

Essay

Old Game, New Rules: Rethinking the Form of Physics

Christian Baumgarten

5244 Birrhard, Switzerland; christian-baumgarten@gmx.net

Academic Editor: Young Suh Kim

Received: 26 February 2016; Accepted: 28 April 2016; Published: 6 May 2016

Abstract: We investigate the modeling capabilities of sets of coupled *classical harmonic oscillators* (CHO) in the form of a modeling game. The application of the simple but restrictive rules of the game lead to conditions for an isomorphism between Lie-algebras and real Clifford algebras. We show that the correlations between two coupled classical oscillators find their natural description in the Dirac algebra and allow to model aspects of special relativity, inertial motion, electromagnetism and quantum phenomena including spin in one go. The algebraic properties of Hamiltonian motion of low-dimensional systems can generally be related to certain types of interactions and hence to the dimensionality of emergent space-times. We describe the intrinsic connection between phase space volumes of a 2-dimensional oscillator and the Dirac algebra. In this version of a phase space interpretation of quantum mechanics the (components of the) spinor wavefunction in momentum space are abstract canonical coordinates, and the integrals over the squared wave function represents second moments in phase space. The wave function in ordinary space-time can be obtained via Fourier transformation. Within this modeling game, 3+1-dimensional space-time is interpreted as a structural property of electromagnetic interaction. A generalization selects a series of Clifford algebras of specific dimensions with similar properties, specifically also 10- and 26-dimensional real Clifford algebras.

Keywords: Hamiltonian mechanics; coupled oscillators; Lorentz transformation; Dirac equation

PACS: 45.20.Jj, 47.10.Df, 41.75, 41.85, 03.65.Pm, 05.45.Xt, 03.30.+p, 03.65.-w, 29.27.-a

1. Introduction

D. Hestenes had the joyful idea to describe physics as a modeling game [1]. We intend to play a modeling game with (ensembles of) classical harmonic oscillators (CHO). The CHO is certainly one of the most discussed and analyzed systems in physics and one of the few exactly solvable problems. One would not expect any substantially new discoveries related to this subject. Nevertheless there are aspects that are less well-known than others. One of these aspects concerns the transformation group of the symplectic transformations of n coupled oscillators, $Sp(2n)$. We invite the reader to join us playing “a modeling game” and to discover some fascinating features related to possible reinterpretations of systems of two (or more) coupled oscillators. We will show that special relativity can be reinterpreted as a transformation theory of the second moments of the abstract canonical variables of coupled oscillator systems (The connection of the Dirac matrices to the symplectic group has been mentioned by Dirac in Reference [2]. For the connection of oscillators and Lorentz transformations (LTs) see also the papers of Kim and Noz [3–5] and references therein. The use of CHOs to model quantum systems has been recently described—for instance—by Briggs and Eisfeld [6–8]). We extend the application beyond pure LTs and show that the Lorentz force can be reinterpreted by the second moments of two coupled oscillators in proper time. Lorentz transformations can be modeled as symplectic transformations [4]. We shall show how Maxwell’s equations find their place within the game.

The motivation for this game is to show that many aspects of modern physics can be understood on the basis of the classical notions of harmonic oscillation if these notions are appropriately reinterpreted.

In Section 2 we introduce the rules of our game, in Section 3 we introduce the algebraic notions of the Hamilton formalism. In Section 4 we describe how geometry emerges from coupled oscillator systems, in Section 5 we describe the use of symplectic transformations and introduce the Pauli- and Dirac algebra. In Section 6 we introduce a physical interpretation of oscillator moments and in Section 7 we relate the phase space of coupled oscillators to the real Dirac algebra. Section 8 contains a short summary.

2. The Rules Of The Game

The first rule of our game is the principle of reason (POR): *No distinction without reason*—we should not add or remove something *specific* (an asymmetry, a concept, a distinction) from our model without having a clear and explicit reason. If there is no reason for a specific asymmetry or choice, then all possibilities are considered equivalently.

