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Review

Dynamical Symmetries and Causality in Non-Equilibrium Phase Transitions

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Abstract: Dynamical symmetries are of considerable importance in elucidating the complex behaviour of strongly interacting systems with many degrees of freedom. Paradigmatic examples are cooperative phenomena as they arise in phase transitions, where conformal invariance has led to enormous progress in equilibrium phase transitions, especially in two dimensions. Non-equilibrium phase transitions can arise in much larger portions of the parameter space than equilibrium phase transitions. The state of the art of recent attempts to generalise conformal invariance to a new generic symmetry, taking into account the different scaling behaviour of space and time, will be reviewed. Particular attention will be given to the causality properties as they follow for co-variant n-point functions. These are important for the physical identification of n-point functions as responses or correlators.

Keywords: Schrödinger algebra; conformal Galilei algebra; ageing algebra; representations; causality; parabolic sub-algebra; holography; physical ageing

1. Introduction

Improving our understanding of the collective behaviour of strongly interacting systems consisting of a large number of strongly interacting degrees of freedom is an ongoing challenge. From the point of view of the statistical physicist, paradigmatic examples are provided by systems undergoing a continuous phase transition, where fluctuation effects render traditional methods such as mean-field approximations inapplicable [1,2]. At the same time, it turns out that these systems can be effectively characterised in terms of a small number of "relevant" scaling operators, such that the net effect of all other physical quantities, the "irrelevant" ones, merely amounts to the generation of corrections to the leading scaling behaviour. From a symmetry perspective, phase transitions naturally acquire some kind of scale-invariance, and it then becomes a natural question whether further dynamical symmetries can be present.

1.1. Conformal Algebra

In equilibrium critical phenomena (roughly, for systems with sufficiently short-ranged, local interactions), scale-invariance can be extended to conformal invariance. In two space dimensions, the generators $\ell_n, \bar{\ell}_n$ should obey the infinite-dimensional algebra

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}, \quad \left[\bar{\ell}_n, \bar{\ell}_m\right] = (n-m)\bar{\ell}_{n+m}, \quad \left[\ell_n, \bar{\ell}_m\right] = 0 \tag{1}$$

for $n, m \in \mathbb{Z}$. The action of these generators on physical scaling operators $\phi(z, \bar{z})$, where complex coordinates $z, \bar{z}$ are used, is conventionally given by the representation [3]

$$\ell_n \phi(z, \bar{z}) \to [\ell_n, \phi(z, \bar{z})] = -\left(z^{n+1}\partial_z + \Delta(n+1)z^n\right)\phi(z, \bar{z}) \tag{2}$$

and similarly for $\bar{\ell}_n$, where the rôles of z and $\bar{z}$ are exchanged. Herein, the conformal weights Δ, $\overline{\Delta}$ are real constants, and related to the scaling dimension $x_\phi = \Delta + \overline{\Delta}$ and the spin $s_\phi = \Delta - \overline{\Delta}$ of the scaling operator ϕ. The representation (2) is an infinitesimal form of the (anti)holomorphic transformations $z \mapsto w(z)$ and $\bar{z} \mapsto \bar{w}(\bar{z})$. The maximal finite-dimensional sub-algebra of Equation (1) is isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \cong \left\langle \ell_{\pm 1,0}, \bar{\ell}_{\pm 1,0} \right\rangle$. It is this conformal sub-algebra only which has an analogue in higher space dimensions $d > 2$. Denoting the Laplace operator by $\mathcal{S} := 4\partial_z\partial_{\bar{z}} = 4\ell_{-1}\bar{\ell}_{-1}$, the conformal invariance of the Laplace equation $\mathcal{S}\phi(z, \bar{z}) = 0$ is expressed through the commutator

$$[\mathcal{S}, \ell_n]\,\phi(z, \bar{z}) = -(n+1)z^n\mathcal{S}\phi(z, \bar{z}) - 4\Delta(n+1)nz^{n-1}\partial_{\bar{z}}\phi(z, \bar{z}) \tag{3}$$

and analogously for $\bar{\ell}_n$. Hence, for vanishing conformal weights $\Delta = \Delta_\phi = 0$ and $\overline{\Delta} = \overline{\Delta}_\phi = 0$, any solution of $\mathcal{S}\phi = 0$ is mapped onto another solution of the same equation. Thermal fluctuations in $2D$ classical critical points or quantum fluctuations in $1D$ quantum critical points (at temperature $T = 0$) modify the conformal algebra (1) to a pair of commuting Virasoro algebras, parametrised by the central charge c. Then Equation (2) retains its validity when the set of admissible operators ϕ is restricted to the set of primary scaling operators (a scaling operator is called quasi-primary if the transformation (2) only holds for the finite-dimensional sub-algebra $\mathfrak{sl}(2, \mathbb{R}) \cong \langle \ell_{\pm 1,0} \rangle$) [4]. In turn, this furnishes the basis for the derivation of conformal Ward identities obeyed by n-point correlation functions $F_n := \langle \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle$ of primary operators $\phi_1 \ldots \phi_n$. Celebrated theorems provide a classification of the Virasoro primary operators from the unitary representations of the Virasoro algebra, for example through the Kac formula for central charges $c < 1$ [5,6]. Novel physical applications are continuously being discovered.

1.2. Schrödinger Algebra

When turning to time-dependent critical phenomena, the theory is far less advanced. One of the best-studied examples is the Schrödinger–Virasoro algebra $\mathfrak{sv}(d)$ in d space dimensions [7,8]

$$
\begin{aligned}
\left[X_n, X_{n'}\right] &= (n-n')X_{n+n'} \quad , \quad \left[X_n, Y_m^{(j)}\right] = \left(\frac{n}{2}-m\right) Y_{n+m}^{(j)} \\
\left[X_n, M_{n'}\right] &= -n' M_{n+n'} \quad , \quad \left[X_n, R_{n'}^{(jk)}\right] = -n' R_{n+n'}^{(jk)} \\
\left[Y_m^{(j)}, Y_{m'}^{(k)}\right] &= \delta^{j,k}\,(m-m')\,M_{m+m'} \quad , \quad \left[R_n^{(jk)}, Y_m^{(\ell)}\right] = \delta^{j,\ell}\, Y_{n+m}^{(k)} - \delta^{k,\ell}\, Y_{n+m}^{(j)} \\
\left[R_n^{(jk)}, R_{n'}^{(\ell i)}\right] &= \delta^{j,i} R_{n+n'}^{(\ell k)} - \delta^{k,\ell} R_{n+n'}^{(ji)} + \delta^{k,i} R_{n+n'}^{(j\ell)} - \delta^{j,\ell} R_{n+n'}^{(ik)}
\end{aligned} \tag{4}
$$

(all other commutators vanish) with integer indices $n, n' \in \mathbb{Z}$, half-integer indices $m, m' \in \mathbb{Z} + \frac{1}{2}$ and $i, j, k, \ell \in \{1, \ldots, d\}$. Casting the generators of $\mathfrak{sv}(d)$ into the four families $X, Y^{(j)}, M, R^{(jk)} = -R^{(kj)}$ makes explicit (i) that the generators X_n form a conformal sub-algebra and (ii) that the families $Y^{(j)}$ and $M, R^{(jk)}$ make up Virasoro primary operators of weight $\frac{3}{2}$ and 1, respectively [7]. Non-trivial central extensions are only possible (i) either in the conformal sub-algebra $\langle X_n \rangle_{n\in\mathbb{Z}}$, where it must be of the form of the Virasoro central charge, or else (ii) in the $\mathfrak{so}(d)$-current algebra $\left\langle R_n^{(jk)} \right\rangle_{n\in\mathbb{Z}}$, where it must be a Kac–Moody central charge [5–7,9]. The maximal finite-dimensional sub-algebra of $\mathfrak{sv}(d)$ is the Schrödinger algebra $\mathfrak{sch}(d) = \left\langle X_{0,\pm 1}, Y_{\pm 1/2}^{(j)}, M_0, R_0^{(jk)} \right\rangle_{j,k=1,\ldots d}$, where M_0 is central. An explicit representation in terms of time-space coordinates $(t, \boldsymbol{r}) \in \mathbb{R} \times \mathbb{R}^d$, acting on a (scalar) scaling operator $\phi(t, \boldsymbol{r})$ of scaling dimension x and of mass $\mathcal{M}$, is given by [7]

$$
\begin{aligned}
X_n &= -t^{n+1}\partial_t - \frac{n+1}{2} t^n \boldsymbol{r}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}} - \frac{\mathcal{M}}{4}(n+1)n t^{n-1}\boldsymbol{r}^2 - \frac{n+1}{2} x t^n \\
Y_m^{(j)} &= -t^{m+1/2}\partial_j - \left(m+\frac{1}{2}\right) t^{m-1/2}\mathcal{M} r_j \\
M_n &= -t^n \mathcal{M} \\
R_n^{(jk)} &= -t^n\left(r_j\partial_k - r_k\partial_j\right) = -R_n^{(kj)}
\end{aligned} \tag{5}
$$

with the abbreviations $\partial_j := \partial/\partial r_j$ and $\boldsymbol{\nabla}_{\boldsymbol{r}} = (\partial_1, \ldots, \partial_d)^{\mathrm{T}}$. These are the infinitesimal forms of the transformations $(t, \boldsymbol{r}) \mapsto (t', \boldsymbol{r}')$, where

$$
\begin{aligned}
X_n :&\quad t = \beta(t'), \quad \boldsymbol{r} = \boldsymbol{r}'\sqrt{\frac{\mathrm{d}\beta(t')}{\mathrm{d}t'}} \\
Y_m :&\quad t = t', \quad \boldsymbol{r} = \boldsymbol{r}' - \alpha(t') \\
R_n :&\quad t = t', \quad \boldsymbol{r} = \mathcal{R}(t')\boldsymbol{r}'
\end{aligned} \tag{6}
$$

where $\alpha(t)$ is an arbitrary time-dependent function, $\beta(t)$ is a non-decreasing function and $\mathcal{R}(t) \in SO(d)$ denotes a rotation matrix with time-dependent rotation angles. The generators M_n do not generate a time-space transformation, but rather produce a time-dependent "phase shift" of the scaling operator ϕ [10].

The dilatations X_0 are the infinitesimal form of the transformations $t \mapsto \lambda^z t$ and $\boldsymbol{r} \mapsto \lambda\boldsymbol{r}$, where $\lambda \in \mathbb{R}_+$ is a constant and z is called the *dynamical exponent*. In the representation (5), one has $z = 2$.

