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Article

An Elementary Derivation of the Matrix Elements of Real Irreducible Representations of $\mathfrak{so}(3)$

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Abstract: Using elementary techniques, an algorithmic procedure to construct skew-symmetric matrices realizing the real irreducible representations of $\mathfrak{so}(3)$ is developed. We further give a simple criterion that enables one to deduce the decomposition of an arbitrary real representation R of $\mathfrak{so}(3)$ into real irreducible components from the characteristic polynomial of an arbitrary representation matrix.

Keywords: real representation; matrix element; tensor product

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1. Introduction

Albeit the fact that the representation theory of semisimple Lie algebras in general, and the orthogonal algebras $\mathfrak{so}(n)$ and their various reals forms in particular, is well known and constitutes nowadays a standard tool in (physical) applications (see, e.g., [1,2] and the references therein), specific results in the literature concerning the explicit matrix construction of the matrices corresponding to real irreducible representations of $\mathfrak{so}\,(n)$ are rather scarce. Even if the structural properties of such representations can be derived from the complex case [3], the inherent technical difficulties arising in the analysis of irreducible representations over the real field make it cumbersome to determine an algorithmic procedure that provides the specific real representation matrices explicitly.

Even for the lowest dimensional case, that of $\mathfrak{so}\,(3)$, the description of real irreducible representations is generally restricted to multiplets of low dimension appearing in specific problems [4]. One interesting work devoted exclusively to the real irreducible representations from the perspective of harmonic analysis is given in [5]. Most of the applications of $\mathfrak{so}(3)$ make use of the angular momentum operators or the Gel'fand–Zetlin formalism, hence describing the states by means of eigenvalues of a complete set of diagonalizable commuting operators. However, for real irreducible representations of $\mathfrak{so}(3)$, corresponding to rotations in the representation space, no such bases of states of this type are possible, as no inner labeling diagonalizable operator over the real numbers can exist, the external being the Casimir operator [6]. In spite of this fact, real representations are of considerable practical importance, as they provide information on the embedding of $\mathfrak{so}(3)$ into other simple algebras and, thus, constitute interesting tools to determine the stability of semidirect sums of Lie algebras [7]. The hierarchy of real irreducible representations of simple Lie algebras is therefore deeply connected to the embedding problem and the branching rules. In this context, it is desirable to develop a simple algorithmic method for the construction of real irreducible representations R of $\mathfrak{so}(3)$ in terms of skew-symmetric matrices, as these correspond naturally to the embedding of $\mathfrak{so}(3)$ as a subalgebra of $\mathfrak{so}(\dim R)$.

In this work, we propose such a procedure, based on the elementary properties of rotation matrices. It is shown that the class of a real irreducible representation R is completely determined by the characteristic polynomial of a matrix in R. This further enables one to deduce the decomposition of an arbitrary real representation of $\mathfrak{so}(3)$ into real irreducible factors from the properties of the characteristic polynomial of a matrix within the representation.

1.1. Real Representations of $\mathfrak{so}(3)$

Recall that for $\mathfrak{sl}(2,\mathbb{C})$, the standard basis is given by $\{h,e,f\}$ with commutators:

$$[h,e]=2e,\ [h,f]=-2f,\ [e,f]=h. \tag{1}$$

Let D_J denote the irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ of dimension $(J+1)$, where $J=0,\frac{1}{2},1,\frac{3}{2},\cdots$. For the basis $\{\mathbf{e}_1,\cdots,\mathbf{e}_{2J+1}\}$ of the representation space, the matrices D_J for the generators h,e,f are easily recovered from the matrix elements:

$$\begin{gathered}\left\langle \mathbf{e}^i\right| D_J(h)\left|\mathbf{e}_j\right\rangle=\delta_i^j(2J+1-2i)\,;\quad \left\langle \mathbf{e}^i\right| D_J(e)\left|\mathbf{e}_j\right\rangle=\delta_{i+1}^j(2J+1-i)\\ \left\langle \mathbf{e}^i\right| D_J(f)\left|\mathbf{e}_j\right\rangle=\delta_i^{j+1}(i-1)\,.\end{gathered} \tag{2}$$

As is well known, the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ admits two real forms, the normal real form $\mathfrak{sl}(2,\mathbb{R})$ obtained by restriction of scalars, as well as the compact real form $\mathfrak{so}(3)$ obtained from the Cartan map:

$$X_1=\frac{\mathrm{i}}{2}h,\ X_2=\frac{1}{2}(e-f)\,,\ X_3=\frac{\mathrm{i}}{2}(e+f) \tag{3}$$

and satisfying the brackets:

$$[X_i,X_j]=\varepsilon_{ijk}X_k,\ 1\leq i,j,k\leq 3. \tag{4}$$

While the matrices of the representation D_J define a real representation of $\mathfrak{sl}(2,\mathbb{R})$ for the compact real form $\mathfrak{so}(3)$, the matrices of D_J are complex, given by:

$$D_J(X_1)=\frac{\mathrm{i}}{2}D_J(h)\,,\ D_J(X_2)=\frac{1}{2}(D_J(e)-D_J(f))\,,\ D_J(X_3)=\frac{\mathrm{i}}{2}(D_J(e)+D_J(f))\,. \tag{5}$$

In many applications, the representation space of D_J is best described by states of the type:

$$|\,\mu,\ J(J+1)\rangle\,,\ \mu=-J,\cdots,J \tag{6}$$

on an appropriate basis, as, e.g., that commonly used in the theory of angular momentum [8]. It must be observed, however, that such bases are not suitable for real representations, as geometric rotation matrices are not diagonalizable over the real field $\mathbb{R}$.

The problem of classifying the real irreducible representations of the compact real forms of semisimple Lie algebras was systematically considered by Cartan and Karpelevich, being later expanded for arbitrary real Lie algebras by Iwahori [9]. According to these works, real representations are distinguished by the decomposition of their complexification. More precisely, if Γ is a real representation of the (real) Lie algebra $\mathfrak{g}$, then:

1. Γ is called of first class, denoted by Γ^{I}, if $\Gamma \otimes_{\mathbb{R}} \mathbb{C}$ is a complex irreducible representation of $\mathfrak{g}$.
2. Γ is called of second class, denoted by Γ^{II}, if $\Gamma \otimes_{\mathbb{R}} \mathbb{C}$ is a complex reducible representation of $\mathfrak{g}$.

