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Article

Conservation Laws and Exact Solutions of a Generalized Zakharov–Kuznetsov Equation

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Abstract: In this paper, we study a generalized Zakharov–Kuznetsov equation in three variables, which has applications in the nonlinear development of ion-acoustic waves in a magnetized plasma. Conservation laws for this equation are constructed for the first time by using the new conservation theorem of Ibragimov. Furthermore, new exact solutions are obtained by employing the Lie symmetry method along with the simplest equation method.

Keywords: generalized Zakharov–Kuznetsov equation; conservation laws; exact solutions; simplest equation method

1. Introduction

Many important phenomena and dynamic processes in physics, applied mathematics and engineering can be described by higher-dimensional extensions of the Korteweg–de Vries (KdV) equation. Zakharov and Kuznetsov successfully proposed one such model [1]. The Zakharov–Kuznetsov (ZK) equation given by:

$$u_t + \alpha u u_x + \beta(u_{xx} + u_{yy})_x = 0 \tag{1}$$

is one of the known two-dimensional generalizations of the KdV equation studied in the literature. The ZK equation governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [1–3]. In [4,5], Wazwaz used the extended tanh method and the sine-cosine ansatz to find solitons and periodic solutions for (1). Moussa [6] also obtained some similarity solutions by using the symmetry group method. Equation (1) was also studied, amongst others, by Peng [7], and he obtained abundant periodic wave solutions using the extended mapping method.

This paper aims to study the generalized Zakharov–Kuznetsov (gZK) equation [4,8]:

$$u_t + \alpha u^n u_x + \beta(u_{xx} + u_{yy})_x = 0 \tag{2}$$

where α, β and n are nonzero arbitrary constants and $u = u(t, x, y)$. In [4], the extended tanh method was employed, and solitons and periodic solutions were derived for (2), which may be helpful to describe wave features in plasma physics. The Cole–Hopf transformation and the first integral technique were used in [8] to obtain complex solutions for Equation (2).

It is of great importance to search for exact solutions of nonlinear partial differential equations (NPDEs), such as the gZK equation, because many physical phenomena are described by NPDEs. Although there is no unique method for finding exact solutions of NLPEs, a great deal of research work has been devoted to developing different methods to solve NLPEs. Some of the methods found in the literature include the inverse scattering transform method [9], Darboux transformation [10], Hirota's bilinear method [11], Bäcklund transformation [12], the multiple exp-function method [13], the (G'/G)-expansion method [14], the sine-cosine method [15], the F-expansion method [16], the exp-function expansion method [17] and the Lie symmetry method [18,19].

There is no doubt that in the study of differential equations, conservation laws play an important role. In fact, conservation laws describe physical conserved quantities, such as mass, energy, momentum and angular momentum, as well as charge and other constants of motion [20,21]. They have been used in investigating the existence, uniqueness and stability of solutions of nonlinear partial differential equations [22–24]. Furthermore, they have been used in the development and use of numerical methods [25,26]. Recently, conservation laws were used to obtain exact solutions of some partial differential equations [27–31]. Thus, it is essential to study the conservation laws of partial differential equations.

The paper is organized as follows: In Section 2, we derive conservation laws of (2) by employing the new conservation law theorem by Ibragimov [32]. In Section 3, we obtain exact solutions of (2) using Lie symmetry analysis and the simplest equation method [33–35]. Finally, concluding remarks are presented in Section 4.

2. Conservation Laws

In this section, the new conservation theorem by Ibragimov [32] will be used to construct conservation laws for (2). To use the conservation theorem by Ibragimov [32], we need to know the Lie point symmetries of (2). Thus, we first compute the symmetries of (2).

