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Article

Q-Conditional Symmetries and Exact Solutions of Nonlinear Reaction–Diffusion Systems

Oleksii Pliukhin

Department of Mathematics, Poltava National Technical Yuriy Kondratyuk University,
24, Pershotravnevyi Prospekt, 36601 Poltava, Ukraine; E-Mail: pliukhin@gmail.com;
Tel.: +38-095-823-29-84

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Abstract: A wide range of reaction–diffusion systems with constant diffusivities that are invariant under Q-conditional operators is found. Using the symmetries obtained, the reductions of the corresponding systems to the systems of ODEs are conducted in order to find exact solutions. In particular, the solutions of some reaction–diffusion systems of the Lotka–Volterra type in an explicit form and satisfying Dirichlet boundary conditions are obtained. An biological interpretation is presented in order to show that two different types of interaction between biological species can be described.

Keywords: Q-conditional symmetry; reaction–diffusion systems; exact solution; Lotka–Volterra system

1. Introduction

In 1952, Alan Turing published his prominent paper [1]. In this paper he proposed the Turing hypothesis of pattern formation. He used reaction–diffusion equations of the form

$$\begin{aligned} \lambda_1 u_t &= (D_1(u)u_x)_x + F(u,v), \\ \lambda_2 v_t &= (D_2(v)v_x)_x + G(u,v) \end{aligned} \tag{1}$$

which are central to the field of pattern formation.

In system (1), F and G are arbitrary smooth functions, $u = u(t,x)$ and $v = v(t,x)$ are unknown functions of the variables t and x, while the subscripts t and x denote differentiation with respect to this variable. Nonlinear system (1) generalizes many well-known nonlinear second-order models used to describe various processes in physics [2], biology [3–5] and ecology [6].

Here we concentrate ourselves on the most important subclass of RD systems with the form of (1), namely that with constant coefficients of diffusivity

$$\begin{aligned} \lambda_1 u_t &= u_{xx} + F(u,v), \\ \lambda_2 v_t &= v_{xx} + G(u,v) \end{aligned} \tag{2}$$

System (2) has been intensely studied using different mathematical methods (see, e.g., [3,4,7] and papers cited therein). All possible Lie symmetries of system (2) have been found, in [8–11]. In particular, Q-conditional symmetries of (2) were found in [12]. Reference [13] also contains some results related with system (2).

System (1) is a natural generalization of the well-known RD equation

$$u_t = [D(u)u_x]_x + F(u) \tag{3}$$

There are many papers devoted to the construction of Q-conditional symmetries for this equation [14–21], starting from the pioneering work in [22]. There is also a non-trivial generalization of these results for the case of the reaction–diffusion–convection equation ([21] and papers cited therein).

In contrast to (3), there are not many results for searching Q-conditional symmetries of system (2). Construction of the Q-conditional symmetries (non-classical symmetries) of such systems is a very difficult task. Only a few papers have been devoted to the search of such symmetries. In [23] the Q-conditional symmetries of the system

$$\begin{aligned} u_t &= (u^k u_x)_x + F(u,v), \\ v_t &= (v^l v_x)_x + G(u,v), \ l^2 + k^2 \neq 0, \end{aligned}$$

have been obtained; in [24] the Q-conditional symmetries of the Lotka–Volterra system

$$\begin{aligned} \lambda_1 u_t &= u_{xx} + u(a_1 + b_1 u + c_1 v), \\ \lambda_2 v_t &= v_{xx} + v(a_2 + b_2 u + c_2 v) \end{aligned} \tag{4}$$

were obtained.

The paper is organized as follows. In Section 2 three theorems are presented which contain the main result for Q-conditional symmetries of system (2). In Section 3, ansätze for all systems and solutions for one of the systems are derived. In Section 4, the solutions for a generalization of the Lotka–Volterra system are obtained and analyzed. Some graphs of the exact solutions are also presented. Finally, we present some conclusions.

2. Main Result

Let us consider the reaction–diffusion system with constant diffusivities: (2). We want to find Q-conditional operators of the form

$$Q = \partial_t + \xi(t,x,u,v)\partial_x + \eta^1(t,x,u,v)\partial_u + \eta^2(t,x,u,v)\partial_v \tag{5}$$

under which system (2) is invariant.

The most general form of the Q-conditional operators is

$$Q = \xi^0(t,x,u,v)\partial_t + \xi^1(t,x,u,v)\partial_x + \eta^1(t,x,u,v)\partial_u + \eta^2(t,x,u,v)\partial_v.$$

In the case $\xi^0(t,x,u,v) \neq 0$, this operator can be reduced to that with $\xi^0(t,x,u,v) = 1$ [25]. So we investigate operator (5).

We write down system (2) in the following form:

$$\begin{aligned} u_{xx} &= \lambda_1 u_t + C^1(u,v),\ \lambda_1 \neq 0, \\ v_{xx} &= \lambda_2 v_t + C^2(u,v),\ \lambda_2 \neq 0 \end{aligned} \tag{6}$$

where $C^1(u,v) = -F(u,v),\ C^2(u,v) = -G(u,v)$.

The determining equations for finding coefficients of operator (5) and functions $C^1(u,v)$, $C^2(u,v)$ from system (6) have the form

$$\begin{aligned}
&1)\ \xi_{uu} = \xi_{vv} = \xi_{uv} = 0,\\
&2)\ \eta^1_{vv} = 0,\\
&3)\ \eta^2_{uu} = 0,\\
&4)\ 2\lambda_1\xi\xi_u + \eta^1_{uu} - 2\xi_{xu} = 0,\\
&5)\ 2\lambda_2\xi\xi_v + \eta^2_{vv} - 2\xi_{xv} = 0,\\
&6)\ (\lambda_1+\lambda_2)\xi\xi_v + 2\eta^1_{uv} - 2\xi_{xv} = 0,\\
&7)\ (\lambda_1+\lambda_2)\xi\xi_u + 2\eta^2_{uv} - 2\xi_{xu} = 0,\\
&8)\ (\lambda_1-\lambda_2)\xi\eta^1_v + 2\eta^1_{xv} - 2\xi_v C^1 - 2\lambda_1\xi_v\eta^1 = 0,\\
&9)\ (\lambda_2-\lambda_1)\xi\eta^2_u + 2\eta^2_{xu} - 2\xi_u C^2 - 2\lambda_2\xi_u\eta^2 = 0,\\
&10)\ \lambda_1(2\xi_u\eta^1 - \xi_t - \xi_v\eta^2 - 2\xi\xi_x) + \lambda_2\xi_v\eta^2 + 3\xi_u C^1 + \xi_v C^2 - 2\eta^1_{xu} + \xi_{xx} = 0,\\
&11)\ \lambda_2(2\xi_v\eta^2 - \xi_t - \xi_u\eta^1 - 2\xi\xi_x) + \lambda_1\xi_u\eta^1 + 3\xi_v C^2 + \xi_u C^1 - 2\eta^2_{xv} + \xi_{xx} = 0,\\
&12)\ \lambda_1(\eta^1_t + \eta^2\eta^1_v + 2\xi_x\eta^1) - \lambda_2\eta^2\eta^1_v + \eta^1 C^1_u + \eta^2 C^1_v - \eta^1_u C^1 + 2\xi_x C^1 - \eta^1_v C^2 - \eta^1_{xx} = 0,\\
&13)\ \lambda_2(\eta^2_t + \eta^1\eta^2_u + 2\xi_x\eta^2) - \lambda_1\eta^1\eta^2_u + \eta^1 C^2_u + \eta^2 C^2_v - \eta^2_u C^1 + 2\xi_x C^2 - \eta^2_v C^2 - \eta^2_{xx} = 0.
\end{aligned} \tag{7}$$