Since the work of Lie [12], and before of Jacobi [13], the Schrödinger algebra is known to be a dynamic symmetry of the the free diffusion equation (and, much later, also of the free Schrödinger equation). Define the Schrödinger operator

$$
\mathcal{S} = 2\mathcal{M}\partial_t - \boldsymbol{\nabla}_{\boldsymbol{r}}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}} = 2M_0 X_{-1} - \boldsymbol{Y}_{-1/2}\cdot\boldsymbol{Y}_{-1/2} \tag{7}
$$

Following Niederer [14], dynamical symmetries of such linear equations are analysed through the commutators of $\mathcal{S}$ with the symmetry Lie algebra. For the case of $\mathfrak{sch}(d)$, the only non-vanishing commutators with $\mathcal{S}$ are

$$[\mathcal{S}, X_0] = -\mathcal{S}, \quad [\mathcal{S}, X_1] = -2t\mathcal{S} - (2x - d)M_0 \tag{8}$$

Hence, any solution ϕ of the free Schrödinger/diffusion equation $\mathcal{S}\phi = 0$ with scaling dimension $x_\phi = \frac{d}{2}$ is mapped onto another solution of the free Schrödinger equation [15]. Finally, from representations such as Equation (5), one can derive Schrödinger–Ward identities in order to compute the form of covariant n-point functions $\langle \phi_1(t_1, \boldsymbol{r}_1) \ldots \phi_n(t_n, \boldsymbol{r}_n)\rangle$. With respect to conformal invariance, one has the important difference that the generator $M_0 = -\mathcal{M}$ is central in the finite-dimensional non-semi-simple Lie algebra $\mathfrak{sch}(d)$. This implies the Bargman super-selection rule [17]

$$(\mathcal{M}_1 + \cdots + \mathcal{M}_n) \ \langle \phi_1(t_1, \boldsymbol{r}_1) \ldots \phi_n(t_n, \boldsymbol{r}_n)\rangle = 0 \tag{9}$$

Physicists' conventions require that "physical masses" $\mathcal{M}_i \geq 0$. It it therefore necessary to define a formal "complex conjugate" ϕ^* of the scaling operator ϕ, such that its mass $\mathcal{M}^* := -\mathcal{M} \leq 0$ becomes negative. Then one may write, e.g., a non-vanishing co-variant two-point function of two quasi-primary scaling operators (up to an undetermined constant of normalisation) [7]

$$\langle \phi_1(t_1, \boldsymbol{r}_1)\phi_2^*(t_2, \boldsymbol{r}_2)\rangle = \delta_{x_1,x_2}\delta(\mathcal{M}_1 - \mathcal{M}_2^*)\,(t_1 - t_2)^{-x_1} \exp\left[-\frac{\mathcal{M}_1}{2}\frac{(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2}{t_1 - t_2}\right] \tag{10}$$

Here and throughout this paper, $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ if $a \neq b$. While Equation (10) looks at first sight like a reasonable heat kernel, a closer inspection raises several questions:

1. Why should it be obvious that the time difference $t_1 - t_2 > 0$, to make the power-law prefactor real-valued ?
2. Given the convention that $\mathcal{M}_1 \geq 0$, the condition $t_1 - t_2 > 0$ is also required in order to have a decay of the two-point function with increasing distance $|\boldsymbol{r}| = |\boldsymbol{r}_1 - \boldsymbol{r}_2| \to \infty$.
3. In applications to non-equilibrium statistical physics, one studies indeed two-point functions of the above type, which are then interpreted as the linear response function of the scaling operator ϕ with respect to an external conjugate field $h(t, \boldsymbol{r})$

$$R(t_1, t_2; \boldsymbol{r}_1, \boldsymbol{r}_2) = \left.\frac{\delta\langle\phi(t_1, \boldsymbol{r}_1)\rangle}{\delta h(t_2, \boldsymbol{r}_2)}\right|_{h=0} = \left\langle \phi(t_1, \boldsymbol{r}_1)\widetilde{\phi}(t_2, \boldsymbol{r}_2)\right\rangle \tag{11}$$

   which in the context of the non-equilibrium Janssen–de Dominicis theory [2] can be re-expressed as a two-point function involving the scaling operator ϕ and its associate response operator $\widetilde{\phi}$. In this physical context, one has a natural interpretation of the "complex conjugate" in terms of the relationship of ϕ and $\widetilde{\phi}$.

   Then, the formal condition $t_1 - t_2 > 0$ simply becomes the causality condition, namely that a response will only arise at a later time $t_1 > t_2$ after the stimulation at time $t_2 \geq 0$.

Hence, it is necessary to inquire under what conditions the causality of Schrödinger-covariant n-point functions can be guaranteed.

1.3. Conformal Galilean Algebra

Textbooks in quantum mechanics show that the Schrödinger equation is the non-relativistic variant of relativistic wave equations, be it the Klein–Gordon equation for scalars or the Dirac equations for spinors. One might therefore expect that the Schrödinger algebra could be obtained by a contraction from the conformal algebra, but this is untrue (although there is a well-known contraction from the Poincaré algebra to the Galilei sub-algebra). Rather, applying a contraction to the conformal algebra, one arrives at a different Lie algebra, which we call here the altern-Virasoro algebra [11,18–20]. $\mathfrak{av}(d) = \left\langle X_n, Y_n^{(j)}, R_n^{(jk)} \right\rangle_{n\in\mathbb{Z}}$ with $j,k = 1,\dots,d$, but which nowadays is often referred to as infinite conformal Galilean algebra. Its non-vanishing commutators can be given as follows

$$
\begin{array}{rcl}
[X_n, X_{n'}] & = & (n-n')X_{n+n'} \quad , \quad [X_n, Y_m^{(j)}] \;=\; (n-m)\, Y_{n+m}^{(j)} \\
{[X_n, R_{n'}^{(jk)}]} & = & -n' R_{n+n'}^{(jk)} \qquad , \quad [R_n^{(jk)}, Y_m^{(\ell)}] \;=\; \delta^{j,\ell} Y_{n+m}^{(k)} - \delta^{k,\ell} Y_{n+m}^{(j)} \\
{[R_n^{(jk)}, R_{n'}^{(\ell i)}]} & = & \delta^{j,i} R_{n+n'}^{(\ell k)} - \delta^{k,\ell} R_{n+n'}^{(ji)} + \delta^{k,i} R_{n+n'}^{(j\ell)} - \delta^{j,\ell} R_{n+n'}^{(ik)}
\end{array}
\tag{12}
$$

An explicit representation as time-space transformation is [21]

$$
\begin{array}{rcl}
X_n & = & -t^{n+1}\partial_t - (n+1)t^n \boldsymbol{r}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}} - n(n+1)t^{n-1}\boldsymbol{\gamma}\cdot\boldsymbol{r} - x(n+1)t^n \\
Y_n^{(j)} & = & -t^{n+1}\partial_j - (n+1)t^n\gamma_j \\
R_n^{(jk)} & = & -t^n\left(r_j\partial_k - r_k\partial_j\right) - t^n\left(\gamma_j\partial_{\gamma_k} - \gamma_k\partial_{\gamma_j}\right) \;=\; -R_n^{(kj)}
\end{array}
\tag{13}
$$

where $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_d)$ is a vector of dimensionful constants, called rapidities, and x is again a scaling dimension. The dynamical exponent $z=1$. The maximal finite-dimensional sub-algebra of $\mathfrak{av}(d)$ is the conformal Galilean algebra $\mathrm{CGA}(d) = \langle X_{\pm1,0}, Y_{\pm1,0}^{(j)}, R_0^{(jk)}\rangle_{j,k=1,\dots,d}$ [11,18,22–27].

A more abstract characterisation of $\mathfrak{av}(1)$ can be given in terms of α-densities $\mathcal{F}_\alpha = \{u(z)(\mathrm{d}z)^\alpha\}$, with the action

$$
f(z)\frac{\mathrm{d}}{\mathrm{d}z}\left(u(z)(\mathrm{d}z)^\alpha\right) = (fu' + \alpha f'u)(z)(\mathrm{d}z)^\alpha \tag{14}
$$

Lemma 1. [33] *One has the isomorphism, where* $\ltimes$ *denotes the semi-direct sum*

$$
\mathfrak{av}(1) \cong \mathrm{Vect}(S^1) \ltimes \mathcal{F}_{-1} \tag{15}
$$

Clearly, it follows that $\mathrm{CGA}(1) \cong \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathcal{F}_{-1}$.

As before, the time-space representation (13) can be used to derive conformal-Galilean Ward identities. For example, the $\mathrm{CGA}(d)$-covariant two-point function takes the form

$$
\langle\phi_1(t_1,\boldsymbol{r}_1)\phi_2(t_2,\boldsymbol{r}_2)\rangle = \delta_{x_1,x_2}\delta_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2}\,(t_1-t_2)^{-2x_1}\exp\left[-\frac{\boldsymbol{\gamma}_1\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)}{t_1-t_2}\right] \tag{16}
$$

Again, at first sight this looks physically reasonable, but several questions must be raised:

1. Why should one have $t_1 - t_2 > 0$ for the time difference, as required to make the power-law prefactor real-valued ?
2. Even for a fixed vector $\boldsymbol{\gamma}_1$ of rapidities, and even if $t_1 - t_2 > 0$ could be taken for granted, how does one guarantee that the scalar product $\boldsymbol{\gamma}_1\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2) > 0$, such that the two-point function decreases as $|\boldsymbol{r}| = |\boldsymbol{r}_1-\boldsymbol{r}_2| \to \infty$?

The finite-dimensional $\text{CGA}(2)$ admits a so-called "exotic" central extension [34,35]. Abstractly, this is achieved by completing the commutators (12) by the following

$$\left[Y_n^{(1)}, Y_m^{(2)}\right] = \delta_{n+m,0}\left(3\delta_{n,0} - 2\right)\Theta, \quad n, m \in \{\pm 1, 0\} \tag{17}$$

with a central generator Θ. This is called the exotic Galilean conformal algebra $\text{ECGA} = \text{CGA}(2) + \mathbb{C}\Theta$ in the physics literature. A representation as time-space transformation of ECGA is, with $n \in \{\pm 1, 0\}$ and $j, k \in \{1, 2\}$ [21,24,36]

$$\begin{aligned}
X_n &= -t^{n+1}\partial_t - (n+1)t^n \boldsymbol{r}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}} - x(n+1)t^n - (n+1)nt^{n-1}\boldsymbol{\gamma}\cdot\boldsymbol{r} - (n+1)n\boldsymbol{h}\cdot\boldsymbol{r} \\
Y_n^{(j)} &= -t^{n+1}\partial_j - (n+1)t^n\gamma_j - (n+1)t^n h_j - n(n+1)\theta\varepsilon_{jk}r_k \\
R_0^{(12)} &= -\left(r_1\partial_2 - r_2\partial_1\right) - \left(\gamma_1\partial_{\gamma_2} - \gamma_2\partial_{\gamma_1}\right) - \frac{1}{2\theta}\boldsymbol{h}\cdot\boldsymbol{h}
\end{aligned} \tag{18}$$

The components of the vector $\boldsymbol{h} = (h_1, h_2)$ satisfy $[h_i, hj] = \varepsilon_{ij}\theta$, where θ is a constant, ε is the totally antisymmetric 2×2 tensor and $\varepsilon_{12} = 1$ [37]. The dynamical exponent $z = 1$. Because of Schur's lemma, the central generator Θ can be replaced by its eigenvalue $\theta \neq 0$. The ECGA-invariant Schrödinger operator is

$$\mathcal{S} = -\theta X_{-1} + \varepsilon_{ij}Y_0^{(i)}Y_{-1}^{(j)} = \theta\partial_t + \varepsilon_{ij}\left(\gamma_i + h_i\right)\partial_j \tag{19}$$

with $x = x_\phi = 1$. The requirement that these representations should be unitary gives the bound $x \geq 1$ [24]. Co-variant n-point functions and their applications have been studied in great detail.

