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On Center, Periphery and Average Eccentricity for the Convex Polytopes

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Abstract: A vertex v is a peripheral vertex in G if its eccentricity is equal to its diameter, and periphery $P(G)$ is a subgraph of G induced by its peripheral vertices. Further, a vertex v in G is a central vertex if $e(v) = rad(G)$, and the subgraph of G induced by its central vertices is called center $C(G)$ of G. Average eccentricity is the sum of eccentricities of all of the vertices in a graph divided by the total number of vertices, i.e., $avec(G) = \{\frac{1}{n}\sum e_G(u); \ \ u \in V(G)\}$. If every vertex in G is central vertex, then $C(G) = G$, and hence, G is self-centered. In this report, we find the center, periphery and average eccentricity for the convex polytopes.

Keywords: eccentricity; center; periphery; average eccentricity

1. Introduction

In the facility location problem, we select a site according to some standard judgment. For example, if we want to find out the exact location for an emergency facility, such as a fire station or a hospital, we reduce the distance between that facility and the area where the emergency happens, and if we are to decide the position for a service facility, like a post office, power station or employment office, we try to reduce the traveling time of all people who have been living in that district. In the construction of a railway line, a pipeline and a superhighway, we will reduce the distance of the constructing unit for the people living in that area. All of these situations illustrate the concept of centrality but each of these three examples deals with different types of centers. Nowadays, centrality questions are being studied with the help of distance and graphs. We shall observe that many kinds of centers are helpful in facility location problems.

The most important and fundamental concept that extends to the whole of graph theory is distance. The distance is applicable in many fields, such as graph operation, extremal problems on connectivity, diameter and isomorphism testing. The theme of distance is used to check the symmetry of graphs. It also provides a base for many useful graph parameters, like radius, diameter, metric dimension, eccentricity, center and periphery, etc.

The eccentricity of the vertices in G has a fundamental importance. Recently, many indices related to eccentricity have been derived, i.e., eccentric connectivity index, adjacent eccentric sum index, Wiener index and eccentric distance sum [1]. The center and periphery is also based on minimum and maximum eccentricity, respectively. W.Goddard and O. R. Oellermann in [2] have shown that if G is an undirected graph, then,

$$rad(G) \leq diam(G) \leq 2rad(G)$$

They also examined the radius and diameter of certain families of graphs in the same paper, as follows:

1. $rad(K_n) = diam(K_n) = 1$ for $n \geq 2$,
2. $rad(C_n) = diam(C_n) = \frac{n}{2}$,
3. $rad(K_{m,n}) = diam(K_{m,n}) = 2$ if m and n is at least two,
4. $rad(P_n) = \frac{n-1}{2}$, $diam(P_n) = n - 1$.

This implies that complete graphs K_n for $n \geq 2$, complete bipartite graphs $K_{m,n}$ where $m, n \geq 2$ and all cycles are self-centered. Jordan [3] determined the diameter of a tree. Bela Bollobas [4] discussed the diameter of random graphs. The radius and diameter of a bridge graph are determined by Martin Farber in [5]. More general results were presented by V. Klee and D. Larman [6] and Bela Bollobas [4]. B. Hedman determined the sharp bounds for the diameter of the clique graph $K(G)$ in terms of the diameter of G. The idea of self-centered graphs is presented and elaborated by Ando, Akiyama and Avis individually [7]. These self-centered graphs are extensively studied in [7–11]. The extremal size of a connected self-centered graph with p vertices and r radius is explained by F. Buckely [12]. The center in maximal outer planar graphs is demonstrated by A. Proskurowski in [13]. Hedetniemi [14] has shown that every graph is the center of some graph. The center of graph G is the full graph if and only if $rad(G) = diam(G)$ [15]. F. Buckely and F. Harary [16] gave the concept of average eccentricity. Average eccentricity is the sum of eccentricities of all of the vertices in a graph divided by the total number of vertices, i.e.,

$$avec(G) = \frac{1}{n} \sum_{u \in V(G)} e_G(u).$$

The upper bounds of average eccentricity are determined by P. Dankelman, W. Goddard and C.S. Swart [17]. Average eccentricity is most important in communication networks. The average eccentricity of Sierpinski graphs S_n^p is determined by Andreas, M. Hinz and Daniele Parisse [18]. Since 1980, the average eccentricity has had a great roll as a molecular descriptor in mathematical chemistry. This is attributed to V.A. Skorobogatov and A.A. Dobrynin [19]. For more details, please see [20–22] and the references therein.

Definition 1. *For a connected graph G, the eccentricity $e(v)$ of a vertex v is its distance to a vertex farthest from v. Thus,*

$$e(v) = Max\{d(u, v) : u \in V(G)\}.$$

Definition 2. *The radius $rad(G)$ of G is the minimum eccentricity among all vertices of G.*

Definition 3. *The diameter $diam(G)$ of G is the maximum eccentricity among all vertices of G.*

Definition 4. *Average eccentricity is the sum of eccentricities of all of the vertices in a graph divided by the total number of vertices, i.e.,*

$$avec(G) = \frac{1}{n} \sum_{u \in V(G)} e_G(u).$$

Definition 5. *A vertex u is eccentric to a vertex v if $d(u, v) = e(v)$.*

Definition 6. *A vertex v is a peripheral vertex in G if its eccentricity is equal to its diameter, and periphery $P(G)$ is a subgraph of G induced by its peripheral vertices. Further, a vertex v in G is a central vertex if $e(v) = rad(G)$, and the subgraph of G induced by its central vertices is called center $C(G)$ of G. If every vertex in G is a central vertex, then $C(G) = G$, and hence, G is self-centered.*

In the present report, we discuss the center, periphery and average eccentricity for families of convex polytope graphs, A_n, S_n and T_n.

2. The Center and Periphery for Convex Polytope A_n

In this section, we determine the center and periphery for convex polytope A_n.

Definition 7. *The graph of convex polytope (double antiprism) A_n can be obtained from the graph of convex polytope R_n by adding new edges $b_{i+1}c_i$, i.e.,*

$$V(A_n) = V(R_n) \text{ and } E(A_n) = E(R_n) \cup \{b_{i+1}c_i : 1 \le i \le n\}.$$

Theorem 1. *For the family of convex polytope A_n, $n = 2k$, $Cen(A_n)$ and $Per(A_n)$ are subgraphs induced by the vertices $(b_1, b_2, ..., b_{2k})$ and $\{a_i \cup c_i : 1 \le i \le 2k\}$, respectively.*

Proof. For all even values of n, select a vertex a_1 on the cycle $(a_1a_2a_3...a_i...a_{2k})$. Then:

$$d(a_1, a_i) = i - 1, \quad 1 \le i \le k + 1 \tag{1}$$

when $i = k + 2$, $d(a_1, a_i) = k - 1$ and for $i = 2k$, $d(a_1, a_i) = 1$.

In addition, for every value of i within $k + 2$ to $2k$, $d(a_1, a_i)$ must lie between $k - 1$ and one, i.e.,

$$d(a_1, a_i) = 2k + 1 - i; \; k + 2 \le i \le 2k.$$

Thus, to find the vertices farthest from a_1 in A_n, consider only $1 \le i \le k + 1$.

As each a_i is adjacent to b_i, b_{i-1} and each b_i adjacent to c_i, c_{i-1}, therefore, (1) implies,

$$d(a_1, b_i) = i, \quad 1 \le i \le k$$

$$d(a_1, c_i) = i + 1, \quad 1 \le i \le k$$

For $k + 1 \le i \le 2k$, consider the cycle $(b_1b_2...b_{k+1}...b_i...b_{2k})$. In this cycle, the distance between b_i and b_{2k} is $2k - i$, and b_{2k} is adjacent to a_1, therefore, the distance between a_1 and b_i is $2k - i + 1$.

$$d(a_1, b_i) = 2k - i + 1, \quad k + 1 \le i \le 2k.$$

Now, consider the cycle $(c_1c_2...c_{k+1}...c_i...c_{2k})$. The distance between c_i and c_{2k-1} is $2k - 1 - i$ and the vertex c_{2k-1} is adjacent to b_{2k} and b_{2k} adjacent to a_1. It shows,

$$d(a_1, c_i) = 2k - i + 1, \quad k + 1 \le i \le 2k - 1.$$

For $i = 2k$, $d(a_1, c_i) = 2$.

Hence, c_k is a vertex farthest from a_1.

$$e(a_1) = k + 1 \tag{2}$$

Thus, the eccentricity of each vertex on inner cycle $(a_1a_2a_3...a_i...a_{2k})$ is $k + 1$.

