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*Article*

# From Conformal Invariance towards Dynamical Symmetries of the Collisionless Boltzmann Equation

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**Abstract:** Dynamical symmetries of the collisionless Boltzmann transport equation, or Vlasov equation, but under the influence of an external driving force, are derived from non-standard representations of the 2D conformal algebra. In the case without external forces, the symmetry of the conformally-invariant transport equation is first generalized by considering the particle momentum as an independent variable. This new conformal representation can be further extended to include an external force. The construction and possible physical applications are outlined.

**Keywords:** conformal invariance; conformal Galilean algebra; Boltzmann equation

## 1. Introduction

The Boltzmann transport equation (BTE) [1–4] furnishes a semi-classical description of the effects of particle transport, including the influence of external forces on the effective single-particle distribution function $f = f(t, \boldsymbol{r}, \boldsymbol{p})$ of a small cell in phase phase, centered at position $\boldsymbol{r}$ and momentum $\boldsymbol{p}$. For a system with identical particles of mass $m$, the Boltzmann equation reads:

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{F} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} . \tag{1}$$

Here, $\mathrm{d}N = f(t, \boldsymbol{r}, \boldsymbol{p}, )\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{p}$ is the number of particles in a cell of phase volume $\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{p}$, centered at position $\boldsymbol{r}$ and momentum $\boldsymbol{p}$ [3]. In addition, $\boldsymbol{F} = \boldsymbol{F}(t, \boldsymbol{r})$ is the force field acting on the particles in the fluid. The term on the right-hand side is added to describe the effect of collisions between particles. It is a statistical term and requires knowledge of the statistics that the particles obey, like the Maxwell–Boltzmann, Fermi–Dirac or Bose–Einstein distributions. In his famous "Stoßzahlansatz" (or hypothesis of molecular chaos), Boltzmann obtained an explicit form for it. In modern notation, for example for an interacting Fermi gas, where a particle from a state with momentum $\boldsymbol{p}$ is scattered to a state with momentum $\boldsymbol{p}'$, whereas a second particle is scattered from a momentum $\boldsymbol{q}$ to a momentum $\boldsymbol{q}'$, the collision term reads:

$$\begin{aligned} \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} &= -\int \mathrm{d}\boldsymbol{p}'\mathrm{d}\boldsymbol{q}\mathrm{d}\boldsymbol{q}'\; w(\{\boldsymbol{p}, \boldsymbol{q}\} \to \{\boldsymbol{p}', \boldsymbol{q}'\}) \\ &\quad \times [f(\boldsymbol{p})f(\boldsymbol{q})(1 - f(\boldsymbol{p}'))(1 - f(\boldsymbol{q}')) - f(\boldsymbol{p}')f(\boldsymbol{q}')(1 - f(\boldsymbol{p}))(1 - f(\boldsymbol{q}))] \end{aligned}$$

where $w(\{\boldsymbol{p}\boldsymbol{q}\} \to \{\boldsymbol{p}'\boldsymbol{q}'\})$ is the normalized transition probability from the two-particle state with momenta $\{\boldsymbol{p}, \boldsymbol{q}\}$ to the state labeled by $\{\boldsymbol{p}', \boldsymbol{q}'\}$. Clearly, solving this widely-studied equation is a very difficult task. It might be hoped that symmetries could be helpful. The equation without the collision term is known as the Vlasov equation [5]. The relationship with Landau damping and a physicists' derivation can be found in [6,7]. In this work, we shall explore a class of symmetries of the (collisionless) BTE.

Throughout, we shall restrict to the $d = 1$ space dimension. (By analogy with other constructions of local scale symmetries (see [8–11] and especially [12] and references therein), we expect a straightforward extension of the results reported here to $d > 1$. Since we shall construct here a finite-dimensional Lie algebra of dynamical conformal symmetries of the $1D$ collisionless BTE, one should indeed expect that an extension to $d > 1$ exists. That symmetry algebra should contain three generators $X_{\pm 1,0}$, along with a vector of generators $\boldsymbol{Y}_n$ and also spatial rotations.) We start from a non-standard representation, isomorphic to the infinite-dimensional Lie algebra of conformal transformations in $d = 2$ dimensions. (For the sake of clarity, we shall adopt the following convention of terminology: the infinite-dimensional Lie algebra $\langle X_n, Y_n\rangle_{n\in\mathbb{Z}}$ will be called a (centerless) "conformal Virasoro algebra". Its maximal finite-dimensional sub-algebra $\langle X_n, Y_n\rangle_{n\in\{-1,0,1\}}$ will be called a "conformal algebra") This Lie algebra is spanned by the generators $\langle X_n, Y_n\rangle_{n\in\mathbb{Z}}$ and can be defined from the commutators [9,12]:

$$[X_n, X_m] = (n - m)X_{n+m}, \quad [X_n, Y_m] = (n - m)Y_{n+m}, \quad [Y_n, Y_m] = \mu(n - m)Y_{n+m} \tag{2}$$

where $\mu$ is a parameter. An explicit realization in terms of time-space transformation is [9,12]:

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - \mu^{-1}[(t + \mu r)^{n+1} - t^{n+1}]\partial_r - (n + 1)xt^n - (n + 1)\frac{\gamma}{\mu}[(t + \mu r)^n - t^n] \\ Y_n &= -(t + \mu r)^{n+1}\partial_r - (n + 1)\gamma(t + \mu r)^n \end{aligned} \tag{3}$$

such that $\mu^{-1}$ can be interpreted as a velocity ("speed of light/sound") and where $x, \gamma$ are constants. (The contraction $\mu \to 0$ of Equation (3) produces the non-semi-simple "altern-Virasoro algebra" $\mathfrak{altv}(1)$ (but without central charges). Its maximal finite-dimensional sub-algebra is the conformal Galilean algebra $\mathfrak{alt}(1) \equiv \textsc{cga}(1)$ [9,13]; see also [8,11]. The $\textsc{cga}$(d) is non-isomorphic to either the standard Galilei algebra or else the Schrödinger algebra.) Writing $X_n = \ell_n + \bar{\ell}_n$ and $Y_n = \mu^{-1}\bar{\ell}_n$, where the generators $\langle \ell_n, \bar{\ell}_n \rangle_{n\in\mathbb{Z}}$ satisfy $[\ell_n, \ell_m] = (n-m)\ell_{n+m}, [\bar{\ell}_n, \bar{\ell}_m] = (n-m)\bar{\ell}_{n+m}, [\ell_n, \bar{\ell}_m] = 0$, it can be seen that, provided $\mu \neq 0$, the above Lie algebra Equation (2) is isomorphic to a pair of Virasoro algebras $\mathfrak{vect}(S^1) \oplus \mathfrak{vect}(S^1)$ with a vanishing central charge. However, this isomorphism does not imply that physical systems described by two different representations of the conformal Virasoro algebra, or the conformal algebra, with commutators Equation (2), were trivially related. For example, it is well known that if one uses the generators of the standard representation of conformal invariance or else the non-standard representation Equation (4) in order to find co-variant two-point functions, the resulting scaling forms are different [9].

