \nu \in \{+, -\}$. Equation (41) can easily be viewed as a generalization of a $\mathcal{N} = 2$ equation. Indeed, setting $\theta_3 = \theta_4 = 0$ and $\Gamma = \frac{1}{\sqrt{2}} A$ in Equation (41), we retrieve the SmKdV Equation (8).

To construct virtual solitons of $\mathcal{N} = 2$ SUSY extensions, we have considered chiral superfields. Here, we propose a generalization of this concept. Indeed, we impose the following constraints on the superfield Γ

$$\hat{D}_+ \Gamma = 0, \quad \check{D}_+ \Gamma = 0 \tag{44}$$

A bosonic superfield Ξ satisfying the chiral conditions Equation (44) has the following general form

$$\begin{aligned}\Xi(x, t; \hat{\theta}_\mu, \check{\theta}_\mu) = u + \hat{\theta}_+\, \xi + \check{\theta}_+\, \eta + \hat{\theta}_+ \hat{\theta}_-\, u_x + \check{\theta}_+ \check{\theta}_-\, u_x + \hat{\theta}_+ \check{\theta}_+\, w \\ + \hat{\theta}_+ \hat{\theta}_- \check{\theta}_+\, \eta_x + \hat{\theta}_+ \check{\theta}_+ \check{\theta}_-\, \xi_x + \hat{\theta}_- \hat{\theta}_+ \check{\theta}_- \check{\theta}_+\, u_{xx}\end{aligned} \tag{45}$$

where $u = u(x, t)$ and $w = w(x, t)$ are complex valued bosonic functions and $\xi = \xi(x, t)$ and $\eta = \eta(x, t)$ are complex valued fermionic functions. The Grassmann variables in Equation (45) are defined as $\hat{\theta}_\pm = \frac{1}{\sqrt{2}}(\theta_1 \pm i\theta_2)$ and $\check{\theta}_\pm = \frac{1}{\sqrt{2}}(\theta_3 \pm i\theta_4)$. Now, using the chirality conditions Equation (44), we have $\hat{D}_+ \hat{D}_-\, \Gamma = \check{D}_+ \check{D}_-\, \Gamma = \Gamma_x$ and Equation (41) reduces to the classical nonlinear PDE

$$\Gamma_t + \Gamma_{xxx} + 12\Gamma_x \Gamma_{xx} + 16\Gamma_x^3 = 0 \tag{46}$$

Equation (46) is, up to a slight change of variable, similar to Equation (13) for the integrable cases $a = 1, 4$. Indeed, we retrieve Equation (13) for $a = 1, 4$ by casting $\Gamma = -\frac{i}{12}(a + 2)\tilde{A}$ in Equation (46).

The above equation can be linearized into the linear dispersive Equation (17) by the change of variable

$$\Gamma(x, t; \hat{\theta}_\mu, \check{\theta}_\mu) = \frac{1}{4} \log\, \Upsilon(x, t; \hat{\theta}_\mu, \check{\theta}_\mu) \tag{47}$$

Thus to obtain solutions of Equation (41), the superfield Υ must satisfy the constraints

$$\Upsilon_t + \Upsilon_{xxx} = 0, \quad \hat{D}_+ \Upsilon = \check{D}_+ \Upsilon = 0 \tag{48}$$

A solution to this system is

$$\Upsilon = 1 + e^{\Psi_1} = 1 + e^{\kappa_1 x - \kappa_1^3 t + \varphi_1(\hat{\theta}_\mu, \check{\theta}_\mu)} \tag{49}$$

where φ is a $\mathcal{N} = 4$ chiral bosonic superfield of the form

$$\varphi_1 = \hat{\theta}_+\, \hat{\zeta}_1 + \check{\theta}_+\, \check{\zeta}_1 + (\hat{\theta}_+ \hat{\theta}_- + \check{\theta}_+ \check{\theta}_-)\, \kappa_1 + \hat{\theta}_+ \check{\theta}_+\, \lambda_1 \tag{50}$$

with $\hat{\zeta}_1^2 = \check{\zeta}_1^2 = 0$ and λ_1 is an even constant. This result can thus be generalized to give a N super virtual soliton solution of the SUSY $\mathcal{N} = 4$ KdV Equation (41) by taking

$$\Upsilon = 1 + \sum_{i=1}^{N} e^{\kappa_i x - \kappa_i^3 t + \varphi_i(\hat{\theta}_\mu, \check{\theta}_\mu)} \tag{51}$$

where the superfields φ_i are defined as in Equation (50) for $i = 1, ..., N$.

It is interesting to note that by setting $\check{\theta}_+ = 0$ in Equation (50), one recovers the superfields Equation (24).

6. Concluding Remarks and Future Outlook

In this paper, we have studied special solutions of supersymmetric extensions of the Burgers, KdV and mKdV equations in a unified way and using a chirality of the superfield A.

We have recovered interacting super soliton solutions (often called KdV type solitons) and an infinite set of rational similarity solutions. To produce such rational solutions, we have used an SUSY extension of the Yablonskii–Vorob'ev polynomials. We have introduce a new representation of the τ-functions to solve the bilinear equations. These τ-functions are $\mathcal{N} = 2$ extensions of classical τ-functions of the mKdV equation. Till now, in the literature, only $\mathcal{N} = 1$ extensions of the τ-functions were given.

We have shown the existence of non-interacting super soliton solutions, called virtual solitons, for the Burgers and SKdV$_a$ ($a = 1, 4$). These special solutions are a direct generalization of the solutions obtained in a recent contribution [5] where N super virtual solitons have been found by setting to zero the fermionic contributions ξ_1 and ξ_2 in the bosonic superfield A given as in Equation (1). We retrieve those solutions by setting $\zeta_i = 0$ in the exponent terms Equation (37). Thus the chirality property, exposed in this paper, has produced a nontrivial fermionic sector for a N super virtual soliton. Furthermore, to obtain such solutions we have related the SUSY equations to linear PDE's showing the true origin of those special solutions.

A $\mathcal{N} = 4$ extension of the KdV equation has been shown to produce a N super virtual soliton solution. The study of $\mathcal{N} = 4$ extensions is quite new to us and we hope in the future to produce a N super soliton solution with interaction terms.

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