In the same way, take cycle $(b_1b_2...b_i....b_{2k})$; the distance between b_1 and b_i in this cycle is,

$$d(b_1, b_i) = i - 1; \; 1 \le i \le k + 1 \tag{3}$$

Each b_i is adjacent to a_i and a_{i+1}. Therefore,

$$d(b_1, a_1) = 1, \quad (b_1, a_2) = 1$$

For $3 \le i \le k + 1$, consider the path $b_1 \to a_2 \to a_3 \to ... \to a_i$. Then, the distance between a_i and a_2 is $i - 2$. a_2 is also adjacent to b_1. Therefore, the distance between b_1 and a_i is as follows,

$$d(b_1, a_i) = i - 1, \quad 3 \leq i \leq k + 1$$

For $k + 2 \leq i \leq 2k$, consider the cycle $(a_1 a_2 ... a_{k+2} ... a_i ... a_{2k})$. The distance between a_i and a_{2k} is $2k - i$. Further, a_{2k} is adjacent to a_1 and a_1 adjacent to b_1. Therefore, the distance between b_1 and a_i is $2k - i + 2$.

$$d(b_1, a_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k$$

Further, b_i is also adjacent to c_i and c_{i-1}; it follows from (3):

$$d(b_1, c_i) = i, \quad 1 \leq i \leq k$$

For $k + 1 \leq i \leq 2k$, consider the cycle $(c_1 c_2 ... c_{k+1} ... c_i ... c_{2k})$. The distance between c_i and c_{2k} is $2k - i$, where c_{2k} is also adjacent to b_1. Therefore,

$$d(b_1, c_i) = 2k - i + 1, \quad k + 1 \leq i \leq 2k$$

Hence, b_{k+1}, a_{k+1} and c_k are the vertices farthest from b_1. Therefore:

$$e(b_1) = k, \tag{4}$$

Hence, each vertex on the middle cycle $(b_1 b_2 ... b_i .... b_{2k})$ has eccentricity k.

Further, to find out the eccentricity of the vertices on the outer cycle $(c_1 c_2 ... c_i ... c_{2k})$, choose a vertex c_1 on this cycle. The distance between c_1 and c_i is $i - 1$.

$$d(c_1, c_i) = i - 1, \tag{5}$$

Each c_i is adjacent to b_i and b_{i+1}, i.e.,

$$d(c_1, b_1) = 1, \tag{6}$$

For $3 \leq i \leq k + 1$, consider the path $c_1 \rightarrow b_2 \rightarrow b_3 \rightarrow ... \rightarrow b_i$. The distance between b_2 and b_i is $i - 2$. As b_2 is adjacent to c_1, therefore, c_1 and b_i has the following distance,

$$d(c_1, b_i) = i - 1, \quad 3 \leq i \leq k + 1$$

For $k + 2 \leq i \leq 2k$, consider the cycle $(b_1 b_2 ... b_{k+2} ... b_i ... b_{2k})$. The distance between b_i and b_{2k} is $2k - i$. As b_{2k} is adjacent to b_1 and b_1 adjacent to c_1, therefore, the distance between c_1 and b_i is $2k - i + 2$.

$$d(c_1, b_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k.$$

Each b_i is also adjacent to a_i and a_{i+1}, using the result of (6),

$$d(c_1, a_1) = 2, \quad d(c_1, a_2) = 2$$

For $3 \leq i \leq k + 2$, consider the path $c_1 \rightarrow b_2 \rightarrow b_3 \rightarrow ... \rightarrow b_{i-1} \rightarrow a_i$. In this path, the distance between b_2 and b_{i-1} is $i - 3$. b_{i-1} is adjacent to a_i and b_2 adjacent to c_1. Therefore, the distance between c_1 and a_i is $i - 1$.

$$d(c_1, a_i) = i - 1, \quad 3 \leq i \leq k + 2$$

For $k + 3 \leq i \leq 2k$, consider the cycle $(a_1 a_2 ... a_{k+2} ... a_i ... a_{2k})$. The distance between a_i and a_{2k} is $2k - i$. a_{2k} is adjacent to a_1, a_1 adjacent to b_1 and b_1 adjacent to c_1. Therefore, the distance between c_1 and a_i is $2k - i + 3$.

$$d(c_1, a_i) = 2k - i + 3, \quad k + 3 \leq i \leq 2k$$

This shows that a_{k+1} is a vertex farthest from c_1. Therefore:

$$e(c_1) = k + 1. \quad (7)$$

Therefore, (2), (4) and (7) imply,

$$diam(A_n) = k + 1 = \frac{n}{2} + 1.$$

and:

$$rad(A_n) = k = \frac{n}{2}.$$

Consequently, Cen(A_n) is a subgraph induced by vertices $(b_1, b_2, ..., b_{2k})$, while the set of vertices $\{a_1, a_2, ..., a_{2k}, c_1, c_2, ..., c_{2k}\}$ is the peripheral vertices. Therefore, the periphery of A_n is the subgraph induced by all of these vertices. □

Theorem 2. *For the family of convex polytope A_n, n is odd.*

$$Cen(A_n) = Per(A_n) = A_n.$$

Proof. Consider, $n = 2k + 1 \quad k \geq 2$. Select vertex a_1 on the cycle $(a_1a_2a_3...a_i...a_{2k+1})$. By using this,

$$d(a_1, a_i) = i - 1, \ 1 \leq i \leq k + 1 \quad (8)$$

while i increases from $k + 2$ to $2k + 1$, $d(a_1, a_i)$ reduces from k to one.

$$d(a_1, a_i) = 2k + 2 - i, \quad k + 2 \leq i \leq 2k + 1$$

Thus, to find the vertices farthest from a_1 in A_n, we have to take only those values of i that lie between one and $k + 1$.

As each a_i is adjacent to b_i, b_{i-1} and each b_i adjacent to c_i, c_{i-1}, therefore, (8) implies,

$$d(a_1, b_i) = i, \quad 1 \leq i \leq k + 1$$

$$d(a_1, c_i) = 1 + i, \quad 1 \leq i \leq k$$

For $k + 2 \leq i \leq 2k + 1$, consider the cycle $(b_1b_2...b_{k+1}...b_i...b_{2k+1})$. In this cycle, the distance between b_i and b_{2k+1} is $2k + 1 - i$, and b_{2k+1} is adjacent to a_1. Therefore, the distance between a_1 and b_i is $2k - i + 2$.

$$d(a_1, b_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k + 1.$$

Now, consider the cycle $(c_1c_2...c_{k+1}...c_i...c_{2k+1})$. The distance between c_i and c_{2k} is $2k - i$. The vertex c_{2k} is adjacent to b_{2k+1} and b_{2k+1} adjacent to a_1. It shows that the distance between a_1 and c_i is $2k - i + 2$.

$$d(a_1, c_i) = 2k - i + 2, \quad k + 1 \leq i \leq 2k.$$

For $i = 2k + 1, \quad d(a_1, c_i) = 2$.

Hence, c_k and b_{k+1} are the vertices farthest from a_1. Therefore:

$$e(a_1) = k + 1 \quad (9)$$

Thus, the eccentricity of each vertex on inner cycle $(a_1a_2a_3...a_i...a_{2k+1})$ is $k + 1$.

Similarly as above, the vertices b_1 and b_i on cycle $(b_1b_2...b_i....b_{2k+1})$ have the distance as,

$$d(b_1, b_i) = i - 1, \ 1 \leq i \leq k + 1 \quad (10)$$

Each b_i is adjacent to a_i and a_{i+1}. Therefore,

$$d(b_1, a_1) = 1, \quad (b_1, a_2) = 1$$

For $3 \leq i \leq k+2$, consider the path $b_1 \to a_2 \to a_3 \to ... \to a_i$. Then, the distance between a_i and a_2 is $i-2$. a_2 is also adjacent to b_1. Therefore, the distance between b_1 and a_i is $i-1$, i.e.,

$$d(b_1, a_i) = i - 1, \quad 3 \leq i \leq k+2$$

For $k+3 \leq i \leq 2k+1$, consider the cycle $(a_1a_2...a_{k+3}...a_i...a_{2k+1})$. The distance between a_i and a_{2k+1} is $2k-i+1$. Further, a_{2k+1} is adjacent to a_1, and a_1 is adjacent to b_1. Therefore, the distance between b_1 and a_i is $2k-i+3$.

$$d(b_1, a_i) = 2k - i + 3, \quad k+3 \leq i \leq 2k+1$$

Further, b_i is also adjacent to c_i and c_{i-1}; it follows from (10):

$$d(b_1, c_i) = i, \quad 1 \leq i \leq k+1$$

For $k+2 \leq i \leq 2k+1$, consider the cycle $(c_1c_2...c_{k+2}...c_i...c_{2k+1})$. The distance between c_i and c_{2k+1} is $2k+1-i$. c_{2k+1} is also adjacent to b_1. Therefore,

$$d(b_1, c_i) = 2k + 2 - i, \quad k+2 \leq i \leq 2k+1$$

Hence, a_{k+2} and c_{k+1} are the vertices farthest from b_1. Therefore:

$$e(b_1) = k + 1. \tag{11}$$

Hence, each vertex on the middle cycle $(b_1b_2...b_i....b_{2k+1})$ has eccentricity $k+1$.