Now, consider the maximal finite-dimensional sub-algebra $\langle X_{\pm 1,0}, Y_{\pm 1,0} \rangle$, which for $\mu \neq 0$, in turn, is isomorphic to the direct sum $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. The explicit realization follows from from Equation (3):

$$\begin{aligned} X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - r\partial_r - x, \quad X_1 = -t^2\partial_t - 2tr\partial_r - \mu r^2\partial_r - 2xt - 2\gamma r \\ Y_{-1} &= -\partial_r, \quad Y_0 = -t\partial_r - \mu r\partial_r - \gamma, \quad Y_1 = -t^2\partial_r - 2\mu tr\partial_r - \mu^2 r^2\partial_r - 2\gamma t - 2\gamma\mu r \end{aligned} \tag{4}$$

Here, the generators $X_{-1}, Y_{-1}$ describe time- and space-translations, $Y_0$ is a (conformal) Galilei transformation (since the commutator $[Y_0, Y_{-1}]$ does not vanish and does not give a central element of the Lie algebra Equation (2), its structure is fundamentally different from algebras containing the usual Galilei algebra as a sub-algebra), $X_0$ gives the dynamical scaling $t \mapsto \lambda t$ of $r \mapsto \lambda r$ (with $\lambda \in \mathbb{R}$), such that the so-called "dynamical exponent" $z = 1$, since both time and space are re-scaled in the same way, and, finally, $X_{+1}, Y_{+1}$ give "special" conformal transformations. In the context of statistical mechanics of conformally-invariant phase transitions, one characterizes co-variant quasi-primary scaling operators through the invariant parameters $(x, \mu, \gamma)$, where $x$ is the scaling dimension.

Finally, the finite-dimensional representation Equation (4) acts as a dynamical symmetry on the equation of motion:

$$\hat{S}\phi(t,r) = (-\mu\partial_t + \partial_r)\phi(t,r) = 0. \tag{5}$$

in the sense that a solution $\phi$ of $\hat{S}\phi = 0$ is mapped onto another solution of the same equation. Indeed, it is easily checked that: $[\hat{S}, Y_{\pm 1,0}] = [\hat{S}, X_{-1}] = 0$ and

$$[\hat{S}, X_0] = -\hat{S}, \quad [\hat{S}, X_1] = -2t\hat{S} + 2(\mu x - \gamma) \tag{6}$$

It follows that for fields $\phi$ with scaling dimensions $x_\phi = x = \gamma/\mu$, the algebra Equation (4) really leaves the solution space of Equation (5) invariant.

In order to return to the Boltzmann equation, we consider Equation (5) in the form:

$$\hat{L}f = (\mu\partial_t + v\partial_r)f(t,r,v) = 0 \tag{7}$$

where $f = f(t,r,v)$ is interpreted as a single-particle distribution function and where we consider $v$ as an additional variable. Equation (7) is a simple Boltzmann equation, without an external force, without a collision term and in one space dimension. From Equation (6), with $v$ fixed (and normalized to $v = 1$), its solution space is conformally invariant. (With respect to Equation (5), $\mu \mapsto -\mu$ was replaced. This change must also be made in the generators Equation (4) and commutators Equation (2)). In Section 2, we shall generalize the above representation of the conformal algebra to the situation with $v$ as a further variable. In Section 3, we shall further extend this to the case when an external force $F = F(t,r,v)$, possibly depending on time, spatial position and velocity, is included. The aim of these calculations is to determine which situations of potential physical interest with a non-trivial conformal symmetry might be identified. This explorative study aims at identifying lines for further study, which might lead later to a more comprehensive understanding of the possible symmetries of Boltzmann equations. Taking into account the collision term is left for future work. We shall concentrate on the $d = 1$ space dimension throughout. Conclusions and final comments are given in Section 4.

## 2. Collisionless Boltzmann Equation without External Forces

In our construction of conformal dynamical symmetries of the 1D collisionless BTE, we shall often meet Lie algebras of a certain structure. These will be isomorphic to the two-dimensional conformal algebra.

**Proposition 1.** *The Lie algebra $\langle X_n, Y_n\rangle_{n\in\mathbb{Z}}$ defined by the commutators:*

$$[X_n, X_m] = (n-m)X_{n+m}, \quad [X_n, Y_m] = (n-m)Y_{n+m}, \quad [Y_n, Y_m] = (n-m)\left(kX_{n+m} + qY_{n+m}\right) \tag{8}$$

*where $k, q$ are constants, is isomorphic to the pair of centerless Viraso algebras $\mathfrak{vect}(S^1) \oplus \mathfrak{vect}(S^1)$.*

**Proof.** For either $k = 0$ or $q = 0$, this is either evident or else has already been seen in Section 1. In the other case, consider the change of basis $X_n = \ell_n + \bar{\ell}_n$ and $Y_n = \alpha\ell_n - \beta\bar{\ell}_n$, where $\ell_n, \bar{\ell}_n$ are two families of commuting generators of $\mathfrak{vect}(S^1)$ and $\alpha$ and $\beta$ are constants, such that $\alpha + \beta \neq 0$. It then follows $k = \alpha\beta$ and $q = \alpha - \beta$. □

This implies in particular the isomorphism of the maximal finite-dimensional sub-algebras, or "conformal algebras" in the terminology chosen here. By definition, this "conformal algebra" obeys the commutators Equation (8), but with $n, m \in \{-1, 0, 1\}$.

Our construction of dynamical symmetries of the Equation (7) follows the lines of the construction of local scale-invariance in time-dependent critical phenomena [9]. The physically-motivated requirements are: First of all it, is clear that the equation is invariant under time-translations:

$$X_{-1} = -\partial_t, \quad [\hat{L}, X_{-1}] = 0 \tag{9}$$

Some kind of dynamical scaling must be present, as well. Its most general form is:

$$X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - x, \quad [\hat{L}, X_0] = -\hat{L}. \tag{10}$$

Whenever the dynamical exponent $z \neq 1$, we shall find an explicit dependence on $v$. In general, we look for a family of generators $X_n$, for which we make the ansatz:

$$X_n = -a_n(t, r, v)\partial_t - b_n(t, r, v)\partial_r - c_n(t, r, v)\partial_v - d_n(t, r, v). \tag{11}$$

We shall find $X_n$ from the following three conditions (throughout, we use the notations $\partial_t f = \dot{f}, \partial_r f = f'$):

1. $X_n$ must be a symmetry for the Equation (7); hence, $[\hat{L}, X_n] = \lambda_n \hat{L}$. This gives:

$$\mu \dot{a}_n + v a_n' + \mu \lambda_n = 0, \quad \mu \dot{b}_n + v b_n' - c_n + \lambda_n v = 0 \qquad (12)$$
$$\mu \dot{c}_n + v c_n' = 0, \quad \mu \dot{d}_n + v d_n' = 0.$$

2. The generator $X_0$ is assumed to be in the Cartan sub-algebra; hence, $[X_n, X_0] = \alpha_{n,0} X_n$. It follows:

$$(1+\alpha_{n,0})a_n - t\dot{a}_1 - \frac{r}{z}a_n' - \frac{1-z}{z}v\partial_v a_n = 0 \qquad (13)$$
$$(1/z+\alpha_{n,0})b_n - t\dot{b}_n - \frac{r}{z}b_n' - \frac{1-z}{z}v\partial_v b_n = 0 \qquad (14)$$
$$((1-z)/z+\alpha_{n,0})c_n - t\dot{c}_1 - \frac{r}{z}c_n' - \frac{1-z}{z}v\partial_v c_n = 0 \qquad (15)$$
$$\alpha_{n,0}d_n - t\dot{d}_n - \frac{r}{z}d_n' - \frac{1-z}{z}v\partial_v d_n = 0. \qquad (16)$$

3. The action of $X_{-1}$ is as a lowering operator; hence, $[X_n, X_{-1}] = \alpha_{n,-1} X_{n-1}$. It follows:

$$\dot{a}_n = \alpha_{n,-1} t, \quad \dot{b}_n = \alpha_{n,-1} r/z \qquad (17)$$
$$\dot{c}_n = \alpha_{n,-1} v(1-z)/z, \quad \dot{d}_n = \alpha_{n,-1} x/z.$$

These conditions, combined with the following initial conditions:

$$a_0 = t, \quad b_0 = \frac{r}{z}, \quad c_0 = \frac{1-z}{z}v, \quad d_0 = x$$
$$a_{-1} = 1, \quad b_{-1} = 0, \quad c_{-1} = 0, \quad d_{-1} = 0. \qquad (18)$$

must be sufficient for the determination of all admissible forms of $X_n$.