Following this distinction, the representations D_J of $\mathfrak{so}(3)$ with $J \in \mathbb{N}$ belong to the first class. This in particular implies the existence of an invertible matrix $U \in GL(2J+1, \mathbb{C})$, such that for $1 \leq k \leq 3$:

$$R_J^I(X_k) = U\, D_J(X_k)\, U^{-1} \tag{7}$$

is a real matrix [9]. For half-integer values $J \in \frac{1}{2}\mathbb{N}$, no such transition matrices U can exist, and in order to obtain a real representation, the dimension of the representation space must be doubled:

$$D_J(X_k) \mapsto D_J^{II}(X_k) = \begin{pmatrix} \mathrm{Re}D_J(a_k) & -\mathrm{Im}D_J(a_k) \\ \mathrm{Im}D_J(a_k) & \mathrm{Re}D_J(a_k) \end{pmatrix}. \tag{8}$$

As a consequence, even dimensional irreducible real representations of $\mathfrak{so}(3)$ only exist for $n = 4q$ with $q \geq 1$ (details on the double-covering $SU(2) \to SO(3)$ can be found, e.g., in [10]).

Albeit not usually referred to in the literature, the class of a real representation of a (simple) Lie algebra is deeply connected to the embedding problem of (complex) semisimple Lie algebras [11]. In particular, it determines whether an algebra is irreducibly embedded into another. Recall that an embedding $j : \mathfrak{s}' \to \mathfrak{s}$ of semisimple Lie algebras is called irreducible if the lowest dimensional irreducible representation Γ of $\mathfrak{s}$ remains irreducible when restricted to $\mathfrak{s}'$ [11]. Irreducible embeddings play an important role in applications, as they allow one to construct bases of a Lie algebra $\mathfrak{s}$ in terms of a basis of irreducibly-embedded subalgebras and irreducible tensor operators [12].

From the analysis of $\mathfrak{so}(3)$ representations, it is straightforward to establish the following embeddings:

1. For $J = 2$, $\mathfrak{so}(3)$ is a maximal subalgebra irreducibly embedded into $\mathfrak{sp}(4) \simeq \mathfrak{so}(5)$.
2. For $J = 3$, $\mathfrak{so}(3)$ is irreducibly embedded into $\mathfrak{so}(7)$ through the chain:

$$\mathfrak{so}(3) \subset G_{2,-14} \subset \mathfrak{so}(7).$$

3. For any integer $J \geq 4$, $\mathfrak{so}(3)$ is a maximal subalgebra irreducibly embedded into $\mathfrak{so}(2J+1)$.
4. For $J = \frac{3}{2}$, $\mathfrak{so}(3)$ is embedded into $\mathfrak{so}(4)$ through the chain:

$$\mathfrak{so}(3) \subset \mathfrak{sp}(4) \subset \mathfrak{su}(4) \subset \mathfrak{so}(7) \subset \mathfrak{so}(8)$$

5. For half-integers $J \geq \frac{5}{2}$, $\mathfrak{so}(3)$ is embedded into $\mathfrak{so}(4J+2)$ through the chain:

$$\mathfrak{so}(3) \subset \mathfrak{sp}(2J+1) \subset \mathfrak{su}(2J+1) \subset \mathfrak{so}(4J+2)$$

In this context, a natural construction of real irreducible representations of $\mathfrak{so}(3)$ should be by means of skew-symmetric matrices that realize these embeddings.

2. Construction of the Matrices $R_J^I(X_k)$

As already observed, for integer J, the representation R_J^I given by (7) is of first class. Therefore, $\mathfrak{so}(3)$ can be represented as a subalgebra of the compact Lie algebra $\mathfrak{so}(2J+1)$. In particular, we can find a transition matrix $U \in GL(2J+1, \mathbb{C})$, such that the matrices:

$$R_J^I(X_k) = U\, D_J(X_k)\, U^{-1} \tag{9}$$

are skew-symmetric for $k = 1, 2, 3$, thus describe the embedding.

The construction of skew-symmetric real matrices $R_J^I(X_k)$ satisfying the similarity Condition (9) is essentially based on the following two properties of the (complex) representation matrices $D_J(X_k)$, the proof of which is straightforward using Equation (5):

Lemma 1. *Let J be a positive integer. The following conditions hold:*

1. *The characteristic and minimal polynomials $p_J(z)$ and $q_J(z)$ of the matrices $D_J(X_k)$ in (5) coincide and are given by:*

$$p_J(z) = q_J(z) = -z\left(z^2+1\right)\left(z^2+4\right)\cdots\left(z^2+J^2\right) \tag{10}$$

for $k = 1, 2, 3$.

2. *In the representation D_J, the Casimir operator C_2 of $\mathfrak{so}(3)$ is given by:*

$$C_2 = D_J(X_1)^2 + D_J(X_2)^2 + D_J(X_3)^2 = -J(J+1)\,\mathrm{Id}_{2J+1}. \tag{11}$$

We show that, up to multiplicative factors, these properties are sufficient to construct skew-symmetric matrices $R_J^I(X_k)$, such that:

$$\left[R_J^I(X_i), R_J^I(X_j)\right] = \varepsilon_{ijk} R_J^I(X_k) \tag{12}$$

holds and Equation (7) is satisfied. In particular, there is no need to consider the transition matrix U explicitly. As a starting point, for any $1 \leq \alpha \leq J$, we define the 2×2 matrices:

$$M_\alpha = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}. \tag{13}$$

We further define the $(2J+1) \times (2J+1)$-block matrix:

$$R_J^I(X_3) = \begin{pmatrix} M_J & & & \\ & \ddots & & \\ & & M_1 & \\ & & & 0 \end{pmatrix}. \tag{14}$$