Further, to find out the eccentricity of the vertices on the outer cycle $(c_1c_2...c_i...c_{2k+1})$, choose a vertex c_1 on this cycle. The distance between c_1 and c_i is $i-1$.

$$d(c_1, c_i) = i - 1, \ 1 \leq i \leq k+1 \tag{12}$$

Each c_i is adjacent to b_i and b_{i+1}. Therefore,

$$d(c_1, b_1) = 1, \ d(c_1, b_2) = 1 \tag{13}$$

For $3 \leq i \leq k+2$, consider the path $c_1 \to b_2 \to b_3 \to ... \to b_i$. The distance between b_2 and b_i is $i-2$. As b_2 is adjacent to c_1, therefore, the distance between c_1 and b_i is $i-1$.

$$d(c_1, b_i) = i - 1, \quad 3 \leq i \leq k+2$$

For $k+3 \leq i \leq 2k+1$, consider the cycle $(b_1b_2...b_{k+3}...b_i...b_{2k+1})$. The distance between b_i and b_{2k+1} is $2k+1-i$. As b_{2k+1} is adjacent to b_1 and b_1 adjacent to c_1, therefore, the distance between c_1 and b_i is $2k-i+3$.

$$d(c_1, b_i) = 2k - i + 3, \quad k+3 \leq i \leq 2k+1.$$

Each b_i is also adjacent to a_i and a_{i+1}, using the result of (13):

$$d(c_1, a_1) = 2, \quad d(c_1, a_2) = 2$$

For $3 \leq i \leq k+2$, consider the path $c_1 \to b_2 \to b_3 \to ... \to b_{i-1} \to a_i$. In this path, the distance between b_2 and b_{i-1} is $i-3$. b_{i-1} is adjacent to a_i, and b_2 is adjacent to c_1. Therefore, the distance between c_1 and a_i is $i-1$.

$$d(c_1, a_i) = i - 1, \quad 3 \leq i \leq k+2$$

For $k+3 \leq i \leq 2k$, consider the cycle $(a_1 a_2 ... a_{k+3} ... a_i ... a_{2k+1})$. The distance between a_i and a_{2k+1} is $2k+1-i$. a_{2k+1} is adjacent to a_1 and a_1 adjacent to b_1. In addition, b_1 is adjacent to c_1. Therefore, the distance between c_1 and a_i is $2k-i+4$.

$$d(c_1, a_i) = 2k - i + 4, \quad k+3 \leq i \leq 2k+1$$

This shows that a_{k+2} and b_{k+2} are the vertices farthest from c_1. Therefore:

$$e(c_1) = k + 1. \tag{14}$$

Consequently, (9), (11) and (14) show the smallest, In addition, the greatest eccentricity of these vertices is $k+1$. Therefore:

$$diam(A_n) = rad(A_n) = k + 1 = \frac{n-1}{2} + 1 = \frac{n+1}{2}.$$

Implies:

$$\text{Cen}(A_n) = \text{Per}(A_n) = A_n.$$

Hence, the family of A_n is self-centered for odd values of n. □

2.1. Average Eccentricity for Convex Polytope A_n

Here, we also are concerned with calculating the average eccentricity for the graph of convex polytope A_n. The average eccentricity of any graph can be calculated by dividing the sum of the eccentricities of all of the vertices to the total number of vertices $(\breve{n})$. There are three circles in the graph of convex polytope A_n, and each circle consists of n vertices. Therefore, A_n has a total of $3n$ vertices; it follows,

$$avec(A_n) = \frac{1}{n} \sum_{u \in V(G)} e_G u \tag{15}$$

By Theorem 1:

$$\begin{aligned} avec(A_n) &= \frac{1}{3 \times (\breve{n})} [n \times \{e(a_1) + e(b_1) + e(c_1)\}] \\ &= \frac{1}{3 \times n} [n \times \{(k+1) + (k) + (k+1)\}] = k + \frac{2}{3} = \frac{n}{2} + \frac{2}{3}. \end{aligned}$$

and by Theorem 2,

$$\begin{aligned} avec(A_n) &= \frac{1}{3 \times n} [n \times \{3(k+1)] \\ &= k + 1 = \frac{n-1}{2} + 1 = \frac{n+1}{2}. \end{aligned}$$

Therefore, we have the following result:

$$avec(A_n) = \begin{cases} \frac{n+1}{2}, & \text{if } n = 2k+1\,; \\ \frac{n}{2} + \frac{2}{3}, & \text{if } n = 2k. \end{cases}$$

2.2. Illustration

Consider the graph of A_8. We have labeled each of its vertices by its eccentricities. The center and periphery are shown in Figures 1 and 2.

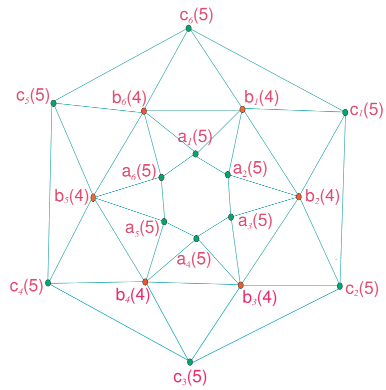

Figure 1. The graph of convex polytope A_8.

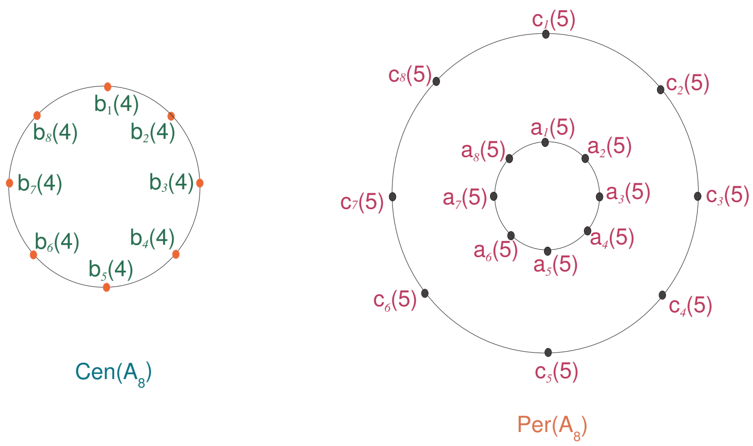

Figure 2. Centrality in the graph of convex polytope A_8.

3. The Center and Periphery for Convex Polytope S_n

Here, we examine the center and periphery for convex polytope S_n.

Definition 8. *The graph of convex polytope (double antiprism) S_n can be obtained from the graph of convex polytope Q_n by adding new edges c_ic_{i+1}, i.e.,*

$$V(S_n) = V(Q_n) \text{ and } E(S_n) = E(Q_n) \cup \{c_ic_{i+1} : 1 \leq i \leq n\}.$$

For our convenience, we identify the cycle induced by the vertices $(a_1, a_2, ..., a_n)$, $(b_1, b_2, ..., b_n)$, $(c_1, c_2, ..., c_n)$ and $(d_1, d_2, ..., d_n)$ as the inner cycle, interior cycle, exterior cycle and outer cycle, respectively.

Theorem 3. *For the family of convex polytope S_n, when n is even, we have:*

$$diam(S_n) = \frac{n}{2} + 1.$$

$$rad(S_n) = \frac{n}{2} + 2.$$

Proof. Suppose, $n = 2k$, $k \geq 2$. Consider the cycle $(a_1a_2....a_i...a_{2k})$. Here, the eccentricity of only one vertex, i.e., a_1, is determined, and due to the symmetry of the graph, all other vertices have the same eccentricity as a_1 on this cycle. Using this cycle,

$$d(a_1, a_i) = i - 1, \; 1 \leq i \leq k + 1. \tag{16}$$

For $k + 2 \leq i \leq 2k$, $d(a_1, a_i)$ varies from $k - 1$ to one, i.e.,

$$d(a_1, a_i) = 2k - i + 1, \quad k + 2 \leq i \leq 2k.$$

Thus, to identify a vertex at maximum distance from a_1 in S_n, take only $1 \leq i \leq k + 1$.