In the special case $n = 1$, we have $\alpha_{1,0} = 1$ and find the most general form of $X_1$ as a symmetry of Equation (7) as follows (the requirement that $\langle X_{\pm 1,0} \rangle$ close into the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ fixes $\alpha_{1,-1} = 2$):

$$X_1 = -a_1(t,r,v)\partial_t - b_1(t,r,v)\partial_r - c_1(t,r,v)\partial_v - d_1(t,r,v) \qquad (19)$$

and

$$a_1(t,r,v) = t^2 + A_{12} r^2 v^{-2} + A_{110} r v^{\frac{2z-1}{1-z}} + A_{100} v^{\frac{2z}{1-z}} \qquad (20)$$
$$b_1(t,r,v) = \frac{2}{z}tr + \left(\frac{A_{12}}{\mu} + \frac{z-2}{z}\mu\right) r^2 v^{-1} + B_{110} r v^{\frac{z}{1-z}} + B_{100} v^{\frac{z+1}{1-z}} \qquad (21)$$
$$c_1(t,r,v) = \frac{2}{z}(1-z)(vt - \mu r) + (B_{110} - \frac{A_{110}}{\mu}) v^{\frac{z}{1-z}} \qquad (22)$$
$$d_1(t,r,v) = \frac{2}{z}xt - \frac{2}{z}\mu x r v^{-1} + D_0 v^{\frac{z}{1-z}} \qquad (23)$$

with a certain set of undetermined constants.

For conformal invariance, a family of generators $Y_n$ must also be found. Its construction is straightforward if the explicit form of $Y_{-1}$ is known. Really, $X_1$ must act as a raising operator, in both hierarchies, such that [9]:

$$[X_1, Y_{-1}] \sim Y_0, \quad [X_1, Y_0] \sim Y_1. \qquad (24)$$

which implies that $[Y_{-1}, [Y_{-1}, X_1]] \sim Y_{-1}$. However, the usual realization of $Y_{-1} = -\partial_r$ as space translations does not work, since if we set all undetermined constants in Equation (19) to zero, one would have $[Y_{-1}, [Y_{-1}, X_1]] \sim v^{-1} Y_{-1}$. It is better to work with the form:

$$Y_{-1} = -v\partial_r. \tag{25}$$

as we shall do from now on.

We first consider the special case, when all of the constants in the expression Equation (19) for $X_1$ vanish:

Case A: $A_{12} = A_{110} = A_{100} = B_{110} = B_{100} = D_0 = 0$.

**Proposition 2.** *The six generators:*

$$\begin{aligned}
X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - \frac{x}{z} \\
X_1 &= -t^2\partial_t - \left(\frac{2}{z}tr + \frac{z-2}{z}\mu r^2 v^{-1}\right)\partial_r - \frac{2(1-z)}{z}(vt - \mu r)\partial_v - \frac{2}{z}xt + \frac{2}{z}\mu x r v^{-1} \\
Y_{-1} &= -v\partial_r, \quad Y_0 = -(tv - \frac{\mu}{z}r)\partial_r - \frac{z-1}{z}\mu v\partial_v + \mu\frac{x}{z} \\
Y_1 &= -\left(t^2 v - \frac{2}{z}\mu t r - \frac{z-2}{z}\mu^2 r^2 v^{-1}\right)\partial_r - \frac{2}{z}(z-1)\mu(vt - \mu r)\partial_v \\
&\quad + \frac{2}{z}\mu x t - \frac{2}{z}\mu^2 x r v^{-1}
\end{aligned} \tag{26}$$

*span a representation of the conformal algebra Equation (2), which acts as dynamical symmetry algebra of the Equation (7), for arbitrary dynamical exponent $z$.*

**Proof.** It is readily checked that the generator Equation (26) satisfies the commutation relations (2), with $\mu \mapsto -\mu$. On the other hand, for any $f = f(t, r, v)$, one has:

$$\begin{aligned}
&[\hat{L}, X_{-1}] = [\hat{L}, Y_{-1}] = [\hat{L}, Y_0] = [\hat{L}, Y_1] = 0 \\
&[\hat{L}, X_0] = -\hat{L}, \quad [\hat{L}, X_1] = -2t\hat{L},
\end{aligned}$$

which establishes the asserted dynamical symmetry. □

Next, we treat the general case, when all of the constants are non-zero:

Case B: $A_{12} \neq 0, A_{110} \neq 0, A_{100} \neq 0, B_{110} \neq 0, B_{100} \neq 0, D_0 \neq 0$.

Then, the generators are modified as follows:

$$\begin{aligned}
\bar{X}_1 &= X_1 + \widetilde{X}_1 \\
\widetilde{X}_1 &= -\left(A_{12} r^2 v^{-2} + A_{110} r v^{\frac{2z-1}{1-z}} + A_{100} v^{\frac{2z}{1-z}}\right)\partial_t \\
&\quad - \left(\frac{A_{12}}{\mu} r^2 v^{-1} + B_{110} r v^{\frac{z}{1-z}} + B_{100} v^{\frac{z+1}{1-z}}\right)\partial_r \\
&\quad - (B_{110} - \frac{A_{110}}{\mu}) v^{\frac{z}{1-z}}\partial_v - D_0 v^{\frac{z}{1-z}},
\end{aligned} \tag{27}$$

$$\begin{aligned}
\bar{Y}_0 &= Y_0 + \widetilde{Y}_0 \\
\widetilde{Y}_0 &= \frac{1}{2}[\widetilde{X}_1, Y_{-1}] \\
&= -(A_{12} r v^{-1} + \frac{1}{2} A_{110} v^{-1+1/(1-z)})\partial_t - \frac{1}{2\mu}(2A_{12} r + A_{110} v^{1/(1-z)})\partial_r. \qquad (28)
\end{aligned}$$

Now, computing:

$$[\bar{Y}_0, Y_{-1}] = -\mu Y_{-1} + A_{12} X_{-1} + \frac{A_{12}}{\mu} Y_{-1} \qquad (29)$$

we conclude that the cases $A_{12} = 0$ and $A_{12} \neq 0$ must be treated separately.

