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Article

Conservation Laws of Discrete Evolution Equations by Symmetries and Adjoint Symmetries

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Abstract: A direct approach is proposed for constructing conservation laws of discrete evolution equations, regardless of the existence of a Lagrangian. The approach utilizes pairs of symmetries and adjoint symmetries, in which adjoint symmetries make up for the disadvantage of non-Lagrangian structures in presenting a correspondence between symmetries and conservation laws. Applications are made for the construction of conservation laws of the Volterra lattice equation.

Keywords: symmetry; adjoint symmetry; conservation law

1. Introduction

Noether's theorem tells us that a symmetry of a differential equation leads to a conservation law of the same equation, if the equation is derived from a Lagrangian, namely, it has a Lagrangian formulation [1,2]. The Lagrangian formulation of the equation is essential for presenting conservation laws from symmetries, and many physically important examples can be found in [1–4]. A natural question arises whether there is any correspondence between symmetries and conservation laws for differential equations not derivable from any Lagrangian. We would, in this paper, like to show that it is possible to give a positive answer to the above question if we adopt adjoint symmetries. More precisely, we want to exhibit that using symmetries and adjoint symmetries together can lead to conservation laws for both Lagrangian and non-Lagrangian equations.

A good attempt to use adjoint symmetries in computing conservation laws of differential equations was made in [5], and the approach utilizes adjoint symmetries, in which an adjoint invariance condition

equivalently requires the existence of an adjoint symmetry. More generally, nonlinear self-adjointness was introduced on the basis of adjoint systems and successfully applied to construction of conservation laws of differential equations [6,7]. In that theory, the nonlinear self-adjointness means that the second set of dependent variables in an adjoint system stands for an adjoint symmetry.

We would like to consider regular differential-difference equations, and so, they can be written as evolution equations. Equations of this type contain difference equations, since any difference equation can be considered as a stationary equation of discrete evolution equations. We will utilize pairs of symmetries and adjoint symmetries to present a direct formula for constructing conservation laws, and thus conserved densities, for evolution equations. Our approach will be used to compute conservation laws for the Volterra lattice equation. The general theory justifies that symmetries really reflect conservation laws. However, the adoption of adjoint symmetries has not attracted much attention within the mathematical physics community. It is expected that our findings could stimulate to expose more mathematical properties of adjoint symmetries as well as develop efficient algorithms for computing adjoint symmetries.

The paper is structured as follows. In Section 2, a general theory will be formulated for constructing conservation laws and thus conserved densities from symmetries and adjoint symmetries for discrete evolution equations. In Section 3, an example will be analyzed, along with new conserved densities. Finally in Section 4, a few of concluding remarks will be given with some discussion.

2. General Theory

We will use a plain language to formulate the framework of our results on the correspondence between conservation laws and pairs of symmetries and adjoint symmetries.

Let the potential vector u be defined by

$$u = (u^1, \cdots, u^q)^T, \ u^i = u^i(n,t), \ 1 \leq i \leq q, \ n = (n_1, \cdots, n_p) \in \mathbb{Z}^p, \ t \in \mathbb{R}, \tag{1}$$

and E_i, $1 \leq i \leq p$, denote the shift operators for the variables n_i, $1 \leq i \leq p$, *i.e.*,

$$(E_i u)(n) = u(n_1, \cdots, n_{i-1}, n_i + 1, n_{i+1}, \cdots, n_p), \ n = (n_1, \cdots, n_p) \in \mathbb{Z}^p, \ 1 \leq i \leq p. \tag{2}$$

we introduce

$$E^\alpha = E_1^{\alpha_1} \cdots E_p^{\alpha_p}, \ E^\alpha u = (E^\alpha u^1, \cdots, E^\alpha u^q) = (u_\alpha^1, \cdots, u_\alpha^q), \ (E^\alpha u^i)(n) = u^i(n + \alpha), \tag{3}$$

where $\alpha = (\alpha_1, \cdots, \alpha_p), n = (n_1, \cdots, n_p) \in \mathbb{Z}^p$, and $n + \alpha = (n_1 + \alpha_1, \cdots, n_p + \alpha_p)$. We denote by $\mathcal{A}$ and $\mathcal{B}$ the space of all local C^∞ functions in n, t, u and $E^\alpha u$ to some finite order α, and the space of all C^∞ functions in n, t, u and $E^\alpha u$ to some finite order α. Moreover, we assume that $\mathcal{A}^r$ and $\mathcal{B}^r$ denote the r-th order tensor products of $\mathcal{A}$ and $\mathcal{B}$, respectively, *i.e.*,

$$\mathcal{A}^r = \underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{r}, \ \mathcal{B}^r = \underbrace{\mathcal{B} \otimes \cdots \otimes \mathcal{B}}_{r}. \tag{4}$$