As each a_i is adjacent to b_i, therefore,

$$d(a_1, b_i) = i, \; 1 \leq i \leq k + 1. \tag{17}$$

For $k + 2 \leq i \leq 2k$, take the interior cycle $(b_1b_2...b_{k+2}...b_i...b_{2k})$. In this cycle, the vertices b_i and b_{2k} are at a distance $2k - i$. Further, b_{2k} is adjacent to b_1 and b_1 adjacent to a_1. Therefore, The distance between a_1 and b_i is $2k - i + 2$.

$$d(a_1, b_i) = 2k - i + 2, \quad k + 2 \leq i \leq 2k.$$

Each b_i is adjacent to c_i and c_{i-1}, by using (17),

$$d(a_1, c_i) = i + 1, \; 1 \leq i \leq k. \tag{18}$$

For $k + 1 \leq i \leq 2k$, consider the exterior cycle $(c_1c_2...c_{k+1}...c_i...c_{2k})$. The vertices c_i and c_{2k} are at a distance $2k - i$. As c_{2k} is adjacent to b_1 and b_1 adjacent to a_1, therefore, a_1 and c_i are at a distance $2k - i + 2$.

$$d(a_1, c_i) = 2k - i + 2, \quad k + 1 \leq i \leq 2k.$$

c_i is also adjacent to d_i, so (18) implies,

$$d(a_1, d_i) = i + 2, \quad 1 \leq i \leq k.$$

For $k + 1 \leq i \leq 2k$, the vertices d_i and d_{2k} are at a distance $2k - i$ in the outer cycle $(d_1d_2...d_{k+1}...d_i...d_{2k})$. As d_{2k} is adjacent to c_{2k}, c_{2k} adjacent to b_1 and b_1 adjacent to a_1, therefore, a_1 and d_i are at a distance $2k - i + 3$.

$$d(a_1, d_i) = 2k - i + 3 \quad k + 1 \leq i \leq 2k.$$

Therefore, $e(a_1) = k + 2$.

In the same manner as above, we calculate the eccentricity of b_1 in the cycle $(b_1b_2...b_i...b_{2k})$. The distance between b_1 and b_i in this cycle is,

$$d(b_1, b_i) = i - 1, \; 1 \leq i \leq k + 1. \tag{19}$$

and,

$$d(b_1, b_i) = 2k + 1 - i, \quad k + 2 \leq i \leq 2k.$$

Therefore, we only take values of i between one and $k + 1$. As each b_i is adjacent to c_i and c_{i-1}, using (19),

$$d(b_1, c_i) = i, \; 1 \leq i \leq k. \tag{20}$$

For $k+1 \leq i \leq 2k$, consider the cycle $(c_1c_2...c_{k+2}...c_i...c_{2k})$. The distance between c_i and c_{2k} is $2k-i$. Since, c_{2k} is adjacent to b_1. Thus,

$$d(b_1, c_i) = 2k - i + 1, \quad k+1 \leq i \leq 2k.$$

Each c_i is also adjacent to d_i. Therefore, (20) shows,

$$d(b_1, d_i) = i + 1, \quad 1 \leq i \leq k.$$

For $k+1 \leq i \leq 2k$, consider the cycle $(d_1d_2...d_i...d_{2k})$. The distance between d_i and d_{2k} is $2k-i$. As d_{2k} is adjacent to c_{2k} and c_{2k} adjacent to b_1, therefore,

$$d(b_1, d_i) = 2k - i + 2, \quad k+1 \leq i \leq 2k.$$

b_i is also adjacent to a_i, i.e.,

$$d(b_1, a_i) = i, \quad 1 \leq i \leq k+1.$$

For $k+2 \leq i \leq 2k$, consider the cycle $(a_1, a_2...a_{k+2}...a_i...a_{2k})$. The distance between the vertices a_i and a_{2k} is $2k-i$. As a_{2k} is adjacent to a_1, a_1 adjacent to b_1, therefore,

$$d(b_1, a_i) = 2k - i + 2, \quad k+2 \leq i \leq 2k.$$

As, d_k and a_{k+1} are farthest from b_1, therefore, $e(b_1) = k+1$. Next, the distance between c_1 and c_i in the cycle $(c_1c_2...c_i...c_{2k})$ is $i-1$.

$$d(c_1, c_i) = i - 1, \ 1 \leq i \leq k+1. \tag{21}$$

Additionally, for $k+2 \leq i \leq 2k$,

$$d(c_1, c_i) = 2k - i + 1, \quad k+2 \leq i \leq 2k$$

Each c_i is adjacent to d_i, from (21):

$$d(c_1, d_i) = i, \quad 1 \leq i \leq k+1.$$

For $k+2 \leq i \leq 2k$, the vertices d_i and d_{2k} are at a distance $2k-i$ in the cycle $(d_1d_2...d_{k+2}...d_i...d_{2k})$. The vertex d_{2k} is adjacent to d_1 and d_1 adjacent to c_1. Therefore,

$$d(c_1, d_i) = 2k - i + 2, \quad k+2 \leq i \leq 2k.$$

Each c_i is adjacent to b_i and b_{i+1}; Equation (17) implies,

$$d(c_1, b_1) = 1, \ d(c_1, b_2) = 1. \tag{22}$$

For $3 \leq i \leq k+1$, b_2 and b_i are at a distance $i-2$ in the path $c_1 \to b_2 \to b_3 \to ... \to b_i$. Again, b_2 is adjacent to c_1; thus, we have:

$$d(c_1, b_i) = i - 1, \ 3 \leq i \leq k+1. \tag{23}$$

For $k+2 \leq i \leq 2k$, consider the cycle $(b_1b_2...b_{k+2}...b_i...b_{2k})$. The distance between b_i and b_{2k} is $2k-i$ in this cycle. As, b_{2k} is adjacent to b_1, b_1 adjacent to c_1. Therefore,

$$d(c_1, b_i) = 2k - i + 2, \quad k+2 \leq i \leq 2k.$$

Since b_i is adjacent to a_i, it follows from (22) that:

$$d(c_1, a_1) = 2 \ , \ d(c_1, a_2) = 2$$

For $3 \leq i \leq k+1$, b_i and b_2 are at a distance $i-2$ in the path $c_1 \to b_2 \to b_3 \to ... \to b_i \to a_i$. The vertex b_i is adjacent to a_i and b_2 adjacent to c_1. Therefore,

$$d(c_1, a_i) = i, \quad 3 \leq i \leq k+1. \quad ...(23)$$

For $k+2 \leq i \leq 2k$, the distance between a_i and a_{2k} is $2k-i$ in the cycle $(a_1 a_2 ... a_{k+2} ... a_i ... a_{2k})$. a_{2k} is again adjacent to a_1, a_1 adjacent to b_1 and b_1 adjacent to c_1. For that reason,

$$d(c_1, a_i) = 2k - i + 3, \quad k+2 \leq i \leq 2k.$$

Consequently, d_{k+1} and a_{k+1} are farthest from c_1. Therefore, $e(c_1) = k+1$.

Next, take a vertex d_1 on the outer cycle. In this cycle $(d_1 d_2 ... d_i ... d_{2k})$,

$$d(d_1, d_i) = i - 1, \ 1 \leq i \leq k+1. \tag{24}$$

Additionally,

$$d(d_1, d_i) = 2k - i + 1, \quad k+2 \leq i \leq 2k.$$

In addition, each d_i is adjacent to c_i,

$$d(d_1, c_i) = i, \quad 1 \leq i \leq k+1.$$

For $k+2 \leq i \leq 2k$, take a cycle $(c_1, c_2 ... c_{k+2} ... c_i ... c_{2k})$. The vertices c_i and c_{2k} are at a distance $2k-i$ in this cycle. In addition, c_{2k} is adjacent to c_1 and c_1 adjacent to d_1. Then,

$$d(d_1, c_i) = 2k - i + 2, \quad k+2 \leq i \leq 2k.$$

Each c_i is adjacent to b_i and b_{i+1}. i.e., $d(d_1, b_1) = 2$ and:

$$d(d_1, b_2) = 2. \tag{25}$$

For $3 \leq i \leq k+1$, the vertices b_2 and b_i are at a distance $i-2$ in the path $d_1 \to c_1 \to b_2 \to b_3 \to ... \to b_i$. b_2 is adjacent to c_1 and c_1 adjacent to d_1 in S_n. Therefore,

$$d(d_1, b_i) = i, \ 3 \leq i \leq k+1, \tag{26}$$

For $k+2 \leq i \leq 2k$, the vertices b_i and b_{2k} are at distance $2k-i$ in the cycle $(b_1, b_2 ... b_{k+2} ... b_i ... b_{2k})$. b_{2k} is adjacent to b_1, b_1 adjacent to c_1 and c_1 adjacent to d_1; for this,

$$d(d_1, b_i) = 2k - i + 3, \quad k+2 \leq i \leq 2k.$$

In addition, b_i is adjacent to a_i. This implies from (25),

$$d(d_1, a_1) = 3 \ , \ d(d_1, a_2) = 3.$$

For $3 \leq i \leq k+1$, the vertices b_2 and b_i are at a distance $i-2$ in the path $d_1 \to c_1 \to b_2 \to b_3 \to ... \to b_i \to a_i$. b_i is adjacent to a_i, b_2 adjacent to c_1 and c_1 adjacent to d_1 in S_n. Therefore,

$$d(d_1, a_i) = i + 1, \quad 3 \leq i \leq k+1.$$

For $k+2 \leq i \leq 2k$, consider the cycle $(a_1, a_2...a_{k+2}...a_i...a_{2k})$. The vertices a_i and a_{2k} are at a distance $2k-i$ in this cycle. a_{2k} is adjacent to a_1, a_1 adjacent to b_1, b_1 adjacent to c_1 and c_1 adjacent to d_1. As a result,

$$d(d_1, a_i) = 2k - i + 4, \quad k+2 \leq i \leq 2k.$$

Consequently,

$$e(d_1) = k + 2.$$

Thus, it is concluded that maximum eccentricity among all of the vertices of S_n is $k+2$, and the minimum eccentricity is $k+1$.