Case B1: $A_{12} = 0$. It follows that the constants in Equation (19) are given by:

$$B_{110} = A_{110}/\mu, \quad A_{100} = \frac{A_{110}^2}{4\mu^2}, \quad B_{100} = \frac{A_{100}}{\mu} = \frac{A_{110}^2}{4\mu^3}, \quad D_0 = 0.$$

**Proposition 3.** *Let $z \neq 1$ and $A_{110}$ be arbitrary constants. Then, the six generators:*

$$\begin{aligned}
\bar{X}_{-1} &= -\partial_t, \quad \bar{X}_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z} v\partial_v - \frac{x}{z} \\
\bar{X}_1 &= -(t^2 + A_{110} r v^{(2z-1)/(1-z)} + \frac{A_{110}^2}{4\mu^2} v^{2z/(1-z)})\partial_t \\
&\quad - \left(\frac{2}{z} tr + \frac{z-2}{z}\mu r^2 v^{-1} + \frac{A_{110}}{\mu} r v^{z/(1-z)} + \frac{A_{110}^2}{4\mu^3} v^{(z+1)/(1-z)}\right)\partial_r \\
&\quad - \frac{2(1-z)}{z}(vt - \mu r)\partial_v - \frac{2}{z} xt + \frac{2}{z}\mu x r v^{-1} \\
\bar{Y}_{-1} &= -v\partial_r \\
\bar{Y}_0 &= -\frac{A_{110}}{2} v^{z/(1-z)}\partial_t - (tv - \frac{\mu}{z} r + \frac{A_{110}}{2\mu} v^{1/(1-z)})\partial_r - \frac{z-1}{z}\mu v\partial_v + \mu\frac{x}{z} \\
\bar{Y}_1 &= -A_{110}(t v^{z/(1-z)} - \mu r v^{(2z-1)/(1-z)})\partial_t \\
&\quad - \left(t^2 v - \frac{2}{z}\mu t r - \frac{z-2}{z}\mu^2 r^2 v^{-1} + \frac{A_{110}}{\mu}(t v^{1/(1-z)} - \mu r v^{z/(1-z)})\right)\partial_r \\
&\quad - \frac{2}{z}(z-1)\mu(vt - \mu r)\partial_v + \frac{2}{z}\mu x t - \frac{2}{z}\mu^2 x r v^{-1} \qquad (30)
\end{aligned}$$

*span a representation of the conformal algebra (the above result of Case A is recovered upon setting $A_{110} = 0$). These generators give more symmetries of Equation (7).*

**Proof.** From the above, the commutator Equation (2) is readily verified, with $\mu \mapsto -\mu$. For the dynamical symmetries, one checks the commutators:

$$\begin{aligned}
[\hat{L}, X_{-1}] &= [\hat{L}, Y_{\pm 1,0}] = 0 \\
[\hat{L}, X_0] &= -\hat{L}, \quad [\hat{L}, X_1] = -(2t + \frac{A_{110}}{\mu} v^{z/(1-z)})\hat{L}.
\end{aligned}$$

which proves the assertion. □

In contrast to the previous Case A, the representation acting only on $(t, r)$, but keeping $v$ as a constant parameter, can no longer be obtained by simply setting $z = 1$. Rather, one must set $A_{110} = 0$ first, and only then, the limit $z \to 1$ is well-defined.

Case B2: $A_{12} \neq 0, A_{110} \neq 0, B_{110} \neq 0, A_{100} \neq 0, B_{100} \neq 0, D_0 \neq 0$.

It turns out that for $A_{12} \neq 0$, the algebra also can be closed, but only if $A_{12} = \mu$ and $A_{110} = 0$ (then, all other constants also vanish).

**Proposition 4.** *Let $z$ be an arbitrary constant. Then, the generators $\langle \mathcal{X}_{\pm 1,0}, \mathcal{Y}_{\pm 1,0} \rangle$, where:*

$$\begin{aligned}
\mathcal{X}_{-1} &= -\partial_t, \quad \mathcal{X}_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - \frac{x}{z} \\
\mathcal{X}_{-1} &= X_{-1}, \quad \mathcal{X}_0 = X_0 \\
\mathcal{X}_1 &= -\left(t^2 + \mu r^2 v^{-2}\right)\partial_t - \left(\frac{2}{z}tr + \frac{z+\mu(z-2)}{z}r^2v^{-1}\right)\partial_r \\
&\quad -\frac{2(1-z)}{z}(vt-\mu r)\partial_v - \frac{2}{z}xt + \frac{2}{z}\mu xrv^{-1} \\
\mathcal{Y}_{-1} &= -v\partial_r \\
\mathcal{Y}_0 &= -\mu rv^{-1}\partial_t - \left(tv - (\frac{\mu}{z}-1)r\right)\partial_r - \frac{z-1}{z}\mu v\partial_v + \mu\frac{x}{z} \\
\mathcal{Y}_1 &= -\mu\left(2trv^{-1} + (1-\mu)r^2v^{-2}\right)\partial_t - \left(t^2v - \frac{2}{z}(z-\mu)tr + \frac{z(1-\mu)-(z-2)\mu^2}{z}r^2v^{-1}\right)\partial_r \\
&\quad -\frac{2}{z}(z-1)\mu(vt-\mu r)\partial_v + \frac{2}{z}\mu xt - \frac{2}{z}\mu^2 xrv^{-1}
\end{aligned} \tag{31}$$

*close into a Lie algebra, with the following non-zero commutation relations:*

$$\begin{aligned}
&[\mathcal{X}_n, \mathcal{X}_{n'}] = (n-n')\mathcal{X}_{n+n'}, \quad [\mathcal{X}_n, \mathcal{Y}_m] = (n-m)\mathcal{Y}_{n+m} \\
&[\mathcal{Y}_m, \mathcal{Y}_{m'}] = (m-m')\left(\mu\mathcal{X}_{m+m'} + (1-\mu)\mathcal{Y}_{m+m'}\right),
\end{aligned} \tag{32}$$

*with $n, n', m, m' \in \{-1, 0, 1\}$. The algebra is isomorphic to the usual conformal algebra Equation (2) and further extends the dynamical symmetries of Equation (7).*

**Proof.** The commutation relation is directly verified. The isomorphism with the conformal algebra follows from Proposition 1. The requirement to have a symmetry algebra of Equation (7) implies a relation between the constants $k, q$ (called $\alpha, \beta$ in Proposition 1) and $\mu$, namely $q = (k-\mu^2)/\mu$. In this case at hand, we have $k = \mu$, $q = 1-\mu$. It is then verified that $[\hat{L}, \mathcal{X}_{-1}] = [\hat{L}, \mathcal{Y}_{-1}] = 0$ and:

$$\begin{aligned}
&[\hat{L}, \mathcal{X}_0] = -\hat{L} \\
&[\hat{L}, \mathcal{X}_1] = -2(t + \frac{r}{z}v^{-1})\hat{L} \\
&[\hat{L}, \mathcal{Y}_0] = -(k/\mu)\hat{L} = -\hat{L} \\
&[\hat{L}, \mathcal{Y}_1] = -2\left(\frac{k}{\mu}t + \frac{k}{z\mu^2}rv^{-1}\right)\hat{L} = -2\left(t + \frac{1}{z\mu}rv^{-1}\right)\hat{L}.
\end{aligned}$$

which proves that these are dynamical symmetries of (7). □

We now ask whether the finite-dimensional representation Equations (26), (30) and (31), with $\mu \neq 0$, acting on functions $f = f(t, r, v)$, and having a dynamical exponent $z \neq 1$, can be extended to representations of an infinite-dimensional conformal Virasoro algebra. The answer turns out to be negative:

**Proposition 5.** *The representation Equations (26), (30) and (31) of the finite-dimensional conformal algebra $\langle X_n, Y_n\rangle_{n\in\{\pm1,0\}}$ with commutator Equation (8) cannot be extended to representations of an infinite-dimensional conformal Virasoro algebra with commutator Equation (8) when $z \neq 1$.*

Similar no-go results have been found before for variants of representations of the Schrödinger and conformal Galilean algebras [14]. On the other hand, for $\mu = 0$, extensions to a representation of a conformal Virasoro algebra with $z \neq 1$ exist [15].