It is obvious that $R_J^I(X_3)$ belongs to $\mathfrak{so}(2J+1)$ and that the minimal and characteristic polynomials of $R_J^I(X_3)$ coincide. These polynomials are given by (10). It follows at once that $R_J^I(X_3)$ is similar to the matrices $D_J(X_k)$ for any $k = 1, 2, 3$. Now, to construct skew-symmetric matrices $R_J^I(X_1)$ and $R_J^I(X_2)$ satisfying (12), we consider block matrices of the type:

$$S = \begin{pmatrix} 0 & A_1 & & & & & \\ B_1 & 0 & A_2 & & & & \\ & B_2 & \ddots & & & & \\ & & & 0 & A_{J-1} & & \\ & & & B_{J-1} & 0 & -v^T \\ & & & & v & 0 \end{pmatrix}, \tag{15}$$

where A_l, B_l are 2×2 real matrices for $1 \leq l \leq J-1$ and $v = (v_1, v_2)$ is a vector. As S is assumed to be a skew-symmetric matrix, for any index l, we have:

$$B_l + A_l^T = 0. \tag{16}$$

The choice of the matrix form is motivated by the fact that each block M_l of $R_J^I(X_3)$ describes a rotation in the two-plane generated by the vectors $\{\mathbf{e}_l, \mathbf{e}_{l+1}\}$. With this block structure, it is straightforward to verify that the commutator of A_3 and S has the following structure:

$$\left[R_J^I(X_3), S\right] = \begin{pmatrix} 0 & C_1 & & & & & \\ D_1 & 0 & C_2 & & & & \\ & D_2 & \ddots & & & & \\ & & & 0 & C_{J-1} & & \\ & & & D_{J-1} & 0 & -w^T \\ & & & & w & 0 \end{pmatrix}, \tag{17}$$

where $w = (-v_2, v_1)$ and for $1 \leq l \leq J-1$ the identities:

$$C_l = M_{J+1-l}A_l - A_l M_{J-l};\ D_l = M_{J-l}B_l - B_l M_{J+1-l}. \tag{18}$$

hold. The matrix $[A_3, S]$ is still skew-symmetric, as can be easily shown using (16) and the skew-symmetry of the (2×2)-matrices M_α. For each l, we have:

$$\begin{aligned} C_l^T + D_l &= A_l^T M_{J+1-l}^T - M_{J-l}^T A_l^T + M_{J-l}B_l - B_l M_{J+1-l} \\ &= B_l M_{J+1-l} - M_{J-l}B_l + M_{J-l}B_l - B_l M_{J+1-l} = 0. \end{aligned} \tag{19}$$

As the matrix S is composed of 2×2-blocks (with the exception of the vector v), the A_l can be essentially of two types: either A_l is a diagonal matrix or it is skew-symmetric. A generic S-matrix will thus depend at most on $3J-1$ parameters. In order to facilitate the computation of representatives to describe the real representation R_J^I, we consider all blocks A_l being of the same type (by a change of basis, an equivalent matrix representative with 2×2-blocks of a different type can be obtained). Without loss of generality, we make the choice:

$$A_l = \begin{pmatrix} 0 & a_l \\ -a_l & 0 \end{pmatrix},\ 1 \leq l \leq J-1. \tag{20}$$

By Equation (16), we have $B_l = A_l$; hence, the matrix S depends on $(J+1)$ parameters. For the commutator matrix $\left[R_J^I(X_3), S\right]$, it now follows at once from (18) that:

$$C_l = \begin{pmatrix} a_l & 0 \\ 0 & a_l \end{pmatrix} = -D_l \tag{21}$$

for any $1 \leq l \leq J-1$. The blocks C_l correspond to the second possible type (diagonal) for the blocks A_l, showing that the result does not depend on the particular form chosen initially for the blocks.

If we now compute the iterated commutator $\left[S, \left[R_J^I(X_3), S\right]\right]$, we obtain a matrix having the same block structure as $R_J^I(X_3)$ and given explicitly by:

$$\left[S, \left[R_J^I(X_3), S\right]\right] = \begin{pmatrix} E_1 & & & \\ & \ddots & & \\ & & E_J & \\ & & & 0 \end{pmatrix}, \tag{22}$$

where

$$E_1 = \begin{pmatrix} 0 & -2a_1^2 \\ 2a_1^2 & 0 \end{pmatrix}; \; E_k = \begin{pmatrix} 0 & 2a_{k-1}^2 - 2a_k^2 \\ -2a_{k-1}^2 + 2a_k^2 & 0 \end{pmatrix}, \; 2 \leq k \leq J-1 \tag{23}$$

and

$$E_J = \begin{pmatrix} 0 & 2a_k^2 - v_1^2 - v_2^2 \\ -2a_k^2 + v_1^2 + v_2^2 & 0 \end{pmatrix}. \tag{24}$$

Assuming that the blocks A_l are given by (16), we define $R_J^I(X_1) = S$. Following Equation (12):

$$R_J^I(X_2) = \left[R_J^I(X_3), R_J^I(X_1)\right]. \tag{25}$$

As a consequence, the matrix on the right hand side of the commutator (22) must coincide with $R_J^I(X_3)$. Comparing the entries leads to the quadratic system:

$$\begin{aligned} J &= 2a_1^2, \\ J-l &= 2\left(a_l^2 - a_{l-1}^2\right), \quad 2 \leq l \leq J-2 \\ v_1^2 + v_2^2 - 2a_{J-1}^2 &= 1. \end{aligned} \tag{26}$$

Up to the sign, the solution to this system is given by:

$$a_l = \pm\sqrt{\frac{2l\,J - l\,(l-1)}{4}}, \; 1 \leq l \leq J-1; \; v_1 = \pm\sqrt{\frac{J\,(J+1)}{2} - v_2^2}, \tag{27}$$

where $v_2 \leq \sqrt{\frac{J(J+1)}{2}}$ is free. This shows that the matrices $R_J^I(X_k)$ transform like the $\mathfrak{so}(3)$ generators (4). As these matrices must satisfy the similarity Condition (7) with the matrices (5), the Casimir operator must have the form (11). In particular, this implies that the following matrix identity must be fulfilled:

$$R_J^I(X_1)^2 + R_J^I(X_2)^2 = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{2J+1} \end{pmatrix}, \tag{28}$$

where

$$\begin{array}{c}\lambda_{2q-1}=\lambda_{2q}=\left(q-1\right)^{2}-J\left(2q-1\right), \quad 1\leq q\leq J\\ \lambda_{2J+1}=-J\left(J+1\right).\end{array} \tag{29}$$