The locality here means that for a function $f(u)$ in $\mathcal{A}$, any value $(f(u))(n)$ is completely determined by the values of u at finitely many points $n \in \mathbb{Z}^p$. The space $\mathcal{A}$ contains functions of polynomial

type, $P(u, Eu, \cdots, E^\alpha u)$, where P is a polynomial in its variables, and the space $\mathcal{B}$ contains non-local functions:

$$(f(u))(n) = \sum_{|\alpha| \leq m} \sum_{k \geq n} (E^\alpha u)(k),\ m \geq 1,\ |\alpha| = |\alpha_1| + \cdots + |\alpha_p|.$$

For a local vector function $X = X(u) = (X_1, \cdots, X_r)^T \in \mathcal{A}^r$, we can compute its Gateaux derivative operator as follows:

$$X' = X'(u) = (V_j(X_i))_{r \times q} = \begin{bmatrix} V_1(X_1) & V_2(X_1) & \cdots & V_q(X_1) \\ V_1(X_2) & V_2(X_2) & \cdots & V_q(X_2) \\ \vdots & \vdots & \ddots & \vdots \\ V_1(X_r) & V_2(X_r) & \cdots & V_q(X_r) \end{bmatrix},\ V_i(X_j) = \sum_{\alpha \in \mathbb{Z}^p} \frac{\partial X_j}{\partial u_\alpha^i} E^\alpha, \quad (5)$$

and its (formal) adjoint operator

$$X'^\dagger = X'^\dagger(u) = (V_i^\dagger(X_j))_{q \times r},\ V_i^\dagger(X_j) = \sum_{\alpha \in \mathbb{Z}^p} E^{-\alpha} \frac{\partial X_j}{\partial u_\alpha^i}, \quad (6)$$

where the action of the operator $E^{-\alpha} f$ on g is given by $(E^{-\alpha} f)g = E^{-\alpha}(fg)$, $f, g \in \mathcal{B}$. The operator $X'^\dagger$ can be explained as an adjoint operator of X', if the inner products

$$\langle Y, Z \rangle = \sum_{n \in \mathbb{Z}^p} \sum_{i=1}^{s} Y_i(n) Z_i(n),\ Y = (Y_1, \cdots, Y_s)^T, Z = (Z_1 \cdots, Z_s)^T \in \mathcal{B}^s,\ s \geq 1, \quad (7)$$

are well defined over some selected space of u. For example, for the adjoint operator $E_i^\dagger = E_i^{-1}$ of each shift operator E_i $(1 \leq i \leq p)$, we have

$$\langle E_i^\dagger Y, Z \rangle = \langle E_i^{-1} Y, Z \rangle = \langle Y, E_i Z \rangle,\ Y, Z \in \mathcal{B}.$$

Let us now consider an discrete evolution equation

$$u_t = K(n, t, u),\ K \in \mathcal{A}^q. \quad (8)$$

its linearized equation and adjoint linearized equation are defined by

$$(\sigma(n, t, u))_t = K'(n, t, u)\sigma(n, t, u),\ \sigma \in \mathcal{B}^q, \quad (9)$$

$$(\rho(n, t, u))_t = -K'^\dagger(n, t, u)\rho(n, t, u),\ \rho \in \mathcal{B}^q, \quad (10)$$

respectively. Here K' and $K'^\dagger$ denotes the Gateaux derivative operator and the adjoint Gateaux derivative operator of K with respect to u, respectively; and f_t denotes the total derivative of f with respect to t.