1.4. Ageing Algebra

The common sub-algebra of $\mathfrak{sch}(d)$ and $\text{CGA}(d)$ is called the ageing algebra $\mathfrak{age}(d) := \langle X_{0,1}, Y_{\pm\frac{1}{2}}^{(j)}, M_0, R_0^{(jk)}\rangle$ with $j, k = 1, \ldots, d$ and does not include time-translations. Starting from the representation (5), only the generators X_n assume a more general form [38]

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n\boldsymbol{r}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}} - \frac{n+1}{2}xt^n - n(n+1)\xi t^n - \frac{n(n+1)}{4}\mathcal{M}t^{n-1}\boldsymbol{r}^2 \tag{20}$$

such that $z = 2$ is kept from Equation (5). When the generator X_n is applied to a scaling operator, the constant ξ describes a second scaling dimension, besides the habitual one denoted here by x, of that scaling operator ϕ. It is an important new aspect of extended dynamical symmetries, far from a stationary state, that at least two distinct scaling dimensions of a given scaling operator ϕ must be introduced. This will be made explicit later through concrete examples.

The invariant Schrödinger operates now becomes $\mathcal{S} = 2\mathcal{M}\partial_t - \partial_r^2 + 2\mathcal{M}t^{-1}\left(x + \xi - \frac{1}{2}\right)$, but without any constraint, neither on x nor on ξ [39]. Co-variant n-point functions can be derived as before [1,38,40], but we shall include these results with those to be derived from more general representations in the next sections. The absence of time-translations is particular appealing for application to dynamical critical phenomena, such as physical ageing, in non-stationary states far from equilibrium, see [1].

In Figure 1 (on page 19 below), the root diagrammes [41] of the Lie algebra (a) $\mathfrak{age}(1)$, (b) $\mathfrak{sch}(1)$ and (c) $\text{CGA}(1)$ are shown, where the generators (roots) are represented by the black dots. This visually illustrates that the Schrödinger and conformal Galilean algebras are not isomorphic, $\mathfrak{sch}(d) \not\cong \text{CGA}(d)$.

Comparing Figure 1a with Figure 1c, a different representation of $\mathrm{CGA}(1)$ can be identified. This representation is spanned by the generators $X_{0,1}, Y_{\pm 1/2}, M_0$ from Equation (5), along with a new generator V_+, and leads to a dynamic exponent $z = 2$ [11]. It is not possible to extend this to a representation of $\mathfrak{av}(1)$ [33]. Explicit expressions of V_+ will be given in Section 4.

This algebra also appears in more systematic approaches, either from a classification of non-relativistic limits of conformal symmetries [42] or else from an attempt to construct all possible infinitesimal local scale transformations [8,18].

1.5. Langevin Equation and Reduction formulæ

In non-equilibrium statistical mechanics [2], one considers often equations under the form of a stochastic Langevin equation, viz. (we use the so-called "model-A" dynamics with a non-conserved order-parameter)

$$2\mathcal{M}\partial_t \phi = \boldsymbol{\nabla}_{\boldsymbol{r}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} \phi - \frac{\delta \mathcal{V}[\phi]}{\delta \phi} + \eta \tag{21}$$

for a physical field ϕ (called the order parameter), and where $\delta/\delta\phi$ stands for a functional derivative. Herein, $\mathcal{V}[\phi]$ is the Ginzburg–Landau potential and η is a white noise, *i.e.*, its formal time-integral is a Brownian motion. In the context of Janssen–de Dominicis theory, see [2], this can be recast as the variational equation of motion of the functional

$$\begin{aligned} \mathcal{J}[\phi,\widetilde{\phi}] &= \mathcal{J}_0[\phi,\widetilde{\phi}] + \mathcal{J}_b[\widetilde{\phi}] \\ \mathcal{J}_0[\phi,\widetilde{\phi}] &= \int_{\mathbb{R}_+\times\mathbb{R}^d} \mathrm{d}t\mathrm{d}\boldsymbol{r}\, \widetilde{\phi} \left(2\mathcal{M}\partial_t\phi - \boldsymbol{\nabla}_{\boldsymbol{r}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}}\phi + \frac{\delta\mathcal{V}[\phi]}{\delta\phi} \right) \\ \mathcal{J}_b[\widetilde{\phi}] &= -T \int_{\mathbb{R}_+\times\mathbb{R}^d} \mathrm{d}t\mathrm{d}\boldsymbol{r}\, \widetilde{\phi}^2(t,\boldsymbol{r}) - \frac{1}{2}\int_{\mathbb{R}^{2d}} \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}'\, \widetilde{\phi}(0,\boldsymbol{r}) c(\boldsymbol{r}-\boldsymbol{r}')\widetilde{\phi}(0,\boldsymbol{r}') \end{aligned} \tag{22}$$

where the term $\mathcal{J}_0[\phi,\widetilde{\phi}]$ contains the deterministic terms coming from the Langevin equation and $\mathcal{J}_b[\widetilde{\phi}]$ contains the stochastic terms generated by averaging over the thermal noise and the initial condition, characterised by an initial correlator $c(\boldsymbol{r})$ [43]. In particular, by adding an external source term $h(t,\boldsymbol{r})\phi(t,\boldsymbol{r})$ to the potential $\mathcal{V}[\phi]$, one can write the two-time linear response function as follows (spatial arguments are suppressed for brevity)

$$R(t,s) = \left.\frac{\delta\langle\phi(t)\rangle}{\delta h(s)}\right|_{h=0} = \int \mathcal{D}\phi\mathcal{D}\widetilde{\phi}\, \phi(t)\widetilde{\phi}(s) e^{-\mathcal{J}[\phi,\widetilde{\phi}]} = \left\langle \phi(t)\widetilde{\phi}(s) \right\rangle \tag{23}$$

with an explicit expression of the average $\langle . \rangle$ as a functional integral.

Theorem 1. [44] *If in the functional $\mathcal{J}[\phi,\widetilde{\phi}] = \mathcal{J}_0[\phi,\widetilde{\phi}] + \mathcal{J}_b[\widetilde{\phi}]$, the part $\mathcal{J}_0$ is Galilei-invariant with non-vanishing masses and $\mathcal{J}_b[\widetilde{\phi}]$ does not contain the field ϕ, then the computation of all responses and correlators can be reduced to averages which only involve the Galilei-invariant part $\mathcal{J}_0$.*

Proof. We illustrate the main idea for the calculation of the two-time response. Define the average $\langle X \rangle_0 = \int \mathcal{D}\phi\mathcal{D}\widetilde{\phi}\, X[\phi] e^{-\mathcal{J}_0[\phi,\widetilde{\phi}]}$ with respect to the functional $\mathcal{J}_0[\phi,\widetilde{\phi}]$. Then, from Equation (23)

$$R(t,s) = \left\langle \phi(t)\widetilde{\phi}(s) e^{-\mathcal{J}_b[\widetilde{\phi}]} \right\rangle_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left\langle \phi(t)\widetilde{\phi}(s)\mathcal{J}_b[\widetilde{\phi}]^k \right\rangle_0 = \left\langle \phi(t)\widetilde{\phi}(s) \right\rangle_0$$

since the Bargman super-selection rule (9) implies that only the term with $k = 0$ remains. Hence the response function $R(t,s) = R_0(t,s)$ is reduced to the expression obtained from the deterministic part $\mathcal{J}_0$ of the action.

Analogous reduction formulæ can be derived for all Galilei-covariant n-point responses and correlators [1,44]. □

This means that one may study the deterministic, noiseless truncation of the Langevin equation and its symmetries, provided that spatial translation- and Galilei-invariance are included therein, in order to obtain the form of the stochastic two-time response functions, as it will be obtained from models, simulations or experiments.

This work is organised as follows. In Section 2, we review several distinct representations of the Schrödinger and conformal Galilean algebras, discuss the associated invariant Schrödinger operators an co-covariant two-point functions. Applications to non-equilibrium statistical mechanics and the non-relativistic AdS/CFT correspondence will be indicated. In Section 3, the dual representations and the extensions to parabolic sub-algebras will be reviewed. In Section 4, it will be shown how to use these, to algebraically derive causality and long-distance properties of co-variant two-point functions. Conclusions are given in Section 5.

2. Representations

We now list several results relevant for the extension of the representations discussed in the introduction. The basic new fact, first observed in [40], is compactly stated as follows.

Proposition 1. *Let γ be a constant and $g(z)$ a non-constant function. Then the generators*

$$\ell_n = -z^{n+1}\partial_z - n\gamma z^n - g(z)z^n \tag{24}$$

obey the conformal algebra $[\ell_n, \ell_m] = (n-m)\ell_{n+m}$ for all $n, m \in \mathbb{Z}$.

The commutator is readily checked. We point out that the rapidity γ serves as a second scaling dimension and the choice of the function $g(z)$ can be helpful to include effects of corrections to scaling into the generators of time-space transformations. Next, we give an example on how these terms in the generators ℓ_n appear in the two-point function, co-variant under the maximal finite-dimensional sub-algebra $\langle \ell_{\pm 1,0}\rangle$.