A routine computation shows that the preceding system is satisfied identically for the values obtained in (27). Therefore, the three matrices $R_J^I\left(X_k\right)$ have (10) as their characteristic and minimal polynomial, and thus, there exists a complex matrix U transforming the matrices (5) onto the real matrices $R_J^I\left(X_k\right)$. We observe that the value of v_2 is not determined by either the commutator (12) or the Condition (28). This parameter is however inessential, as it merely indicates the possibility of considering linear combinations of the matrices $R_J^I\left(X_1\right)$ and $R_J^I\left(X_2\right)$. In fact, taking the case $J=1$, the realization above gives the matrices:

$$R_J^I\left(X_2\right)=\begin{pmatrix} & & -\sqrt{1-v_2^2}\\ & & v_2\\ \sqrt{1-v_2^2} & -v_2 & \end{pmatrix}, \quad R_J^I\left(X_1\right)=\begin{pmatrix} & & v_2\\ & & \sqrt{1-v_2^2}\\ -v_2 & -\sqrt{1-v_2^2} & \end{pmatrix}. \tag{30}$$

For $v_2=0$, these matrices reduce to the standard rotation matrices in $\mathbb{R}^3$ corresponding to the adjoint representation of $\mathfrak{so}\left(3\right)$. For this reason, in the following, we set $v_2=0$ without loss of generality. As the signs in (27) can further be chosen freely, we make the following choice:

$$a_l=\sqrt{\frac{2l\,J-l\left(l-1\right)}{4}},\ 1\leq l\leq J-1;\ v_1=\sqrt{\frac{J\left(J+1\right)}{2}}. \tag{31}$$

The matrices $R_J^I\left(X_k\right)$ constructed with these values satisfy Equation (7) and clearly belong to $\mathfrak{so}\left(2J+1\right)$, showing that the linear map:

$$\varphi_J:\mathfrak{so}\left(3\right)\rightarrow\mathfrak{so}\left(2J+1\right); \quad X_k\mapsto R_J^I\left(X_k\right) \tag{32}$$

defines a Lie algebra homomorphism and an irreducible embedding. We observe that choosing different signs for the parameters a_l gives rise to an embedding belonging to the same conjugation class in $\mathfrak{so}\left(2J+1\right)$.

Let $\{\mathbf{e}_1,\cdots,\mathbf{e}_{2J+1}\}$ denote a basis of the representation space of the real representation R_J^I. Further, let $\left[\frac{n}{2}\right]$ denote the integer part of $\frac{n}{2}$. Then, the matrix elements are easily described in terms of the coefficients in (31) as:

$$\begin{aligned}\left\langle\mathbf{e}^k\right|R_J^I\left(X_1\right)\left|\mathbf{e}_l\right\rangle=&\left(\frac{1+(-1)^{k-1}}{2}\right)\left(\delta_{k+3}^{l}\,a_{\left(\left[\frac{k+1}{2}\right]\right)}+\delta_{k}^{l+1}a_{\left(\left[\frac{k-1}{2}\right]\right)}\right)-\left(a_J+\sqrt{\frac{J^2+J}{2}}\right)\times\\ &\left(\delta_{2J+1}^{l}\delta_{k}^{2J}-\delta_{2J}^{l}\delta_{k}^{2J+1}\right)-\left(\frac{1+(-1)^{k}}{2}\right)\left(\delta_{k+1}^{l}a_{\left(\left[\frac{k}{2}\right]\right)}+\delta_{k}^{l+3}a_{\left(\left[\frac{k-2}{2}\right]\right)}\right).\end{aligned} \tag{33}$$

$$\left\langle\mathbf{e}^k\right|R_J^I\left(X_2\right)\left|\mathbf{e}_l\right\rangle=\delta_{k+2}^{l}\,a_{\left(\left[\frac{k+1}{2}\right]\right)}-\delta_{k}^{l+2}a_{\left(\left[\frac{k-1}{2}\right]\right)}-\left(a_J+\sqrt{\frac{J^2+J}{2}}\right)\left(\delta_{2J+1}^{l}\delta_{k}^{2J-1}-\delta_{2J-1}^{l}\delta_{k}^{2J+1}\right). \tag{34}$$

$$\left\langle\mathbf{e}^k\right|R_J^I\left(X_3\right)\left|\mathbf{e}_l\right\rangle=\frac{\left(1+(-1)^{k}\right)\delta_{k}^{l+1}\left(2J+2-k\right)+\left((-1)^{k}-1\right)\delta_{l}^{k+1}\left(2J+1-k\right)}{4}, \tag{35}$$

where $1 \leq k, l \leq 2J+1$.

The first non-trivial case for which the method applies is $J=2$ in dimension five. According to (5), the complex matrices of the irreducible representation D_2 are given by the diagonal matrix $D_2\left(X_1\right) = \Delta\left(2\,\mathrm{i},\, \mathrm{i}, 0, -\mathrm{i}, -2\,\mathrm{i}\right)$ and:

$$D_2\left(X_2\right) = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{3}{2} & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix},\; D_2\left(X_3\right) = \begin{pmatrix} 0 & 2\,\mathrm{i} & 0 & 0 & 0 \\ \frac{\mathrm{i}}{2} & 0 & \frac{3\mathrm{i}}{2} & 0 & 0 \\ 0 & \mathrm{i} & 0 & \mathrm{i} & 0 \\ 0 & 0 & \frac{3\mathrm{i}}{2} & 0 & \frac{\mathrm{i}}{2} \\ 0 & 0 & 0 & 2\,\mathrm{i} & 0 \end{pmatrix}.$$

In this form, however, the matrices are not skew-symmetric, and hence, the properties of the representation are not easily recognized. Using the matrix elements deduced in (33)–(35), we can easily construct the corresponding real matrices $R_2^I\left(X_k\right)$. Their explicit expression is:

$$R_2^I\left(X_1\right) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & -\sqrt{3} & 0 \end{pmatrix},\; R_2^I\left(X_2\right) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & -\sqrt{3} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \end{pmatrix},\; R_2^I\left(X_3\right) = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{36}$$