The second rule is the principle of variation (POV): We postulate that change is immanent to all fundamental quantities in our game. From these two rules, we take that the mathematical object of our theory is a list (n-tuple) of quantities (variables) ψ , each of which varies at all times.

The third rule is the principle of *objectivity* (POO): Any law within this game refers to measurements, defined as comparison of quantities (object properties) in relation to other object properties of the same type (*i.e.*, unit). Measurements require reference standards (rulers). A measurement is objective if it is based on (properties of) the objects of the game. This apparent self-reference is unavoidable, as it models the *real* situation of physics as experimental science. Since all fundamental objects (quantities) in our model *vary at all times*, the only option to construct a constant quantity that might serve as a ruler, is given by *constants of motion* (COM). Hence the principle of objectivity requires that measurement standards are derived from constants of motion.

This third rule implies that the fundamental variables can not be directly measured, but only functions of the fundamental variables of the same dimension (unit) of a COM. Thus the model has two levels: The level of the fundamental variable list ψ , which is experimentally not directly accessible and a level of *observables* which are (as we shall argue) even moments of the fundamental variables ψ .

2.1. Discussion of the Rules

E.T. Jaynes wrote that “Because of their empirical origins, QM and QED are not physical theories at all. In contrast, Newtonian celestial mechanics, Relativity, and Mendelian genetics are physical theories, because their mathematics was developed by reasoning out the consequences of clearly stated physical principles from which constraint the possibilities”. And he continues “To this day we have no constraining principle from which one can deduce the mathematics of QM and QED; [...] In other words, the mathematical system of the present quantum theory is [...] unconstrained by any physical principle” [9]. This remarkably harsh criticism of quantum mechanics raises the question of what we consider to be a physical principle. Are the rules of our game physical principles? We believe that they are no substantial physical principles but *formal* first principles, they are *preconditions* of a sensible theory. They contain no immediate physical content, but they define the *form* or the *idea* of physics.

It is to a large degree immanent to science and specifically to physics to presuppose the existence of *reason*: Apples do not fall down by chance—there is a reason for this tendency. Usually this belief in reason implies the belief in causality, *i.e.*, that we can also (at least in principle) explain why a specific apple falls at a specific time, but practically this latter belief can rarely be confirmed experimentally and therefore remains to some degree metaphysical. Thus, if, as scientists, we postulate that things have reason, then this is not a *physical* principle but a precondition, a first principle.

The second rule (POV), is specific to the form (or idea) of physics, e.g., that it is the *sense* of physics to *recognize the pattern* of motion and to *predict future*. Therefore the notion of time in the form of change is indeed immanent to the physical description of reality.

The principle of objectivity (POO) is immanent to the very idea of physics: A measurement is the comparison of properties of objects with compatible properties of reference objects, e.g., requires “constant” rulers. Hence the rules of the game are to a large degree unavoidable: They follow from the very form of physics and therefore certain laws of physics are not substantial results of a physical theory. For instance a consistent “explanation” of the stability of matter is impossible *as we presumed it already within the idea of measurement*. More precisely: if this presumption does *not* follow within the framework of a physical theory, then the theory is flawed, since it can not reproduce its own presumptions.

Einstein wrote with respect to relativity that “It is striking that the theory (except for the four-dimensional space) introduces two kinds of things, *i.e.*, (1) measuring rods and clocks; (2) all other things, e.g., the electromagnetic field, the material point, *etc.* This, in a certain sense, is inconsistent; strictly speaking, measuring rods and clocks should emerge as solutions of the basic equations [...], not, as it were, as theoretically self-sufficient entities”. [10]. The more it may surprise that the stability of matter can not be obtained from classical physics as remarked by Elliott H. Lieb: “A fundamental paradox of classical physics is why matter, which is held together by Coulomb forces, does not collapse” [11]. This single sentence seems to rule out the possibility of a fundamental classical theory and uncovers the uncomfortable situation of theoretical physics today: Despite the overwhelming experimental and technological success, there is a deep-seated confusion concerning the theoretical foundations. Our game is therefore a meta-experiment. It is not the primary goal to find “new” laws of nature or new experimental predictions, but it is a conceptional “experiment” that aims to further develop our understanding of the consequences of principles: which ones are *really* required to derive central “results” of contemporary physics. In this short essay final answers can not be given, but maybe some new insights are possible.