Proposition 2. *If $\phi(z)$ is a quasi-primary scaling operator under the representation (24) of the conformal algebra $\langle \ell_{\pm 1,0}\rangle$, its co-variant two-point function is, where φ_0 is a normalisation constant*

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \varphi_0\, \delta_{\gamma_1,\gamma_2}\, (z_1 - z_2)^{-\gamma_1-\gamma_2}\Gamma_1(z_1)\Gamma_2(z_2), \quad \Gamma_i(z) := z^{\gamma_i} \exp\left(-\int_1^z \mathrm{d}\zeta\, \frac{g(\zeta)}{\zeta}\right) \tag{25}$$

Proof. For brevity, denote $F(z_1, z_2) = \langle \phi_1(z_1)\phi_2(z_2)\rangle$. Then the co-variance of F is expressed by the three Ward identities, with $\partial_i := \partial/\partial z_i$

$$\begin{aligned}
\ell_{-1}F &= \left(-\partial_1 - \partial_2 + \gamma_1 z_1^{-1} + \gamma_2 z_2^{-1} - g(z_1)z_1^{-1} - g(z_2)z_2^{-1}\right) F = 0 \\
\ell_0 F &= \left(-z_1\partial_1 - z_2\partial_2 - g(z_1) - g(z_2)\right) F = 0 \\
\ell_1 F &= \left(-z_1^2\partial_1 - z_2^2\partial_2 - \gamma_1 z_1 - \gamma_2 z_2 - g(z_1)z_1 - g(z_2)z_2\right) F = 0
\end{aligned}$$

Rewrite the correlator as $F(z_1, z_2) = \Gamma_1(z_1)\Gamma_2(z_2)\Psi(z_1, z_2)$. Then the function $\Psi(z_1, z_2)$ satisfies

$$\begin{aligned} (-\partial_1 - \partial_2)\,\Psi &= 0 \\ (-z_1\partial_1 - z_2\partial_2 - \gamma_1 - \gamma_2)\,\Psi &= 0 \\ \left(-z_1^2\partial_1 - z_2^2\partial_2 - 2\gamma_1 z_1 - 2\gamma_2 z_2\right)\Psi &= 0 \end{aligned}$$

which are the standard Ward identities of the representation (2) of conformal invariance, where the γ_i take the rôle of the conformal weights. The resulting function Ψ is well-known [45]. □

One can now generalise the representation (5) of the Schrödinger–Virasoro algebra $\mathfrak{sv}(d)$.

Proposition 3. *If one replaces in the representation (5) the generator X_n as follows*

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n \boldsymbol{r}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}} - \frac{n+1}{2}xt^n - n(n+1)\xi t^n - \Xi(t)t^n - \frac{n(n+1)}{4}\mathcal{M}t^{n-1} \tag{26}$$

where x, ξ are constants and $\Xi(t)$ is an arbitrary (non-constant) function, then the commutators (4) of the Lie algebra $\mathfrak{sv}(d)$ are still satisfied.

This result was first obtained, for the maximal finite-dimensional sub-algebra $\mathfrak{sch}(d)$, by Minic, Vaman and Wu [40], who also further take the dependence on the mass $\mathcal{M}$ into account and write down terms of order $\mathrm{O}(1/\mathcal{M})$ and $\mathrm{O}(1)$ in $v(t)$ explicitly. We extend this observation to $\mathfrak{sv}(d)$, but do not trace the dependence in $\mathcal{M}$ explicitly, although one could re-introduce it, if required. The proof is immediate, since all modifications of the generator X_n merely depend on the time t and none of the other generators of $\mathfrak{sv}(d)$ changes t. For the sub-algebra $\mathfrak{age}(d) \subset \mathfrak{sch}(d)$, the representation (20) is a special case, with arbitrary ξ, but with $\Xi(t) = 0$.

It is obvious that similar extensions of the representations of time-space transformation of the other algebras, especially $\mathfrak{av}(d)$, its finite-dimensional sub-algebra $\mathrm{CGA}(d)$ or the exotic algebra ECGA apply.

Proposition 4. *Consider the representation (5), but with the generators X_n replaced by Equation (26), of the ageing algebra $\mathfrak{age}(d)$ and the Schrödinger algebra $\mathfrak{sch}(d)$. The invariant Schrödinger operator has the form*

$$\mathcal{S} = 2\mathcal{M}\partial_t - \boldsymbol{\nabla}_{\boldsymbol{r}}^2 + 2\mathcal{M}v(t), \quad v(t) = \frac{x+\xi-d/2}{t} + \frac{\Xi(t)}{t} \tag{27}$$

such that a solution of $\mathcal{S}\phi = 0$ is mapped onto another solution of the same equation. For the algebra $\mathfrak{age}(d)$, there is no restriction, neither on x, nor on ξ, nor on $\Xi(t)$. For the algebra $\mathfrak{sch}(d)$, one has the additional condition $x = \frac{d}{2} - 2\xi$.

Proof. To shorten the calculations, we restrict here to $d = 1$. It is enough to restrict attention to the generators $X_{\pm1,0}$, and we must reproduce Equation (8) in this more general setting. We first look at $\mathfrak{age}(1)$. Consideration of X_0 gives $t\dot{v}(t) + v - \dot{\Xi}(t) = 0$ and considering X_1 gives $x + \xi - \frac{1}{2} + \Xi(t) + t\dot{\Xi}(t) - 2tv(t) - t^2\dot{v}(t) = 0$, where the dot denotes the derivative with respect to t. The second relation can be simplified to $x + \xi - \frac{1}{2} + \Xi(t) - tv(t) = 0$ which gives the assertion. Going over to $\mathfrak{sch}(1)$, the condition $[\mathcal{S}, X_{-1}] = 0$ leads to $\xi/t^2 + \dot{\Xi}(t)/t - \Xi(t)/t^2 - \dot{v}(t) = 0$. This is only compatible with the result found before for $\mathfrak{age}(1)$, if $\xi = -x - \xi + \frac{1}{2}$, hence $x = \frac{1}{2} - 2\xi$, as claimed. □

Example 1. *For a physical illustration of the meaning of the explicitly time-dependent terms in the Schrödinger operator (27), we consider the growth of an interface [46]. One may imagine that an interface can be created by randomly depositing particle onto a substrate. The height of this interface will be described by a function $h(t,\boldsymbol{r})$. One usually works in a co-moving coordinate system such that the average height $\langle h(t,\boldsymbol{r})\rangle = 0$ which we shall assume from now on. Then physically interesting quantities are either the interface width $w(t) = \langle h(t,\boldsymbol{r})^2\rangle \sim t^{\beta}$, which for sufficiently long times t defines the* growth exponent *β, or else two-time height-height correlators $C(t,s;\boldsymbol{r}) = \langle h(t,\boldsymbol{r})h(s,\boldsymbol{0})\rangle$ or two-time response functions $R(t,s;\boldsymbol{r}) = \left.\frac{\delta\langle h(t,\boldsymbol{r})\rangle}{\delta j(s,\boldsymbol{0})}\right|_{j=0}$, with respect to an external deposition rate $j(t,\boldsymbol{r})$. Their scaling behaviour is described by several non-equilibrium exponents [1,2]. Herein, spatial translation-invariance was assumed for the sake of simplicity of the notation.*

Physicists have identified several universality classes *of interface growth, see e.g., [2,46]. For the* Edwards–Wilkinson *universality class, h is simply assumed to be a continuous function in space. Its equation of motion for the height is just a free Schrödinger equation with an additional white noise. A distinct universality class is given by the celebrated* Kardar–Parisi–Zhang equation *which contains an additional term, quadratic in $\boldsymbol{\nabla}_{\boldsymbol{r}} h$. A lattice realisation may be obtained by requiring that the heights only take integer values such that the height difference on two neighbouring sites, such that $|\boldsymbol{r}_1-\boldsymbol{r}_2| = a$ where a is the lattice constant, is restricted to $h(t,\boldsymbol{r}_1) - h(t,\boldsymbol{r}_2) = \pm 1$. An intermediate universality class is the one of the* Arcetri model*, where the strong restriction of the Kardar-Parisi-Zhang model is relaxed in that h is taken to be a real-valued function, but subject to the constraint that the sum of its* slopes *$\sum_{\boldsymbol{r}}\langle \boldsymbol{\nabla}_{\boldsymbol{r}} h(t,\boldsymbol{r})^2\rangle \stackrel{!}{=} \mathcal{N}$, where $\mathcal{N}$ is the number of sites of the lattice [47] (this is just one of the many conditions automatically satisfied in lattice realisations of the Kardar–Parisi–Zhang universality class). Schematically, in the continuum limit, the slopes $u_a(t,\boldsymbol{r}) = \partial h(t,\boldsymbol{r})/\partial r_a$ in the Arcetri model satisfy a Langevin equation*

$$\partial_t u_a(t,\boldsymbol{r}) = \Delta_{\boldsymbol{r}} u_a(t,\boldsymbol{r}) + \mathfrak{z}(t) u_a(t,\boldsymbol{r}) + \frac{\partial}{\partial r_a}\eta(t,\boldsymbol{r}) \tag{28}$$

$\Delta_{\boldsymbol{r}}$ is the spatial Laplacian and η is a white noise. The constraint on the slopes can be cast into a simple form by defining

$$g(t) = \exp\left(-2\int_0^t \mathrm{d}\tau\, \mathfrak{z}(\tau)\right) \tag{29}$$

which can be shown to obey a Volterra integral equation

$$dg(t) = 2f(t) + 2T\int_0^t \mathrm{d}\tau\, f(t-\tau)g(\tau), \quad f(t) = d\frac{e^{-4t}I_1(4t)}{4t}\left(e^{-4t}I_0(4t)\right)^{d-1} \tag{30}$$

where T is the "temperature" defined by the second moment of the white noise and the I_n are modified Bessel functions. This model is exactly soluble [47] and the exponents of the (non-stationary) interface growth are distinct from the Edwards–Wilkinson (if $d < 2$) and the Kardar–Parisi–Zhang universality classes.

It turns out that for all dimensions $d > 0$, there is a "critical temperature" $T_c(d) > 0$ such that for $T \leq T_c(d)$, long-range correlations build up. For example, $T_c(1) = 2$ and $T_c(2) = 2\pi/(\pi-2)$. For $T \leq T_c(d)$, the long-time solution of Equation (30) becomes $g(t) \sim t^{-F}$ as $t \to \infty$. This is compatible with the large-time behaviour $\mathfrak{z}(t) \sim t^{-1}$ of the Lagrange multiplier in Equation (28).

Hence, recalling Theorem 1, it is enough to concentrate on the deterministic part. This is given by the Schrödinger operator (27). Therein, the first term in the potential $v(t)$, of order $1/t$, represents the asymptotic behaviour of the Arcetri model; whereas the term described by $\Xi(t)/t$ takes into account the finite-time corrections to this leading scaling behaviour [48].