System (7) is an over-determined system of partial differential equations and there are no any general method for solving of such systems [26,27]. Thus, we were not able to find the general solution of system (7), hence we have solved it with conditions

$$\xi = \xi(u,v),\ \eta^i = \eta^i(u,v),\ i = 1,2 \tag{8}$$

Solving Equations 1)–3) of system (7), we obtain

$$\xi = au + bv + c,\ \eta^1 = p^1(u)v + q^1(u),\ \eta^2 = p^2(v)u + q^2(v) \tag{9}$$

where $a,\ b,\ c$ are arbitrary constants, $p^1,\ p^2,\ q^1,\ q^2$ are arbitrary smooth functions. Substituting (9) into 6), 7) from (7) and splitting the obtained equations with respect to the powers of u and v, we arrive at the system

$$\begin{aligned} &a^2(\lambda_1+\lambda_2) = 0,\ b^2(\lambda_1+\lambda_2) = 0, \\ &(\lambda_1+\lambda_2)a(bv+c) + 2p^2_v = 0,\ (\lambda_1+\lambda_2)b(au+c) + 2p^1_u = 0 \end{aligned} \tag{10}$$

Obviously, that solutions of first pair of equations of (10) will be $\lambda_2 = -\lambda_1$, or $a = b = 0$.

Let us consider the case $\lambda_2 = -\lambda_1$ (the case $a = b = 0$ will be considered later). In this case we obtain $p^1 = const = \alpha_1,\ p^2 = const = \alpha_2$. Substituting (9) $p^1 = \alpha_1,\ p^2 = \alpha_2$ into Equations $4)$ and $5)$ of system (7) and splitting with respect to the powers of u and v, we arrive at

$$2\lambda_1 ab = 0,\ q^1_{uu} + 2\lambda_1 a(au + c) = 0,\ q^2_{vv} - 2\lambda_1 b(bv + c) = 0 \tag{11}$$

Since $\lambda_1 \neq 0$, we conclude that $ab = 0$. Consider the case $a = 0,\ b \neq 0$ (the case $b = 0$, $a \neq 0$ is symmetrical). From Equation $8)$, we obtain $\alpha_2 = 0$. Substituting (9) with the specified coefficients, namely

$$\xi = bv + c,\ \eta^1 = \alpha_1 v + q^1,\ \eta^2 = q^2,$$

into Equations $10),\ 11)$ of system (7), we arrive at

$$b(C^2 - 2\lambda_1 q^2) = 0,\ b(3C^2 - 2\lambda_1 q^2) = 0 \tag{12}$$

Substituting $q^2 = 0$, obtained from (12), into the third equation of system (11), we obtain $\lambda_1 b(bv+c) = 0$, that is $\lambda_1 b = 0$, but that contradicts the above restrictions.

Thus, in the case $\lambda_2 = -\lambda_1$ we do not obtain any Q-conditional operator of the form (5).

Consider the case $a = b = 0$. In this case, from Equations $4),\ 5),\ 6)$ and $7)$ of system (7), we obtain

$$p^i = const = \alpha_i,\ i = 1, 2,\ q^1 = \beta_1 u + \gamma_1,\ q^2 = \beta_2 v + \gamma_2,$$

where $\beta_i, \gamma_i,\ i = 1, 2$ are the arbitrary constants. Thus, expressions (9) take the form

$$\xi = c,\ \eta^1 = \alpha_1 v + \beta_1 u + \gamma_1,\ \eta^2 = \alpha_2 u + \beta_2 v + \gamma_2 \tag{13}$$

Substituting (13) into Equations $8)$ and $9)$ of system (7), we arrive at

$$c\alpha_1(\lambda_2 - \lambda_1) = 0,\ c\alpha_2(\lambda_2 - \lambda_1) = 0 \tag{14}$$

Solving the system of algebraic Equations (14), we obtain three solutions $\lambda_2 = \lambda_1,\ \alpha_1 = \alpha_2 = 0$ and $c = 0$, therefore we obtain three cases. Let us consider all these cases.

Theorem 1. *In the cases $\lambda_2 = \lambda_1$ or $\eta^1_v = \eta^2_u = 0$ with conditions (8), the system of determining equations for finding of the Q-conditional operators of the form (5) for system (6) coincide with the system of determining equations for finding Lie operators.*

Proof. Substituting (13), with $\lambda_2 = \lambda_1$, into system (7) we find that Equations $1) - 11)$ are transformed into identities, and Equations $12)$ and $13)$ take the form

$$\eta^1 C^1_u + \eta^2 C^1_v - \eta^1_u C^1 - \eta^1_v C^2 = 0, \quad \eta^1 C^2_u + \eta^2 C^2_v - \eta^2_u C^1 - \eta^2_v C^2 = 0 \tag{15}$$

In [11] the determining equations for finding of Lie symmetries with condition $\lambda_2 = \lambda_1$ are written down in explicit form. Substituting conditions (8) into these equations, we see that the result is completely identical to Equations (15).

Substituting (13), with $\alpha_1 = \alpha_2 = 0$ into system (7), we see that Equations $1) - 11)$ also transform into identities, and equations $12)$ and $13)$ take the form

$$\eta^1 C^1_u + \eta^2 C^1_v - \eta^1_u C^1 = 0, \quad \eta^1 C^2_u + \eta^2 C^2_v - \eta^2_v C^2 = 0. \tag{16}$$

Comparing equations (16) with equations for finding of Lie symmetries of system (6) with conditions (8) from [9], we see that they are completely identical. □

Thus, in the following we assume that $\lambda_1 \neq \lambda_2,\ \alpha_1^2 + \alpha_2^2 \neq 0$.