2.1. Lie Point Symmetries of (2)

The vector field:

$$X = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \xi^3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u}$$

is a Lie point symmetry of (2) if:

$$X^{[3]}[u_t + \alpha u^n u_x + \beta(u_{xx} + u_{yy})_x] = 0$$

on Equation (2). Expanding Equation (3) and then splitting on the derivatives of u, we obtain the following overdetermined system of linear partial differential equations:

$$\begin{aligned}
&\xi^3_t = 0,\ \ \xi^2_y = 0,\ \ \xi^3_x = 0,\ \ \xi^3_u = 0,\ \ \xi^2_u = 0,\ \ \xi^1_x = 0,\\
&\xi^1_y = 0,\ \ \xi^1_u = 0,\ \ \xi^2_{xx} = 0,\ \ \eta_{xu} = 0,\ \ \eta_{uu} = 0,\ \ \xi^3_{yy} - 2\eta_{yu} = 0,\\
&\beta u\eta_{yyu} - u\xi^2_t + 2\alpha u^{n+1}\xi^2_x + n\alpha u^n\eta = 0,\\
&\beta u\eta_{yyu} - u\xi^2_t + 2\alpha u^{n+1}\xi^3_y + n\alpha u^n\eta = 0,\\
&\beta\eta_{xxx} + \beta\eta_{xyy} + \alpha u^n\eta_x + \eta_t = 0,\\
&\beta u\eta_{yyu} - u\xi^2_t + n\alpha u^n\eta + \alpha u^{n+1}\xi^1_t - \alpha u^{n+1}\xi^2_x = 0.
\end{aligned}$$

Solving the above system of partial differential equations, one obtains the following four Lie point symmetries:

$$X_1 = \frac{\partial}{\partial t},\quad X_2 = \frac{\partial}{\partial x},\quad X_3 = \frac{\partial}{\partial y},\quad X_4 = 3nt\frac{\partial}{\partial t} + nx\frac{\partial}{\partial x} + ny\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}$$

2.2. Application of the New Conservation Theorem

The gZK equation together with its adjoint equation are given by:

$$F \equiv u_t + \alpha u^n u_x + \beta u_{xxx} + \beta u_{xyy} = 0 \tag{3a}$$

$$F^* \equiv v_t + \alpha v_x u^n + \beta v_{xxx} + \beta v_{xyy} = 0 \tag{3b}$$

The third-order Lagrangian for the system of Equations (3a) and (3b) is given by:

$$L = v(u_t + \alpha u^n u_x + \beta u_{xxx} + \beta u_{xyy}) \tag{4}$$

which can be reduced to the second-order Lagrangian:

$$L = v(u_t + \alpha u^n u_x) - \beta v_x u_{xx} - \beta v_x u_{yy} \tag{5}$$

We have the following four cases:

(i) We first consider the Lie point symmetry $X_1 = \partial_t$ of (2). Corresponding to this symmetry, the Lie characteristic functions are $W^1 = -u_t$ and $W^2 = -v_t$. Thus, by using the Ibragimov theorem [32], the components of the conserved vector associated with the symmetry $X_1 = \partial_t$ are given by:

$$C^t_1 = \alpha v u^n u_x - \beta v_x(u_{xx} + u_{yy}),$$

$$\begin{aligned} C_1^x &= \beta v_x u_{tx} + \beta v_t(u_{xx} + u_{yy}) - \alpha v u^n u_t - \beta u_t v_{xx}, \\ C_1^y &= \beta v_x u_{ty} - \beta u_t v_{xy}. \end{aligned}$$

(ii) Likewise, the Lie point symmetry $X_2 = \partial_x$ has the Lie characteristic functions $W^1 = -u_x$ and $W^2 = -v_x$. Invoking Ibragimov's theorem, we obtain the conserved vector, whose components are:

$$\begin{aligned} C_2^t &= -v u_x, \\ C_2^x &= \beta v_x u_{xx} - \beta u_x v_{xx} + v u_t, \\ C_2^y &= \beta v_x u_{xy} - \beta u_x v_{xy}. \end{aligned}$$