Therefore

$$diam(S_n) = k + 2 = \frac{n}{2} + 2.$$

$$rad(S_n) = k + 1 = \frac{n}{2} + 1.$$

□

The following corollary is straightforward.

Corollary 1. *The center and periphery for the family of convex polytope* (S_n)*, when n is even, are subgraphs induced by all of the central vertices* $\{b_1, b_2, ..., b_i, ..., b_{2k}, c_1, c_2, ..., c_i, ..., c_{2k}\}$ *and peripheral vertices* $\{a_1, a_2, ..., a_i, ..., a_{2k}, d_1, d_2, ..., d_i, ..., d_{2k}\}$ *of* S_n*, respectively.*

Now, we find out the radius and diameter of S_n, when n is odd.

Theorem 4. *When n is odd, the family of convex polytope* S_n *has the radius and diameter as,*

$$diam(S_n) = \frac{n-1}{2} + 3,$$

$$rad(S_n) = \frac{n-1}{2} + 2.$$

Proof. Let $n = 2k+1$, $k \geq 2$. Consider the cycle $(a_1 a_2....a_i...a_{2k+1})$, and select a vertex a_1 in it. It is clear that,

$$d(a_1, a_i) = i - 1, \quad 1 \leq i \leq k+1$$

$$d(a_1, a_i) = 2k + 2 - i, \; k+2 \leq i \leq 2k+1, \tag{27}$$

Thus, the equations above lead to the proof including only $1 \leq i \leq k+1$ in order to find a vertex having the greatest distance from a_1 in S_n. Since each a_i is adjacent to b_i, therefore, (27) implies that:

$$d(a_1, b_i) = i, \; 1 \leq i \leq k+1. \tag{28}$$

For $k+2 \leq i \leq 2k+1$, the vertices b_i and b_{2k+1} are at a distance $2k-i+1$ in the cycle $(b_1 b_2...b_{k+2}...b_i...b_{2k+1})$. b_{2k+1} is adjacent to b_1 and b_1 adjacent to a_1. Therefore, The distance between a_1 and b_i is $2k-i+3$.

$$d(a_1, b_i) = 2k + 3 - i, \quad k+2 \leq i \leq 2k+1.$$

Again, each b_i is adjacent to c_i and c_{i-1}, by using (28).

$$d(a_1, c_i) = i + 1, \; 1 \leq i \leq k+1. \tag{29}$$

For $k+2 \leq i \leq 2k+1$, the distance between the vertices c_i and c_{2k+1} is $2k+1-i$ in the cycle $(c_1c_2...c_{k+1}...c_i...c_{2k+1})$. Since, c_{2k+1} is adjacent to b_1 and b_1 adjacent to a_1, therefore, a_1 and c_i are at a distance $2k-i+3$.

$$d(a_1, c_i) = 2k - i + 3, \quad k+2 \leq i \leq 2k+1.$$

In addition, c_i is adjacent to d_i, therefore, (29) shows,

$$d(a_1, d_i) = i + 2, \quad 1 \leq i \leq k+1.$$

For $k+2 \leq i \leq 2k+1$, the vertices d_i and d_{2k+1} are at a distance $2k+1-i$ in the cycle $(d_1d_2...d_{k+1}...d_i...d_{2k+1})$. In addition, d_{2k+1} is adjacent to c_{2k+1}, c_{2k+1} adjacent to b_1 and b_1 adjacent to a_1. Therefore, a_1 and d_i are at a distance $2k-i+4$.

$$d(a_1, d_i) = 2k - i + 4, \quad k+2 \leq i \leq 2k+1.$$

As a result, d_{k+1} is farthest from a_1; therefore, $e(a_1) = k+3$.

In order to find out the eccentricity of the vertices on the cycle $(b_1b_2...b_i...b_{2k+1})$, the distance between b_1 and b_i in this cycle is $i-1$.

$$d(b_1, b_i) = i - 1, \ 1 \leq i \leq k+1. \tag{30}$$

In addition,

$$d(b_1, b_i) = 2k - i + 2, \quad k+2 \leq i \leq 2k+1.$$

Further, each b_i is adjacent to c_i and c_{i-1}, therefore, (30) shows,

$$d(b_1, c_i) = i, \ 1 \leq i \leq k+1. \tag{31}$$

For $k+2 \leq i \leq 2k+1$, consider the cycle $(c_1c_2...c_{k+2}...c_i...c_{2k+1})$. The distance between c_i and c_{2k+1} is $2k-i+1$. Since, c_{2k+1} is adjacent to b_1, thus,

$$d(b_1, c_i) = 2k - i + 2, \quad k+2 \leq i \leq 2k+1.$$

Each c_i is also adjacent to d_i. It is shown from (31),

$$d(b_1, d_i) = i + 1, \quad 1 \leq i \leq k+1.$$

For $k+2 \leq i \leq 2k+1$, consider the cycle $(d_1d_2...d_i...d_{2k+1})$. The distance between d_i and d_{2k+1} is $2k-i+1$. As d_{2k+1} is adjacent to c_{2k+1} and c_{2k+1}adjacent to b_1, therefore,

$$d(b_1, d_i) = 2k - i + 3, \quad k+2 \leq i \leq 2k+1.$$

b_i is also adjacent to a_i, i.e.,

$$d(b_1, a_i) = i, \quad 1 \leq i \leq k+1.$$

For $k+2 \leq i \leq 2k+1$, consider the cycle $(a_1a_2...a_{k+2}...a_i...a_{2k+1})$. The vertices a_i and a_{2k+1} is $2k+1-i$. As a_{2k+1} is adjacent to a_1, $a_1$adjacent to b_1, therefore,

$$d(b_1, a_i) = 2k - i + 3, \quad k+2 \leq i \leq 2k+1.$$

Since, d_{k+1} is a vertex farthest from b_1, therefore, $e(b_1) = k+2$.

Next, the distance between c_1 and c_i in the cycle $(c_1c_2...c_i...c_{2k+1})$ is,

$$d(c_1, c_i) = i - 1, \ 1 \leq i \leq k+1. \tag{32}$$

Additionally,

$$d(c_1, c_i) = 2k - i + 2, \quad k + 2 \le i \le 2k + 1.$$

Each c_i is adjacent to d_i, from (32):

$$d(c_1, d_i) = i, \quad 1 \le i \le k + 1.$$

For $k + 2 \le i \le 2k + 1$, the vertices d_i and d_{2k+1} are at a distance $2k + 1 - i$ in the cycle $(d_1 d_2 ... d_{k+2} ... d_i ... d_{2k+1})$. The vertex d_{2k+1} is adjacent to d_1 and d_1 adjacent to c_1. Therefore,

$$d(c_1, d_i) = 2k - i + 3, \quad k + 2 \le i \le 2k + 1.$$

Each c_i is adjacent to b_i and b_{i+1}; Equation (28) implies,

$$d(c_1, b_1) = 1, \ d(c_1, b_2) = 1. \tag{33}$$

For $3 \le i \le k + 2$, b_2 and b_i are at a distance $i - 2$ in the path $c_1 \to b_2 \to b_3 \to ... \to b_i$. Again, b_2 is adjacent to c_1; thus, we have:

$$d(c_1, b_i) = i - 1, \ 3 \le i \le k + 2. \tag{34}$$

For $k + 3 \le i \le 2k + 1$, consider the cycle $(b_1 b_2 ... b_{k+1} ... b_i ... b_{2k+1})$. The distance between b_i and b_{2k+1} is $2k + 1 - i$ in this cycle. As b_{2k+1} is adjacent to b_1 and b_1 adjacent to c_1, therefore,

$$d(c_1, b_i) = 2k - i + 3, \quad k + 3 \le i \le 2k + 1.$$

Since b_i is adjacent to a_i, it follows from (33) that:

$$d(c_1, a_1) = 2 \ , \ d(c_1, a_2) = 2$$

For $3 \le i \le k + 2$, b_i and b_2 are at a distance $i - 2$ in the path $c_1 \to b_2 \to b_3 \to ... \to b_i \to a_i$. The vertex b_i is adjacent to a_i, b_2 adjacent to c_1. Therefore,

$$d(c_1, a_i) = i, \quad 3 \le i \le k + 2.$$

For $k + 3 \le i \le 2k + 1$, the distance between a_i and a_{2k+1} in the cycle $(a_1 a_2 ... a_{k+2} ... a_i ... a_{2k+1})$. a_{2k+1} is again adjacent to a_1, a_1 adjacent to b_1 and b_1 adjacent to c_1. For that reason,

$$d(c_1, a_i) = 2k - i + 4, \quad k + 3 \le i \le 2k + 1.$$

Consequently, a_{k+2} is a vertex farthest from c_1. Therefore, $e(c_1) = k + 2$.