Example 2. *We give a different illustration of the new representations of $\mathfrak{age}(d)$ with $\xi \neq 0$ (and $\Xi(t) = 0$). Although we shall not be able to write down explicitly the invariant Schrödinger operator of the form specified in Equation (27), this example makes it clear that the domain of application of these representations extends beyond the context of that single differential equation.*

The physical context involved will be the kinetic Ising model with Glauber dynamics. *The statistical mechanics of the Ising model can be described in terms of discrete "spins" $\sigma_i = \pm 1$, attached to each site i of a lattice. In one spatial dimension, one associates to each configuration $\sigma = \{\sigma_1, \ldots, \sigma_{\mathcal{N}}\}$ of spins an energy (hamiltonian) $\mathcal{H} = -\sum_{i=1}^{\mathcal{N}} \sigma_i \sigma_{i+1}$, with periodic boundary conditions $\sigma_{\mathcal{N}+1} = \sigma_1$. The dynamics of these spins is described in terms of a Markov process, such that the "time" $t \in \mathbb{N}$ is discrete. At each time-step, a single spin σ_i is randomly selected and is updated according to the Glauber rates (also referred to as "heat-bath rule") [52]. These are specified in terms of the probabilites*

$$P\left(\sigma_i(t+1) = \pm 1\right) = \frac{1}{2}\left[1 \pm \tanh\left(\frac{1}{T}\left(\sigma_{i-1}(t) + \sigma_{i+1}(t) + h_i(t)\right)\right)\right] \tag{31}$$

where the constant T is the temperature and $h_i(t)$ is a time-dependent external field. From these probabilites alone, the time-evolution of the average of any local observable, such as the time-dependent magnetisation or magnetic correlators, can be evaluated analytically [52]. In one spatial dimension, and at temperature $T = 0$, the model displays dynamical scaling and the exactly-known magnetic two-time correlator and response take a simple form. In the scaling limit $t, s \to \infty$ with t/s being kept fixed, one has [53–55]

$$C(t,s) = \langle \sigma_i(t)\sigma_i(s)\rangle = \frac{2}{\pi}\arctan\sqrt{\frac{2}{t/s-1}} \tag{32}$$

$$R(t,s) = \left.\frac{\delta\langle\sigma_i(t)\rangle}{\delta h_i(s)}\right|_{h=0} = \frac{1}{\sqrt{2}\,\pi}\frac{1}{\sqrt{s(t-s)}} \tag{33}$$

This is independent of the initical conditions (which merely enter into corrections to scaling), hence these results should be interpreted as being relevant to a critical point at $T_c = 0$ [1].

As a first observation, we remark that the form Equation (33) of the auto-response function $R(t,s) = R(t,s;0)$ is not compatible with the prediction Equation (10) of Schrödinger-invariance. This means that the representation (5) of the Schrödinger-algebra $\mathfrak{sch}(1)$, with time-translation-invariance included, is too restrictive to account for the phenomenology of the relaxational behaviour, far from a stationary state, of the one-dimensional Glauber–Ising model [56].

In order to explain the exact results Equations (32) and (33) in terms of the representation (20) of $\mathfrak{age}(d)$, one first generalises the prediction Equation (10) of the Schrödinger algebra as follows, up to normalisation [44]

$$R(t,s;\boldsymbol{r}) = \left\langle \phi(t,\boldsymbol{r})\widetilde{\phi}(s,\boldsymbol{0})\right\rangle = R(t,s)\exp\left[-\frac{\mathcal{M}}{2}\frac{\boldsymbol{r}^2}{t-s}\right]$$
$$= \delta_{x+2\xi,\widetilde{x}+2\widetilde{\xi}}\,\delta(\mathcal{M}+\widetilde{\mathcal{M}})\, s^{-(x+\widetilde{x})/2}\left(\frac{t}{s}\right)^{\xi+F}\left(\frac{t}{s}-1\right)^{-x-2\xi}\exp\left[-\frac{\mathcal{M}}{2}\frac{\boldsymbol{r}^2}{t-s}\right] \tag{34}$$

with $F := \frac{1}{2}(\widetilde{x}-x)+\widetilde{\xi}-\xi$. Herein, ϕ denotes the order parameter, with scaling dimensions x,ξ and of mass $\mathcal{M}>0$. The response field $\widetilde{\phi}$, with scaling dimensions $\widetilde{x},\widetilde{\xi}$ and mass $\widetilde{\mathcal{M}}=-\mathcal{M}<0$ takes over the rôle of the "complex conjugate" in Equation (10), but now time-translation-invariance is no longer required. Spatial translation-invariance is implicitly admitted. Comparison of the auto-response $R(t,s)$ with the exact result Equation (33) leads to the identifications $x=\frac{1}{2}$, $\widetilde{x}=0$, $\xi=0$ and $\widetilde{\xi}=\frac{1}{4}$. Remarkably, only the second scaling dimension $\widetilde{\xi}$ of the response scaling operator $\widetilde{\phi}$ does not vanish—a feature also observed numerically in models such as directed percolation or the Kardar–Parisi–Zhang Equation, see [57–59] for details.

On the other hand, along the lines of Theorem 1, the autocorrelator at the critical point $T=T_c$ can be expressed as an integral of a "noiseless" three-point response, up to normalisation [7]

$$C(t,s)=\int_{\mathbb{R}_+\times\mathbb{R}^d}\mathrm{d}u\mathrm{d}\boldsymbol{R}\,\left\langle\phi(t,\boldsymbol{y})\phi(s,\boldsymbol{y})\widetilde{\phi}^2(u,\boldsymbol{R}+\boldsymbol{y})\right\rangle_0 \tag{35}$$

Ageing-invariance fixes this three-point function up to a certain undetermined scaling function. Herein, one considers $\widetilde{\phi}^2$ as a new composite operator with scaling dimensions $2\widetilde{x}_2, 2\widetilde{\xi}_2$. Up to normalisation, the autocorrelator becomes (assuming $t>s$ for definiteness) [38]

$$C(t,s) = s^{1+d/2-x-\widetilde{x}_2}\left(\frac{t}{s}\right)^{x+F}\left(\frac{t}{s}-1\right)^{\widetilde{x}_2+2\widetilde{\xi}_2-x-2\xi-d/2}$$
$$\times\int_0^1\mathrm{d}v\, v^{2\widetilde{\xi}_2-F}\left[\left(\frac{t}{s}-v\right)(1-v)\right]^{d/2-\widetilde{x}_2-2\widetilde{\xi}_2}\boldsymbol{\Psi}\left(\frac{t/s+1-2v}{t/s-1}\right)$$
$$= C_0\int_0^1\mathrm{d}v\, v^{2\mu}\left[\left(\frac{t}{s}-v\right)(1-v)\right]^{2\mu}\left(\frac{t}{s}+1-2v\right)^{2\mu} \tag{36}$$

where in the second line we recognised that the scaling function can be described in terms of the single parameter $\mu=\xi+\widetilde{\xi}_2$ and there remains an undetermined scaling function $\boldsymbol{\Psi}$. Furthermore, the autocorrelator scaling function should be non-singular as $t\to s$. This implies $\boldsymbol{\Psi}(w)\sim w^{\widetilde{x}_2-x-4\xi-d/2+\mu}$ for $w\gg 1$. The most simple case arises when this form remains valid for all w. Using the values of the scaling exponents identified from the autoresponse $R(t,s)$ before, the exact $1D$ Glauber–Ising autocorrelator Equation (32) is recovered from Equation (36), with the choice $\mu=-\frac{1}{4}$ and $C_0=2/\sqrt{\pi}$ [38].

Although the discrete nature of the Ising spins does not permit to recognise explicitly the continuum equation of motion in the form Equation (27) (the underlying field theory of the model is a free-fermion theory, and not a free-boson theory as in the first example [60]), this illustrates the necessity of the second scaling dimension ξ, of the representation (20) of $\mathfrak{age}(1)$. For $d\geq 2$ dimensions, there is no known analytical solution and one must turn to numerical simulations. The available evidence suggests that the second scaling dimensions $\xi+\widetilde{\xi}\neq 0$ at criticality, at least for dimensions $d<d^*=4$,

the upper critical dimension. For details and a review of further examples, see [1].

How the choice of the representation can affect the physical interpretation, is further illustrated by considering a "lattice" representation rather than the usually employed "continuum" representation of the Schrödinger algebra $\mathfrak{sch}(1)$. In Table 1, we list the generators of the "continuum" representation (5) along with the one of the "lattice" representation. Herein, the non-linear functions of the derivative ∂_r are understood to stand for their Taylor expansions. The origin of the name of a "lattice" representation can be understood when considering the generator $Y_{-1/2}$ of "spatial translations", which reads explicitly

$$Y_{-1/2}f(t,r) = -\frac{1}{a}\Big(f(t,r+a/2) - f(t,r-a/2)\Big) \tag{37}$$

It is suggestive to interpret this as a discretised symmetric lattice derivative operator, with a as a lattice constant, although the X_n, Y_m are still generators of infinitesimal transformations.

Table 1. The "lattice" representations of the Schrödinger algebra $\mathfrak{sch}(1)$, and its "continuum" representation, to which it reduces in the limit $a \to 0$ [61].

Generator	Continuum	Lattice
X_{-1}	$-\partial_t$	$-\partial_t$
X_0	$-t\partial_t - \frac{1}{2}r\partial_r$	$-t\partial_t - \frac{1}{a\cosh(\frac{a}{2}\partial_r)} r\sinh(\frac{a}{2}\partial_r)$
X_1	$-t^2\partial_t - tr\partial_r - \frac{1}{2}\mathcal{M}r^2$	$-t^2\partial_t - \frac{2t}{a\cosh(\frac{a}{2}\partial_r)} r\sinh(\frac{a}{2}\partial_r) - \frac{\mathcal{M}}{2}\left(\frac{1}{\cosh(\frac{a}{2}\partial_r)}r\right)^2$
$Y_{-1/2}$	$-\partial_r$	$-\frac{2}{a}\sinh(\frac{a}{2}\partial_r)$
$Y_{1/2}$	$-t\partial_r - \mathcal{M}r$	$-\frac{2t}{a}\sinh(\frac{a}{2}\partial_r) - \frac{\mathcal{M}}{\cosh(\frac{a}{2}\partial_r)}r$
M_0	$-\mathcal{M}$	$-\mathcal{M}$

The Schrödinger operator has, in the "lattice" representation, the following form

$$\mathcal{S} = 2\mathcal{M}\partial_t - \frac{1}{a^2}\left(e^{a\partial_r} + e^{-a\partial_r} - 2\right) \tag{38}$$

and the equation $\mathcal{S}\phi = 0$ could be viewed as a "lattice analogue" of a free Schrödinger equation.