**Proof.** Since for the finite-dimensional representations Equations (26), (30) and (31), we have:

$$[X_n, X_{n'}] = (n-n')X_{n+n'}, \quad [X_n, Y_m] = (n-m)Y_{n+n'}, \quad n, n', m = 0, \pm 1$$

we suppose that this must be valid for all admissible $n, m \in \mathbb{Z}$. Now, using the condition Equation (17) for $n = 2$, a conformal Virasoro algebra should contain a new generator $X_2$. Starting from the most general form, $X_2 = -a_2(t,r,v)\partial_t - b_2(t,r,v)\partial_r - c_2(t,r,v)\partial_v - d_2(t,r,v)$ we find that the coefficients are obtained from:

$$\begin{aligned} a_2 &= t^3 + a_{21}(r,v), \quad b_2 = \frac{3}{z}t^2 r + 3\frac{z-2}{z}\mu t r^2 v^{-1} + b_{21}(r,v) \\ c_2 &= 3\frac{1-z}{z}(vt^2/2 - \mu r t) + c_{21}(r,v), \quad d_2 = \frac{3}{z}xt^2 - \frac{6}{z}\mu x + d_{21}(r,v), \end{aligned}$$

where $a_{21}(r,v), b_{21}(r,v), c_{21}(r,v), d_{21}(r,v)$ are unknown functions of their arguments, but do no longer depend on the time $t$. We want to satisfy $[X_2, Y_{-1}] = 3Y_1$. However, when calculating:

$$\begin{aligned} [X_2, Y_{-1}] &= [-a_2\partial_t - b_2\partial_r - c_2\partial_v - d_2, -v\partial_r] = \\ &= 3Y_1 - va'_{21}\partial_t - (3\frac{1-z}{2z}t^2 v - \frac{3}{z}(1-z)\mu t r + vb'_{21} - c_{21} + 3\frac{z-2}{z}\mu^2 r^2 v^{-1})\partial_r \\ &\quad -(vc'_{21} + 3\frac{1-z}{z}\mu(tv - 2\mu r))\partial_v - vd'_{21} - \frac{6}{z}\mu\gamma r v^{-1} \end{aligned}$$

we see that closure is not possible for $z \neq 1$. Indeed, although the dependence on $r, v$ of the functions $a_{21}, b_{21}, c_{21}, d_{21}$ can be chosen to satisfy the above closure condition, the $t$-dependence cannot be absorbed into these functions. Hence, our new representation Equations (26), (30) and (31) of the conformal algebra Equation (8) are necessarily finite-dimensional. □

## 3. Symmetry Algebra of Collisionless Boltzmann Equation with an Extra Force Term

We write the collisionless Boltzmann equation in the form:

$$\hat{B}f(t,r,v) = (\mu\partial_t + v\partial_r + F(t,r,v)\partial_v)\, f(t,r,v) = 0. \tag{33}$$

We want to determine the admissible forms of an external force $F(t,r,v)$, such that Equation (33) is invariant under a representation of the conformal algebra Equation (8). The unknown representation must include the "force" term and, in particular, for $F(t,r,v) = 0$, it should coincide with the representations of conformal algebra obtained in the previous section.

The idea of the construction is similar to the one used in Section 2. First, we impose invariance under basic symmetries:

- From invariance under time translation $X_{-1} = -\partial_t$, it follows:

$$[X_{-1}, \hat{B}] = -\dot{F} = 0 \to F = F(r, v) \tag{34}$$

- From invariance under dynamical scaling $X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - \frac{x}{z}$, we obtain that:

$$[\hat{B}, X_0] = -\hat{B}, \tag{35}$$

  if $F(r, v)$ satisfies the equation $(r\partial_r + (1-z)v\partial_v - (1-2z))F(r, v) = 0$, with solution:

$$F(r, v) = r^{1-2z}\varphi(r^{z-1}v), \tag{36}$$

  where $\varphi(u)$ is an arbitrary function of the scaling variable $u := r^{z-1}v$.

It turns out that for the following calculations, it is more convenient to make a change of independent variables $(t, r, v) \mapsto (t, r, u)$. In the new variables, the generator of dynamical scaling just reads:

$$X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{x}{z}. \tag{37}$$

Next, in order to be specific, we make the following ansatz for the analogue of space translations. (Indeed, we might also require to find $Y_{-1}$ from the conditions to be (i) a symmetry of Boltzmann equation and (ii) to form a closed Lie algebra with the other basic symmetries $X_{-1,0}$. Such requirements lead to a system of differential equations, and the ansatz Equation (38) is a particular solution of this system, which has the special property that the Boltzmann operator can be linearly expressed $\hat{B} = -\mu X_{-1} - Y_{-1}$ by the generators. We believe this to be a natural auxiliary hypothesis):

$$Y_{-1} = -r^{1-z}u\partial_r - r^{-z}\Phi(u)\partial_u, \quad \Phi(u) = (z-1)u^2 + \varphi(u). \tag{38}$$

In the same coordinate system, the collisionless Boltzmann equation becomes:

$$\hat{B}f(t, r, u) = \left(\mu\partial_t + r^{1-z}u\partial_r + r^{-z}\Phi(u)\partial_u\right) f(t, r, u) = 0. \tag{39}$$

Here, some comments are in order. In the structure of Boltzmann Equation (39), as well as in the form Equation (38) of the modified space translations, $Y_{-1}$ enters an unknown function $\Phi(t, r, u)$. Therefore, the form of $X_1$ cannot be found only from its commutator with the other generators $X_n$, but the constraints form the entire conformal algebra must be used, as well as the requirement that $X_1$ and $Y_{0,1}$ are dynamical symmetries of Equation (39):

$$[\hat{B}, X_1] = \lambda_{X_1}(t, r, v)\hat{B}, \quad [\hat{B}, Y_0] = \lambda_{Y_0}(t, r, v)\hat{B}, \quad [\hat{B}, Y_1] = \lambda_{Y_1}(t, r, v)\hat{B}. \tag{40}$$

In fact, commuting the unknown generators $X_1, Y_0, Y_1$ with $X_{-1}$ and $X_0$, we can fix the $t$- and $r$-dependence of the yet undetermined functions that occur in them:

$$\begin{aligned}
Y_0 &= -r^z a_0(u)\partial_t - (r^{1-z}u + rb_0(u))\partial_r - (r^{-z}\Phi(u)t + c_0(u))\partial_u - d_0(u) \\
X_1 &= -(t^2 + r^{2z}a_{12}(u))\partial_t - ((2/z)tr + r^{z+1}b_{12}(u))\partial_r \\
&\quad -r^z c_{12}(u)\partial_u - (2/z)xt - r^z d_{12}(u) \\
Y_1 &= -\left(2tr^z a_0(u) + r^{2z}A(u)\right)\partial_t - \left(t^2 r^{1-z}u + 2trb_0(u) + r^{z+1}B(u)\right)\partial_r \\
&\quad -\left(t^2 r^{-z}\Phi(u) + 2tc_0(u) + r^z C(u)\right)\partial_u + (2/z)\mu xt - r^z D(u),
\end{aligned} \tag{41}$$

with the four functions:

$$\begin{aligned} A(u) &= 2zb_0a_{12} + c_0a'_{12} - za_0b_{12} - a'_0c_{12}, \quad C(u) = zb_0c_{12} + c_0c'_{12} - c'_0c_{12} - a_{12}\Phi \\ B(u) &= \frac{2}{z}a_0 + zb_0b_{12} + c_0b'_{12} - ua'_{12} - b'_0c_{12}, \quad D(u) = \frac{2}{z}xa_0 + zb_0d_{12} + c_0d'_{12}. \end{aligned} \tag{42}$$

In particular, looking for a representation of the analog of the extended Galilei algebra $\langle X_{-1}, X_0, Y_{-1}, Y_0\rangle$, we find that the unknown functions $a_0(u), b_0(u), c_0(u), d_0(u)$ must satisfy the system:

$$zua_0(u) + \Phi(u)a'_0(u) - k = 0 \tag{43}$$

$$zub_0(u) + \Phi(u)b'_0(u) - c_0(u) - qu = 0 \tag{44}$$

$$\Phi'(u)c_0 - \Phi(u)c'_0(u) + (q - zb_0)\Phi = 0 \tag{45}$$

$$\Phi(u)d'_0(u) = 0 \tag{46}$$

Because of Equation (46), one must distinguish two cases:

1. $\Phi(u) = 0$, when $d_0(u)$ can be arbitrary
2. $\Phi(u) \neq 0$, when $d_0(u) = d_0 =$ cste. is a constant.