(iii) The Lie point symmetry $X_3 = \partial_y$ has the Lie characteristic functions $W^1 = -u_y$ and $W^2 = -v_y$, and using Ibragimov's theorem, the components of the conserved vector are:

$$\begin{aligned} C_3^t &= -v u_y, \\ C_3^x &= \beta v_x u_{xy} + \beta v_y(u_{xx} + u_{yy}) - \beta u_y v_{xx} - \alpha v u_n u_y, \\ C_3^y &= v(u_t + \alpha u^n u_x) - \beta v_x u_{xx} - \beta u_y v_{xy}. \end{aligned}$$

(iv) Finally, the Lie point symmetry $X_4 = 3nt\partial_t + nx\partial_x + ny\partial_y - 2u\partial_u$ gives $W^1 = -(2u + 3ntu_t + nxu_x + nyu_y)$ and $W^2 = (2-2n)v - 3ntv_t - nxv_x - nyv_y$, and so, the associated conserved vector has components:

$$\begin{aligned} C_4^t &= v(3\alpha ntu^n u_x - 2u - nxu_x - nyu_y) - 3\beta ntv_x(u_{xx} + u_{yy}), \\ C_4^x &= v(nxu_t - 2\alpha u^{n+1} - 3\alpha ntu^n u_t - \alpha nyu^n u_y) + \beta nyv_y(u_{xx} + u_{yy}) \\ &\quad +3\beta ntv_t(u_{xx} + u_{yy}) + \beta v_x(2u_x + 3ntu_{tx} + nu_x + nxu_{xx} + nyu_{xy}) \\ &\quad -\beta v_{xx}(2u + 3ntu_t + nxu_x + nyu_y) - 2\beta v(u_{xx} + u_{yy}) + 2\beta nv(u_{xx} + u_{yy}), \\ C_4^y &= \beta v_x(2u_y + 3ntu_{ty} + nxu_{xy} + nu_y - nyu_{xx}) - \beta v_{xy}(2u + 3ntu_t + nxu_x + nyu_y) \\ &\quad +nyv(u_t + \alpha u^n u_x). \end{aligned}$$

3. Exact Solutions

In this section, we obtain exact solutions of (2) using firstly its Lie point symmetries and, secondly, by employing the simplest equation method.

3.1. Exact Solutions of (2) Using Its Lie Point Symmetries

First of all, we utilize the linear combination of the three translation symmetries, namely $X = X_1 + \nu X_2 + X_3$, and reduce the gZK Equation (2) to a PDE in two independent variables. The associated Lagrange system is:

$$\frac{dt}{1} = \frac{dx}{\nu} = \frac{dy}{1} = \frac{du}{0}$$

which yields the following three invariants:

$$f = t - y, \quad g = x - \nu y, \quad \theta = u \tag{6}$$

By considering θ as the new dependent variable and f and g as new independent variables, the gZK Equation (2) transforms to:

$$\theta_f + \alpha\theta^n\theta_g + \beta(\nu^2+1)\theta_{ggg} + 2\beta\nu\theta_{fgg} + \beta\theta_{ffg} = 0 \tag{7}$$

which is a nonlinear PDE in two independent variables. Further symmetry reduction of (7) can be done by using its symmetries. Equation (7) has the two translational symmetries, *viz.*,

$$\Gamma_1 = \frac{\partial}{\partial f}, \quad \Gamma_2 = \frac{\partial}{\partial g}$$