Definition 2.1. *A vector field $\sigma \in \mathcal{B}^q$ is called a symmetry of the discrete evolution Equation (8), if it satisfies the linearized Equation (9) when u solves Equation (8). A vector field $\rho \in \mathcal{B}^q$ is called an adjoint symmetry of the discrete evolution equation Equation (8), if it satisfies the adjoint linearized Equation (10) when u solves (8).*

It is easy to see that two local vector fields $\sigma \in \mathcal{A}^q$ and $\rho \in \mathcal{A}^q$ are a symmetry and an adjoint symmetry of the discrete evolution Equation (8), if and only if they satisfy

$$\frac{\partial \sigma(n,t,u)}{\partial t} = K'(n,t,u)\sigma(n,t,u) - \sigma'(n,t,u)K(n,t,u), \tag{11}$$

$$\frac{\partial \rho(n,t,u)}{\partial t} = -K'^{\dagger}(n,t,u)\rho(n,t,u) - \rho'(n,t,u)K(n,t,u), \tag{12}$$

respectively, when u solves Equation (8). Here σ' and ρ' are the Gateaux derivative operators of σ and ρ, and $\frac{\partial}{\partial t}$ denotes the partial derivative with respect to t.

Definition 2.2. *If a relation*

$$h_t = \sum_{i=1}^{p}(E_i - 1)f_i, \ h, f_i \in \mathcal{B}, \ 1 \leq i \leq p, \tag{13}$$

holds when u solves the discrete evolution Equation (8), then Equation (13) is called a conservation law of Equation (8), h a conserved density of Equation (8), and $f = (f_1, \cdots, f_p)^T$ a conversed flux of Equation (8) corresponding to h. A conserved density $h \in \mathcal{B}$ is called trivial, if there exist $g_i \in \mathcal{B}$, $1 \leq i \leq p$, such that $h = \sum_{i=1}^{p}(E_i - 1)g_i$ holds on the solution set.

A conserved quantity means a quantity which does not vary with respect to time t on the solution set. A quantity defined by $\mathcal{I} = \sum_{n \in \mathcal{Z}^p} I(n)$, $I = I(n,t,u) \in \mathcal{B}$, is called a functional. We would first like to show a relation between conserved quantities and adjoint symmetries.

Proposition 2.1. *Let $\mathcal{I} = \mathcal{I}(n,u)$ be a functional which does not depend explicitly on time t. Then, $\mathcal{I}$ is a conserved quantity of a discrete evolution equation $u_t = K$, $K = K(n,t,u) \in \mathcal{B}^q$, if and only if its variational derivative $\frac{\delta \mathcal{I}}{\delta u}$ is an adjoint symmetry of the same equation.*

Proof: Let $\mathcal{S}$ be the Schwartz space and the Gateaux derivative of an object $P = P(u)$ with respect to Y be defined by

$$P'[Y] = P'(u)[Y] = \frac{\partial P(n,t,u+\varepsilon Y)}{\partial \varepsilon}\bigg|_{\varepsilon=0}, \ Y \in \mathcal{B}^q.$$

We denote the variational derivative $\frac{\delta \mathcal{I}}{\delta u}$ by G, and so, the definition of variational derivatives tells that

$$\mathcal{I}'[Y] = \langle G, Y \rangle, \ Y \in \mathcal{S}^q,$$

where the inner product is defined as in Equation (7). Then the differentiability of functions of C^{∞} class guarantees that on the solution set, we have

$$\mathcal{I}_t = \frac{\partial \mathcal{I}}{\partial t} + \frac{\partial \mathcal{I}(n, u+\varepsilon u_t)}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \langle G, u_t \rangle = \langle G, K \rangle.$$

It now follows that

$$\begin{aligned}
(\mathcal{I}_t)'[Y] &= \langle G'[Y], K \rangle + \langle G, K'[Y] \rangle \\
&= \langle G'^{\dagger}K, Y \rangle + \langle K'^{\dagger}G, Y \rangle \\
&= \langle G'K, Y \rangle + \langle K'^{\dagger}G, Y \rangle \\
&= \langle \frac{\partial G}{\partial t} + G'K + K'^{\dagger}G, Y \rangle, \ Y \in \mathcal{S}^q,
\end{aligned}$$

where $G'^\dagger = G'$ and $\frac{\partial G}{\partial t} = 0$ were used. This last equality implies that $\mathcal{I}$ is conseved if and only if G is an adjoint symmetry. ∎

We use a conventional assumption below: an empty product of shift operators E^{α_i}, $\alpha_i \in \mathbb{Z}$, $1 \le i \le p$, is understood to be the identity operator. For example, the operator $\Pi_{i=k}^l E_i^2$ implies the identity operator, when $k > l$. We would now like to prove the following lemma, in order to give a direct formula for constructing conservation laws of the discrete evolution Equation (8).