2.2. What about Space-Time?

A theory has to make the choice between postulate and proof. If a 3+1 dimensional space-time is presumed, then it cannot be proven within the same theoretical framework. More precisely, the value of such a proof remains questionable. This is a sufficient reason to avoid postulates concerning the dimensionality of space-time. Another, even stronger, reason to avoid a direct postulate of space-time and its geometry has been given above: The fundamental variables that we postulated, can not be directly measured. This excludes space-time coordinates as primary variables (which can be directly measured), but with it almost all other apriori assumed concepts like velocity, acceleration, momentum, energy and so on. At some point these concepts certainly have to be introduced, but we suggest an approach to the formation of concepts that differs from the Newtonian axiomatic method. The POR does not allow to introduce distinctions between the fundamental variables into coordinates and momenta without reason. Therefore we are forced to use an interpretational method, which one might summarize as *function follows form*. We shall first derive equations and then we shall interpret the equations according to some formal criteria. This implies that we have to refer to already existing notions if we want to identify quantities according to their appearance within a certain formalism. The consequence for the game is, that we have to show how to give rise to *geometrical* notions: If we do not postulate space-time then we have to suggest a method to construct it.

A consequence of our conception is that both, objects and fields have to be identified with dynamical structures, as there is simply nothing else available. This fits to the framework of structure preserving (symplectic) dynamics that we shall derive from the described principles.

3. Theory of Small Oscillations

In this section we shall derive the theory of coupled oscillators from the rules of our game. According to the POO there exists a function (COM) $\mathcal{H}(\psi)$ such that (Let us first (for simplicity) assume that $\frac{\partial \mathcal{H}}{\partial t} = 0$):

$$\frac{d\mathcal{H}}{dt} = \sum_k \frac{\partial \mathcal{H}}{\partial \psi_k} \dot{\psi}_k = 0 \quad (1)$$

or in vector notation

$$\frac{d\mathcal{H}}{dt} = (\nabla_{\psi} \mathcal{H}) \cdot \dot{\psi} = 0 \quad (2)$$

The simplest solution is given by an arbitrary skew-symmetric matrix \mathcal{X} :

$$\dot{\psi} = \mathcal{X} \nabla_{\psi} \mathcal{H} \quad (3)$$

Note that it is only the *skew-symmetry* of \mathcal{X} , which ensures that it is always a solution to Equation (2) and which ensures that \mathcal{H} is constant. If we now consider a state vector ψ of dimension k , then there is a theorem in linear algebra, which states that for *any* skew-symmetric matrix \mathcal{X} there exists a non-singular matrix \mathbf{Q} such that we can write [12]:

$$\mathbf{Q}^T \mathcal{X} \mathbf{Q} = \text{diag}(\eta_0, \eta_0, \eta_0, \dots, 0, 0, 0) \quad (4)$$

where η_0 is the matrix

$$\eta_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (5)$$

If we restrict us to orthogonal matrices \mathbf{Q} , then we may still write

$$\mathbf{Q}^T \mathcal{X} \mathbf{Q} = \text{diag}(\lambda_0 \eta_0, \lambda_1 \eta_0, \lambda_2 \eta_0, \dots, 0, 0, 0) \quad (6)$$

In both cases we may leave away the zeros, since they correspond to non-varying variables, which would be in conflict with the second rule of our modeling game. Hence $k = 2n$ must be even and the square matrix \mathcal{X} has the dimension $2n \times 2n$. As we have no specific reason to assume asymmetries between the different degrees of freedom (DOF), we have to choose all $\lambda_k = 1$ in Equation (6) and return to Equation (4) without zeros and define the block-diagonal so-called *symplectic unit matrix* (SUM) γ_0 :

$$\mathbf{Q}^T \mathcal{X} \mathbf{Q} = \text{diag}(\eta_0, \eta_0, \dots, \eta_0) \equiv \gamma_0 \quad (7)$$