These matrices are linear combinations of the basis elements of the compact orthogonal Lie algebra $\mathfrak{so}\left(5\right)$, hence defining an embedding $\mathfrak{so}\left(3\right) \subset \mathfrak{so}\left(5\right)$. If, moreover, $\{\mathbf{e}_1, \cdots, \mathbf{e}_5\}$ denotes the canonical basis of the representation space, we can easily check that:

$$\begin{gathered} R_2^I\left(X_1\right)\mathbf{e}_1 = -\mathbf{e}_4, \quad R_2^I\left(X_2\right)\mathbf{e}_1 = -\mathbf{e}_3, \quad R_3^I\left(X_1\right)\mathbf{e}_1 = 2\mathbf{e}_2, \\ \left(R_2^I\left(X_1\right)\right)^2\mathbf{e}_1 = -\mathbf{e}_1 + \sqrt{3}\mathbf{e}_5,\; \left(R_2^I\left(X_2\right)\right)^2\mathbf{e}_1 = -\mathbf{e}_1 - \sqrt{3}\mathbf{e}_5, \end{gathered}$$

showing that the action of $\mathfrak{so}\left(3\right)$ is actually irreducible. It is routine to check that for $1 \leq j \leq 3$, the similarity relation $R_2^I\left(X_j\right) = U\, D_2\left(X_j\right)\, U^{-1}$ is satisfied for the transition matrix:

$$U = \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & \sqrt{3} & 0 & \frac{\sqrt{3}}{2} \\ 0 & \frac{-2\mathrm{i}}{\sqrt{3}} & 0 & \frac{-2\mathrm{i}}{\sqrt{3}} & 0 \\ 0 & \frac{-2}{\sqrt{3}} & 0 & \frac{2}{\sqrt{3}} & 0 \\ \frac{\mathrm{i}}{\sqrt{3}} & 0 & 0 & 0 & \frac{-\mathrm{i}}{\sqrt{3}} \\ -\frac{1}{2} & 0 & 1 & 0 & -\frac{1}{2} \end{pmatrix}.$$

3. Construction of the Matrices $R_J^{II}\left(X_k\right)$

In contrast to the case of integer J, the matrices $D_J^{II}\left(X_k\right)$ are already given over the reals, as a consequence of the dimension doubling in the representation space. It is straightforward to see that the matrices $D_J^{II}\left(X_k\right)$ can be written in terms of tensor products as:

$$D_J^{II}\left(X_k\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathrm{Re}D_J\left(X_k\right) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \mathrm{Im}D_J\left(X_k\right). \tag{37}$$

We observe that $D_J^{II}\left(X_1\right)$ is skew-symmetric by construction, as $D_J\left(X_1\right)$ is diagonal with purely imaginary entries. In general, however, $D_J^{II}\left(X_2\right)$ and $D_J^{II}\left(X_3\right)$ are not skew-symmetric, and therefore, the representation is not given in terms of elements belonging to the (compact) Lie algebra $\mathfrak{so}\left(4J+2\right)$. The two properties required to construct the skew-symmetric matrices realizing the representation $R_{\frac{J}{2}}^{II}$

are again the characteristic polynomial and the eigenvalue of the Casimir operator. The procedure to find such matrices is formally very similar to the previous case, up to the necessary modifications due to the tensor product (37). For this reason, we merely indicate the mains steps, skipping the detailed computations.

For any $\frac{J}{2} \in \frac{1}{2}\mathbb{N}$, the characteristic and minimal polynomials of $D_J^{II}\left(X_k\right)$ are respectively given by:

$$p_J\left(z\right) = \frac{1}{2^{2J+2}}\left(1+4z^2\right)^2\left(9+4z^2\right)^2\cdots\left(J^2+4z^2\right)^2,\ q_J\left(z\right) = \sqrt{p_J\left(x\right)}. \tag{38}$$

The eigenvalue of the Casimir operator on such a representation is given by:

$$C_2(D_J^{II}) = -\frac{J\left(J+1\right)}{4}\,\mathrm{Id}_{2J+1}. \tag{39}$$

In this case, the 2×2-matrices to start from are of the type:

$$N_\beta = \begin{pmatrix} 0 & -\frac{\beta}{2} \\ \frac{\beta}{2} & 0 \end{pmatrix} \tag{40}$$

where $1 \leq \beta \leq J$ is an odd integer. With these blocks, we define the $(4J+2)\times(4J+2)$-block matrix:

$$R_J^{II}\left(X_3\right) = \begin{pmatrix} N_J & & & & & \\ & \ddots & & & & \\ & & N_1 & & & \\ & & & -N_1 & & \\ & & & & \ddots & \\ & & & & & -N_J \end{pmatrix}. \tag{41}$$

For this rotation matrix, it is easy to verify that the characteristic and minimal polynomials satisfy Equation (38). Next, we consider matrices of the type:

$$S = \begin{pmatrix} 0 & A_1 & & & \\ -A_1^T & 0 & A_2 & & \\ & -A_2^T & \ddots & & \\ & & & 0 & A_J \\ & & & -A_J^T & 0 \end{pmatrix}, \tag{42}$$

where the A_l are 2×2-matrices. We observe that, without loss of generality, these can be taken as in (16). Repeating the same argument as for the integer case, the commutator $\left[R_{\frac{J}{2}}^{II}(X_3), S\right]$ is a skew-symmetric matrix having the same block structure as (42). We thus define the matrix $R_{\frac{J}{2}}^{II}(X_1) = S$ and also $R_{\frac{J}{2}}^{II}(X_2) = \left[R_{\frac{J}{2}}^{II}\left(X_3\right), S\right]$. Developing explicitly the commutators of these matrices, it can be proven easily that the A_l-blocks satisfy the constraint:

$$A_l + A_{J-l} = 0, 1 \leq l \leq \left[\frac{J}{2}\right]. \tag{43}$$

Hence, the number of parameters for a generic matrix S is bounded by $3\left(\left[\frac{J}{2}\right]+1\right)$. Now, imposing the condition $\left[S, \left[R_{\frac{J}{2}}^{II}\left(X_3\right), S\right]\right] = R_{\frac{J}{2}}^{II}\left(X_3\right)$, we are again led to a quadratic system in the coefficients of

N_β and A_l. In this case, however, the solution can be computed up to the sign, and no free parameters appear (this is a consequence of the constraint (43)).