The combination $\Gamma_1 + k\Gamma_2$, of the two symmetries Γ_1 and Γ_2, for an arbitrary constant k, yields the two invariants:

$$z = g - kf \quad \text{and} \quad W = \theta$$

which gives rise to a group invariant solution $W = W(z)$. Consequently, using these invariants, (7) is transformed into the third-order nonlinear ODE:

$$\beta(1+(\nu-k)^2)W''' + \alpha W^n W' - kW' = 0 \tag{8}$$

The integration of (8) yields

$$\beta(1+(\nu-k)^2)W'' + \frac{\alpha}{n+1}W^{n+1} - kW = 0 \tag{9}$$

where the constant of integration has been taken to be zero, because we are looking for soliton solutions. Equation (9) can be integrated easily by first multiplying it by W'. We then obtain the first-order variables separable equation:

$$\frac{\beta(1+(\nu-k)^2)}{2}W'^2 + \frac{\alpha}{(n+1)(n+2)}W^{n+2} - \frac{k}{2}W^2 = 0 \tag{10}$$

which can be integrated easily. After integrating and reverting back to the original variables, we obtain the following group-invariant solution of the gZK Equation (2) for arbitrary values of n in the form:

$$u(t,x,y) = \left(\frac{k(n+1)(n+2)}{2\alpha}\right)^{\frac{1}{n}} \operatorname{sech}^{\frac{2}{n}}(R) \tag{11}$$

where

$$R = \frac{n\sqrt{k}(C_1+z)}{2\sqrt{\beta(1+(\nu-k)^2)}}$$
$$z = x - kt - (\nu-k)y$$

Note that (11) represents a non-topological soliton solution. A sketch of the solution (11) with $n = 2$, $\alpha = 2$, $k = 5$, $\nu = 1$, $\beta = 1$, $t = 0$ and $C_1 = 1$ is given in Figure 1.

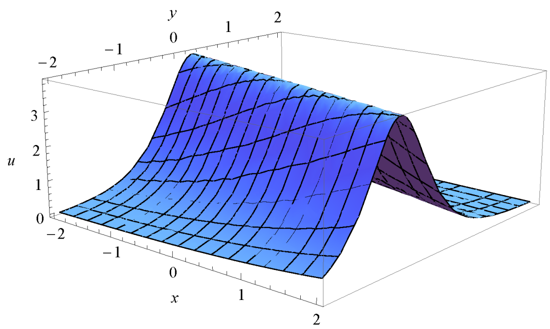

Figure 1. Profile of the solution of (11).

3.2. Exact Solutions of (2) Using the Simplest Equation Method

In this subsection, we use the simplest equation method [33–35] to solve the nonlinear third-order ODE (8) for $n = 1, 2$. The simplest equations that we use here are the Bernoulli equation:

$$H'(z) = aH(z) + bH^2(z) \tag{12}$$

and the Riccati equation:

$$G'(z) = aG^2(z) + bG(z) + c \tag{13}$$

where a, b and c are constants [36]. We look for solutions of the nonlinear ODE (8) that are of the form:

$$W(z) = \sum_{i=0}^{M} A_i (G(z))^i$$

where $G(z)$ satisfies the Bernoulli equation or Riccati equation, M is a positive integer that can be determined by a balancing procedure and $A_0, \cdots, A_M$ are parameters to be determined.

The solution of Bernoulli Equation (12) we use here is given by:

$$H(z) = a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}$$

where C is a constant of integration. For the Riccati Equation (13), the solutions to be used are:

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right] \tag{14}$$

and

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)} \tag{15}$$

with $\theta = \sqrt{b^2 - 4ac}$ and C is a constant of integration.

3.2.1. Solutions of (2) Using the Bernoulli Equation as the Simplest Equation

n* = *1

In this case, the balancing procedure yields $M = 2$ and solutions of (8) are of the form:

$$W(z) = A_0 + A_1 G + A_2 G^2$$

We insert this value of $W(z)$ in (8). Then, using the Bernoulli Equation (12) and, thereafter, equating the coefficients of powers of G^i to zero, we obtain an algebraic system of five equations in terms of A_0, A_1, A_2, namely:

$$\begin{aligned}
24\,b^3\beta\,k^2A_2 - 48\,b^3\beta\,k\nu\,A_2 + 24\,b^3\beta\,\nu^2A_2 + 24\,b^3\beta\,A_2 + 2\,\alpha\,bA_2{}^2 &= 0,\\
a^3\beta\,k^2A_1 - 2\,a^3\beta\,k\nu\,A_1 + a^3\beta\,\nu^2A_1 + a^3\beta\,A_1 + a\alpha\,A_0A_1 - akA_1 &= 0,\\
54\,ab^2\beta\,k^2A_2 - 108\,ab^2\beta\,k\nu\,A_2 + 54\,ab^2\beta\,\nu^2A_2 + 6\,b^3\beta\,k^2A_1 - 12\,b^3\beta\,k\nu\,A_1 &\\
+6\,b^3\beta\,\nu^2A_1 + 54\,ab^2\beta\,A_2 + 6\,b^3\beta\,A_1 + 2\,a\alpha\,A_2{}^2 + 3\,\alpha\,bA_1A_2 &= 0,\\
38\,a^2b\beta\,k^2A_2 - 76\,a^2b\beta\,k\nu\,A_2 + 38\,a^2b\beta\,\nu^2A_2 + 12\,ab^2\beta\,k^2A_1 - 24\,ab^2\beta\,k\nu\,A_1 &\\
+12\,ab^2\beta\,\nu^2A_1 + 38\,a^2b\beta\,A_2 + 12\,ab^2\beta\,A_1 + 3\,a\alpha\,A_1A_2 + 2\,\alpha\,bA_0A_2 &\\
+\alpha\,bA_1{}^2 - 2\,bkA_2 &= 0,\\
8\,a^3\beta\,k^2A_2 - 16\,a^3\beta\,k\nu\,A_2 + 8\,a^3\beta\,\nu^2A_2 + 7\,a^2b\beta\,k^2A_1 - 14\,a^2b\beta\,k\nu\,A_1 &\\
+7\,a^2b\beta\,\nu^2A_1 + 8\,a^3\beta\,A_2 + 7\,a^2b\beta\,A_1 + 2\,a\alpha\,A_0A_2 + a\alpha\,A_1{}^2 &\\
+\alpha\,bA_0A_1 - 2\,akA_2 - bkA_1 &= 0.
\end{aligned}$$

With the aid of Maple, we solve the above system and obtain:

$$\begin{aligned}
&A_0 = \frac{1}{\alpha}\{2a^2\beta k\nu - a^2\beta\nu^2 - a^2\beta - a^2\beta k^2 + k\},\ A_1 = \frac{1}{\alpha}\{12ab\beta\left(2k\nu - \nu^2 - k^2 - 1\right)\},\\
&A_2 = \frac{1}{\alpha}\{12b^2\beta\left(2k\nu - \nu^2 - k^2 - 1\right)\}.
\end{aligned}$$

Therefore, the solution of (2), for $n = 1$ is given by:

$$\begin{aligned}
u(t,x,y) =& A_0 + aA_1\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}\\
&+ A_2a^2\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2
\end{aligned} \tag{16}$$

where $z = x - kt - (\nu - k)y$ and C is an arbitrary constant of integration.

***n* = 2**

The balancing procedure yields $M = 1$, so the solutions of (8) take the form:

$$W(z) = A_0 + A_1G \tag{17}$$

As before, substituting (17) into (8), we obtain the algebraic system of equations:

$$\begin{aligned}
6\,b^3\beta\,k^2A_1 - 12\,b^3\beta\,k\nu\,A_1 + 6\,b^3\beta\,\nu^2A_1 + \alpha\,bA_1{}^3 + 6\,b^3\beta\,A_1 &= 0,\\
a^3\beta\,k^2A_1 - 2\,a^3\beta\,k\nu\,A_1 + a^3\beta\,\nu^2A_1 + a^3\beta\,A_1 + a\alpha\,A_0{}^2A_1 - akA_1 &= 0,\\
12\,ab^2\beta\,k^2A_1 - 24\,ab^2\beta\,k\nu\,A_1 + 12\,ab^2\beta\,\nu^2A_1 + a\alpha\,A_1{}^3 + 12\,ab^2\beta\,A_1 &\\
+2\,\alpha\,bA_0A_1{}^2 &= 0,\\
7\,a^2b\beta\,k^2A_1 - 14\,a^2b\beta\,k\nu\,A_1 + 7\,a^2b\beta\,\nu^2A_1 + 7\,a^2b\beta\,A_1 + 2\,a\alpha\,A_0A_1{}^2 &\\
+\alpha\,bA_0{}^2A_1 - bkA_1 &= 0,
\end{aligned}$$