In the second case, taking Equations (44) and ( 45) together, we obtain an equation for $b_0(u)$. It is:

$$\Phi^2(u)b''_0(u) + zu\Phi(u)b'_0(u) + (2z\Phi(u) - zu\Phi'(u))b_0(u) - 2s\Phi(u) = 0, \tag{47}$$

and has in general two independent solution: $b_{01}(u), b_{02}(u)$. It follows that, for a given arbitrary value of $\Phi(u) \neq 0$, we have in general two distinct realizations of the analogue of Galilei transformation; and consequently, also two realizations of the analogue of the Galilei algebra. By construction, these are Lie algebras of symmetries of the collisionless Boltzmann Equation (39) (with $\lambda_{Y_0} = -k/\mu = -(\mu + q)$):

$$\begin{aligned} &[Y_0, X_{-1}] = Y_{-1}, \quad [X_0, X_{-1}] = X_{-1}, \\ &[Y_0, Y_{-1}] = \frac{k - \mu^2}{\mu}Y_{-1} + kX_{-1}. \end{aligned} \tag{48}$$

Next, we include the generators of special conformal transformation $X_1$ and $Y_1$ to the extended Galilei algebras Equation (48) just constructed. We must also satisfy the other commutators of the conformal algebra Equation (8). Furthermore, the generators of the representation we are going to construct are dynamical symmetries of the collisionless Boltzmann equation (we use the commutators $[Y_1, Y_0] = KX_1 + QY_1$ and $[Y_1, Y_{-1}] = k_0X_0 + q_0Y_0$ in order to establish a relation between the constants $k, q$ and $K, Q, k_0, q_0$). We find:

$$\lambda_{X_1}(t, r, u) = -2t - (r^z/\mu)(2zua_{12} + \Phi(u)a'_{12}(u)) = -2t - 2r^z a_0(u)/\mu \tag{49}$$

for the eigenvalue and:

$$c_{12}(u) = (2/z)\mu - (u/\mu)(2za_{12}(u) + \Phi(u)a'_{12}(u)) + (2zub_{12} + \Phi(u)b'_{12}(u)) \tag{50}$$

$$zuc_{12}(u) + \Phi c'_{12}(u) - c_{12}(u)\Phi'(u) + zb_{12}(u)\Phi(u) - 2c_0(u) = 0 \tag{51}$$

$$zud_{12}(u) + \Phi(u)d'_{12}(u) + (2/z)\mu x = 0 \tag{52}$$

$$\begin{aligned}&\Phi^2(u)b''_{12}(u) + 3zu\Phi(u)b'_{12}(u) + z[2zu^2 + 3\Phi(u) - 2u\Phi'(u)]b_{12}(u)\\ &\quad - (u/\mu)\Phi^2(u)a''_{12}(u) - [3zu^2 + 2\Phi(u)](\Phi/\mu)a'_{12}(u)\\ &\quad - [zu^2 + 3\Phi(u) - u\Phi'(u)](2zu/\mu)a_{12}(u) + (2\mu/z)(zu - \Phi'(u)) = 0\end{aligned} \tag{53}$$

$$2zua_{12}(u) + \Phi(u)a'_{12}(u) - 2a_0(u) = 0 \tag{54}$$

$$2zub_{12}(u) + \Phi(u)b'_{12}(u) - c_{12}(u) - 2b_0(u) = 0 \tag{55}$$

$$b_0(u) = (u/\mu)a_0(u) - \mu/z \tag{56}$$

$$c_0(u) = (\Phi/\mu)a_0(u) \tag{57}$$

$$d_0(u) = \text{cste.} = -\mu x/z. \tag{58}$$

$$k_0 = \alpha_0 k = 2k, \quad q_0 = \alpha_0 q = 2q$$

$$2zuA(u) + \Phi(u)A'(u) - 2qa_0(u) = 0 \tag{59}$$

$$2zuB(u) + \Phi(u)B'(u) - C(u) - 2(k/z + qb_0(u)) = 0 \tag{60}$$

$$zuC(u) + \Phi(u)C'(u) - \Phi'(u)C(u) + z\Phi(u)B(u) - 2qc_0(u) = 0 \tag{61}$$

$$zuD(u) + \Phi(u)D'(u) - (2x/z)(k - \mu q) = 0. \tag{62}$$

$$K = k, \quad Q = q \tag{63}$$

$$(q - 2zb_0)A(u) - c_0A'(u) + za_0(u)B(u) + a'_0(u)C(u) + ka_{12}(u) - 2a_0^2 = 0 \tag{64}$$

$$(q - zb_0)B(u) - c_0B'(u) + uA(u) + b'_0C(u) + kb_{12}(u) - 2a_0(u)b_0(u) = 0 \tag{65}$$

$$(q - zb_0 + c'_0(u))C(u) - c_0C'(u) + \Phi(u)A(u) + kc_{12}(u) - 2a_0(u)c_0(u) = 0 \tag{66}$$

$$(q - zb_0)D(u) - c_0D'(u) + kd_{12}(u) + \frac{2a_0(u)\mu x}{z} = 0 \tag{67}$$

$$2z(b_{12}(u)A(u) - a_{12}(u)B(u)) + c_{12}(u)A'(u) - a'_{12}(u)C(u) + 2a_0(u)a_{12}(u) = 0 \tag{68}$$

$$(2/z)A(u) - c_{12}(u)B'(u) + b'_{12}C(u) - 2b_0(u)a_{12}(u) = 0 \tag{69}$$

$$(zb_{12}(u) - c'_{12}(u))C(u) + c_{12}C'(u) - zc_{12}(u)B(u) + 2c_0(u)a_{12}(u) = 0 \tag{70}$$

$$\begin{aligned}&(2x/z)(\mu a_{12}(u) + A(u)) + zd_{12}(u)B(u) + d'_{12}(u)C(u)\\ &\quad - zb_{12}D(u) - c_{12}(u)D'(u) = 0\end{aligned} \tag{71}$$

The system of Equations (43)–(45) and (50)–(71) must give a solution for the unknown functions $a_0(u), b_0(u), c_0(u), d_0(u), a_{12}(u), b_{12}(u), c_{12}(u), d_{12}(u)$. Of course, it is possible that several of the above equations are equivalent. Because of this fact, although the above system might look to be over-determined, we have not yet been able to produce an explicit solution without making an auxiliary assumption. A classification of all solutions of the above system is left as an open problem. We shall now describe some examples of solutions of this large system.