Let us consider the case $c = 0$, which is on the one hand the most interesting and on the other the most difficult. In this case, (13) takes the form

$$\xi = 0,\ \eta^1 = \alpha_1 v + \beta_1 u + \gamma_1,\ \eta^2 = \alpha_2 u + \beta_2 v + \gamma_2 \tag{17}$$

Equations $1) - 11)$ satisfy expressions (17) and Equations $12),\ 13)$ take the form

$$\begin{aligned}
&(\alpha_1 v + \beta_1 u + \gamma_1)C^1_u + (\alpha_2 u + \beta_2 v + \gamma_2)C^1_v - \beta_1 C^1 - \alpha_1 C^2\\
&= \alpha_1(\lambda_2 - \lambda_1)(\alpha_2 u + \beta_2 v + \gamma_2),\\
&(\alpha_1 v + \beta_1 u + \gamma_1)C^2_u + (\alpha_2 u + \beta_2 v + \gamma_2)C^2_v - \alpha_2 C^1 - \beta_2 C^2\\
&= \alpha_2(\lambda_1 - \lambda_2)(\alpha_1 v + \beta_1 u + \gamma_1)
\end{aligned} \tag{18}$$

Thus, we can formulate the following theorem.

Theorem 2. *The nonlinear reaction–diffusion system (6) is Q-conditionally invariant under operator (5) with coefficients (17) if and only if the nonlinearities $C^1,\ C^2$ are the solutions of linear system (18).*

To find the general solution of system (18), one need to analyze two cases $\alpha_2 = 0$ and $\alpha_2 \neq 0$. In the case $\alpha_2 = 0$, system (18) takes the form

$$\begin{aligned}
&(\alpha_1 v + \beta_1 u + \gamma_1)C^1_u + (\beta_2 v + \gamma_2)C^1_v = \beta_1 C^1 + \alpha_1 C^2 + \alpha_1(\lambda_2 - \lambda_1)(\beta_2 v + \gamma_2),\\
&(\alpha_1 v + \beta_1 u + \gamma_1)C^2_u + (\beta_2 v + \gamma_2)C^2_v = \beta_2 C^2
\end{aligned} \tag{19}$$

Since $\alpha_1 \neq 0$, renaming $C^1 \to \alpha_1 C^1$, $u \to \alpha_1 u$ and $\gamma_1 \to \alpha_1 \gamma_1$, and taking into account that with any coefficients β_1, β_2 we can remove the parameter γ_1 using linear substitutions of $u,\ v$, system (19) reduces to the form

$$\begin{aligned}
&(v + \beta_1 u)C^1_u + (\beta_2 v + \gamma_2)C^1_v = \beta_1 C^1 + C^2 + (\lambda_2 - \lambda_1)(\beta_2 v + \gamma_2),\\
&(v + \beta_1 u)C^2_u + (\beta_2 v + \gamma_2)C^2_v = \beta_2 C^2
\end{aligned} \tag{20}$$

One notes a particular solution of system (20), of the form

$$C^1_{part} = \frac{1}{2}(\lambda_2 - \lambda_1)(v + \beta_1 u),\ C^2_{part} = \frac{1}{2}(\lambda_1 - \lambda_2)(\beta_2 v + \gamma_2) \tag{21}$$

Now to construct the general solution of (20), we need to solve the corresponding homogeneous system, that is

$$\begin{aligned}
&(v + \beta_1 u)C^1_u + (\beta_2 v + \gamma_2)C^1_v = \beta_1 C^1 + C^2,\\
&(v + \beta_1 u)C^2_u + (\beta_2 v + \gamma_2)C^2_v = \beta_2 C^2
\end{aligned} \tag{22}$$

As a result, the following statement was proved.

Theorem 3. *Reaction–diffusion system (6) is Q-conditionally invariant under operator (5) with conditions (8), and* $\eta_u^2 = 0$*, if and only if the system and corresponding operator have one of the seven following forms (moreover* $\lambda_2 \neq \lambda_1$*):*

$$\begin{aligned} &u_{xx} = \lambda_1 u_t + g(\omega) v \ln(v) + h(\omega) v + \tfrac{1}{2}(\lambda_2 - \lambda_1)(v + \beta_1 u), \\ &v_{xx} = \lambda_2 v_t + \beta_1 g(\omega) v + \tfrac{1}{2}(\lambda_1 - \lambda_2)\beta_1 v, \\ &Q = \partial_t + (v + \beta_1 u)\partial_u + \beta_1 v \partial_v, \\ &\omega = v^{-1} \exp(\tfrac{\beta_1 u}{v}),\ \beta_1 \neq 0 \end{aligned} \tag{23}$$

$$\begin{aligned} &u_{xx} = \lambda_1 u_t + h(\omega) v^{\frac{\beta_1}{\beta_2}} + g(\omega) v + \tfrac{1}{2}(\lambda_2 - \lambda_1)(v + \beta_1 u), \\ &v_{xx} = \lambda_2 v_t + (\beta_2 - \beta_1) g(\omega) v + \tfrac{1}{2}(\lambda_1 - \lambda_2)\beta_2 v, \\ &Q = \partial_t + (v + \beta_1 u)\partial_u + \beta_2 v \partial_v, \\ &\omega = v^{-\frac{\beta_1}{\beta_2}}((\beta_1 - \beta_2)u + v),\ \beta_1\beta_2(\beta_1 - \beta_2) \neq 0 \end{aligned} \tag{24}$$

$$\begin{aligned} &u_{xx} = \lambda_1 u_t + u g(v) + h(v), \\ &v_{xx} = \lambda_2 v_t + v g(v), \\ &Q = \partial_t + v \partial_u \end{aligned} \tag{25}$$

$$\begin{aligned} &u_{xx} = \lambda_1 u_t + g(\omega) v + h(\omega) + \tfrac{1}{2}(\lambda_2 - \lambda_1) v, \\ &v_{xx} = \lambda_2 v_t + g(\omega) + \tfrac{1}{2}(\lambda_1 - \lambda_2), \\ &Q = \partial_t + v \partial_u + \partial_v, \\ &\omega = 2u - v^2 \end{aligned} \tag{26}$$

$$\begin{aligned} &u_{xx} = \lambda_1 u_t + g(\omega) v + h(\omega) + \tfrac{1}{2}(\lambda_2 - \lambda_1) v, \\ &v_{xx} = \lambda_2 v_t + \beta_2 g(\omega)\,(v + \tau_2) + \tfrac{1}{2}\beta_2(\lambda_1 - \lambda_2)(v + \tau_2), \\ &Q = \partial_t + v \partial_u + \beta_2 (v + \tau_2)\partial_v, \\ &\omega = \beta_2 u - v + \tau_2 \ln(v + \tau_2),\ \beta_2 \neq 0 \end{aligned} \tag{27}$$