These few basic rules thus lead us directly to Hamiltonian mechanics: Since the state vector has even dimension and due to the form of γ_0 , we can interpret ψ as an ensemble of n classical DOF—each DOF represented by a canonical pair of coordinate and momentum: $\psi = (q_1, p_1, q_2, p_2, \dots, q_n, p_n)^T$. In this notation and after the application of the transformation \mathbf{Q} , Equation (3) can be written in form of the Hamiltonian equations of motion (HEQOM):

$$\begin{aligned} \dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial q_i} \end{aligned} \quad (8)$$

The validity of the HEQOM is of fundamental importance as it allows for the use of the results of Hamiltonian mechanics, of statistical mechanics and thermodynamics—but without the intrinsic presupposition that the q_i have to be understood as positions in real space and the p_i as the corresponding canonical momenta. This is legitimate as the theory of canonical transformations is *independent from any specific physical interpretation of what the coordinates and momenta represent physically*. As no other interpretation is at hand, we say that these canonical pairs are coordinates q_i, p_i in an

abstract phase space and they are canonical coordinates and momenta only due to the *form* of the HEQOM. The choice of the specific form of γ_0 is for $n > 1$ DOF not unique. It could for instance be written as

$$\gamma_0 \equiv \eta_0 \otimes \mathbf{1}_{n \times n} \quad (9)$$

which corresponds a state vector of the form

$$\psi = (q_1, \dots, q_n, p_1, \dots, p_n)^T$$

or by

$$\gamma_0 \equiv \mathbf{1}_{n \times n} \otimes \eta_0 \quad (10)$$

as in Equation (7). Therefore we are forced to make an arbitrary choice (But we should keep in mind, that other “systems” with a different choice are possible. If we can not exclude their existence, then they should exist as well. With respect to the form of the SUM, we suggest that different “particle” types (different types of fermions for instance) have a different SUM). But in all cases the SUM γ_0 must be skew-symmetric and have the following properties:

$$\begin{aligned} \gamma_0^T &= -\gamma_0 \\ \gamma_0^2 &= -\mathbf{1} \end{aligned} \quad (11)$$

which also implies that γ_0 is orthogonal and has unit determinant. Note also that all eigenvalues of γ_0 are purely imaginary. However, once we have chosen a specific form of γ_0 , we have specified a set of canonical pairs (q_i, p_i) within the state vector. This choice fixes the set of possible canonical (structure preserving) transformations.

Now we write the Hamiltonian $\mathcal{H}(\psi)$ as a Taylor series, we remove the rule-violating constant term and cut it after the second term. We do not claim that higher terms may not appear, but we delay the discussion of higher orders to a later stage. All this is well-known in the theory of small oscillations. There is only one difference to the conventional treatment: We have no direct macroscopic interpretation for ψ and following our first rule we have to write the second-order Hamiltonian $\mathcal{H}(\psi)$ in the most general form:

$$\mathcal{H}(\psi) = \frac{1}{2} \psi^T \mathcal{A} \psi \quad (12)$$

where \mathcal{A} is only restricted to be *symmetric* as all non-symmetric terms *do not contribute* to \mathcal{H} . Since it is not unlikely to find more than a single constant of motion in systems with multiple DOFs, we distinguish systems with singular matrix \mathcal{A} from those with a positive or negative definite matrix \mathcal{A} . Positive definite matrices are favoured in the sense that they allow to identify \mathcal{H} with the amount of a substance or an amount of energy (It is immanent to the concept of substance that it is understood as something positive semidefinite).

Before we try to interpret the elements in \mathcal{A} , we will explore some general algebraic properties of the Hamiltonian formalism. If we plug Equations (12) into (3), then the equations of motion can be written in the general form

$$\dot{\psi} = \gamma_0 \mathcal{A} \psi = \mathbf{F} \psi \quad (13)$$

The matrix $\mathbf{F} = \gamma_0 \mathcal{A}$ is the product of the symmetric (positive semi-definite) matrix \mathcal{A} and the skew-symmetric matrix γ_0 . As known from linear algebra, the trace of such products is zero:

$$\text{Tr}(\mathbf{F}) = 0 \quad (14)$$

Pure harmonic oscillation of ψ is described by matrices \mathbf{F} with purely imaginary eigenvalues and those are the only stable solutions [12]. Note that Equation (13) may represent a tremendous amount of different types of systems—all linearly coupled systems in any dimension, chains or d -dimensional

lattices of linear coupled oscillators and wave propagation (However the linear approximation does not allow for the description of the transport of heat).