Lemma 2.1. *Let f and g be two C^∞ functions in variables $n_1, \cdots, n_p$.   Then for any $\alpha = (\alpha_1, \cdots \alpha_p) \in \mathbb{Z}^p$, we have*

$$
\begin{aligned}
& fE^\alpha g - (E^{-\alpha}f)g = f(E_1^{\alpha_1} \cdots E_p^{\alpha_p}g) - (E_1^{-\alpha_1} \cdots E_p^{-\alpha_p}f)g \\
= {} & \sum_{i=1}^p (E_i - 1)\Big[\sum_{\beta_i=1}^{\alpha_i}(E_1^{-\alpha_1} \cdots E_{i-1}^{-\alpha_{i-1}}E_i^{-\beta_i}f)(E_i^{\alpha_i-\beta_i}E_{i+1}^{\alpha_{i+1}} \cdots E_p^{\alpha_p}g) \\
& \qquad\qquad - \sum_{\beta_i=1}^{-\alpha_i}(E_1^{-\alpha_1} \cdots E_{i-1}^{-\alpha_{i-1}}E_i^{-\alpha_i-\beta_i}f)(E_i^{-\beta_i}E_{i+1}^{\alpha_{i+1}} \cdots E_p^{\alpha_p}g) \Big],
\end{aligned}
\tag{14}
$$

where the value of an empty sum is conventionally zero.

Proof: First note that we have

$$
\begin{aligned}
& aE_i^k b - (E_i^{-k}a)b \\
= {} & (E_i - 1)\Big[\sum_{l=1}^k (E_i^{-l}a)(E_i^{k-l}b) - \sum_{l=1}^{-k}(E_i^{-k-l}a)(E_i^{-l}b) \Big] \\
= {} & \begin{cases} (E_i - 1)\displaystyle\sum_{l=1}^k (E_i^{-l}a)(E_i^{k-l}b), & k \ge 0, \\[2.5em] (E_i - 1)\displaystyle\sum_{l=1}^{-k}(E_i^{-k-l}a)(E_i^{-l}b), & k < 0, \end{cases}
\end{aligned}
\tag{15}
$$

for any $k \in \mathbb{Z}$ and any two C^∞ functions a and b in variables $n_1, \cdots, n_p$. Then we decompose that

$$
\begin{aligned}
& fE^\alpha g - (E^{-\alpha}f)g = f(E_1^{\alpha_1} \cdots E_p^{\alpha_p}g) - (E_1^{-\alpha_1} \cdots E_p^{-\alpha_p}f)g \\
= {} & \sum_{i=1}^p \Big[(E_1^{-\alpha_1} \cdots E_{i-1}^{-\alpha_{i-1}}f)(E_i^{\alpha_i} \cdots E_p^{\alpha_p}g) - (E_1^{-\alpha_1} \cdots E_i^{-\alpha_i}f)(E_{i+1}^{\alpha_{i+1}} \cdots E_p^{\alpha_p}g) \Big].
\end{aligned}
$$

It now follows from Equation (15) that each term in the above sum can be computed as follows:

$$
\begin{aligned}
& (E_1^{-\alpha_1} \cdots E_{i-1}^{-\alpha_{i-1}}f)(E_i^{\alpha_i} \cdots E_p^{\alpha_p}g) - (E_1^{-\alpha_1} \cdots E_i^{-\alpha_i}f)(E_{i+1}^{\alpha_{i+1}} \cdots E_p^{\alpha_p}g) \\
= {} & (E_1^{-\alpha_1} \cdots E_{i-1}^{-\alpha_{i-1}}f)[E_i^{\alpha_i}(E_{i+1}^{\alpha_{i+1}} \cdots E_p^{\alpha_p}g)] \\
& - [E_i^{-\alpha_i}(E_1^{-\alpha_1} \cdots E_{i-1}^{-\alpha_{i-1}}f)](E_{i+1}^{\alpha_{i+1}} \cdots E_p^{\alpha_p}g) \\
= {} & (E_i - 1)\Big[\sum_{\beta_i=1}^{\alpha_i}(E_1^{-\alpha_1} \cdots E_{i-1}^{-\alpha_{i-1}}E_i^{-\beta_i}f)(E_i^{\alpha_i-\beta_i}E_{i+1}^{\alpha_{i+1}} \cdots E_p^{\alpha_p}g) \\
& \qquad\qquad - \sum_{\beta_i=1}^{-\alpha_i}(E_1^{-\alpha_1} \cdots E_{i-1}^{-\alpha_{i-1}}E_i^{-\alpha_i-\beta_i}f)(E_i^{-\beta_i}E_{i+1}^{\alpha_{i+1}} \cdots E_p^{\alpha_p}g) \Big], \quad 1 \le i \le p,
\end{aligned}
$$

where the value of an empty sum is conventionally zero. This allows us to conclude that the equality Equation (14) holds for any $\alpha = (\alpha_1, \cdots, \alpha_p) \in \mathbb{Z}^p$. The proof is finished. ∎