whose solution is:

$$A_0 = \pm\sqrt{\frac{3k}{\alpha}},\ A_1 = \pm\frac{2b}{a}\sqrt{\frac{3k}{\alpha}}\ \beta = \frac{2k}{a^2\left(2k\nu - \nu^2 - k^2 - 1\right)}.$$

Therefore, the solutions of (2) for $n = 2$ are given by:

$$u(t,x,y) = A_0 + aA_1\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\} \tag{18}$$

where $z = x - kt - (\nu - k)y$ and C is an arbitrary constant of integration. A sketch of the solution (18) is given in Figure 2.

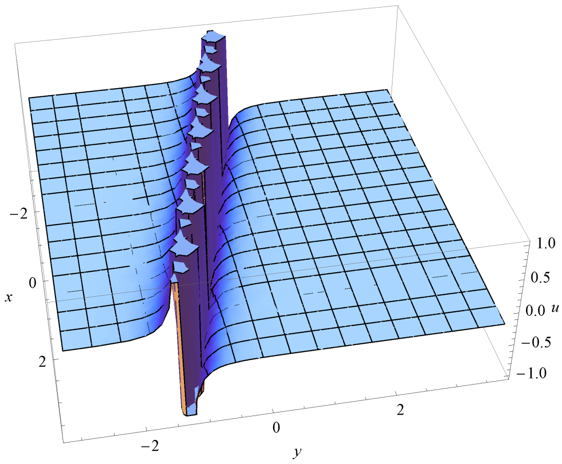

Figure 2. Profile of the solution of (18).

3.2.2. Solutions of (2) Using the Riccati Equation as the Simplest Equation

n* = *1

For this case, the balancing procedure gives $M = 2$, and so, (14) becomes:

$$W(z) = A_0 + A_1 G + A_2 G^2 \tag{19}$$

The insertion of this value of $W(z)$ into (8) and making use of the Riccati Equation (13) yields the following algebraic system of equations in terms of A_0, A_1, A_2:

$$\begin{aligned}
24\,a^3\beta\,k^2 A_2 - 48\,a^3\beta\,k\nu\,A_2 + 24\,a^3\beta\,\nu^2 A_2 + 24\,a^3\beta\,A_2 + 2\,a\alpha\,{A_2}^2 &= 0,\\
6\,a^3\beta\,k^2 A_1 - 12\,a^3\beta\,k\nu\,A_1 + 6\,a^3\beta\,\nu^2 A_1 + 54\,a^2 b\beta\,k^2 A_2 - 108\,a^2 b\beta\,k\nu\,A_2&\\
+54\,a^2 b\beta\,\nu^2 A_2 + 6\,a^3\beta\,A_1 + 54\,a^2 b\beta\,A_2 + 3\,a\alpha\,A_1 A_2 + 2\,\alpha\,b{A_2}^2 &= 0,\\
2\,a\beta\,c^2 k^2 A_1 - 4\,a\beta\,c^2 k\nu\,A_1 + 2\,a\beta\,c^2\nu^2 A_1 + b^2\beta\,ck^2 A_1 - 2\,b^2\beta\,ck\nu\,A_1&\\
+b^2\beta\,c\nu^2 A_1 + 6\,b\beta\,c^2 k^2 A_2 - 12\,b\beta\,c^2 k\nu\,A_2 + 6\,b\beta\,c^2\nu^2 A_2 + 2\,a\beta\,c^2 A_1&\\
+b^2\beta\,cA_1 + 6\,b\beta\,c^2 A_2 + \alpha\,cA_0 A_1 - ckA_1 &= 0,\\
12\,a^2 b\beta\,k^2 A_1 - 24\,a^2 b\beta\,k\nu\,A_1 + 12\,a^2 b\beta\,\nu^2 A_1 + 40\,a^2\beta\,ck^2 A_2 - 80\,a^2\beta\,ck\nu\,A_2&
\end{aligned}$$