Making, e.g., the choice of skew-symmetric blocks A_l and fixing the positive sign for the solution of the quadratic system, the matrix elements of $R^{II}_{\frac{J}{2}}(X_k)$ for $k = 1, 2, 3$ are given by the formulae:

$$\left\langle \mathbf{e}^k \right| R^{II}_{\frac{J}{2}}(X_1) \left|\mathbf{e}_l\right\rangle = \left(\frac{1+(-1)^{k-1}}{2}\right)\left(\delta^l_{k+3}\, a_{\left(\left[\frac{k+1}{2}\right]\right)} + \delta^{l+1}_k a_{\left(\left[\frac{k-1}{2}\right]\right)}\right) - \left(\frac{1+(-1)^k}{2}\right)\times \left(\delta^l_{k+1} a_{\left(\left[\frac{k}{2}\right]\right)} + \delta^{l+3}_k a_{\left(\left[\frac{k-2}{2}\right]\right)}\right) \tag{44}$$

$$\left\langle \mathbf{e}^k \right| R^{II}_{\frac{J}{2}}(X_2) \left|\mathbf{e}_l\right\rangle = \delta^l_{k+2}\, a_{\left(\left[\frac{k+1}{2}\right]\right)} - \delta^{l+2}_k a_{\left(\left[\frac{k-1}{2}\right]\right)}. \tag{45}$$

$$\left\langle \mathbf{e}^k \right| R^{II}_{\frac{J}{2}}(X_3) \left|\mathbf{e}_l\right\rangle = \frac{\left(1+(-1)^k\right)\delta^{l+1}_k (2J+2-k) + \left((-1)^k - 1\right)\delta^{k+1}_l (2J+1-k)}{4} \tag{46}$$

As a byproduct of the method, we remark that the matrix elements (33)–(35), as well as those in (44)–(46) provide a prescription to realize the Lie algebra $\mathfrak{so}(3)$ in terms of vectors fields in $\mathbb{R}^{2J+1}$ and $\mathbb{R}^{4J+2}$, respectively. More specifically, if M is the representation matrix of an element $Y \in \mathfrak{so}(3)$, the associated vector field $\widehat{Y}$ is given by:

$$\widehat{Y} := \left\langle \mathbf{e}^k \right| M \left|\mathbf{e}_l\right\rangle x_k \frac{\partial}{\partial x_l}. \tag{47}$$

4. Tensor Products of Real Irreducible Representations

While the tensor products of complex representations of $\mathfrak{so}(3)$ are well known and easily found by means of the formula:

$$D_J \otimes D_{J'} = D_{J+J'} \oplus \cdots \oplus D_{|J-J'|}, \tag{48}$$

for the tensor products of the real irreducible representations, the preceding formula is generally no longer valid, due to the division into the first and second class [8]. As a consequence, in general, such a tensor product will not be always multiplicity free, *i.e.*, the irreducible real representations appearing in the decomposition may have multiplicity greater than one. This is easily seen using the corresponding complexification, to which Formula (48) applies. A simple computation shows that for the tensor products of real irreducible representations R^I_J and $R^{II}_{\frac{J'}{2}}$ of $\mathfrak{so}(3)$, three possibilities are given:

1. $J, J' \in \mathbb{N}$ and $J \geq J'$:

$$R^I_J \otimes R^I_{J'} = \sum_{\alpha=0}^{2J'} R^I_{J+J'-\alpha}. \tag{49}$$

   The tensor product is multiplicity free, and the irreducible factors are all of Class I. This actually corresponds exactly to the tensor product of the complex representations D_J.

2. $J \in \mathbb{N}$, $J' \equiv 1 \pmod 2$:

$$R^I_J \otimes R^{II}_{\frac{J'}{2}} = \sum_{\alpha=0}^{2J'} R^{II}_{\frac{|2J+J'-\alpha|}{2}}. \tag{50}$$

   The irreducible factors are all of Class II and have multiplicity one; hence, the product is also multiplicity free.

3. $J, J' \equiv 1 \pmod 2$:

$$R^{II}_{\frac{J}{2}} \otimes R^{II}_{\frac{J'}{2}} = \sum_{\alpha=0}^{2J'} 4\, R^{I}_{\frac{|J+J'-\alpha|}{2}}. \tag{51}$$

As expected, in this case, the irreducible factors are all of Class I, and the tensor product is not multiplicity free. All factors have the same multiplicity $\lambda = 4$.

As follows from (38) when compared to (10), given an arbitrary matrix of a real irreducible representation of $\mathfrak{so}(3)$, its class can be immediately deduced from the characteristic polynomial. Actually, a stronger assertion can be obtained using this property. The main fact in this context is that the representation matrices of the three generators X_1, X_2, X_3 of $\mathfrak{so}(3)$ have the same characteristic and minimal polynomials. This enables us to determine easily the characteristic polynomial for any linear combination $X = \sum_{k=1}^{3} \lambda_k X_k$ and any real irreducible representation:

1. If R^I_J is a representation of first class, then $R^I_J(X)$ has characteristic polynomial:

$$p_J(z) = -z \prod_{\alpha=1}^{J} \left(z^2 + \xi\, \alpha^2\right), \tag{52}$$

where $\xi = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. Moreover, the minimal polynomial satisfies $q_J(z) = p_J(z)$.

2. If $R^{II}_{\frac{J}{2}}$ is a representation of the second class, then $R^{II}_{\frac{J}{2}}(X)$ has characteristic polynomial:

$$p_{\frac{J}{2}}(z) = \frac{1}{2^{2J+2}} \prod_{\beta=0}^{\frac{J-1}{2}} \left(4z^2 + \xi\, (2\beta+1)^2\right)^2 \tag{53}$$

where $\xi = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. In this case, $q_{\frac{J}{2}}(z) = \sqrt{p_{\frac{J}{2}}(z)}$.

It is worthy to be observed that the quadratic factor (z^2+1) must appear in any representation with integer J, while $(4z^2+1)^2$ appears for any half-integer. This implies that the common factor ξ can be easily found from the corresponding characteristic polynomial when the latter is rewritten taking into account (10) and (38). This fact further enables us to deduce the decomposition of an arbitrary real representation of $\mathfrak{so}(3)$ by simply analyzing the characteristic polynomial of a matrix within this representation. Let us inspect this fact more closely.