It is also of interest to write down the co-variant two-point functions. The extension of Equation (10) reads, up to a normalisation constant [61]

$$\Phi(t,n) := \langle \phi_1(t_1,r_1)\phi_2^*(t_2,r_2)\rangle = \delta_{\mathcal{M}_1,\mathcal{M}_2}\delta_{x_1,x_2}\, t^{1/2-x_1}\, e^{-t} I_n(t) \tag{39}$$

where I_n is again a modified Bessel function, and with the abbreviations

$$t = \frac{t_1 - t_2}{\mathcal{M}_1 a^2}, \quad n = \frac{r_1 - r_2}{a} \tag{40}$$

Herein, both r_1 and r_2 must be integer multiples of the "lattice constant" a. In the limit $a \to 0$, all these results reduce to those of the "continuum" representation, discussed in Section 1. Again, although at first sight this looks as a physically reasonable Green's function on an infinite chain [62], the same questions as raised in relation with Equation (10) should be addressed. The extensions discussed in the above propositions 2–4 can be readily added, since those only concern the time-dependence of the generators.

All representations of the Schrödinger algebra discussed so far have the dynamical exponent $z = 2$, which fixes the dilatations $t \mapsto \lambda^z t$ and $r \mapsto \lambda r$. This can be changed, however, by admitting "non-local" representations. We shall write them here, for the case $z = \nu \in \mathbb{N}$, in the form given for the sub-algebra $\mathfrak{age}(1)$, when the generators read [63]

$$\begin{aligned} X_0 &= -\frac{\nu}{2}t\partial_t - \frac{1}{2}r\partial_r - \frac{x}{2} \\ X_1 &= \left(-\frac{\nu}{2}t^2\partial_t - tr\partial_r - (x+\xi)t\right)\partial_r^{\nu-2} - \frac{\mathcal{M}}{2}r^2 \\ Y_{-1/2} &= -\partial_r, \quad Y_{1/2} = -t\partial_r^{\nu-1} - \mathcal{M}r, \quad M_0 = -\mathcal{M} \end{aligned} \tag{41}$$

and reduce to Equation (5) for $z = \nu = 2$. Clearly, these generators (especially $X_1, Y_{1/2}$) cannot be interpreted as infinitesimal transformations on time-space coordinates (t, r) and cannot be seen as mimicking a finite transformation, as was still possible with the "lattice" representation given in Table 1. In [63], a possible interpretation as transformation of distribution functions of (t, r) was explored, but the issue is not definitely settled.

Proposition 5. [63] *For any $\nu \in \mathbb{N}$, the generators (41) of the algebra $\mathfrak{age}(d)$ satisfy the commutators (4) in $d = 1$ spatial dimensions, but with the only exception*

$$[X_1, Y_{1/2}] = \frac{\nu-2}{2}t^2\partial_r^{\nu-3}\mathcal{S} \tag{42}$$

where the Schrödinger operator $\mathcal{S}$ is given by

$$\mathcal{S} = \nu\mathcal{M}\partial_t - \partial_r^\nu + 2\mathcal{M}\left(x + \xi + \frac{\nu-1}{2}\right)t^{-1} \tag{43}$$

These indeed generate a dynamical symmetry on the space of solutions of the equation $\mathcal{S}\phi = 0$, since the only non-vanishing commutators of $\mathcal{S}$ with the generators (41) are

$$[\mathcal{S}, X_0] = -\frac{\nu}{2}\mathcal{S}, \quad [\mathcal{S}, X_1] = -\nu t\partial_r^{n-2}\mathcal{S} \tag{44}$$

Verifying the required commutators is straightforward (but there is no known extension to a representation of $\mathfrak{av}(1)$). It is possible to generalise this construction to dimensions $d > 1$ and to generic dynamical exponents $z \in \mathbb{R}_+$, but this would require the introduction of fractional derivatives into the generators [39,64]. Formally, one can also derive the form of co-variant two-point functions $F(t_1, t_2; r_1, r_2) = \langle\phi_1(t_1, r_1)\phi_2^*(t_2, r_2)\rangle$.

Proposition 6. [63] *For $\nu \in \mathbb{N}$, a two-point function F, covariant under the non-local representation (41) of the Lie algebra $\mathfrak{age}(1)$, defined on the solution space of $\mathcal{S}\phi = 0$, where $\mathcal{S}$ is the Schrödinger operator (43), has the form $F = \delta(\mathcal{M}_1 - \mathcal{M}_2^*)\, t_2^{-(x_1+x_2)/\nu} F(u, v, r)$, where*

$$\begin{aligned} F(u,v,r) &= (v-1)^{-\frac{2}{\nu}[(x_1+x_2)/2+\xi_1+\xi_2-\nu+2]} v^{-\frac{1}{\nu}[x_2-x_1+2\xi_2-\nu+2]} f\left(ru^{-1/\nu}\right), \quad \nu \text{ even} \\ F(u,v,r) &= (v+1)^{-\frac{2}{\nu}[(x_1+x_2)/2+\xi_1+\xi_2-\nu+2]} v^{-\frac{1}{\nu}[x_2-x_1+2\xi_2-\nu+2]} f\left(ru^{-1/\nu}\right), \quad \nu \text{ odd} \end{aligned} \tag{45}$$

the function $f(y)$ *satisfies the equation* $\mathrm{d}^{\nu-1}f(y)/\mathrm{d}y^{\nu-1} + \mathcal{M}_1 y f(y) = 0$*, and with the variables* $r = r_1 - r_2$*,* $v = t_1/t_2$ *and*

$$u = t_1 - t_2 \quad \text{if } \nu \text{ is even} \quad , \quad u = t_1 + t_2 \quad \text{if } \nu \text{ is odd} \tag{46}$$

The set of admissible functions $f(y)$ will have to be restricted by imposing physically reasonable boundary conditions, especially $\lim_{y\to\infty} f(y) = 0$. The value $z = \nu$ of the dynamical exponent is obvious.

Again, one should inquire into the behaviour when $r \to \infty$. Furthermore, one observes that the interpretation of u depends on whether ν is even or odd. In the first case, the co-variant two-point functions could be a physical two-time response function, while in the second case, it looks more like a two-time correlator, since it is symmetric symmetry under the exchange of the two scaling operators.

All representations considered here are scalar. It is possible to consider multiplets of scaling operators. In the case of conformal invariance, one should formally replace the conformal weight Δ by a matrix [65–69]. New structures are only found if that matrix takes a Jordan form. Analogous representations can also be considered for the Schrödinger and conformal Galilean algebras and their sub-algebras. Then, it becomes necessary to consider simultaneously the scaling dimensions x, ξ and the rapidities $\boldsymbol{\gamma}$ as matrices [36,70–73]. From the Lie algebra commutators it can then be shown that these characteristic elements of the scaling operators are simultaneously Jordan [36]. Several applications to non-equilibrium relaxation phenomena have been explored in the literature [57,58,74,75], see [59] for a review.

3. Dual Representations

In order to understand how the causality and the large-distance behaviour of the co-variant two-point functions can be derived algebraically, it is helpful to go over to a dual description. The new dual coordinate ζ is related to either the scalar mass $\mathcal{M}$ for the Schrödinger algebra (this was first noted by Giulini [76] for the case of its Galilei-subalgebra) or else to the vector of rapidities $\boldsymbol{\gamma}$ for the conformal Galilei algebra. It will therefore be scalar or vector, respectively. The dual fields are [11,77]

$$\begin{aligned}
\hat{\phi}(\zeta, t, \boldsymbol{r}) &:= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}\mathcal{M}\, e^{\mathrm{i}\mathcal{M}\zeta} \phi_{\mathcal{M}}(t, \boldsymbol{r}), \quad \text{for } \mathfrak{sch}(d), \mathfrak{age}(d) \\
\hat{\phi}(\boldsymbol{\zeta}, t, \boldsymbol{r}) &:= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{\gamma}\, e^{\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{\zeta}} \phi_{\boldsymbol{\gamma}}(t, \boldsymbol{r}), \quad \text{for } \mathrm{CGA}(d)
\end{aligned} \tag{47}$$

For the sake of notational simplicity, we shall almost always restrict to the one-dimensional case, although we shall quote some final results for a generic dimension d.

3.1. Schrödinger Algebra

From Proposition 3, the dual generators of the Schrödinger–Virasoro algebra take the form (with $j, k = 1, \ldots, d$)

$$\begin{aligned}
X_n &= \frac{\mathrm{i}}{4}(n+1)n\, t^{n-1} r^2 \partial_\zeta - t^{n+1}\partial_t - \frac{n+1}{2} t^n r \partial_r - \frac{n+1}{2} x t^n - n(n+1)\xi t^n - \Xi(t) t^n \\
Y_m &= \mathrm{i}\left(m + \frac{1}{2}\right) t^{m-1/2} r \partial_\zeta - t^{m+1/2}\partial_r \\
M_n &= \mathrm{i} t^n \partial_\zeta
\end{aligned} \tag{48}$$

with $n \in \mathbb{Z}$ and $m \in \mathbb{Z}+\frac{1}{2}$. This acts on a $(d+2)$-dimensional space, with coordinates $\zeta, t, \boldsymbol{r}$. According to Proposition 3, not only the finite-dimensional sub-algebra $\mathfrak{age}(1)$, but also the finite-dimensional sub-algebra $\mathfrak{sch}(1)$ [40] generates dual dynamical symmetries of the Schrödinger operator

$$\mathcal{S} = -2\mathrm{i}\partial_\zeta \partial_t - \partial_r^2 - 2\mathrm{i}\left(x+\xi-\frac{1}{2}\right)t^{-1}\partial_\zeta \tag{49}$$

Co-variant dual three-point functions have been derived explicitly [40].

In the context of the non-relativistic AdS/CFT correspondence, also referred to as non-relativistic holography by string theorists, see [74,78] and references therein, one rather considers a $(d+3)$-dimensional space, with coordinates $Z, \zeta, t, \boldsymbol{r}$. The time-space transforming parts of the Schrödinger–Virasoro generators read (generalising Son [79], who restricted himself to the finite-dimensional sub-algebra $\mathfrak{sch}(d)$)

$$\begin{aligned} X_n &= \frac{\mathrm{i}}{4}(n+1)n\, t^{n-1}\left(\boldsymbol{r}^2+Z^2\right) - t^{n+1}\partial_t - \frac{n+1}{2}t^n\left(\boldsymbol{r}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}} + Z\partial_Z\right) \\ Y_m^{(j)} &= \mathrm{i}\left(m+\frac{1}{2}\right)t^{m-1/2} r_j \partial_\zeta - t^{m+1/2}\partial_{r_j} \\ M_n &= \mathrm{i}t^n\partial_\zeta \\ R_n^{(jk)} &= -t^n\left(r_j\partial_{r_k} - r_k\partial_{r_j}\right) \end{aligned} \tag{50}$$

Clearly, the variable Z distinguishes the bulk from the boundary at $Z = 0$. Heuristically, if one replaces $Z\partial_Z \mapsto x$ and then sets $Z = 0$, one goes back from Equation (50) to Equation (48), with $\xi = 0$ and $\Xi(t) = 0$.