One quickly derives from the properties of γ_0 and \mathcal{A} that

$$\mathbf{F}^T = \mathcal{A}^T \gamma_0^T = -\mathcal{A} \gamma_0 = \gamma_0^2 \mathcal{A} \gamma_0 = \gamma_0 \mathbf{F} \gamma_0 \quad (15)$$

Since any square matrix can be written as the sum of a symmetric and a skew-symmetric matrix, it is nearby to also consider the properties of products of γ_0 with a skew-symmetric real square matrices \mathcal{B} . If $\mathbf{C} = \gamma_0 \mathcal{B}$, then

$$\mathbf{C}^T = \mathcal{B}^T \gamma_0^T = \mathcal{B} \gamma_0 = -\gamma_0^2 \mathcal{B} \gamma_0 = -\gamma_0 \mathbf{C} \gamma_0 \quad (16)$$

Symmetric $2n \times 2n$ -matrices contain $2n(2n+1)/2$ different matrix elements and skew-symmetric ones $2n(2n-1)/2$ elements, so that there are ν_s linear independent matrix elements in \mathcal{A}

$$\nu_s = n(2n+1) \quad (17)$$

and ν_c matrix elements in \mathcal{B} with

$$\nu_c = n(2n-1) \quad (18)$$

In the theory of linear Hamiltonian dynamics, matrices of the form of \mathbf{F} are known as ‘‘Hamiltonian’’ or ‘‘infinitesimal symplectic’’ and those of the form of \mathbf{C} as ‘‘skew-Hamiltonian’’ matrices. This convention is a bit odd as \mathbf{F} does not appear in the Hamiltonian and it is in general not symplectic. Furthermore the term ‘‘Hamiltonian matrix’’ has a different meaning in quantum mechanics - in close analogy to \mathcal{A} . But it is known that this type of matrix is closely connected to symplectic matrices as every symplectic matrix is a matrix exponential of a matrix \mathbf{F} [12]. We consider the matrices as defined by Equations (15) and (16) as too important and fundamental to have no meaningful and unique names: Therefore we speak of a **symplex** (plural *symplexes*), if a matrix holds Equation (15) and of a **cosymplex** if it holds Equation (16).

Symplectic Motion and Second Moments

So what is a symplectic matrix anyway? The concept of symplectic transformations is a specific formulation of the theory of canonical transformations. Consider we define a new state vector (or new coordinates) $\phi(\psi)$ -with the additional requirement, that the transformation is reversible. Then the Jacobian matrix of the transformation is given by

$$\mathbf{J}_{ij} = \begin{pmatrix} \frac{\partial \phi_i}{\partial \psi_j} \end{pmatrix} \quad (19)$$

and the transformation is said to be symplectic, if the Jacobian matrix holds [12]

$$\mathbf{J} \gamma_0 \mathbf{J}^T = \gamma_0 \quad (20)$$

Let us see what this implies in the linear case:

$$\begin{aligned} \mathbf{J} \psi &= \mathbf{J} \mathbf{F} \mathbf{J}^{-1} \mathbf{J} \psi \\ \tilde{\psi} &= \mathbf{J} \psi \\ \tilde{\dot{\psi}} &= \mathbf{J} \mathbf{F} \mathbf{J}^{-1} \tilde{\psi} \\ \dot{\tilde{\psi}} &= \tilde{\mathbf{F}} \tilde{\psi} \end{aligned} \quad (21)$$

and-by the use of Equation (20) one finds that $\tilde{\mathbf{F}}$ is still a symplex:

$$\begin{aligned}
 \tilde{\mathbf{F}}^T &= (\mathbf{J}^{-1})^T \mathbf{F}^T \mathbf{J}^T \\
 \tilde{\mathbf{F}}^T &= (\mathbf{J}^{-1})^T \gamma_0 \mathbf{F} \gamma_0 \mathbf{J}^T \\
 \tilde{\mathbf{F}}^T &= -\gamma_0^2 (\mathbf{J}^{-1})^T \gamma_0 \mathbf{F} \mathbf{J}^{-1} \gamma_0 \\
 \tilde{\mathbf{F}}^T &= -\gamma_0 \mathbf{J} \gamma_0^2 \mathbf{F} \mathbf{J}^{-1} \gamma_0 \\
 \tilde{\mathbf{F}}^T &= \gamma_0 \mathbf{J} \mathbf{F} \mathbf{J}^{-1} \gamma_0 \\
 \tilde{\mathbf{F}}^T &= \gamma_0 \tilde{\mathbf{F}} \gamma_0
 \end{aligned} \tag{22}$$

Hence a symplectic transformation is first of all a similarity transformation, but secondly, it preserves the structure of all involved equations. Therefore the transformation is said to be *canonical* or *structure preserving*. The distinction between canonical and non-canonical transformations can therefore be traced back to the skew-symmetry of γ_0 and the symmetry of \mathcal{A} - both of them consequences of the rules of our physics modeling game.

Recall that we argued that the matrix \mathcal{A} should be symmetric *because* skew-symmetric terms do not contribute to the Hamiltonian. Let us have a closer look what this means. Consider the matrix of second moments Σ that can be build from the variables ψ :

$$\Sigma \equiv \langle \psi \psi^T \rangle \tag{23}$$

in which the angles indicate some (yet unspecified) sort of average. The equation of motion of this matrix is given by

$$\begin{aligned}
 \dot{\Sigma} &= \langle \dot{\psi} \psi^T \rangle + \langle \psi \dot{\psi}^T \rangle \\
 \dot{\Sigma} &= \langle \mathbf{F} \psi \psi^T \rangle + \langle \psi \psi^T \mathbf{F}^T \rangle
 \end{aligned} \tag{24}$$

Now, as long as \mathbf{F} does not depend on ψ , we obtain

$$\begin{aligned}
 \dot{\Sigma} &= \mathbf{F} \Sigma + \Sigma \mathbf{F}^T \\
 \dot{\Sigma} &= \mathbf{F} \Sigma + \Sigma \gamma_0 \mathbf{F} \gamma_0 \\
 (\dot{\Sigma} \gamma_0) &= \mathbf{F} (\Sigma \gamma_0) - (\Sigma \gamma_0) \mathbf{F} \\
 \dot{\mathbf{S}} &= \mathbf{F} \mathbf{S} - \mathbf{S} \mathbf{F}
 \end{aligned} \tag{25}$$

where we defined the new matrix $\mathbf{S} \equiv \Sigma \gamma_0$. For completeness we introduce the “adjunct” spinor $\bar{\psi} = \psi^T \gamma_0$ so that we may write

$$\mathbf{S} = \langle \psi \bar{\psi} \rangle \tag{26}$$

Note that \mathbf{S} is also a symplex. The matrix \mathbf{S} (*i.e.*, all second moments) is constant, iff \mathbf{S} and \mathbf{F} commute.

Now we define an *observable* to be an operator \mathbf{O} with a (potentially) non-vanishing expectation value, defined by:

$$\langle \mathbf{O} \rangle \equiv \langle \bar{\psi} \mathbf{O} \psi \rangle = \langle \psi^T \gamma_0 \mathbf{O} \psi \rangle \tag{27}$$

Thus, if the product $\gamma_0 \mathbf{O}$ is *not* skew-symmetric, *i.e.*, contains a product of γ_0 with a symmetric matrix \mathcal{B} , then the expectation value is potentially non-zero:

$$\langle \mathbf{O} \rangle \equiv \langle \psi^T \gamma_0 (\gamma_0 \mathcal{B}) \psi \rangle = -\langle \psi^T \mathcal{B} \psi \rangle \tag{28}$$