Let

$$R = \mu_0 R^I_0 \oplus \mu_1 R^I_{J_1} \oplus \cdots \oplus \mu_r R^I_{J_r} \oplus \nu_1 R^{II}_{\frac{J'_1}{2}} \oplus \cdots \oplus \nu_s R^{II}_{\frac{J'_s}{2}} \tag{54}$$

be the decomposition of R into real irreducible factors, where μ_k, ν_l are positive integers, such that:

$$\dim R = \sum_{k=0}^{r} \mu_k (2J_k + 1) + \sum_{l=1}^{s} \nu_l (2J'_l + 2) \tag{55}$$

holds and $J_k, J'_l \neq 0$ for $k, l \neq 0$. Without loss of generality, we can suppose that $J_1 < J_2 < \cdots < J_r$ and $J'_1 < J'_2 < \cdots < J'_s$. The polynomial $p(z)$ of $R(X)$ thus factorizes as the product:

$$p(z) = p_0^{\mu_0}(z)\, p_{J_1}^{\mu_1}(z) \cdots p_{J_r}^{\mu_r}(z)\; p_{\frac{J'_1}{2}}(z)^{\nu_1} \cdots p_{\frac{J'_s}{2}}(z)^{\nu_s}. \tag{56}$$

As follows from (52) and (53), there exists a common factor ξ in all quadratic factors of $p\left(z\right)$. For $0\leq\sigma\leq r$ and $1\leq\tau\leq s$, define further:

$$m_{\sigma}=\sum_{k=\sigma}^{r}\mu_{k};\; n_{\tau}=2\sum_{l=\tau}^{s}\nu_{l}. \tag{57}$$

Expanding the polynomial $p\left(z\right)$, we obtain the expression:

$$p\left(z\right)=-z^{m_0}\left(\prod_{\alpha=1}^{J_1}\left(z^2+\xi\,\alpha^2\right)\right)^{m_1}\left(\prod_{\alpha=1+J_1}^{J_2}\left(z^2+\xi\,\alpha^2\right)\right)^{m_2}\cdots\left(\prod_{\alpha=J_{r-1}+1}^{J_r}\left(z^2+\xi\,\alpha^2\right)\right)^{m_r}\times$$
$$\prod_{l=1}^{s}2^{-(2J+2)\nu_l}\left(\prod_{\beta=0}^{\frac{J_1'-1}{2}}\left(4z^2+\xi\,\left(2\beta+1\right)^2\right)\right)^{n_1}\cdots\left(\prod_{\beta=\frac{J_{s-1}'-1}{2}}^{\frac{J_s'-1}{2}}\left(4z^2+\xi\,\left(2\beta+1\right)^2\right)\right)^{n_s}. \tag{58}$$

Starting from the polynomial (58), we can go backwards and deduce the precise decomposition (54) of R by merely inspecting the multiplicities of the different quadratic factors. In practice, the coefficients of the polynomial simplify, so that the factor ξ must be first deduced from the quadratic real irreducible factors, having in mind that for irreducible representations of the first class and second class, they are of the form given in (52) and (53). On the other hand, the values $J_1,\cdots,J_r$ and $\frac{J_1'}{2},\cdots,\frac{J_s'}{2}$ of the irreducible factors are uniquely determined as the highest values in the quadratic factors $\left(z^2+\rho^2\right)$ and $\left(4z^2+\omega^2\right)$ preceding a variation in the multiplicity. Therefore, the number of irreducible factors in the decomposition of R is given by the number of different multiplicities of the quadratic factors and that of z. The corresponding multiplicity of each irreducible factor of R is easily obtained by the following prescription:

1. The multiplicity of z, given by m_0, indicates the number of irreducible factors of Class I.
2. The multiplicity of $R^I_{J_r}$ is given by m_r, whereas the multiplicity of $R^I_{J_k}$ is given by m_k-m_{k+1} for $r-1\geq k\geq 1$.
3. The multiplicity of the trivial representation R^I_0 is given by m_0-m_1.
4. The multiplicity of $R^{II}_{\frac{J_s'}{2}}$ is given by $\frac{1}{2}n_s$, whereas the multiplicity of $R^{II}_{\frac{J_l'}{2}}$ is given by $\frac{n_l-m_{l+1}}{2}$ for $s-1\geq l\geq 1$.

This proves that the essential information concerning the real irreducible factors of a real representation is codified in the factorization of the characteristic polynomial of an arbitrary matrix. This proves the following criterion:

Theorem 2. *Let R be an arbitrary real representation of $\mathfrak{so}\left(3\right)$ and $X\in\mathfrak{so}\left(3\right)$. Then, the decomposition of R as the sum of real irreducible representations is completely determined by the characteristic polynomial $p\left(z\right)$ of the matrix $R\left(X\right)$.*

As an example that illustrates the method, suppose that the matrix X belonging to a real representation R of $\mathfrak{so}(3)$ has characteristic polynomial:

$$p(z)=\lambda\left(25+2z^2\right)^8\left(225+2z^2\right)^6\left(625+2z^2\right)^4\left(1225+2z^2\right)^2, \tag{59}$$

where $\lambda \neq 0$. The exponents are $n_1 = 8$, $n_2 = 6$, $n_3 = 4$ and $n_4 = 2$; thus, it follows at once that R must be a sum of four irreducible factors of Class II, as z does not appear in the factorization of $p(z)$ into real irreducible factors. Taking into account Expression (38), the polynomial can be rewritten as:

$$p(z) = 2^{-20}\lambda\left(50+4z^2\right)^8\left(450+4z^2\right)^6\left(1250+4z^2\right)^4\left(2450+4z^2\right)^2, \tag{60}$$

Hence, we can extract the common factor $\xi = 50$. The values of J for the irreducible components are:

$$J_1^2 = 1, \quad J_2^2 = \frac{450}{50} = 9, \quad J_3^2 = \frac{1250}{50} = 25, \quad J_4^2 = \frac{2450}{50} = 49. \tag{61}$$

On the other hand, $\nu_{J_4} = 1$, $\nu_3 = \frac{n_3-n_4}{2} = 1$, $\nu_2 = \frac{n_2-n_3}{2} = 1$ and $\nu_1 = \frac{n_1-n_2}{2} = 1$, showing that X is a matrix belonging to the representation $R^{II}_{\frac{1}{2}} \oplus R^{II}_{\frac{3}{2}} \oplus R^{II}_{\frac{5}{2}} \oplus R^{II}_{\frac{7}{2}}$.

5. Conclusions

By means of elementary techniques of Lie algebras and matrix theory, explicit formulae to construct real matrices of real irreducible representations of the first and second class of the compact Lie algebra $\mathfrak{so}(3)$ have been obtained. The procedure is based on the important observation that, as a consequence of the Cartan map (3), the representation matrices of the $\mathfrak{so}(3)$-generators in an irreducible representation have the same characteristic and minimal polynomial, a fact that is not true on the usual Cartan–Weyl basis. This enables us to characterize the class of a real representation according to the structure of these polynomials. Using the latter enables one to construct skew-symmetric matrices for any irreducible real representation. The real matrices so constructed actually realize the embedding of $\mathfrak{so}(3)$ into the compact Lie algebras $\mathfrak{so}(2J+1)$ and $\mathfrak{so}(4J+2)$, respectively, depending on whether J is an integer or half-integer and, hence, corresponding to matrices of the representation subduced by the restriction of the defining representation of the orthogonal Lie algebras. As an application of the method, it has been shown that for an arbitrary real representation R of $\mathfrak{so}(3)$, the decomposition of R into irreducible factors can be deduced from the characteristic polynomial of an arbitrary matrix in the representation. This provides in particular a useful practical criterion to determine whether a given matrix belongs to an irreducible real representation.