Following Aizawa and Dobrev [78,80], the passage between the boundary and the bulk is described in terms of the eigenvalues of the quartic Casimir operator of the Schrödinger algebra $\mathfrak{sch}(1)$ [81]

$$C_4 = \left(4M_0X_0 - Y_{-1/2}Y_{1/2} - Y_{1/2}Y_{-1/2}\right)^2 - 2\left\{2M_0X_{-1} - Y_{-1/2}^2, 2M_0X_1 - Y_{1/2}^2\right\} \tag{51}$$

such that in the representation Equation (5), which lives on the boundary $Z = 0$, one has the eigenvalue $c_4 = c_4(x) := \mathcal{M}^2(2x-1)(2x-5)$. Since $c_4(x) = c_4(3-x)$, two scaling operators with scaling dimensions x and $3-x$ will be related. In order to formulate the holographic principle, which prescribes the mapping of a boundary scaling operators φ to a bulk scaling operator ϕ, a necessary condition is the eigenvalue equation (in the bulk)

$$C_4\phi(Z,\zeta,t,r) = c_4(x)\phi(Z,\zeta,t,r) \tag{52}$$

The other condition is the expected limiting behaviour when the boundary is approached

$$\phi(Z,\zeta,t,r) \stackrel{Z\to 0}{\longrightarrow} Z^\alpha \varphi(\zeta,t,r), \quad \alpha = x, 3-x \tag{53}$$

Lemma 2. [80] *For the Schrödinger algebra in $d = 1$ space dimension, the holographic principle takes the form*

$$\phi(Z,\chi) = \int \mathrm{d}^3\chi'\, S_\alpha(Z,\chi-\chi')\varphi(\chi') \tag{54}$$

where $\chi = (\zeta,t,r)$ is a label for a three-dimensional coordinate, $\mathrm{d}^3\chi = \mathrm{d}\zeta\mathrm{d}t\mathrm{d}r$ and

$$S_\alpha(Z,\chi) = \left[\frac{4Z}{-2\zeta t + r^2}\right]^\alpha \tag{55}$$

and where $\alpha = x$ or $\alpha = 3-x$.

Proof. We merely outline the main ideas. First, construct the Green's function in the bulk, by solving

$$(C_4 - c_4(x))\, G(Z,\chi; Z',\chi') = Z'^4\, \delta(Z-Z')\delta^3(\chi-\chi')$$

In terms of the invariant variable

$$u := \frac{4ZZ'}{(Z+Z')^2 - 2(\zeta-\zeta')(t-t') + (r-r')^2}$$

the Casimir operator becomes $C_4 = 4u^2(1-u)\partial_u^2 - 8u\partial_u + 5$, hence $G = G(u)$. Next, the ansatz $G(u) = u^\alpha \bar{G}(u)$ reduces the eigenvalue equation to a standard hyper-geometric equation, with solutions expressed in terms of the hyper-geometric function ${}_2F_1$. Finally, $S_\alpha(Z,\chi-\chi') = \lim_{Z'\to 0} Z'^{-\alpha}G(u)$ leads to the assertion. □

We refer to the literature for the non-relativistic reduction and the derivation of invariant differential equations [78,80]. The consequences of passing to the more general representations with $\xi \neq 0$ and $\Xi(t) \neq 0$ [40] remain to be studied.

3.2. Conformal Galilean Algebra I

Starting from Equation (48), a dual representation of the conformal Galilean algebra $\mathrm{CGA}(1)$ with $z = 2$ is found if (i) the generator X_{-1} is dropped, (ii) the generator X_1 is taken as follows and (iii) and adds a new generator V_+ [11]

$$\begin{aligned} X_1 &= \mathrm{i}r^2\partial_\zeta - t^2\partial_t - tr\partial_r - (x+\xi)\,t \\ V_+ &= -\zeta r\partial_\zeta - tr\partial_t - \left(\mathrm{i}\zeta t + \frac{r^2}{2}\right)\partial_r - (x+\xi)\,r \end{aligned} \tag{56}$$

They are dynamical symmetries of the dual Schrödinger operator Equation (49).

3.3. Conformal Galilean Algebra II

Another dual representation of the algebra $\mathrm{CGA}(d)$ is given by (with $j,k = 1,\ldots,d$)

$$\begin{aligned} X_n &= +\mathrm{i}(n+1)nt^{n-1}\boldsymbol{r}\cdot\partial_{\boldsymbol{\zeta}} - t^{n+1}\partial_t - (n+1)t^n\boldsymbol{r}\cdot\partial_{\boldsymbol{r}} - (n+1)xt^n \\ Y_n^{(j)} &= -t^{n+1}\partial_{r_j} + \mathrm{i}(n+1)t^n\partial_{\zeta_j} \\ R_n^{(jk)} &= -t^n\left(r_j\partial_{r_k} - r_k\partial_{r_j}\right) - t^n\left(\zeta_j\partial_{\zeta_k} - \zeta_k\partial_{\zeta_j}\right) \end{aligned} \tag{57}$$

In contrast with the representations studied so far, there are no central generators $\sim \partial_{\zeta_j}$.

The dualisation of the "lattice" representation and the non-local representations discussed in Section 2 proceeds analogously and will not be spelt out in detail here.

3.4. Parabolic Sub-Algebras

The other important ingredient is understood by considering the root diagrams of these non-semi-simple Lie algebras, see Figure 1. Therein, it is in particular illustrated that the complexified versions of these algebras are all sub-algebras of the complex Lie algebra B_2, in Cartan's notation [11].

In particular, it is possible to add further generators in the Cartan sub-algebra in order to obtain an extension to a *maximal parabolic sub-algebra*. A parabolic sub-algebra is the sub-algebra of "positive" generators, which from a root diagramme can be identified by simply placing a straight line through the center (a.k.a. the Cartan sub-algebra). By definition, all generators which are not on the left of that line are called "positive" [41]. In Figure 1, we illustrate this for the three maximal parabolic sub-algebras. The notion of "maxima" does depend here on the precise definition of "positivity". For a generic slope, see Figure 1a, both the generators X_{-1} and V_+ are non-positive, and one has the maximal parabolic sub-algebra $\widetilde{\mathfrak{age}}(1) = \mathfrak{age}(1) + \mathbb{C}N$. This sub-algebra is indeed maximal as a *parabolic* sub-algebra: for example an extension to a Schrödinger algebra by including the time-translations X_{-1} would no longer be parabolic, according to the specific definition of "positivity" used in this specific context. If a different definition of "positivity" is used, and the slope is now taken to be exactly unity, X_{-1} is included into the positive generators, see Figure 1b, and we have the maximal parabolic sub-algebra $\widetilde{\mathfrak{sch}}(1) = \mathfrak{sch}(1) + \mathbb{C}N$. Finally, and with yet a different definition of "positivity", where the slope is now infinite, see Figure 1c, one has the maximal parabolic sub-algebra $\widetilde{\mathrm{CGA}}(1) = \mathrm{CGA}(1) + \mathbb{C}N$. The Weyl symmetries of the root diagramme of B_2 [41] imply that any other maximal and non-trivial sub-algebra of B_2 is isomorphic to one of the three already given. For a formal proof, see [11].

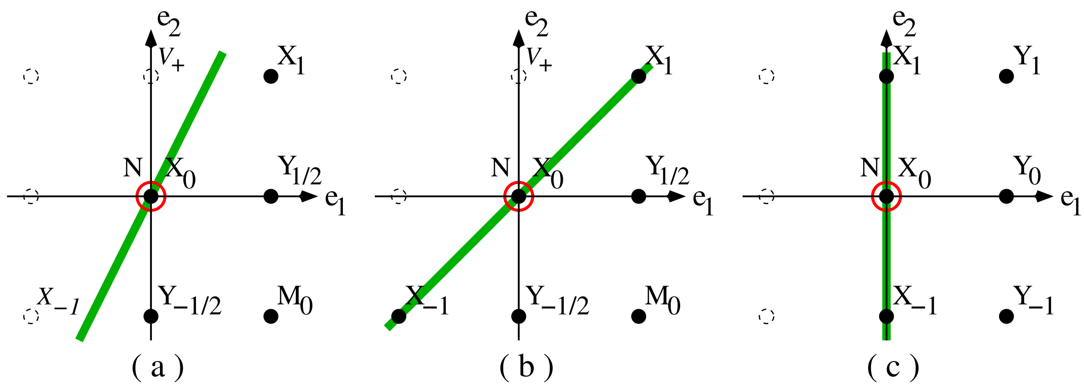

Figure 1. Root diagrammes of the Lie algebras (**a**) $\mathfrak{age}(1)$; (**b**) $\mathfrak{sch}(1)$ and (**c**) $\mathrm{CGA}(1)$. The generators are represented by the black filled dots. The red circles indicate the extra generator N which extends these algebras to maximal parabolic sub-algebras of the complex Lie algebra B_2. The thick green line indicates the separation between positive and non-positive roots.

It remains to construct the operator N explicitly, for each representation. We collect the results, coming from different sources [11,40,77].

Proposition 7. *Consider the dual representations (48) of the Schrödinger–Virasoro algebra, the* $z = 2$ *dual representation (56) of the conformal Galilean algebra* $\mathrm{CGA}(1)$*, the* $z = 1$ *dual representation (57) of* $\mathrm{CGA}(d)$ *and the dualisation of the non-local representation (41) of* $\mathfrak{age}(1)$*, dualised with respect to the* *mass $\mathcal{M}$. There is a generator N which extends these representations to representations of the associated maximal parabolic sub-algebra. The explicit form of the generator N is as follows:*

$$N = \begin{cases} \zeta\partial_\zeta - t\partial_t + \xi' & \text{representation (48) of } \mathfrak{sch}(d) \\ \zeta\partial_\zeta - t\partial_t + \xi & \text{representation (56) of } \mathrm{CGA}(1) \\ -\zeta\partial_\zeta - r\partial_r - \xi & \text{representation of } \mathrm{CGA}(d) \text{ as constructed in (57)} \\ \zeta\partial_\zeta - t\partial_t + \xi' & \text{dualised non-local representation (41) of } \mathfrak{age}(d) \end{cases} \tag{58}$$

Herein, ξ is the second scaling dimension and ξ' is a constant. These generators give dynamical symmetries of the Schrödinger operators $\mathcal{S}$ associated with each representation.

4. Causality

It turns out that the maximal parabolic sub-algebras are the smallest Lie algebras which permit unambiguous statements on the causality of co-variant two-point functions. For illustration, we shall concentrate on the dual representations (48) of $\mathfrak{sch}(d)$ and (57) of $\mathrm{CGA}(d)$.