Next, take a vertex d_1 on the outer cycle. In this cycle, $(d_1 d_2 ... d_i ... d_{2k+1})$,

$$d(d_1, d_i) = i - 1, \ 1 \le i \le k + 1. \tag{35}$$

$d(d_1, d_i)$ starts to decrease for $k + 2 \le i \le 2k + 1$ as,

$$d(d_1, d_i) = 2k - i + 2, \quad k + 2 \le i \le 2k + 1.$$

Each d_i is adjacent to c_i,

$$d(d_1, c_i) = i, \quad 1 \le i \le k + 1.$$

For $k+2 \leq i \leq 2k+1$, take a cycle $(c_1c_2...c_{k+2}...c_i...c_{2k+1})$. The vertices c_i and c_{2k+1} are at a distance $2k+1-i$ in this cycle. In addition, c_{2k+1} is adjacent to c_1 and c_1 adjacent to d_1. Then,

$$d(d_1, c_i) = 2k - i + 3, \quad k+2 \leq i \leq 2k+1.$$

Each c_i is adjacent to b_i and b_{i+1}, i.e., $d(d_1, b_1) = 2$ and

$$d(d_1, b_2) = 2. \tag{36}$$

For $3 \leq i \leq k+2$, the vertices b_2 and b_i are at a distance $i-2$ in the path $d_1 \to c_1 \to b_2 \to b_3 \to ... \to b_i$. b_2 is adjacent to c_1 and c_1 adjacent to d_1 in S_n. Therefore,

$$d(d_1, b_i) = i, \ 3 \leq i \leq k+2. \tag{37}$$

For $k+3 \leq i \leq 2k+1$, consider the cycle $(b_1b_2...b_{k+2}...b_i...b_{2k+1})$. The vertices b_i and b_{2k+1} are $2k+1-i$. b_{2k+1} is adjacent to b_1, b_1 adjacent to c_1 and c_1 adjacent to d_1; for this,

$$d(d_1, b_i) = 2k - i + 4, \quad k+3 \leq i \leq 2k+1.$$

In addition, b_i is adjacent to a_i. Therefore, (36) implies,

$$d(d_1, a_1) = \ , \quad d(d_1, a_2) = 3.$$

For $3 \leq i \leq k+2$, the vertices b_2 and b_i are at a distance $i-2$ in the path $d_1 \to c_1 \to b_2 \to b_3 \to ... \to b_i \to a_i$. b_i is adjacent to a_i, b_2 adjacent to c_1 and c_1 adjacent to d_1 in S_n. Therefore,

$$d(d_1, a_i) = i + 1, \quad 3 \leq i \leq k+2.$$

For $k+3 \leq i \leq 2k+1$, consider the cycle $(a_1a_2...a_{k+2}...a_i...a_{2k+1})$. The vertices a_i and a_{2k+1} are $2k+1-i$. a_{2k+1} is adjacent to a_1, a_1 adjacent to b_1, b_1 adjacent to c_1 and c_1 adjacent to d_1. As a result,

$$d(d_1, a_i) = 2k - i + 5, \quad k+3 \leq i \leq 2k+1.$$

This means,

$$e(d_1) = k + 3.$$

It shows that the maximum and minimum eccentricity among all of the vertices of S_n are $k+3$ and $k+2$, respectively. Therefore:

$$diam(S_n) = k + 3 = \frac{n-1}{2} + 3.$$

$$rad(S_n) = k + 2 = \frac{n-1}{2} + 2.$$

□

Thus, we can summarize the above results as,

Corollary 2. *The center for the family of convex polytope $S(n)$ is a subgraph induced by all of the vertices of the interior and exterior cycles, and the periphery is the subgraphs induced by all of the peripheral vertices $\{a_1, a_2, ..., a_i, ..., a_{2k}, d_1, d_2, ..., d_i, ..., d_{2k}\}$ of S_n, respectively.*

3.1. Average Eccentricity for Convex Polytopes S_n

Here, the average eccentricity for the family of S_n is being determined. The graph of S_n consist of four major circles, and there are n vertices in each circle. Therefore, the total number of vertices in S_n (i.e., $\acute{n}$) is $4n$; it follows,

$$avec(S_n) = \frac{1}{\acute{n}} \sum_{u \in V(G)} e_G u$$

By Theorem 3:

$$\begin{aligned} avec(S_n) &= \frac{1}{4 \times (\acute{n})}[n \times \{e(a_1) + e(b_1) + e(c_1) + e(d_1\}] \\ &= \frac{1}{4 \times n}[n \times \{(k+2) + (k+1) + (k+1) + (k+2)\}] \\ &= \frac{1}{4 \times n}[2n \times \{(k+2) + (k+1)\}] \\ &= \frac{1}{2}[2k+3] \\ &= k + \frac{3}{2} \\ &= \frac{n+3}{2}. \end{aligned}$$

and by Theorem 4,

$$\begin{aligned} avec(S_n) &= \frac{1}{4 \times n}[n \times \{(k+3) + (k+2) + (k+2) + (k+3)\}] \\ &= \frac{1}{4 \times n}[2n \times \{(k+3) + (k+2)\}] \\ &= \frac{1}{2}[2k+5] \\ &= k + \frac{5}{2} \\ &= \frac{n-1}{2} + \frac{5}{2} \\ &= \frac{n+4}{2}. \end{aligned}$$

Therefore, we have the following result:

$$avec(S_n) = \begin{cases} \frac{n+3}{2}, & \text{for all even values of } n \\ \frac{n+4}{2}, & \text{for all odd values of } n. \end{cases}$$

3.2. Illustration

Consider the graph of S_6. Its center and periphery are shown in Figures 3 and 4.

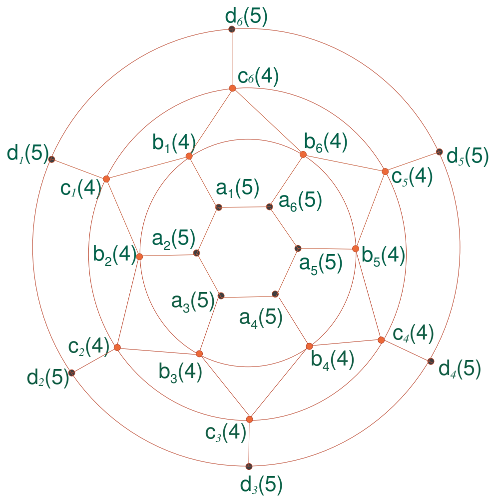

Figure 3. The graph of convex polytope S_6.

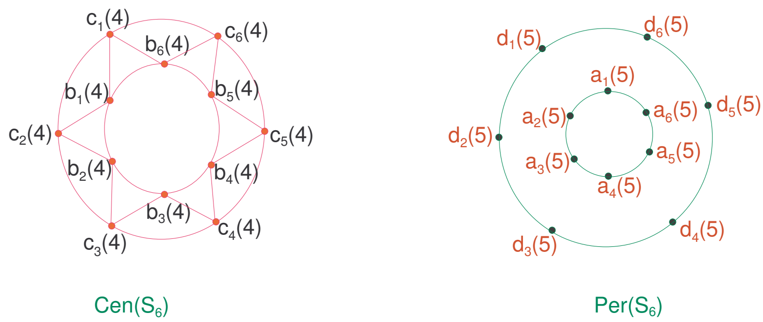

Figure 4. Centrality for S_6.

4. The Center and Periphery for Convex Polytopes T_n

Here, we established the center and periphery for T_n and show that T_n is not self-centered.

Definition 9. *The graph of convex polytope T_n can be obtained from the graph of convex polytope Q_n by adding new edges . It consist of three-sided faces, five-sided faces and n-sided faces. $a_{i+1}b_i$, i.e., $V(T_n) = V(Q_n)$ and $E(T_n) = E(Q_n) \bigcup \{a_{i+1}b_i : 1 \leq i \leq n\}$.*

This section begins with the following theorem on T_n.