Theorem 2.1. *Let* $\sigma = (\sigma_1, \cdots, \sigma_q)^T \in \mathcal{B}^q$ *and* $\rho = (\rho_1, \cdots, \rho_q)^T \in \mathcal{B}^q$ *be a symmetry and an adjoint symmetry of the discrete evolution Equation (8), respectively. Then we have a conservation law of the discrete evolution Equation (8):*

$$(\sigma^T \rho)_t = (\sum_{i=1}^{q} \sigma_i \rho_i)_t$$

$$= \sum_{k=1}^{p}(E_k - 1)\sum_{i,j=1}^{q}\sum_{\alpha \in \mathbb{Z}^p}\Big[\sum_{\beta_k=1}^{\alpha_k}(E_1^{-\alpha_1}\cdots E_{k-1}^{-\alpha_{k-1}}E_k^{-\beta_k}\rho_i\frac{\partial K_i}{\partial u_\alpha^j})(E_k^{\alpha_k-\beta_k}E_{k+1}^{\alpha_{k+1}}\cdots E_p^{\alpha_p}\sigma_j)$$

$$-\sum_{\beta_k=1}^{-\alpha_k}(E_1^{-\alpha_1}\cdots E_{k-1}^{-\alpha_{k-1}}E_k^{-\alpha_k-\beta_k}\rho_i\frac{\partial K_i}{\partial u_\alpha^j})(E_k^{-\beta_k}E_{k+1}^{\alpha_{k+1}}\cdots E_p^{\alpha_p}\sigma_j)\Big], \qquad (16)$$

where $\alpha = (\alpha_1, \cdots, \alpha_p)$, *and the value of an empty sum is conventionally zero. Therefore,* $\sigma^T \rho$ *is a conserved density of the discrete evolution Equation (8).*

Proof: Let us compute that

$$(\sigma^T \rho)_t = \sigma_t^T \rho + \sigma^T \rho_t = \rho^T \sigma_t + \sigma^T \rho_t$$

$$= \rho^T K'\sigma - \sigma^T K'^\dagger \rho = \sum_{i,j=1}^{q}(\rho_i V_j(K_i)\sigma_j - \sigma_j V_j^\dagger(K_i)\rho_i)$$

$$= \sum_{i,j=1}^{q}\sum_{\alpha \in \mathbb{Z}^p}(\rho_i\frac{\partial K_i}{\partial u_\alpha^j}E^\alpha \sigma_j - \sigma_j E^{-\alpha}\rho_i\frac{\partial K_i}{\partial u_\alpha^j}).$$

By using Lemma 2.1, for all $1 \leq i, j \leq q$, $\alpha \in \mathbb{Z}^p$, we have

$$\rho_i\frac{\partial K_i}{\partial u_\alpha^j}E^\alpha \sigma_j - \sigma_j E^{-\alpha}\rho_i\frac{\partial K_i}{\partial u_\alpha^j}$$

$$= \sum_{k=1}^{p}(E_k - 1)\Big[\sum_{\beta_k=1}^{\alpha_k}(E_1^{-\alpha_1}\cdots E_{k-1}^{-\alpha_{k-1}}E_k^{-\beta_k}\rho_i\frac{\partial K_i}{\partial u_\alpha^j})(E_k^{\alpha_k-\beta_k}E_{k+1}^{\alpha_{k+1}}\cdots E_p^{\alpha_p}\sigma_j)$$

$$-\sum_{\beta_k=1}^{-\alpha_k}(E_1^{-\alpha_1}\cdots E_{k-1}^{-\alpha_{k-1}}E_k^{-\alpha_k-\beta_k}\rho_i\frac{\partial K_i}{\partial u_\alpha^j})(E_k^{-\beta_k}E_{k+1}^{\alpha_{k+1}}\cdots E_p^{\alpha_p}\sigma_j)\Big],$$

where the value of an empty sum is conventionally zero. If then follows that the equality Equation (16) is true, and thus, $\sigma^T \rho$ is a conserved density of the discrete evolution Equation (8). The proof is finished. ∎

This theorem tells us that a pair of a symmetry and an adjoint symmetry naturally yields a conservation law. Moreover, the expression of the conserved density is only dependent on the pair of a symmetry and an adjoint symmetry, but the expression of the conserved flux is dependent on the pair of a symmetry and an adjoint symmetry as well as the underlying equation.