$$\begin{aligned} &u_{xx} = \lambda_1 u_t + (v + u) h(v) - g(v), \\ &v_{xx} = \lambda_2 v_t + g(v), \\ &Q = \partial_t + \beta_1 (v + u)\partial_u,\ \beta_1 \neq 0 \end{aligned} \tag{28}$$

$$\begin{aligned} &u_{xx} = \lambda_1 u_t + h(\omega) \exp(\tfrac{\beta_1}{\gamma_2} v) - g(\omega) + \tfrac{1}{2}(\lambda_2 - \lambda_1)(v + \beta_1 u), \\ &v_{xx} = \lambda_2 v_t + \beta_1 g(\omega) + \tfrac{1}{2}\gamma_2(\lambda_1 - \lambda_2), \\ &Q = \partial_t + (v + \beta_1 u)\partial_u + \gamma_2 \partial_v, \\ &\omega = \exp(-\tfrac{\beta_1}{\gamma_2} v)\,(\beta_1^2 u + \beta_1 v + \gamma_2),\ \beta_1\gamma_2 \neq 0 \end{aligned} \tag{29}$$

Proof. To prove this theorem, it is necessary and sufficient to construct the general solution of system (22) for all possible ratios between parameters $\beta_1,\ \beta_2,\ \gamma_2$. To do this we need to investigate the following seven cases:

1. $\beta_1\beta_2 \neq 0,\ \beta_1 = \beta_2$;
2. $\beta_1\beta_2 \neq 0,\ \beta_1 \neq \beta_2$;
3. $\beta_2 = 0,\ \beta_1 = 0,\ \gamma_2 = 0$;
4. $\beta_2 = 0,\ \beta_1 = 0,\ \gamma_2 \neq 0$;
5. $\beta_2 \neq 0,\ \beta_1 = 0$.

6. $\beta_2 = 0,\ \beta_1 \neq 0,\ \gamma_2 = 0;$
7. $\beta_2 = 0,\ \beta_1 \neq 0,\ \gamma_2 \neq 0.$

These cases take into account all possibilities that arise when we solve system (22). Let us consider these cases.

Case 1. Solving the second equation of (22), we get $C^2 = \beta_1 v g(\omega)$ and $\omega = v^{-1}\exp\left(\frac{\beta_1 u}{v}\right)$. So the first equation of (22) reduces to an ODE for finding of the function C^1:

$$C^1_v - \frac{C^1}{v} = g(\omega).$$

Solving it, we get that $C^1 = g(\omega)v\ln(v) + vh(\omega)$. Taking into account the expressions for C^1, C^2, ω, obtained above, C^1_{part}, C^2_{part} from Formulas (21) and restrictions (obtained above), finally we arrive at the reaction–diffusion system and the Q-conditional operator listed in (23) of Theorem 3.

Cases 2–7. Considering similarly these cases and using simple renamings, we arrive at systems and operators (24)–(29) of Theorem 3. □

In the case $\alpha_2 \neq 0$ we should also assume that $\alpha_1 \neq 0$, otherwise we obtain the case $\alpha_2 = 0$ up to renaming. We seek a solution of system (18) of the form

$$C^1 = r_1 u + r_2 v + r_3,\ C^2 = s_1 u + s_2 v + s_3 \tag{30}$$

Substituting (30) into (18), we obtain the system of algebraic equations

$$\begin{aligned}
&\alpha_2 r_2 - \alpha_1 s_1 + \alpha_1\alpha_2\left(\lambda_1 - \lambda_2\right) = 0,\\
&\alpha_1 r_1 + \left(\beta_2 - \beta_1\right) r_2 - s_2\alpha_1 + \alpha_1\beta_2\left(\lambda_1 - \lambda_2\right) = 0,\\
&\gamma_1 r_1 + \gamma_2 r_2 - \beta_1 r_3 - \alpha_1 s_3 + \alpha_1\gamma_2\left(\lambda_1 - \lambda_2\right) = 0,\\
&\alpha_2 r_1 + \left(\beta_2 - \beta_1\right) s_1 - s_2\alpha_2 + \alpha_2\beta_1\left(\lambda_1 - \lambda_2\right) = 0,\\
&\alpha_2 r_3 - \gamma_1 s_1 - \gamma_2 s_2 + \beta_2 s_3 + \alpha_2\gamma_1\left(\lambda_1 - \lambda_2\right) = 0
\end{aligned} \tag{31}$$

Solving system (31), we arrive at two possibilities depending on $\Delta = \alpha_1\alpha_2 - \beta_1\beta_2$:

I) $\Delta = 0$,

$$r_1 = \tfrac{\beta_1}{\alpha_2}s_1 + \beta_1\left(\lambda_2 - \lambda_1\right),\ r_2 = \tfrac{\alpha_1}{\alpha_2}s_1 + \alpha_1\left(\lambda_2 - \lambda_1\right),$$
$$r_3 = \tfrac{\alpha_2\gamma_1 + \beta_2\gamma_2}{\alpha_2^2}s_1 - \tfrac{\beta_2}{\alpha_2}s_3 + \gamma_1\left(\lambda_2 - \lambda_1\right), s_2 = \tfrac{\beta_2}{\alpha_2}s_1.$$

II) $\Delta \neq 0$,

$$r_1 = \tfrac{(\beta_1 - \beta_2)}{\alpha_2}s_1 + s_2 + \beta_1\left(\lambda_2 - \lambda_1\right),\ r_2 = \tfrac{\alpha_1}{\alpha_2}s_1 + \alpha_1\left(\lambda_2 - \lambda_1\right),$$
$$r_3 = \tfrac{\left(\Delta + \beta_2^2\right)\gamma_1 - \alpha_1\beta_2\gamma_2}{\alpha_2\Delta}s_1 + \tfrac{\alpha_1\gamma_2 - \beta_2\gamma_1}{\Delta}s_2 + \gamma_1\left(\lambda_2 - \lambda_1\right),\ s_3 = \tfrac{\alpha_1\gamma_2 - \beta_2\gamma_1}{\Delta}s_1 + \tfrac{\alpha_2\gamma_1 - \beta_1\gamma_2}{\Delta}s_2.$$

In Case I) $s_1 = s_3 = 0$, we obtain the solution of system (18)

$$C^1_{part} = \left(\lambda_2 - \lambda_1\right)\left(\frac{\beta_1}{\alpha_2}\left(\alpha_2 u + \beta_2 v\right) + \gamma_1\right),\ C^2_{part} = 0 \tag{32}$$

In Case II) $s_1 = s_2 = 0$ we obtain the solution of system (18)

$$C^1_{part} = \left(\lambda_2 - \lambda_1\right)\left(\alpha_1 v + \beta_1 u + \gamma_1\right),\ C^2_{part} = 0 \tag{33}$$