$$
\begin{aligned}
+40\,a^2\beta\,c\nu^2 A_2 + 38\,ab^2\beta\,k^2 A_2 - 76\,ab^2\beta\,k\nu\,A_2 + 38\,ab^2\beta\,\nu^2 A_2 + 12\,a^2 b\beta\,A_1 & \\
+40\,a^2\beta\,cA_2 + 38\,ab^2\beta\,A_2 + 2\,a\alpha\,A_0A_2 + a\alpha\,{A_1}^2 + 3\,\alpha\,bA_1A_2 & \\
+2\,\alpha\,c{A_2}^2 - 2\,akA_2 &= 0, \\
8\,ab\beta\,ck^2 A_1 - 16\,ab\beta\,ck\nu\,A_1 + 8\,ab\beta\,c\nu^2 A_1 + 16\,a\beta\,c^2k^2 A_2 - 32\,a\beta\,c^2k\nu\,A_2 & \\
+16\,a\beta\,c^2\nu^2 A_2 + b^3\beta\,k^2 A_1 - 2\,b^3\beta\,k\nu\,A_1 + b^3\beta\,\nu^2 A_1 + 14\,b^2\beta\,ck^2 A_2 & \\
-28\,b^2\beta\,ck\nu\,A_2 + 14\,b^2\beta\,c\nu^2 A_2 + 8\,ab\beta\,cA_1 + 16\,a\beta\,c^2 A_2 + b^3\beta\,A_1 & \\
+14\,b^2\beta\,cA_2 + \alpha\,bA_0A_1 + 2\,\alpha\,cA_0A_2 + \alpha\,c{A_1}^2 - bkA_1 - 2\,ckA_2 &= 0, \\
8\,a^2\beta\,ck^2 A_1 - 16\,a^2\beta\,ck\nu\,A_1 + 8\,a^2\beta\,c\nu^2 A_1 + 7\,ab^2\beta\,k^2 A_1 - 14\,ab^2\beta\,k\nu\,A_1 & \\
+7\,ab^2\beta\,\nu^2 A_1 + 52\,ab\beta\,ck^2 A_2 - 104\,ab\beta\,ck\nu\,A_2 + 52\,ab\beta\,c\nu^2 A_2 + 8\,b^3\beta\,k^2 A_2 & \\
-16\,b^3\beta\,k\nu\,A_2 + 8\,b^3\beta\,\nu^2 A_2 + 8\,a^2\beta\,cA_1 + 7\,ab^2\beta\,A_1 + 52\,ab\beta\,cA_2 + 8\,b^3\beta\,A_2 & \\
+a\alpha\,A_0A_1 + 2\,\alpha\,bA_0A_2 + \alpha\,b{A_1}^2 + 3\,\alpha\,cA_1A_2 - akA_1 - 2\,bkA_2 &= 0.
\end{aligned}
$$

The solution of the above system using Maple gives:

$$
\begin{aligned}
A_0 &= \frac{1}{\alpha}\{16a\beta ck\nu - 8a\beta c\nu^2 - 8a\beta c - 8a\beta ck^2 - \beta b^2\nu^2 - \beta b^2 - \beta b^2k^2 + 2\beta b^2k\nu + k\}, \\
A_1 &= \frac{1}{\alpha}\{12ab\beta\left(2k\nu - \nu^2 - k^2 - 1\right)\},\;\; A_2 = \frac{1}{\alpha}\{12a^2\beta\left(2k\nu - \nu^2 - k^2 - 1\right)\}.
\end{aligned}
$$

Consequently, the solutions of (2) are:

$$
\begin{aligned}
u(t,x,y) =& A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta(z+C)\right)\right\} \\
&+ A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta(z+C)\right)\right\}^2
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
u(t,x,y) =& A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\} \\
&+ A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2
\end{aligned}
\tag{21}
$$

where $z = x - kt - (\nu - k)y$ and C is an arbitrary constant of integration.