This means that only the symplex-part of an operator is “observable”, while cosymplexes yield a vanishing expectation value. Hence Equation (25) delivers the blueprint for the general definition of observables. Furthermore we find in the last line the constituting equation for Lax pairs [13]. Peter Lax has shown that for such pairs of operators \mathbf{S} and \mathbf{F} that obey Equation (25) there are the following constants of motion

$$\text{Tr}(\mathbf{S}^k) = \text{const} \tag{29}$$

for arbitrary integer $k > 0$. Since \mathbf{S} is a symplectic and therefore by definition the product of a symmetric matrix and the skew-symmetric γ_0 , Equation (29) is always zero and hence trivially true for $k = 1$. The same is true for any odd power of \mathbf{S} , as it can be easily shown that any odd power of a symplectic is again a symplectic (see Equation (35)), so that the only non-trivial general constants of motion correspond to even powers of \mathbf{S} , which implies that all observables are functions of even powers of the fundamental variables.

To see the validity for $k > 1$ we have to consider the general algebraic properties of the trace operator. Let λ be an arbitrary real constant and τ be a real parameter, then

$$\begin{aligned}\mathrm{Tr}(\mathbf{A}) &= \mathrm{Tr}(\mathbf{A}^T) \\ \mathrm{Tr}(\lambda \mathbf{A}) &= \lambda \mathrm{Tr}(\mathbf{A}) \\ \frac{d}{d\tau} \mathrm{Tr}(\mathbf{A}(\tau)) &= \mathrm{Tr}\left(\frac{d\mathbf{A}}{d\tau}\right) \\ \mathrm{Tr}(\mathbf{A} + \mathbf{B}) &= \mathrm{Tr}(\mathbf{A}) + \mathrm{Tr}(\mathbf{B}) \\ \mathrm{Tr}(\mathbf{A} \mathbf{B}) &= \mathrm{Tr}(\mathbf{B} \mathbf{A})\end{aligned}\quad (30)$$

It follows that

$$\begin{aligned}0 &= \mathrm{Tr}(\mathbf{A} \mathbf{B} - \mathbf{B} \mathbf{A}) \\ 0 &= \mathrm{Tr}(\mathbf{A}^n \mathbf{B} - \mathbf{A}^{n-1} \mathbf{B} \mathbf{A}) \\ 0 &= \mathrm{Tr}[\mathbf{A}^{n-1} (\mathbf{A} \mathbf{B} - \mathbf{B} \mathbf{A})]\end{aligned}\quad (31)$$

From the last line of Equation (31) it follows with $\frac{d\mathbf{A}}{d\tau} = \lambda (\mathbf{A} \mathbf{B} - \mathbf{B} \mathbf{A})$

$$\frac{d}{d\tau} \mathrm{Tr}(\mathbf{A}^n) = 0 \quad (32)$$

Remark: This conclusion is not limited to symplectics.

However for single spinors ψ and the corresponding second moments $\mathbf{S} = \Sigma \gamma_0 = \psi \psi^T \gamma_0$ we find:

$$\begin{aligned}\mathrm{Tr}(\mathbf{S}^k) &= \mathrm{Tr}[\psi \psi^T \gamma_0 \cdots \psi \psi^T \gamma_0] \\ &= \mathrm{Tr}[\psi (\psi^T \gamma_0 \cdots \psi \psi^T \gamma_0)] \\ &= \mathrm{Tr}[(\psi^T \gamma_0 \cdots \psi \psi^T \gamma_0) \psi] \\ &= \mathrm{Tr}[(\psi^T \gamma_0 \psi) \cdots (\psi^T \gamma_0 \psi)] = 0\end{aligned}\quad (33)$$

since each single factor $(\psi^T \gamma_0 \psi)$ vanishes due to the skew-symmetry of γ_0 . Therefore the constants of motion as derived from Equation (29) are non-zero only for even k and *after averaging over some kind of distribution* such that $\mathbf{S} = \langle \psi \psi^T \gamma_0 \rangle$ has non-zero eigenvalues as in Equation (34) below.