Furthermore, we must solve the homogeneous system

$$\begin{array}{l}(\alpha_1 v+\beta_1 u+\gamma_1)C^1_u+(\alpha_2 u+\beta_2 v+\gamma_2)C^1_v=\beta_1 C^1+\alpha_1 C^2,\\ (\alpha_1 v+\beta_1 u+\gamma_1)C^2_u+(\alpha_2 u+\beta_2 v+\gamma_2)C^2_v=\alpha_2 C^1+\beta_2 C^2\end{array} \quad (34)$$

Let us consider Case I). Using the condition $\Delta = 0$ for system (34), we get

$$\begin{array}{l}(\frac{\beta_1}{\alpha_2}(\alpha_2 u+\beta_2 v)+\gamma_1)C^1_u+(\alpha_2 u+\beta_2 v+\gamma_2)C^1_v=\frac{\beta_1}{\alpha_2}(\alpha_2 C^1+\beta_2 C^2),\\ (\frac{\beta_1}{\alpha_2}(\alpha_2 u+\beta_2 v)+\gamma_1)C^2_u+(\alpha_2 u+\beta_2 v+\gamma_2)C^2_v=\alpha_2 C^1+\beta_2 C^2\end{array} \quad (35)$$

Multiplying the second equation of (35) by $-\frac{\beta_1}{\alpha_2}$, adding to the first and renaming $u \to u - \frac{\gamma_2}{\alpha_2}$, $\gamma_1 \to \gamma_1 + \frac{\beta_1\gamma_2}{\alpha_2}$, we arrive at

$$\left(\frac{\beta_1}{\alpha_2}(\alpha_2 u+\beta_2 v)+\gamma_1\right)\left(C^1-\frac{\beta_1}{\alpha_2}C_2\right)_u+(\alpha_2 u+\beta_2 v)\left(C^1-\frac{\beta_1}{\alpha_2}C_2\right)_v=0 \quad (36)$$

Using the substitution

$$C^1 = S(u,v)+\frac{\beta_1}{\alpha_2}C_2 \quad (37)$$

we obtain the equation

$$\left(\frac{\beta_1}{\alpha_2}(\alpha_2 u+\beta_2 v)+\gamma_1\right)S_u+(\alpha_2 u+\beta_2 v)S_v=0 \quad (38)$$

Solving Equation (38), we arrive at three subcases:
1) $\beta_2=-\beta_1,\ \gamma_1=0,\ S=S(\omega),\ \omega=\alpha_2 u-\beta_1 v;$
2) $\beta_2=-\beta_1,\ \gamma_1\neq 0,\ S=S(\omega),\ \omega=(\alpha_2 u-\beta_1 v)^2-2\alpha_2\gamma_1 v;$
3) $\beta_2\neq-\beta_1,\ S=S(\omega),\ \omega=\alpha_2 u-\beta_1 v-\frac{\alpha_2\gamma_1}{\beta_1+\beta_2}\ln\left|\alpha_2 u+\beta_2 v+\frac{\alpha_2\gamma_1}{\beta_1+\beta_2}\right|.$

Substituting (37) together with the function S from subcase 1) into the second equation of (35), we obtain

$$(\alpha_2 u-\beta_1 v)\left(\alpha_2 {C_v}^2+\beta_1 {C_u}^2\right)=\alpha_2^2 f(\omega) \quad (39)$$

Solving (39), using (37), (32) and renaming $u\to\beta_1 u,\ v\to\alpha_2 v$ we obtain the system

$$\begin{array}{l}u_{xx}=\lambda_1 u_t+f(\omega)u+\beta_1(g(\omega)-\lambda_1)(u-v),\\ v_{xx}=\lambda_2 v_t+f(\omega)v+\beta_1(g(\omega)-\lambda_2)(u-v),\ \omega=u-v,\end{array}$$

Q-conditionally invariant under the operator

$$Q=\partial_t+\beta_1(u-v)(\partial_u+\partial_v).$$

Similarly, for subcase 2), we arrive at the system

$$\begin{array}{l}u_{xx}=\lambda_1 u_t+(f(\omega)-\lambda_1)(\beta_1(u-v)+\gamma_1)+g(\omega),\\ v_{xx}=\lambda_2 v_t+\beta_1(f(\omega)-\lambda_2)(u-v)+g(\omega),\ \omega=(u-v)^2-2\frac{\gamma_1}{\beta_1}v,\ \gamma_1\neq 0\end{array}$$

and the operator

$$Q=\partial_t+(\beta_1(u-v)+\gamma_1)\partial_u+\beta_1(u-v)\partial_v.$$

In the subcase 3), we obtain the system

$$u_{xx} = \lambda_1 u_t + k(f(\omega) - \lambda_1) + \beta_1(g(\omega) - \lambda_1)\,(u+v)\,,$$
$$v_{xx} = \lambda_2 v_t - k(f(\omega) - \lambda_2) + \beta_2(g(\omega) - \lambda_2)\,(u+v)\,,$$
$$\omega = \beta_2 u - \beta_1 v - k\ln|u+v|,\ k = \frac{\gamma_1}{\beta_1+\beta_2},$$

and the operator

$$Q = \partial_t + (\beta_1(u+v)+k)\partial_u + (\beta_2(u+v)-k)\partial_v.$$

Examination of Case II) is highly nontrivial and will be reported in another paper.

3. Ansätze and Exact Solutions of the Reaction–Diffusion System

Using standard procedures, we obtain ansätze for all operators of Theorem 3. Substituting these anzätze in the corresponding reaction–diffusion systems, we obtain the reduction systems of equations. All anzätze and reduction systems are presented in Table 1.

Table 1. Ansätze and reduction systems of Theorem 3.