Theorem 5. *The diameter for the family of convex polytope T_n is,*

$$diam(T_n) = \begin{cases} \frac{n}{2} + 2, & \text{for } n = 2k; \\ \frac{n-1}{2} + 2, & \text{for } n = 2k+1. \end{cases}$$

In addition, its radius,

$$rad(T_n) = \begin{cases} \frac{n}{2} + 1, & \text{for } n \text{ to be even;} \\ \frac{n+1}{2}, & \text{for } n \text{ to be odd.} \end{cases}$$

Proof. Consider, $n = 2k$, $k \geq 2$. Choose take cycle $(a_1a_2...a_i...a_{2k})$. In this cycle:

$$d(a_1, a_i) = i - 1, \ 1 \leq i \leq k+1. \tag{38}$$

For $k+2 \leq i \leq 2k$, the distance between a_1 and a_i decreases from $k-1$ to one, i.e.,

$$d(a_1, a_i) = 2k - i + 1, \quad k+2 \leq i \leq 2k.$$

Therefore, we must considered $1 \leq i \leq k+1$ in order to find the distance of a vertex a_1 from a vertex farthest from it in T_n.

In the graph of T_n, each a_i is adjacent to b_i and b_{i-1}; thus, (38) implies,

$$d(a_1, b_i) = i, \ 1 \leq i \leq k. \tag{39}$$

For $k+1 \leq i \leq 2k$, the vertices b_i and b_{2k} are at a distance $2k - i$ in the cycle $(b_1 b_2 ... b_{k+1} ... b_i ... b_{2k})$. In addition, b_{2k} is adjacent to a_1. Therefore, The distance between a_1 and b_i is $2k - i + 1$.

$$d(a_1, b_i) = 2k - i + 1 \quad k+1 \leq i \leq 2k.$$

Further, each b_i is adjacent to c_i and c_{i-1}, using (39).

$$d(a_1, c_1) = 2, d(a_1, c_{2k}) = 2$$

for $2 \leq i \leq k$, consider path $a_1 \to b_1 \to b_2 \to ... \to b_i \to c_i$. The distance between b_1 and b_i is $i - 1$. Each b_i is adjacent to c_i and b_1 adjacent to a_1. Therefore,

$$d(a_1, c_i) = i + 1, \ 2 \leq i \leq k. \tag{40}$$

Next, for $k+1 \leq i \leq 2k-1$, consider the cycle $(b_1 b_2 ... b_{i+1} ... b_{2k})$. The vertices b_{2k} and b_{i+1} are at a distance $2k - i - 1$. Further, b_{2k} is adjacent to a_1 and b_{i+1} adjacent to c_i. Therefore,

$$d(a_1, c_i) = 2k - i + 1, \quad k+1 \leq i \leq 2k-1.$$

c_i is also adjacent to d_i. Therefore, (40) implies

$$d(a_1, d_i) = i + 2, \quad 1 \leq i \leq k.$$

For $k+1 \leq i \leq 2k$, the vertices d_i and d_{2k} are at a distance $2k - i$ in the cycle $(d_1 d_2 ... d_i ... d_{2k})$. In addition, each d_{2k} is adjacent to c_{2k}, c_{2k} adjacent to b_1 and b_1 adjacent to a_1; therefore,

$$d(a_1, d_i) = 2k + 3 - i, \quad k+1 \leq i \leq 2k.$$

Hence, d_k is a vertex at the largest distance from a_1. Therefore, $e(a_1) = k + 2$.

Next, continue this for cycle $(b_1 b_2 ... b_i ... b_{2k})$; we choose a vertex b_1, such that,

$$d(b_1, b_i) = i - 1, \ 1 \leq i \leq k+1. \tag{41}$$

The distance between b_1 and b_i decreases from $k-1$ to one, when i increases from $k+2$ to $2k$.

$$d(b_1, b_i) = 2k - i + 1, \quad k+2 \leq i \leq 2k.$$

In addition, each b_i is adjacent to a_i and a_{i+1}.

$$d(b_1, a_1) = 1, \quad d(b_1, a_2) = 1$$

and when $3 \leq i \leq k+1$, consider the path $b_1 \to a_2 \to a_3 \to ... \to a_i$. a_2 and a_i are at a distance $i-2$ in this path, and a_2 is adjacent to b_1; therefore,

$$d(b_1, a_i) = i-1, \quad 3 \leq i \leq k+1.$$

For $k+2 \leq i \leq 2k$, consider the cycle $(a_1a_2...a_{k+2}...a_i...a_{2k})$. The distance between a_i and a_{2k} is $2k-i$. As a_{2k} is adjacent to a_1 and a_1 adjacent to b_1, therefore,

$$d(b_1, a_i) = 2k-i+2, \quad k+2 \leq i \leq 2k.$$

In addition, b_i is adjacent to c_i and c_{i-1}; using (41), we have:

$$d(b_1, c_i) = i, \ 1 \leq i \leq k. \tag{42}$$

For $k+1 \leq i \leq 2k$, consider the path $b_1 \to b_{2k} \to b_{2k-1} \to ... \to b_{i+1} \to c_i$. The distance between b_{2k} and b_{i+1} is $2k-i-1$. Further, b_{2k} is adjacent to b_1. In addition, b_{i+1} is adjacent to c_i. Therefore, the distance between b_1 and c_i is $2k-i+1$.

$$d(b_1, c_i) = 2k-i+1, \quad k+1 \leq i \leq 2k$$

Further, c_i is adjacent to d_i; hence, (42) shows,

$$d(b_1, d_i) = i+1, \quad 1 \leq i \leq k.$$

For $k+1 \leq i \leq 2k$, the vertices d_i and d_{2k} are at a distance $2k-i$ in the cycle $(d_1d_2...d_{k+2}...d_i...d_{2k})$. The vertex d_{2k} is adjacent to c_{2k} and c_{2k} adjacent to b_1. Therefore,

$$d(b_1, d_i) = 2k-i+2, \quad k+1 \leq i \leq 2k.$$

Hence, d_k is a vertex farthest from b_1. Therefore, $e(b_1) = k+1$

Next, to find out the eccentricity of the vertices $\{c_i, 1 \leq i \leq 2k\}$, take a vertex c_1 among all c_i's, and each c_i is adjacent to b_i, b_{i+1}, i.e.,

$$d(c_1, b_1) = 1, \quad , \quad d(c_1, b_2) = 1$$

and when $3 \leq i \leq k+1$, consider the path $c_1 \to b_2 \to b_3 \to ... \to b_i$. b_2 and b_i are at distance $i-2$, and again, b_2 is adjacent to c_1; thus, we have:

$$d(c_1, b_i) = i-1, \ 3 \leq i \leq k+1. \tag{43}$$

For $k+2 \leq i \leq 2k$, consider the cycle $(b_1b_2...b_{k+2}...b_i...b_{2k})$. The distance between b_i and b_{2k} is $2k-i$ in this cycle. As b_{2k} is adjacent to b_1 and b_1 adjacent to c_1, therefore,

$$d(c_1, b_i) = 2k-i+2, \quad k+2 \leq i \leq 2k.$$

Moreover, b_i is adjacent to a_i and a_{i+1}; it follows from (43) that:

$$d(c_1, a_1) = 2 \ , \ d(c_1, a_2) = 2, \ d(c_1, a_3) = 2$$

For $4 \leq i \leq k+2$, a_i and a_2 are at a distance $i-3$ in the path $c_1 \to b_2 \to a_3 \to ... \to a_i$. Furthermore, a_2 is adjacent to b_2 and b_2 adjacent to c_1. Thus,

$$d(c_1, a_i) = i-1, \ 4 \leq i \leq k+2. \tag{44}$$

For $k+3 \leq i \leq 2k$, the distance between a_i and a_{2k} in the cycle $(a_1a_2...a_{k+3}...a_i...a_{2k})$ is $2k-i$, and a_{2k} is adjacent to a_1, a_1 adjacent to b_1 and b_1 adjacent to c_1. For that reason,

$$d(c_1, a_i) = 2k - i + 3, \quad k+3 \leq i \leq 2k.$$

Again, c_i is adjacent to d_i. Hence,

$$d(c_1, d_i) = i, \ \ 1 \leq i \leq k+1$$

For $k+2 \leq i \leq 2k$, the vertices d_{2k} and d_i are at a distance $2k-i$ in the cycle $(d_1d_2...d_{k+2}...d_i...d_{2k})$, and d_{2k} is adjacent to d_1 and d_1 adjacent to c_1 in T_n. Therefore,

$$d(c_1, d_i) = 2k - i + 2, \quad k+2 \leq i \leq 2k$$

In order to find the distance between c_1 and $c_i, 1 \leq i \leq k+1$, consider the path $c_1 \to b_2 \to b_3 ... \to b_i \to c_i$. The distance between b_2 and b_i is i-2, and b_i is adjacent to c_i and b_2 adjacent to c_1. Therefore,

$$d(c_1, c_i) = i, \quad 1 \leq i \leq k+1.$$

For more values of i, $d(c_1, c_i)$ begins to reduce as,

$$d(c_1, c_i) = 2k - i + 2, \quad k+2 \leq i \leq 2k$$

This means that a_{k+2}, c_{k+1} and d_{k+1} are the vertices farthest from c_1. Therefore, $e(c_1) = k+1$.
Now, we find the eccentricities of the vertices on the cycle $(d_1d_2...d_i...d_{2k})$. In this cycle,

$$d(d_1, d_i) = i - 1, \quad 1 \leq i \leq k+1.$$

For $k+2 \leq i \leq 2k$, the distance between d_1 and d_i decreases from $k-1$ to one.