If for two functions $h_1, h_2 \in \mathcal{B}$, there exist functions $g_i \in \mathcal{B}$, $1 \leq i \leq p$, such that

$$h_1 - h_2 = \sum_{i=1}^{p}(E_i - 1)g_i, \qquad (17)$$

then we say that h_1 is equivalent to h_2, denoted by $h_1 \sim h_2$. Obviously, this is an equivalence relation, and can be used in classifying conserved densities. A trivial conserved density is equivalent to zero.

3. Applications to the Volterra Lattice Equation

Now we go on to illustrate by an example rich structures of the conservation laws resulted from symmetries and adjoint symmetries.

Let us consider the Volterra lattice equation [8]:

$$u_t = K(u) = u(E^{-1}u - Eu), \ u = u(n,t), \ n \in \mathbb{Z}, \ t \in \mathbb{R}. \tag{18}$$

Its linearized equation and adjoint linearized equation read

$$\sigma_t = (E^{-1}u - Eu)\sigma + uE^{-1}\sigma - uE\sigma, \tag{19}$$

$$\rho_t = (Eu - E^{-1}u)\rho - E(u\rho) + E^{-1}(u\rho), \tag{20}$$

respectively. There exist infinitely many symmetries [9]:

$$K_i = \Phi^i(u(E^{-1}u - Eu)), \ i \geq 0, \tag{21}$$

$$\tau_i = \Phi^i(t[K_0, u] + u) = \Phi^i(tK_0 + u) = \Phi^i(tu(E^{-1}u - Eu) + u), \ i \geq 0, \tag{22}$$

where the hereditary recursion operator Φ is defined by

$$\Phi = u(1 + E^{-1})(-(Eu)E^2 + u)(E - 1)^{-1}u^{-1}.$$

Moreover, these symmetries constitute a Lie algebra [9]:

$$[K_i, K_i] = 0, \ [K_i, \tau_j] = (i + 1)K_{i+j}, \ [\tau_i, \tau_j] = (i - j)\tau_{i+j}, \ i, j \geq 0, \tag{23}$$

where $[K, S]$ is defined by $[K, S] = K'S - S'K$, K' and S' being the corresponding Gateaux derivative operators.

By an inspection, the function defined by

$$S_0 = u^{-1} \tag{24}$$

is an adjoint symmetry of the Volterra lattice Equation (18). This means that S_0 satisfies the adjoint linearized Equation (20) while u solves Equation (18). The proof just needs a simple and direct computation:

$$(S_0)_t = -u^{-2}u_t, \ (Eu - E^{-1}u)S_0 - E(uS_0) + E^{-1}(uS_0) = (Eu - E^{-1}u)u^{-1},$$

from which it directly follows that if u solves Equation (18), S_0 satisfies Equation (20). Now, using the principle in Theorem 2.1, we have infinitely many conserved densities, $u^{-1}K_i$ and $u^{-1}\tau_i$, $i \geq 0$. In particular, since we have

$$S_0\tau_1 = S_0(tK_1 + \Phi u) \sim u^{-1}\Phi u,$$

$$(\Phi u)(n) = u(n)[-(n + 2)u(n + 1) - u(n) + (n - 1)u(n - 1)],$$

we obtain a nontrivial local conserved density of the Volterra lattice Equation (18):

$$(h_1)(n) = -(n+2)u(n+1) - u(n) + (n-1)u(n-1). \tag{25}$$

This generates a conservation law

$$h_{1t} = (E-1)[([n]+1)u(Eu) - ([n]-1)(E^{-2}u)(E^{-1}u)], \tag{26}$$

where $[n]$ is the operator

$$([n]f)(m) = mf(m), \ f \in \mathcal{B}, \ m \in \mathbb{Z}.$$

Except $S_0\tau_1 = S_0(tK_0 + u) \sim 1$, all other products of the same type, $u^{-1}\tau_i, \ i \geq 2$, give us nontrivial nonlocal conserved densities of the Volterra lattice Equation (18). But all $u^{-1}K_i, \ i \geq 0$, are trivial conserved densities, which can be seen directly from the recursion structure of symmetries [10].