No.	Ansätze	Systems of ODEs
(23)	$u=(t+\psi)\,e^{\beta_1 t+\varphi}$	$\varphi''+(\varphi')^2-\beta_1\left(g\left(e^{\beta_1\psi-\varphi}\right)+\frac{\lambda_1+\lambda_2}{2}\right)=0$
	$v=e^{\beta_1 t+\varphi}$	$\psi''+2\varphi'\psi'+g\left(e^{\beta_1\psi-\varphi}\right)(\beta_1\psi-\varphi)-h\left(e^{\beta_1\psi-\varphi}\right)-\frac{\lambda_1+\lambda_2}{2}=0$
(24)	$u=\psi\left(e^{\beta_2 t+\varphi}\right)^{\frac{\beta_1}{\beta_2}}-\frac{e^{\beta_2 t+\varphi}}{\beta_1-\beta_2}$	$\varphi''+(\varphi')^2+g\left((\beta_1-\beta_2)\,\psi\right)(\beta_1-\beta_2)-\beta_2\frac{\lambda_1+\lambda_2}{2}=0$
	$v=e^{\beta_2 t+\varphi}$	$\psi''+\frac{2\beta_1}{\beta_2}\varphi'\psi'+\frac{\beta_1}{\beta_2}\left(\frac{\beta_1}{\beta_2}-1\right)\psi\left((\varphi')^2-\beta_2 g\left((\beta_1-\beta_2)\,\psi\right)\right)-h\left((\beta_1-\beta_2)\,\psi\right)=0$
(25)	$u=\varphi t+\psi$	$\varphi''-\varphi g(\varphi)=0$
	$v=\varphi$	$\psi''-g(\varphi)\psi-h(\varphi)-\lambda_1\varphi=0$
(26)	$u=\frac{1}{2}t^2+\varphi t+\psi$	$\varphi''-g\left(2\psi-\varphi^2\right)-\frac{\lambda_1+\lambda_2}{2}=0$
	$v=t+\varphi$	$\psi''-g\left(2\psi-\varphi^2\right)\varphi-h\left(2\psi-\varphi^2\right)-\frac{\lambda_1+\lambda_2}{2}\varphi=0$
(27)	$u=\frac{1}{\beta_2}e^{\beta_2 t+\varphi}-\tau_2 t+\psi$	$\varphi''+(\varphi')^2-\beta_2\,g\left(\tau_2(\varphi+1)+\beta_2\psi\right)-\frac{\lambda_1+\lambda_2}{2}\beta_2=0$
	$v=e^{\beta_2 t+\varphi}-\tau_2$	$\psi''-h\left(\tau_2(\varphi+1)+\beta_2\psi\right)+\tau_2 g\left(\tau_2(\varphi+1)+\beta_2\psi\right)+\frac{\lambda_1+\lambda_2}{2}\tau_2=0$
(28)	$u=\psi e^{\beta_1 t}-\varphi$	$\varphi''-g(\varphi)=0$
	$v=\varphi$	$\psi''-\left(h(\varphi)+\beta_1\lambda_1\right)\psi=0$
(29)	$u=\psi e^{\beta_1 t}-\frac{\gamma_2 t}{\beta_1}-\frac{\varphi}{\beta_1}-\frac{\gamma_2}{\beta_1^2}$	$\varphi''-g\left(\beta_1^2\psi e^{-\frac{\beta_1\varphi}{\gamma_2}}\right)\beta_1-\frac{(\lambda_1+\lambda_2)}{2}\gamma_2=0$
	$v=\gamma_2 t+\varphi$	$\psi''-e^{\frac{\beta_1\varphi}{\gamma_2}}h\left(\beta_1^2\psi e^{-\frac{\beta_1\varphi}{\gamma_2}}\right)-\beta_1\frac{\lambda_1+\lambda_2}{2}\psi=0$

It is impossible to find the general solution of the systems from Table 1 for arbitrary functions g and h. However, if we correctly specify these functions we can find the solutions of these systems.

System (27) is the most interesting one from the point of view of applicability. Let us consider system (27) with $\tau_2 = 0,\ g(\omega) = b_1\omega + b_0 - \frac{\lambda_1+\lambda_2}{2},\ h(\omega) = a_2\omega^2 + a_1\omega + a_0$. In this case, the reduction system has the form

$$\varphi'' + (\varphi')^2 - \beta_2(b_1\beta_2\psi + b_0) = 0 \qquad (40)$$

$$\psi'' - a_2\beta_2^2\psi^2 - a_1\beta_2\psi - a_0 = 0 \qquad (41)$$

The solution of Equation (41) has the form

$$\psi = k = const \qquad (42)$$

Substituting (42) into (40), we arrive at

$$\varphi'' + (\varphi')^2 = A,\ A = \beta_2 (b_1 \beta_2 k + b_0) \tag{43}$$

The solutions of Equation (43) depend on the parameter A. Solving Equation (43) we get three different solutions (up to transformations $x \to x + C_1,\ C_1 = const$)

$$\varphi = \begin{cases} \ln|Cx|,\ A = 0; \\ \ln|C\cosh(\sqrt{A}x)|,\ A > 0; \\ \ln|C\cos(\sqrt{-A}x)|,\ A < 0. \end{cases}$$

Substituting φ and (42) into corresponding ansatz from Table 1, and renaming $C \to \beta_2 C$, we arrive at the exact solutions

$$\begin{gathered} u = Ce^{\beta_2 t}x + k,\ v = C\beta_2 e^{\beta_2 t}x, \\ u = Ce^{\beta_2 t}\cosh\left(\sqrt{A}x\right) + k,\ v = C\beta_2 e^{\beta_2 t}\cosh\left(\sqrt{A}x\right) \\ u = Ce^{\beta_2 t}\cos\left(\sqrt{-A}x\right) + k,\ v = C\beta_2 e^{\beta_2 t}\cos\left(\sqrt{-A}x\right) \end{gathered} \tag{44}$$

of the reaction–diffusion system

$$\begin{aligned} u_{xx} &= \lambda_1 u_t + (v - u\beta_2)(-a_1 + a_2 v - a_2\beta_2 u) + (b_0 - \lambda_1^* - b_1 v + b_1\beta_2 u)v + a_0, \\ v_{xx} &= \lambda_2 v_t + \beta_2 (b_0 - \lambda_2^* - b_1 v + b_1 \beta_2 u) v \end{aligned} \tag{45}$$

where k is the solution of the equation $a_2\beta_2^2 k^2 + a_1\beta_2 k + a_0 = 0$.

4. Solutions and Their Properties of Some Generalization of the Lotka–Volterra System

Let us consider in detail the case $A < 0$. Renaming $\beta_2 = -\frac{B_2}{C_2},\ b_1 = -\frac{C_2^2}{B_2},\ b_0 = \frac{A_2C_2 + B_2\lambda_2}{B_2}$, $a_0 = -e_0,\ a_1 = \frac{A_1C_2}{B_2},\ a_2 = -\frac{B_1C_2^2}{B_2^2}$, we obtain the exact solution

$$u = Ce^{-\frac{B_2}{C_2}t}\cos\left(\sqrt{-A}x\right) + k,\ v = -\frac{B_2 C}{C_2}e^{-\frac{B_2}{C_2}t}\cos\left(\sqrt{-A}x\right) \tag{46}$$

where $A = -\frac{kB_2C_2 + A_2C_2 + B_2\lambda_2}{C_2}$, and k is the solution of $B_1k^2 + A_1k + e_0 = 0$, of the reaction–diffusion system

$$\begin{aligned} \lambda_1 u_t &= u_{xx} + u(A_1 + B_1 u + C_1 v) + e_2 v^2 + e_1 v + e_0, \\ \lambda_2 v_t &= v_{xx} + v(A_2 + B_2 u + C_2 v) \end{aligned} \tag{47}$$

where $C_1 = \left(\frac{2B_1}{B_2} - 1\right)C_2,\ e_2 = \frac{(B_1 - B_2)C_2^2}{B_2^2},\ e_1 = \frac{(A_1 - A_2)C_2}{B_2} + \lambda_1^* - \lambda_2^*$.