$$d(d_1, d_i) = 2k + 1 - i, \quad k+2 \leq i \leq 2k.$$

As d_i adjacent to c_i:

$$d(d_1, c_i) = i, \ 1 \leq i \leq k+1. \tag{45}$$

When i increases from $k+2$ to $2k$, the distance between d_i and d_{2k} is $2k-i$ in the cycle $(d_1d_2...d_{k+2}...d_i...d_{2k})$. In addition, d_{2k} is adjacent to d_1 and d_i adjacent to c_i. Thus,

$$d(d_1, c_i) = 2k - i + 2, \quad k+2 \leq i \leq 2k.$$

As each c_i is adjacent to b_i, b_{i+1}.

$$d(d_1, b_1) = 2 \ \ , \ d(d_1, b_2) = 2, \tag{46}$$

For $3 \leq i \leq k+1$, consider a path $d_1 \to c_1 \to b_2 \to b_3 \to ... \to b_i$. b_2 and b_i are at a distance $i-2$, and b_2 is adjacent to c_1 and c_1 adjacent to d_1 in T_n. Therefore,

$$d(d_1, b_i) = i, \quad 3 \leq i \leq k+1$$

For $k+2 \leq i \leq 2k$, consider the cycle $(b_1b_2...b_{k+2}...b_i...b_{2k})$. The distance between the vertices b_i and b_{2k} is $2k-i$. b_{2k} is adjacent to b_1, b_1 adjacent to c_1 and c_1 adjacent to d_1; for that reason,

$$d(d_1, b_i) = 2k - i + 3, \quad k+2 \leq i \leq 2k.$$

In addition, b_i is adjacent to a_i and a_{i+1}. Therefore, (46) implies,

$$d(d_1, a_1) = 3, \quad d(d_1, a_2) = 3, \quad d(d_1, a_3) = 3$$

For $4 \leq i \leq k+2$, the vertices a_3 and a_i are at a distance $i-3$ in the path $d_1 \rightarrow c_1 \rightarrow b_2 \rightarrow a_3 \rightarrow a_4 \rightarrow ... \rightarrow a_i$. Further, a_3 is adjacent to b_2, b_2 adjacent to c_1 and c_1 adjacent to d_1 in T_n. Therefore,

$$d(d_1, a_i) = i, \quad 4 \leq i \leq k+2.$$

For $k+3 \leq i \leq 2k$, consider the cycle $(a_1a_2...a_{k+3}...a_i...a_{2k})$. The distance between the vertices a_i and a_{2k} is $2k-i$. a_{2k} is adjacent to a_1 and a_1 adjacent to b_1. Further, b_1 adjacent to c_1 and c_1 adjacent to d_1. As a result,

$$d(d_1, a_i) = 2k - i + 4, \quad k+3 \leq i \leq 2k.$$

This shows that a_{k+2} is at the highest distance from d_1. Therefore, $e(d_1) = k+2$.

Thus, it is concluded that the maximum eccentricity among all of the vertices of T_n is $k+2$, and $k+1$ is the minimum eccentricity. Therefore, $\text{diam}(T_n) = k+2 = \frac{n}{2}+2$.

$$rad(T_n) = k+1 = \frac{n}{2}+1.$$

For odd n, the proof is analogous to the case discussed above and omitted. □

Corollary 3. *The center of T_n, when n is even, is the subgraph induced by the central vertices $\{b_i \cup c_i : 1 \leq i \leq n\}$, while the periphery is the subgraph induced by the vertices of inner and outer cycles.*

4.1. Average Eccentricity for Convex Polytopes T_n

There are four circles in the graph of T_n, and each circle has n vertices. The average eccentricity for the graph of convex polytope T_n can be found out by dividing sum of eccentricities of all vertices on each circle to its total number of vertices. Therefore,

$$avec(T_n) = \frac{1}{\acute{n}} \sum_{u \in V(G)} e_G u$$

By Theorem 5:

$$\begin{aligned} avec(T_n) &= \frac{1}{4 \times n}[n \times \{(k+2) + (k+1) + (k+1) + (k+2)] \\ &= \frac{1}{4 \times n}[2n \times \{(k+2) + (k+1)\}] = \frac{1}{2}[2k+3] = k + \frac{3}{2} = \frac{n+3}{2}. \end{aligned}$$

and by Theorem 5,

$$\begin{aligned} avec(T_n) &= \frac{1}{4 \times n}[n \times \{(k+2) + (k+2) + (k+1) + (k+2)\}] \\ &= \frac{1}{4 \times n}[3n \times (k+2) + n \times (k+1)] = \frac{1}{4}[4k+7] = \frac{1}{4}[4(\frac{n-1}{2}) + 7] = \frac{n}{2} + \frac{5}{4}. \end{aligned}$$

Therefore, we get the following immediate result:

$$avec(T_n) = \begin{cases} \frac{n}{2} + \frac{5}{4}, & \text{if } n = 2k+1\,; \\ \frac{n+3}{2}, & \text{if } n = 2k\,. \end{cases}$$

4.2. Illustration

Consider the graph T_6. The center and periphery for T_6 are shown in Figures 5 and 6.

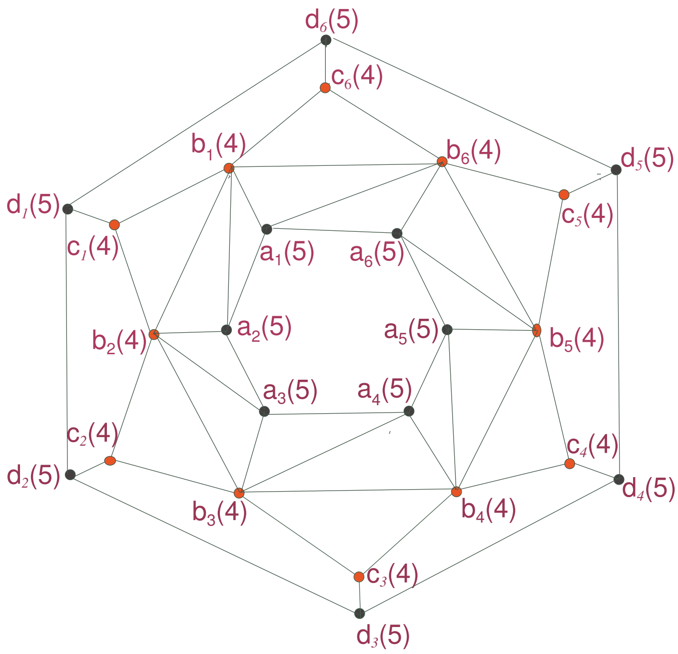

Figure 5. The graph of convex polytope T_6.

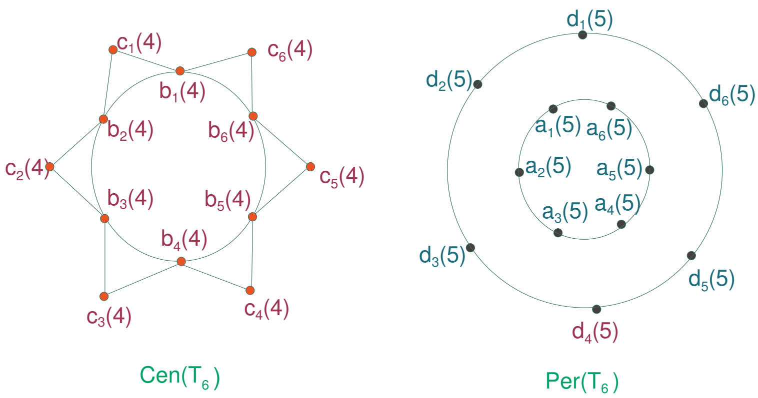

Figure 6. Centrality for T_6.

5. Concluding Remarks

In summary, we have studied the center and periphery of three types of families of convex polytopes via finding a subgraph induced by central and peripheral vertices. The predetermined facts about the eccentricity, radius and diameter of graphs play an important role in order to find the center and periphery for specific families of graphs; the average eccentricity of the above families of graphs has also been demonstrated.

6. Open Problems

This paper consist of the center and periphery for families of convex polytope graphs. This is an open problem for new researchers to find the center and periphery for others families of graphs, such as the corona product, composition product and lexicographic product of families of graphs.

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