Based on Proposition 2.1, the Hamiltonian formulation of the Volterra soliton hierarchy [9] guarantees that the Volterra lattice Equation (18) has a hierarchy of adjoint symmetries:

$$\rho_i = \Psi^i 1 = \frac{\delta \mathcal{H}_i}{\delta u}, \ \Psi = \Phi^\dagger = u^{-1}(E-1)^{-1}(-(Eu)E^2 + u)(1 + E^{-1})u, \ i \geq 0, \tag{27}$$

which generate, by Theorem 2.1, the conserved densities: $\rho_i K_j, \ \rho_i \tau_j, \ i, j \geq 0$. The nontrivial conserved density generated from $\rho_0\tau_1$ is, due to $K_1 \sim 0$, equivalent to

$$(h_2)(n) = (\Phi u)(n) = u(n)[-(n+2)u(n+1) - u(n) + (n-1)u(n-1)], \tag{28}$$

whose associated conservation law reads

$$h_{2t} = (E-1)\{([n]+1)[(E^{-1}u)u^2 + (E^{-1}u)u(Eu)] - ([n]-1)[(E^{-2}u)^2u + (E^{-2}u)(E^{-1}u)u]\}. \tag{29}$$

Moreover, we see that all symmetries, $K_i = \rho_0 K_i$ and $\tau_i = \rho_0 \tau_i, \ i \geq 0$, are also conserved densities, and that the conserved densities $\rho_i\tau_0 \sim \rho_i u, \ i \geq 0$, present all standard ones associated with the Hamiltonian functionals $\{\mathcal{H}_i | i \geq 0\}$ [9], upon noting

$$u\frac{\delta \mathcal{H}}{\delta u} = u\sum_{\alpha \in \mathcal{Z}} E^{-\alpha}\frac{\partial H}{\partial (E^\alpha u)} \sim \sum_{\alpha \in \mathcal{Z}}(E^\alpha u)\frac{\partial H}{\partial (E^\alpha u)} = H, \ \mathcal{H} = \sum_{n \in \mathcal{Z}} H(n), \ H \in \mathcal{A}.$$

The Volterra lattice Equation (18) also has the following Lax pair [11]:

$$U(u, \lambda) = \begin{bmatrix} 0 & u \\ -1 & \lambda \end{bmatrix}, \ V(u, \lambda) = \begin{bmatrix} \lambda^2 - u & -\lambda E^{-1}u \\ \lambda & -E^{-1}u \end{bmatrix}, \tag{30}$$

which means that the Volterra lattice Equation (18) is equivalent to the discrete zero curvature equation $U_t = (EV)U - UV$. Let $\lambda_s, \ 1 \leq s \leq N$, be arbitrary constants, and introduce N replicas of the spectral problems:

$$E\phi^{(s)} = U(u, \lambda_s)\phi^{(s)}, \ \phi_t^{(s)} = V(u, \lambda_s)\phi^{(s)}, \ \phi^{(s)} = (\phi_{1s}, \phi_{2s})^T, \ 1 \leq s \leq N, \tag{31}$$

and N replicas of the adjoint spectral problems:

$$E\psi^{(s)} = (U^T)^{-1}(u, \lambda_s)\psi^{(s)}, \ \psi_t^{(s)} = -V^T(u, \lambda_s)\psi^{(s)}, \ \psi^{(s)} = (\psi_{1s}, \psi_{2s})^T, \ 1 \leq s \leq N. \tag{32}$$

Then, we have a class of adjoint symmetries represented in terms of eigenfunctions and adjoint eigenfunctions [11]:

$$T_0 = \frac{1}{u}(P_2^T A Q_1 + P_2^T Q_2), \tag{33}$$

where the matrix A, and the two N dimensional vector functions P_i and Q_i are defined by

$$A = \operatorname{diag}(\lambda_1, \cdots, \lambda_N),\ P_i = (\phi_{i1}, \cdots, \phi_{iN})^T,\ Q_i = (\psi_{i1}, \cdots, \psi_{iN})^T,\ 1 \le i \le 2. \tag{34}$$

Now, Theorem 2.1 guarantees that we have the conserved densities: $T_0 K_i$ and $T_0 \tau_i$, $i \ge 0$. This particularly gives the following two conserved densities:

$$T_0 K_0 = (E^{-1}u - Eu)(P_2^T A Q_1 + P_2^T Q_2), \tag{35}$$

$$T_0 \tau_0 = t(E^{-1}u - Eu)(P_2^T A Q_1 + P_2^T Q_2) + (P_2^T A Q_1 + P_2^T Q_2). \tag{36}$$