We finally remark that the realizations in terms of vector fields (47) that are deduced from the matrix elements (33)–(35), as well as those in (44)–(46), are potentially of interest in the context of point symmetries of ordinary differential equations. Systems of ordinary differential equations have been exhaustively studied by means of the Lie method (see, e.g., [13–15] and the references therein), albeit for systems containing arbitrary functions as parameters, there still remains some work to be done. In this context, indirect approaches as that developed in [16] characterizing systems in terms of specific realizations of Lie algebras constitute an alternative procedure that can be useful for applications.

As an elementary application of the real representations of $\mathfrak{so}(3)$ to the Lie symmetry method, consider the representation R^I_J for $J = 2$. Using the prescription given in (47), the vector fields in $\mathbb{R}^5$ associated with the matrices (36) are the following:

$$\begin{array}{l} \widehat{X}_1 = -x_4\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} + \left(x_1 - \sqrt{3}x_5\right)\frac{\partial}{\partial x_4} + \sqrt{3}x_4\frac{\partial}{\partial x_5}, \\ \widehat{X}_2 = -x_3\frac{\partial}{\partial x_1} - x_4\frac{\partial}{\partial x_2} + \left(x_1 + \sqrt{3}x_5\right)\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial x_4} - \sqrt{3}x_3\frac{\partial}{\partial x_5}, \\ \widehat{X}_3 = 2x_2\frac{\partial}{\partial x_1} - 2x_1\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4}. \end{array} \tag{62}$$

Now, let $\Phi(t) \neq 0$ be an arbitrary function, and consider the equations of motion:

$$\ddot{x}_i = \Phi(t) \frac{\partial V}{\partial x_i}, \ 1 \leq i \leq 5 \tag{63}$$

associated with the Lagrangian:

$$L = \frac{1}{2}\left(\dot{x}_1^2 + \cdots + \dot{x}_5^2\right) + \Phi(t)\ V(x_1, \cdots, x_5), \tag{64}$$

where $V(x_1, \cdots, x_5) = \alpha_{i_1 \cdots i_5} x_1^{i_1} \cdots x_5^{i_5}$ is a homogeneous cubic polynomial. After some computation, it can be shown that the preceding vector fields are point symmetries of (63) only if $V(x_1, \cdots, x_5)$ has the following form:

$$V(x_1, \cdots, x_5) = \alpha\left(6\left(x_1^2 + x_2^2\right)x_5 + 3\left(\sqrt{3}x_1 - x_5\right)x_3^2 - 3\left(\sqrt{3}x_1 + x_5\right)x_4^2 - 2x_5^3 + 6\sqrt{3}x_2x_3x_4\right),$$

where $\alpha \in \mathbb{R}$. The realization (62) of $\mathfrak{so}(3)$ obtained from the representation R_2^I further imposes some restrictions on the existence of additional point symmetries. A generic point symmetry $Z = \xi(t, \mathbf{x}) \frac{\partial}{\partial t} + \eta^j(t, \mathbf{x}) \frac{\partial}{\partial x_j}$ of (63) has components:

$$\begin{aligned}
\xi(t, \mathbf{x}) = &\ b_4 t^2 + b_5 t + b_6, \\
\eta^1(t, \mathbf{x}) = &\ -b_1 x_4 + b_2 x_3 + 2b_3 x_2 + b_4 t\, x_1 + \tfrac{1}{2} b_5 x_1 + b_7 x_1, \\
\eta^2(t, \mathbf{x}) = &\ b_1 x_3 + b_2 x_4 - 2b_3 x_1 + b_4 t\, x_2 + \tfrac{1}{2} b_5 x_2 + b_7 x_2, \\
\eta^3(t, \mathbf{x}) = &\ -b_1 x_2 - b_2\left(x_1 + \sqrt{3}x_5\right) + b_3 x_4 + b_4 t\, x_3 + \tfrac{1}{2} b_5 x_3 + b_7 x_3, \\
\eta^4(t, \mathbf{x}) = &\ b_1\left(x_1 - \sqrt{3}x_5\right) - b_2 x_2 - b_3 x_3 + b_4 t\, x_4 + \tfrac{1}{2} b_5 x_4 + b_7 x_4, \\
\eta^5(t, \mathbf{x}) = &\ \sqrt{3} b_1 x_4 + \sqrt{3} b_2 x_3 + b_4 t\, x_5 + \tfrac{1}{2} b_5 x_5 + b_7 x_5
\end{aligned} \tag{65}$$

where the coefficients $b_4, \cdots, b_7$ are subjected to the constraint:

$$(10b_4 t + 5b_5 + 2b_7)\,\Phi(t) + \left(2b_4 t^2 + 2b_5 t + 2b_6\right)\frac{d\Phi}{dt} = 0. \tag{66}$$

It follows that for non-constant generic functions $\Phi(t)$, the symmetry algebra is isomorphic to $\mathfrak{so}(3)$, whereas if $\Phi(t)$ satisfies the separable ordinary differential Equation (66), at most two additional point symmetries can be found. It is easily verified that if the system possesses five point symmetries (these are determined by the coefficients b_6 and $b_7 = -\frac{5}{2}b_5$, corresponding to the time translation and a scaling symmetry, respectively), then $\Phi(t)$ is necessarily a constant. It may be observed that, in any case, the symmetries generating the $\mathfrak{so}(3)$-subalgebra are also Noether symmetries. We thus conclude that for functions $\Phi(t)$ not satisfying the constraint (66), the algebras of point and Noether symmetries coincide.

For the remaining values of J, a similar ansatz as the previous one can be applied to obtain criteria that ensure that a non-linear system of ordinary differential equations exhibits an exact $\mathfrak{so}(3)$-symmetry. Work in this direction is currently in progress.

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Conflicts of Interest

The author declares no conflict of interest.

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