Example 1: Let $\Phi(u) = 0$. This case seems to be quite simple, provided it is compatible with our system. From Equation (43), we obtain:

$$a_0(u) = \frac{k}{z}u^{-1} \tag{72}$$

Using this value of $a_0(u)$ from Equations (52)–(54) and (56)–(58), we directly obtain:

$$\begin{aligned} b_0 &= \text{cste.} = \frac{k}{z\mu} - \frac{\mu}{z}, \quad c_0(u) = 0, \quad d_0 = \text{cste.} = -\frac{\mu}{z}x, \qquad (73) \\ a_{12}(u) &= \frac{k}{z^2}u^{-2}, \quad b_{12}(u) = \frac{1}{\mu z^2}(k-\mu^2)u^{-1}, \quad c_{12}(u) = 0, \quad d_{12}(u) = -\frac{2\mu x}{z^2}u^{-1}. \qquad (74) \end{aligned}$$

When we substitute the above results in relation Equation (42), we also find:

$$\begin{aligned} A(u) &= \frac{k}{\mu z^2}(k-\mu^2)u^{-2}, \quad B(u) = \frac{1}{\mu^2 z^2}\left(k(k-\mu^2)+\mu^4\right)u^{-1} \\ C(u) &= 0, \quad D(u) = \frac{2\mu^2 x}{z^2}u^{-1}. \end{aligned} \tag{75}$$

One can now verify that the above results for the functions $a_0(u), b_0(u), c_0(u), d_0(u)$ and $a_{12}(u), b_{12}(u), c_{12}(u), d_{12}(u), A(u), B(u), C(u), D(u)$ satisfy all equations of the above system. Now, we can finally write the algebra generators:

$$\begin{aligned} X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{x}{z} \\ X_1 &= -\left(t^2 + \frac{k}{z^2}r^{2z}u^{-2}\right)\partial_t - \left(\frac{2}{z}tr + \frac{k-\mu^2}{z^2\mu}r^{z+1}u^{-1}\right)\partial_r - \frac{2}{z}xt + \frac{2\mu x}{z^2}r^z u^{-1}, \\ Y_{-1} &= -r^{1-z}u\partial_r, \\ Y_0 &= -\frac{k}{z}r^z u^{-1}\partial_t - \left(tr^{1-z}u + \frac{k-\mu^2}{z\mu}r\right)\partial_r + \frac{\mu x}{z}, \\ Y_1 &= -\left(\frac{2k}{z}tr^z u^{-1} + \frac{k(k-\mu^2)}{z^2\mu}r^{2z}u^{-2}\right)\partial_t \\ &\quad - \left(t^2 r^{1-z}u + 2\frac{k-\mu^2}{z\mu}tr + \frac{k(k-\mu^2)+\mu^4}{z^2\mu^2}r^{z+1}u^{-1}\right)\partial_r + \frac{2}{z}\mu xt - \frac{2\mu^2 x}{z^2}r^z u^{-1}. \end{aligned} \tag{76}$$

We return to the original variables via the change $(t, r, u) \mapsto (t, r, v)$, done through the substitutions $u \to r^{z-1}v$ and $\partial_r \to \partial_r + (1-z)r^{-1}v\partial_v$. Finally, we have the following representation of a conformal symmetry algebra of the collisionless Boltzmann Equation (33):

$$\begin{aligned} X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - \frac{x}{z} \\ X_1 &= -\left(t^2 + \frac{k}{z^2}r^2 v^{-2}\right)\partial_t - \left(\frac{2}{z}tr + \frac{k-\mu^2}{z^2\mu}r^2 v^{-1}\right)\partial_r \\ &- (1-z)\left(\frac{2}{z}tv + \frac{k-\mu^2}{z^2\mu}r\right)\partial_v - \frac{2}{z}xt + \frac{2\mu x}{z^2}rv^{-1}, \\ Y_{-1} &= -v\partial_r - (1-z)r^{-1}v^2\partial_v, \\ Y_0 &= -\frac{k}{z}rv^{-1}\partial_t - \left(tv + \frac{k-\mu^2}{z\mu}r\right)\partial_r - (1-z)\left(tr^{-1}v^2 + \frac{k-\mu^2}{z\mu}v\right)\partial_v + \frac{\mu x}{z}, \\ Y_1 &= -\left(\frac{2k}{z}trv^{-1} + \frac{k(k-\mu^2)}{z^2\mu}r^2 v^{-2}\right)\partial_t - \left(t^2 v + 2\frac{k-\mu^2}{z\mu}tr + \frac{k(k-\mu^2)+\mu^4}{z^2\mu^2}r^2 v^{-1}\right)\partial_r \\ &\quad -(1-z)\left(t^2 r^{-1}v^2 + 2\frac{k-\mu^2}{z\mu}tv + \frac{k(k-\mu^2)+\mu^4}{z^2\mu^2}r\right)\partial_v + \frac{2}{z}\mu xt - \frac{2\mu^2 x}{z^2}rv^{-1}. \end{aligned} \tag{77}$$

**Proposition 6.** *The generator Equation (77) close into the following Lie algebra:*

$$
\begin{aligned}
&[X_n, X_{n'}] = (n - n')X_{n+n'}, \quad [X_n, Y_m] = (n - m)Y_{n+m} \\
&[Y_m, Y_{m'}] = (m - m')\left(kX_{m+m'} + \frac{k - \mu^2}{\mu}Y_{m+m'}\right),
\end{aligned} \tag{78}
$$

*for* $n, n', m, m' \in \{-1, 0, 1\}$ *and for an arbitrary dynamical exponent* $z$. *They give a representation of the finite-dimensional conformal algebra, which acts as a dynamical symmetry algebra of the Boltzmann equation in the form:*

$$
\hat{B}f(t, r, v) = (\mu\partial_t + v\partial_r + (1 - z)r^{-1}v^2\partial_v)f(t, r, v) = 0. \tag{79}
$$

**Proof.** The commutation relation Equation (78) is directly checked. From the commutators $[\hat{B}, X_{-1}] = [\hat{B}, Y_{-1}] = 0$ and:

$$
\begin{aligned}
&[\hat{B}, X_0] = -\hat{B}, \quad [\hat{B}, X_1] = -2\left(t + \frac{k}{z\mu}rv^{-1}\right)\hat{B} \\
&[\hat{B}, Y_0] = -(k/\mu)\hat{B}, \quad [\hat{B}, Y_1] = -2\left(\frac{k}{\mu}t + \frac{k}{z\mu^2}rv^{-1}\right)\hat{B}.
\end{aligned}
$$

it is seen that they generate dynamical symmetries. □

<u>Example 2: Let $k = 0$</u>. In this case, $\Phi(u)$ left arbitrary, which leads to $a_0 = 0$ from Equation (43) and:

$$
b_0 = \text{cste.} = -\mu/z, \quad c_0 = 0, \quad d_0 = -\mu x/z \tag{80}
$$

Then, from Equation (42), we obtain:

$$
\begin{aligned}
A(u) &= -2\mu a_{12}(u), \quad B(u) = -\mu b_{12}(u) - ua'_{12}(u) \\
C(u) &= -\mu c_{12}(u) - a_{12}(u)\Phi(u), \quad D(u) = -\mu d_{12}.
\end{aligned} \tag{81}
$$

However, when substituting in Equations (64)–(67), taking also into account that $q = -\mu$, we find that $A(u) = a_{12}(u) = 0$. Then, it is easy to check that the condition Equations (68)–(71) are fulfilled. This allows us to formulate:

**Proposition 7.** *Let* $\Phi(u) = (z - 1)u^2 + \varphi(u)$. *Consider the generators:*

$$
\begin{aligned}
X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1 - z}{z}v\partial_v - \frac{x}{z} \\
X_1 &= -t^2\partial_t - \left(\frac{2}{z}tr + r^{z+1}b_{12}(u)\right)\partial_r \\
&- (1 - z)\left(\frac{2}{z}tv + r^z vb_{12}(u) + \frac{r^{1-2z}}{1 - z}c_{12}(u)\right)\partial_v - \frac{2}{z}xt - r^z d_{12}(u), \\
Y_{-1} &= -v\partial_r - (1 - z)\left(r^{-1}v^2 + \frac{r^{1-2z}}{1 - z}\Phi(u)\right)\partial_v = -v\partial_r - r^{1-2z}\varphi(u)\partial_v, \\
Y_0 &= -\left(tv - \frac{\mu}{z}r\right)\partial_r - (1 - z)\left(\frac{r^{1-2z}}{1 - z}\varphi(u)t - \frac{\mu}{z}v\right)\partial_v + \frac{\mu x}{z}, \\
Y_1 &= -\left(t^2 v - 2\frac{\mu}{z}tr - \mu r^{z+1}b_{12}(u)\right)\partial_r + \frac{2}{z}\mu xt + \mu r^z d_{12}(u) \\
&-(1 - z)\left(t^2\frac{r^{1-2z}}{1 - z}\varphi(u) - \frac{2}{z}\mu tv - n\mu r^z vb_{12}(u) - \mu\frac{r^{1-2z}}{1 - z}c_{12}(u)\right)\partial_v,
\end{aligned} \tag{82}
$$