***n* = 2**

The balancing procedure yields $M = 1$, so the solutions of (8) are of the form:

$$W(z) = A_0 + A_1 G$$

Substituting (22) into (8) and using the Riccati equation [36], we obtain the following algebraic system of equations:

$$
\begin{aligned}
-6bA_1c^3\nu + 3aA_1^3c = 0, \\
6aA_1^2A_0c + 3aA_1^3d - 12bA_1c^2d\nu = 0,
\end{aligned}
$$

$$3aA_1A_0^2e - 2bA_1ce^2\nu - A_1e\nu - bA_1d^2e\nu = 0,$$
$$-8bA_1cde\nu - A_1d\nu + 6aA_1^2A_0e - bA_1d^3\nu + 3aA_1A_0^2d = 0,$$
$$-A_1c\nu + 3aA_1^3e - 7bA_1cd^2\nu + 3aA_1A_0^2c + 6aA_1^2A_0d - 8bA_1c^2e\nu = 0.$$

Solving the above algebraic equations, one obtains:

$$A_0 = \pm\frac{b}{2\sqrt{\alpha}}\sqrt{6\beta(2k\nu - \nu^2 - k^2 - 1)},\ A_1 = \pm\frac{a}{\sqrt{\alpha}}\sqrt{6\beta(2k\nu - \nu^2 - k^2 - 1)},$$
$$a = \frac{b^2\beta\,k^2 - 2\,b^2\beta\,k\nu + b^2\beta\,\nu^2 + b^2\beta + 2\,k}{4\,\beta\,c\,(k^2 - 2\,\nu\,k + \nu^2 + 1)}$$

Hence, we have the following solutions of (2) for $n = 2$:

$$u(x,y,t) = A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta(z+C)\right)\right\} \tag{22}$$

and

$$u(t,x,y) = A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\} \tag{23}$$

where $z = x - kt - (\nu - k)y$ and C is an arbitrary constant of integration. A sketch of the solution (23) is given in Figure 3.

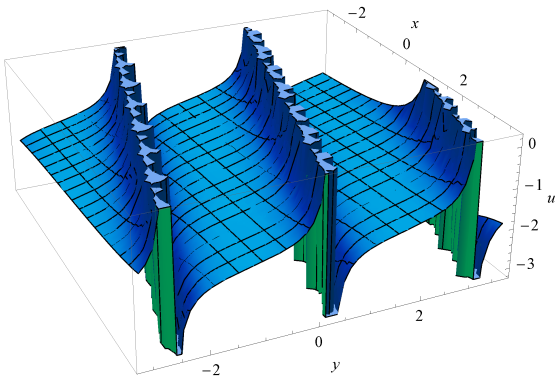

Figure 3. Profile of the solution of (23).

4. Concluding Remarks

In this paper, we studied the generalized Zakharov–Kuznetsov Equation (2). We derived the conservation laws of this equation by using the new conservation theorem by Ibragimov. Moreover, the Lie point symmetries of (2) were obtained and were used in conjunction with the simplest equation method to obtain exact solutions of the generalized Zakharov–Kuznetsov equation. The solutions obtained here are new and more general than the ones obtained before in [4] and [8]. Furthermore, the importance of the conservation laws has been emphasized in the Introduction.

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Author Contributions

D.M.M. and C.M.K. worked together in the derivation of the mathematical results. Both authors read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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