Proposition 8. [11,77] *Consider the co-variant dual two-point functions. For the dual representation (48) of $\widetilde{\mathfrak{sch}}(d)$, it has the form, up to a normalisation constant*

$$\widehat{F}(\zeta, t, \boldsymbol{r}) = \langle \widehat{\phi}(\zeta, t, \boldsymbol{r})\widehat{\phi}^*(0, 0, \mathbf{0})\rangle = \delta_{x_1,x_2}\, |t|^{-x_1} \left(\frac{2\zeta t + \mathrm{i}\boldsymbol{r}^2}{|t|}\right)^{-x_1-\xi_1'-\xi_2'} \tag{59}$$

and where translation-invariance in $\zeta, t, \boldsymbol{r}$ was used. For the dual representation (57) of $\widetilde{\mathrm{CGA}}(1)$, one has, up to a normalisation constant

$$\widehat{F}(\zeta_+, t, r) = \langle \widehat{\phi}_1(\zeta_1, t, r)\widehat{\phi}_2(\zeta_2, 0, 0)\rangle = \delta_{x_1,x_2}|t|^{-2x_1} \left(\zeta_+ + \frac{\mathrm{i}r}{t}\right)^{-\xi_1-\xi_2} \tag{60}$$

and where $\zeta_+ = \frac{1}{2}(\zeta_1 + \zeta_2)$.

This is easily verified by insertion into the respective Ward identities which express the co-variance. Finally, we formulate precisely the spatial long-distance and co-variance properties of these two-point functions.

Theorem 2. [11] *With the convention that masses $\mathcal{M} \geq 0$ of scaling operators ϕ should be non-negative, and if $\frac{1}{2}(x_1 + x_2) + \xi_1' + \xi_2' > 0$, the full two-point function, co-variant under the representation (5) of the parabolically extended Schrödinger algebra $\widetilde{\mathfrak{sch}}(d)$, has the form*

$$\langle \phi(t, \boldsymbol{r})\phi^*(0, \mathbf{0})\rangle = \delta(\mathcal{M} - \mathcal{M}^*)\, \delta_{x_1,x_2}\, \Theta(t)\, t^{-x_1}\, \exp\left[-\frac{\mathcal{M}}{2}\frac{\boldsymbol{r}^2}{t}\right] \tag{61}$$

where the Θ-function expresses the causality condition $t > 0$, and up to a normalisation constant which depends only the mass $\mathcal{M} \geq 0$.

Proof. This follows directly from Equation (59). Carrying out the inverse Fourier transform and using the translation-invariance in the dual coordinate ζ, one recovers the habitual two-point function multiplied by an integral representation of the Θ-function. □

The treatment of the conformal Galilean algebra requires some further preparations, following Akhiezer [82] (Chapter 11).

Definition 1. *Let $\mathbb{H}_+$ be the upper complex half-plane $w = u + \mathrm{i}v$ with $v > 0$. A function $g : \mathbb{H}_+ \to \mathbb{C}$ is said to be in the* Hardy class H_2^+*, written as $g \in H_2^+$, if (i) $g(w)$ is holomorphic in $\mathbb{H}_+$ and (ii) if it satisfies the bound*

$$M^2 := \sup_{v>0} \int_{\mathbb{R}} \mathrm{d}u \, |g(u + \mathrm{i}v)|^2 < \infty \tag{62}$$

Analogously, for functions $g : \mathbb{H}_- \to \mathbb{C}$ one defines the Hardy class H_2^-, where $\mathbb{H}_-$ is the lower complex half-plane and the supremum in Equation (62) is taken over $v < 0$.

Lemma 3. [82] *If $g \in H_2^\pm$, then there are square-integrable functions $\mathcal{G}_\pm \in L^2(0,\infty)$ such that for $v > 0$ one has the integral representation*

$$g(w) = g(u \pm \mathrm{i}v) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}\gamma \; e^{\pm \mathrm{i}\gamma w} \, \mathcal{G}_\pm(\gamma) \tag{63}$$

We shall use Equation (63) as follows. First, consider the case $d = 1$. Fix $\lambda := r/t$. Now, recall Equation (60) and write $\widehat{F} = |t|^{-2x_1} \widehat{f}(u)$, with $u = \zeta_+ + \mathrm{i}r/t$. We shall re-write this as follows:

$$\widehat{f}(\zeta_+ + \mathrm{i}\lambda) =: f_\lambda(\zeta_+) \tag{64}$$

and concentrate on the dependence on ζ_+.

Proposition 9. [77] *Let $\xi := \frac{1}{2}(\xi_1 + \xi_2) > \frac{1}{4}$. If $\lambda > 0$, then $f_\lambda \in H_2^+$ and if $\lambda < 0$, then $f_\lambda \in H_2^-$.*

Proof. The holomorphy of f_λ being obvious, we merely must verify the bound (62). Let $\lambda > 0$. Clearly, $|f_\lambda(u + \mathrm{i}v)| = \left|(u + \mathrm{i}(v + \lambda))^{-2\xi}\right| = (u^2 + (v + \lambda)^2)^{-\xi}$. Hence, computing explicitly the integral,

$$M^2 = \sup_{v>0} \int_{\mathbb{R}} \mathrm{d}u \, |f_\lambda(u + \mathrm{i}v)|^2 = \frac{\sqrt{\pi}\,\Gamma(2\xi - \frac{1}{2})}{\Gamma(2\xi)} \sup_{v>0} (v + \lambda)^{1-4\xi} < \infty$$

since the integral converges for $\xi > \frac{1}{4}$. For $\lambda < 0$, the argument is similar. □

We can now formulate the second main result.

Theorem 3. [77] *The full two-point function, co-variant under the representation (13) of the parabolically extended conformal Galilean algebra $\widetilde{\mathrm{CGA}}(d)$, has the form*

$$\langle \phi_1(t, \boldsymbol{r}) \phi_2(0, \boldsymbol{0}) \rangle = \delta_{x_1,x_2} \delta(\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2) \, |t|^{-2x_1} \exp\left[-2 \left| \frac{\boldsymbol{\gamma}_1 \cdot \boldsymbol{r}}{t} \right| \right] \tag{65}$$

where the normalisation constant only depends on the absolute value of the rapidity vector $\boldsymbol{\gamma}_1$.

Proof. Since the final result is rotation-invariant, because of the representation (13), it is enough to consider the case $d = 1$. Let $\lambda > 0$. From Equation (63) of Lemma 3 we have

$$\sqrt{2\pi}\,\widehat{f}(\zeta_+ + \mathrm{i}\lambda) = \int_0^\infty \mathrm{d}\gamma_+ \, e^{\mathrm{i}(\zeta_+ + \mathrm{i}\lambda)\gamma_+} \widehat{\mathcal{F}_+}(\gamma_+) = \int_{\mathbb{R}} \mathrm{d}\gamma_+ \, \Theta(\gamma_+) \, e^{\mathrm{i}(\zeta_+ + \mathrm{i}\lambda)\gamma_+} \widehat{\mathcal{F}_+}(\gamma_+)$$

Now return from the dual two-point function $\widehat{F}$ to the original one. Let $\zeta_\pm := \frac{1}{2}(\zeta_1 \pm \zeta_2)$. We find, using also that $x_1 = x_2$

$$\begin{aligned}
F &= \frac{|t|^{-2x_1}}{\pi\sqrt{2\pi}} \int_{\mathbb{R}^2} \mathrm{d}\zeta_+ \mathrm{d}\zeta_-\, e^{-\mathrm{i}(\gamma_1+\gamma_2)\zeta_+} e^{-\mathrm{i}(\gamma_1-\gamma_2)\zeta_-} \int_{\mathbb{R}} \mathrm{d}\gamma_+\, \Theta(\gamma_+)\widehat{\mathcal{F}_+}(\gamma_+) e^{-\gamma_+\lambda} e^{\mathrm{i}\gamma_+\zeta_+} \\
&= \frac{|t|^{-2x_1}}{\pi\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}\gamma_+\, \Theta(\gamma_+)\widehat{\mathcal{F}_+}(\gamma_+) e^{-\gamma_+\lambda} \int_{\mathbb{R}} \mathrm{d}\zeta_-\, e^{-\mathrm{i}(\gamma_1-\gamma_2)\zeta_-} \int_{\mathbb{R}} \mathrm{d}\zeta_+\, e^{\mathrm{i}(\gamma_+-\gamma_1-\gamma_2)\zeta_+} \\
&= \delta(\gamma_1-\gamma_2)\Theta(\gamma_1) F_{0,+}(\gamma_1) e^{-2\gamma_1\lambda} |t|^{-2x_1}
\end{aligned}$$

where in the last line, two δ-functions were used and $F_{0,+}$ contains the unspecified dependence on the positive constant γ_1. An analogous argument applies for $\lambda < 0$. □

5. Conclusions

Results on relaxation phenomena in non-equilibrium statistical physics and the associated dynamical symmetries, scattered over many sources in the literature, have been reviewed. By analogy with conformal invariance which applies to equilibrium critical phenomena, it is tempting to try to extend the generically satisfied dynamical scaling to a larger set of dynamical symmetries. If this is possible, one should obtain a set of co-variance conditions, to be satisfied by physically relevant n-point functions. In contrast to equilibrium critical phenomena, it turned out that in non-equilibrium systems, each scaling operator must be characterised at least in terms of two independent scaling dimensions.

A straightforward realisation of this programme was seen to lead to difficulties for a consistent physical interpretation, related to the requirement of a physically sensible large-distance behaviour. This is related to the fact that writing down simple Ward identities for the n-point functions, one implicitly assumes that these n-point functions depend holomorphically on their time-space arguments, see e.g., [83]. However, the constraint of causality, required for a reasonable two-time response function $R(t_1, t_2)$, renders $R(t_1, t_2)$ non-holomorphic in the time difference $t_1 - t_2$. As a possible solution of this difficulty, we propose to go over to dual representations with respect to either the "masses" or the "rapidities", which are physically dimensionful parameters of the representations of the dynamical symmetry algebras considered. If, furthermore, the dynamical symmetry algebras can be extended to a maximal parabolic sub-algebra of a semi-simple complex Lie algebra, then causality conditions, which also guarantee the requested fall-off at large distances, can be derived.

This suggests that the dual scaling operators, rather than the original ones, might possess interesting holomorphic properties which should be further explored. This observation might also become of interest in further studies of the holographic principle.

Specifically, we considered representations of (i) the Schrödinger algebra $\mathfrak{sch}(d)$, where the co-variant two-point functions Equation (61) have the causality properties of two-time linear response functions and also representations of (ii) the conformal Galilean algebra $\mathrm{CGA}(d)$, where the two-point functions Equation (65) have the symmetry properties of a two-time correlator [84].

Although this has not yet been done explicitly, we expect that the techniques reviewed here can be readily extended to several physically distinct representations of these algebras, see e.g., [85–88] for examples [89].

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Conflicts of Interest

The author declares no conflict of interest.

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