*where* $c_{12}(u) = 2zub_{12}(u) + ((z-1)u^2 + \varphi(u))b'_{12}(u) + 2\mu/z$ *and* $\varphi(u), b_{12}(u), d_{12}(u)$ *satisfy:*

$$\begin{aligned} [(z-1)u^2 + \varphi(u)]^2 b''_{12}(u) + 3zu[(z-1)u^2 + \varphi(u)]b'_{12}(u) & \\ +z[(z+1)u^2 - 2u\varphi'(u) + 3\varphi(u)]b_{12}(u) + [(2-z)u - \varphi'(u)]2\mu/z &= 0 \quad (83) \\ zud_{12}(u) + [(z-1)u^2 + \varphi(u)]d'_{12}(u) + 2\mu x/z &= 0. \quad (84) \end{aligned}$$

*For any triplet* $(\varphi(u), b_{12}(u), d_{12}(u))$*, which gives a solution of the system Equations (83) and (84), the generator Equation (82) close into the following Lie algebra:*

$$\begin{aligned} &[X_n, X_{n'}] = (n-n')X_{n+n'}, \quad [X_n, Y_m] = (n-m)Y_{n+m} \\ &[Y_m, Y_{m'}] = -\mu\,(m-m')Y_{m+m'}, \end{aligned} \quad (85)$$

*for* $n, n', m, m' \in \{-1, 0, 1\}$ *and for an arbitrary constant* $z$*. Equation (82) is a representation of the finite-dimensional conformal algebra and acts as the dynamical symmetry algebra of the Vlasov–Boltzmann equation, with a quite general "force" term:*

$$\hat{B}f(t,r,v) = (\mu\partial_t + v\partial_r + r^{1-2z}\varphi(u)\partial_v)f(t,r,v) = 0.$$

**Proof.** The commutators are satisfied for $k = 0$ and $q = -\mu$ if condition Equations (83) and (84) are fulfilled. Under the same conditions, the symmetries are proven by the relations:

$$\begin{aligned} &[\hat{B}, X_{-1}] = [\hat{B}, Y_{-1}] = [\hat{B}, Y_0] = [\hat{B}, Y_1] = 0 \\ &[\hat{B}, X_0] = -\hat{B}, \quad [\hat{B}, X_1] = -2t\hat{B}. \end{aligned}$$

□

In particular, if we implement the physical requirement that the "force" term should depend only on the positions $r$, that is $\varphi(u) = \varphi_0 = $ cste., we can compute explicitly the representation of the algebra Equation (82). To do this, one must find a solution of the system:

$$\begin{aligned} [(z-1)u^2 + \varphi_0]^2 b''_{12}(u) + 3zu[(z-1)u^2 + \varphi_0]b'_{12}(u) & \\ +z[(z+1)u^2 + 3\varphi_0]b_{12}(u) + 2\mu\frac{2-z}{z}u &= 0 \quad (86) \\ zud_{12}(u) + [(z-1)u^2 + \varphi_0]d'_{12}(u) + 2\mu x/z &= 0. \quad (87) \end{aligned}$$

The solution of the second equation is relatively simple, even for an arbitrary $z$:

$$d_{12}(u) = -\delta_0[(z-1)u^2 + \varphi_0]^{\frac{z}{2(1-z)}} \int_{\mathbb{R}} \mathrm{d}u\, [(z-1)u^2 + \varphi_0]^{\frac{z-2}{2(1-z)}}, \quad \delta_0 = \text{cste.} \quad (88)$$

The solution of the Equation (86) for an arbitrary $z$ can be expressed in terms of hypergeometric functions, but we shall not give its explicit form here. However, for $z = 2$, the system Equations (86) and ( 87) have an elementary solution:

$$\begin{aligned} b_{12}(u) &= b_{120}\frac{u}{(u^2+\varphi_0)^2} + b_{121}\frac{u^2-\varphi_0}{(u^2+\varphi_0)^2}, \quad b_{120} = \text{cste.}, b_{121} = \text{cste.} \\ d_{12}(u) &= -\mu x\frac{u}{u^2+\varphi_0}. \end{aligned} \quad (89)$$

Substituting this into the generator Equation (82) for $z = 2$ gives a finite-dimensional representation of the dynamical conformal symmetry of a collisionless Boltzmann equation of the form:

$$\hat{B}f(t,r,v) = (\mu\partial_t + v\partial_r + \varphi_0 r^{-3}\partial_v)f(t,r,v) = 0. \tag{90}$$

## 4. Conclusions

In this work, we have described the results of the first exploration of dynamical symmetries of collisionless Vlasov–Boltzmann transport equations. Our main finding is that these equations admit conformal dynamical symmetries, although it does not seem to be possible to extend this to infinite-dimensional conformal Virasoro symmetries, not even in the case of $d = 1$ space dimensions. These conformal symmetries are new representations of the conformal algebra and are inequivalent to the standard representation, which is habitually used in conformal field-theory descriptions of equilibrium critical phenomena. Our first class of new symmetries was found by admitting the momentum $p$ (or equivalently, the velocity $v = p/\mu$) as an additional independent variable, leading to the representations Equations (26), (30) and (31). The second class of symmetries also allowed for external driving forces $F(t,r,v)$, and it has been one of the questions of which types of forces should be compatible with conformal invariance. As an example, we have seen that time-independent forces $F(r,v) = r^{1-2z}\varphi(r^{z-1}v)$, with an arbitrary scaling function $\varphi$, are admissible and lead to the general representation Equation (82). However, the solutions of the associated system of equations for the coefficients have not yet been classified and the complete content of these representations remains to be worked out in the future.

Some intuition can be gleaned from some examples. We have written down the explicit representations for the force $F(r,v) = (1-z)r^{-1}v^2$, with an $z > 1$ arbitrary Equation (77) and for $F(r,v) = \varphi_0\, r^{1-2z}$ Equations (82), (86), (87) with an arbitrary $z > 1$. In the later case, which could be related to physical situations, we have given the explicit representation of the conformal algebra for $z = 2$, when $F(r,v) = \varphi_0 r^{-3}$, Equations (82) and (89). Having identified these symmetries, the next step would be to use these to find either exact solutions [16] or else to use the algebra representations for fixing the form of co-variant $n$-point correlation functions, in analogy to time-dependent critical phenomena; see, e.g., [12].

The results derived here can be used as a starting point to derive forms of the transition rates $w$ in the collision terms, which would be compatible with the dynamical symmetries of the collision-free equations. This kind of approach would be analogous to the one used for finding dynamical symmetries of non-linear Schrödinger equations; see, e.g., [17,18]. We hope to return to this elsewhere.

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## Author Contributions

This work was performed in close scientific collaboration of S.S. and M.H., during the scientific visits mentioned above.

## Conflicts of Interest

The authors declare no conflict of interest.

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