System (47) is the generalized Lotka–Volterra system. With $e_2 = e_1 = e_0 = 0$ system (47) becomes the classical Lotka–Volterra system

$$\begin{aligned} \lambda_1 u_t &= u_{xx} + u(A_1 + B_1 u + C_1 v), \\ \lambda_2 v_t &= v_{xx} + v(A_2 + B_2 u + C_2 v) \end{aligned} \tag{48}$$

Note that exact solutions of the form (46) for the classical Lotka–Volterra system (48) have been found in [24].

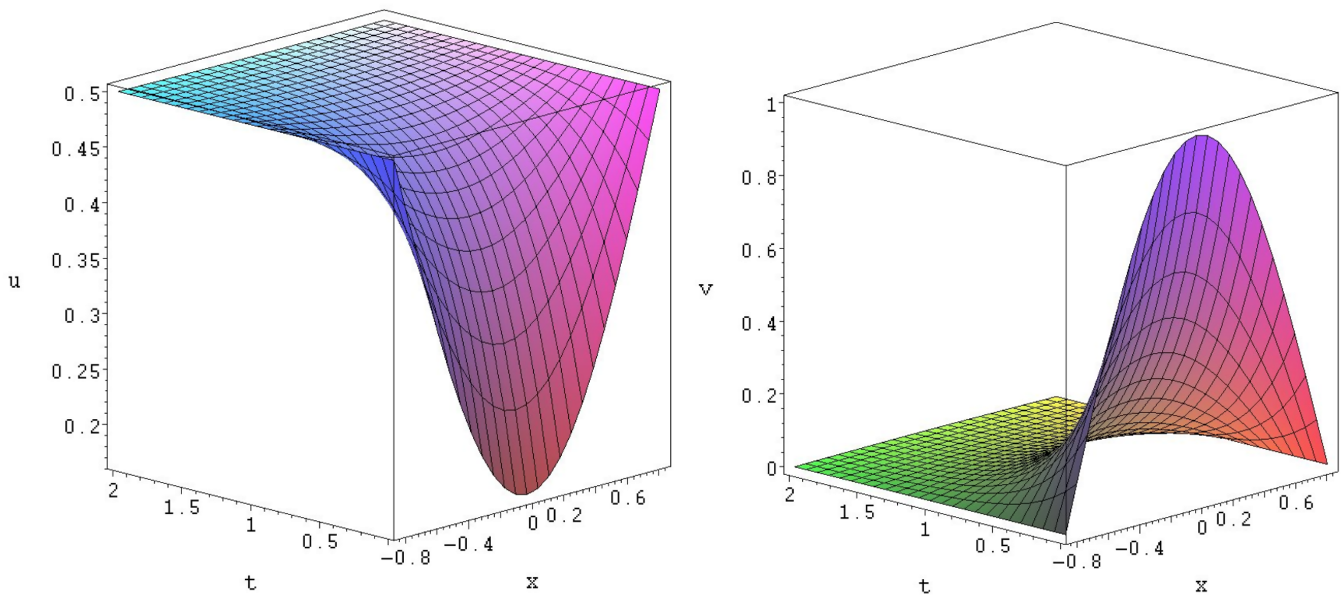

Figure 1. Exact solution to (51).

System (48) can be obtained from system (47) with $e_0 = 0,\ B_1 = B_2,\ A_1 = \frac{A_2C_2+\left(\lambda_2^*-\lambda_1^*\right)B_2}{C_2}$. Also, the coefficients of (46) and (48) must satisfy the equation $k\left(k + \frac{A_2}{B_2} + \frac{\lambda_2^*-\lambda_1^*}{C_2}\right) = 0$.

It is well known [3] that three main kinds of interactions between two biological species are simulated by system (48):

(i) predator u–prey v interaction,

(ii) competition of the species,

(iii) mutualism or symbiosis.

It turns out that solution (46) can describe the predator-prey interaction on the space interval $[-l, l]$, (here $l = \frac{\pi}{2\sqrt{-A}}$) provided that

$$B_2 < 0,\ C_2 < 0,\ C < 0,\ k > |C| \tag{49}$$

One can easily check that solution (46) is non-negative, bounded in the domain $\Omega = \{(t, x) \in (0, +\infty) \times (-l, l)\}$ and satisfies the given Dirichlet boundary conditions, *i.e.*,

$$u|_{x=-l} = k,\ v|_{x=-l} = 0,\ u|_{x=l} = k,\ v|_{x=l} = 0 \tag{50}$$

Choosing the coefficients $\lambda_1 = 2,\ \lambda_2 = 1,\ A_2 = 1,\ B_2 = -1,\ C_2 = -\frac{1}{3},\ B_1 = 0,\ C = -\frac{1}{3},\ e_0 = 1$, gives that $A_1 = -2,\ C_1 = \frac{1}{3},\ k = \frac{1}{2}$. Thus, from solution (46) we obtain the solution

$$u = \frac{1}{2} - \frac{1}{3}e^{-3t}\cos(\sqrt{\frac{7}{2}}x),\ v = e^{-3t}\cos(\sqrt{\frac{7}{2}}x) \tag{51}$$

of the system

$$\begin{array}{l} 2u_t = u_{xx} + u\left(-2 + \frac{v}{3}\right) + \frac{v^2}{9} + 1, \\ v_t = v_{xx} + v\left(1 - u - \frac{v}{3}\right) \end{array} \tag{52}$$

which can describe predator u–prey v interaction, as its coefficients satisfy the conditions for this type of the interaction [3]. System (52) is some generalization of the Lotka–Volterra system (48) with additional nonlinearity $\frac{v^2}{9} + 1$ in the first equation.