Again by Theorem 2.1, their corresponding fluxes read

$$-(P_2^T A Q_1 + P_2^T Q_2)E^{-1}K_0 - (E(P_2^T A Q_1 + P_2^T Q_2))K_0, \tag{37}$$

$$-(P_2^T A Q_1 + P_2^T Q_2)E^{-1}\tau_0 - (E(P_2^T A Q_1 + P_2^T Q_2))\tau_0, \tag{38}$$

respectively. The involvement of eigenfunctions and adjoint eigenfunctions exhibits strong nonlocality of this kind of conservation laws.

4. Concluding Remarks

We have established a direct approach for constructing conservation laws and thus conserved densities of discrete evolution equations, whether the evolution equations are derivable from a Lagrangian or not. Our approach utilizes pairs of symmetries and adjoint symmetries, in which adjoint symmetries make up for the disadvantage of non-Lagrangian structures in establishing a correspondence between symmetries and conservation laws. The approach has been applied to the generation of conserved densities of the Volterra lattice equation.

We remark that on one hand, all evolution equations become the first-order ordinary differential equations (ODEs), when there is no spatial shift appeared in the equations. On the other hand, if we remove the time derivative, *i.e.*, there is no time involved in evolution equations, then we immediately obtain conservation laws for difference equations. Moreover, our results pave a way to construct conservation laws from symmetries for self-adjoint discrete evolution equations or difference equations, since symmetries are also adjoint symmetries of self-adjoint discrete evolution equations or difference equations. We also point out that our idea of constructing conservation laws by symmetries and adjoint symmetries is quite similar to that of carrying out binary nonlinearization under symmetry constraints [12,13]. In the theory of binary nonlinearization [12,13], adjoint symmetries are used to establish a balance with non-Lie adjoint symmetries generated from both spectral problems and adjoint spectral problems.

There are many other approaches or theories on conservation laws of differential equations and differential-difference equations. In the case of integrable equations, Hamiltonian formulations [14,15] generate usual conservation laws associated with Hamiltonian functionals, and a bi-Hamiltonian

formulation presents a recurrence relation of conserved densities in the resulting conservation laws [16]. Moreover, Lax pairs are used to make Riccati type equations that ratios of eigenfunctions need to satisfy, and series expansions of solutions of the resulting Riccati type equations around the spectral parameter yield conservation laws (see, e.g., [17,18]). For continuous evolution equations, adjoint symmetries are also called conserved covariants [19], and it is recognized that the product of a symmetry and an adjoint symmetry presents a conserved density (see, e.g., [19–21]) and that a functional $\mathcal{I}$ is conserved if and only if its variational derivative $\frac{\delta \mathcal{I}}{\delta u}$ is an adjoint symmetry (see, e.g., [13,20,21]).

There exists a geometrical theory to deal with adjoint symmetries of the second-order ODEs [22]. Adjoint symmetries are also used to show separability of finite-dimensional Hamiltonian systems (see, e.g., [23]) and links to integrating factors of the second-order ODEs (see, e.g., [24]). An important character for symmetries is the existence of Lie algebraic structures, and such Lie algebraic structures can be resulted from a kind of Lie algebras of Lax operators corresponding to symmetries (see [25–27] for the continuous case and [9,28] for the discrete case). Therefore, we are curious about whether there exist any Lie algebraic structures for adjoint symmetries and what kind of Lie algebraic structures we can have if they exist. We finally make a remark about this question. A natural binary operation for adjoint symmetries could be taken as

$$\llbracket \rho_1, \rho_2 \rrbracket = (\rho_1')^\dagger \rho_2 - (\rho_2')^\dagger \rho_1,$$

where $(\rho_1')^\dagger$ and $(\rho_2')^\dagger$ denote their adjoint Gateaux derivative operators. However, this doesn't keep the space of adjoint symmetries closed. For example, for the Volterra lattice Equation (18), we have two adjoint symmetries $\rho_1 = 1$ and $\rho_2 = u^{-1}$, which generate

$$\llbracket \rho_1, \rho_2 \rrbracket = \llbracket 1, u^{-1} \rrbracket = u^{-2},$$

but this function u^{-2} is not an adjoint symmetry of the Volterra lattice equation.

Acknowledgments

Conflicts of Interest

The author declares no conflict of interest.

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