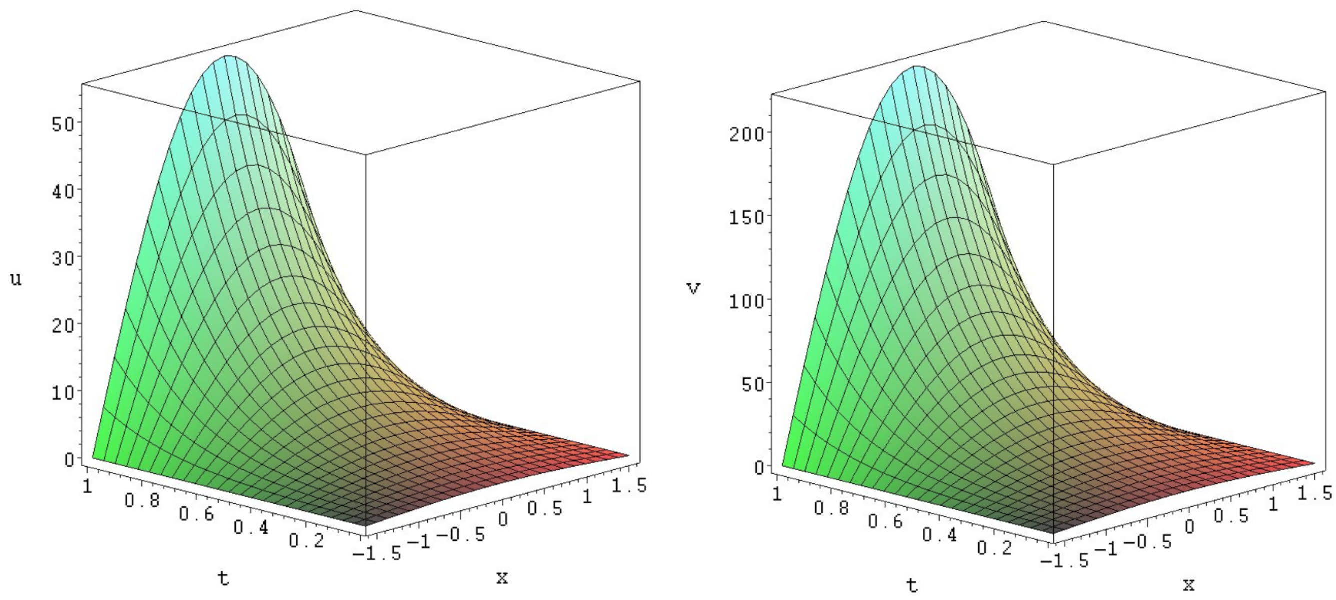

Figure 2. Exact solution of (55).

Solution (51) satisfies Dirichlet boundary conditions (50) with $l = \frac{\pi}{\sqrt{14}},\ k = \frac{1}{2}$.

As an example, we present solution (51) in Figure 1. This solution can describe the predator u–prey v interaction between the species u and v when population of predator u becomes $\frac{1}{2}$ and prey eventually dies, *i.e.*, $(u, v) \to (\frac{1}{2}, 0)$ as $t \to +\infty$.

Choosing coefficients $\lambda_1 = \frac{3}{2},\ \lambda_2 = \frac{5}{4},\ A_2 = 1,\ B_2 = -4,\ C_2 = -\frac{1}{2},\ B_1 = 1,\ C = -\frac{1}{4},\ e_0 = 0$, we get $A_1 = -1,\ C_1 = \frac{3}{4},\ k = 1$. Renaming $t \to \frac{1}{4}t$, from solution (46) we obtain the solution

$$u = 1 - \frac{1}{4}e^{-2t}\cos\left(\sqrt{7}x\right),\ v = 2e^{-2t}\cos\left(\sqrt{7}x\right) \tag{53}$$

of the system

$$\begin{aligned} 6u_t &= u_{xx} + u\left(-1 + u + \tfrac{3}{4}v\right) + \tfrac{5}{64}v^2, \\ 5v_t &= v_{xx} + v\left(1 - 4u - \tfrac{v}{2}\right) \end{aligned} \tag{54}$$

which can also describe the predator u–prey v interaction, as its coefficients satisfy conditions for this type of interaction [3]. System (52) is some generalization of Lotka–Volterra system (48) with additional nonlinearity $\frac{5v^2}{64}$ in the first equation.

Solution (53) satisfies Dirichlet boundary conditions (50) with $l = \frac{\pi}{2\sqrt{7}}$ and $k = 1$. This solution can describe the predator u–prey v interaction between the species u and v when population of predator u becomes 1 and prey eventually die, *i.e.*, $(u, v) \to (1, 0)$ as $t \to +\infty$.

If we consider system (48) with solution (46), then we obtain the solution that can describe competition of the species. Such a solution is presented in [24].

Also, system (47) can describe mutualism—or symbiosis—of two species. Choosing the coefficients $\lambda_1 = 2,\ \lambda_2 = 1,\ A_2 = 5,\ B_2 = 2,\ C_2 = -\frac{1}{2},\ B_1 = -\frac{1}{10},\ k = 0,\ C = 1,\ e_0 = 0$, we obtain $A_1 = 9,\ C_1 = \frac{11}{20}$. So, from solution (46) we obtain the solution

$$u = e^{4t}\cos(x),\ v = 4e^{4t}\cos(x) \tag{55}$$

of the system

$$\begin{aligned} 2u_t &= u_{xx} + u\left(9 - \tfrac{u}{10} + \tfrac{11v}{20}\right) - \tfrac{21v^2}{160}, \\ v_t &= v_{xx} + v\left(5 - \tfrac{v}{2} + 2u\right) \end{aligned} \quad (56)$$

which is a generalization of Lotka–Volterra system (48) with additional nonlinearity $-\frac{21}{160}v^2$ in the first equation.

Solution (55) satisfies Dirichlet boundary conditions (50) with $l = \frac{\pi}{2}$, $k = 0$. As an example, we present solution (55) in Figure 2. Solution (55) can describe the type of the interaction between the species u and v when both populations grow unboundedly, *i.e.*, $(u, v) \to (+\infty, +\infty)$ if $t \to +\infty$.

5. Conclusions

In this paper, the nonlinear RD system (2) was examined in order to find the Q-conditional operators under which this system is invariant and to construct exact solutions. Because the system of differential equations (7) is too complicated, we were unable (and believe it is not possible) to find all the solutions of the determining system (7) and thence to find all possible Q-conditional operators. We have found the Q–conditional operators with restrictions (8) (in the case $\eta_u^2 = 0$ we have found all possible systems and operators, in the case $\eta_u^2 \neq 0$ we have presented some examples) with respect to which the reaction–diffusion system of equations with constant diffusion (2) is invariant. All these operators are given in Theorem 3 of Section 2. In Section 3 the ansätze for all Q-conditional operators of Theorem 3 and the reduction systems are constructed. Section 4 contains the solutions of some generalization of the Lotka–Volterra system. These solutions are analyzed in order to present of biological interpretation. Some graphs of obtained solutions are also presented. It is shown that the obtained solutions satisfy Dirichlet boundary conditions, which are typical for biological interpretation.

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Conflicts of Interest

The author declares no conflict of interest.

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