The symmetric matrix $2n \times 2n$ -matrix Σ (and also \mathcal{A}) is positive definite, if it can be written as a product $\Sigma = \Psi \Psi^T$, where Ψ is a non-singular matrix of size $2n \times m$ with $m \geq 2n$.

For $n = m/2 = 1$, the form of Ψ may be chosen as

$$\begin{aligned}\Psi &= \frac{1}{\sqrt{q^2+p^2}} \begin{pmatrix} q & -p \\ p & q \end{pmatrix} = \frac{1}{\sqrt{q^2+p^2}} (\mathbf{1}\psi, \eta_0 \psi) \\ \Rightarrow \Sigma &= \Psi \Psi^T = \Psi^T \Psi = \mathbf{1} \\ \mathbf{S} &= \gamma_0\end{aligned}\quad (34)$$

so that for $k = 2$ the average of two “orthogonal” column-vectors ψ and $\eta_0 \psi$ gives a non-zero constant of motion via Lax pairs as $\gamma_0^2 = -\mathbf{1}$.

These findings have some consequences for the modeling game. The first is that we have found constants of motion-though some of them are physically meaningful only for a non-vanishing volume in phase space, *i.e.*, by the combination of several spinors ψ . Secondly, a stable state $\dot{\mathbf{S}} = 0$ implies that the matrix operators forming the Lax pair have the same eigenvectors: a density distribution in phase space (as described by the matrix of second moments) is stable if it is adapted or *matched* to the

simplex \mathbf{F} . The phase space distribution as represented by \mathbf{S} and the driving terms (the components of \mathbf{F}) must fit to each other in order to obtain a stable “eigenstate”. But we also found a clear reason, why generators (of symplectic transformations) are always observables and vice versa: Both, the generators as well as the observables are symplectics of the same type. There is a one-to-one correspondence between them, not only as *generators of infinitesimal transformations*, but also algebraically.

Furthermore, we may conclude that (anti-) commutators are an essential part of “classical” Hamiltonian mechanics and secondly that the matrix \mathbf{S} has the desired properties of observables: Though \mathbf{S} is based on continuously varying fundamental variables, it is constant, if it commutes with \mathbf{F} , and it varies otherwise (In accelerator physics, Equation (25) describes the envelope of a beam in linear optics. The matrix of second moments Σ is a covariance matrix-and therefore our modeling game is connected to probability theory exactly when observables are introduced).

Hence it appears sensible to take a closer look on the (anti-) commutation relations of (co-) symplectics and though the definitions of (co-) symplectics are quite plain, the (anti-) commutator algebra that emerges from them has a surprisingly rich structure. If we denote symplectics by \mathbf{S}_k and cosymplectics by \mathbf{C}_k , then the following rules can quickly be derived:

$$\left. \begin{array}{l} \mathbf{S}_1 \mathbf{S}_2 - \mathbf{S}_2 \mathbf{S}_1 \\ \mathbf{C}_1 \mathbf{C}_2 - \mathbf{C}_2 \mathbf{C}_1 \\ \mathbf{C} \mathbf{S} + \mathbf{S} \mathbf{C} \\ \mathbf{S}^{2n+1} \end{array} \right\} \Rightarrow \text{symplex}$$

$$\left. \begin{array}{l} \mathbf{S}_1 \mathbf{S}_2 + \mathbf{S}_2 \mathbf{S}_1 \\ \mathbf{C}_1 \mathbf{C}_2 + \mathbf{C}_2 \mathbf{C}_1 \\ \mathbf{C} \mathbf{S} - \mathbf{S} \mathbf{C} \\ \mathbf{S}^{2n} \\ \mathbf{C}^n \end{array} \right\} \Rightarrow \text{cosymplex} \quad (35)$$

This *Hamiltonian* algebra of (anti-)commutators is of fundamental importance insofar as we derived it in a few steps from first principles (*i.e.*, the rules of the game) and it defines the structure of Hamiltonian dynamics in phase space. The distinction between symplectics and cosymplectics is also the distinction between observables and non-observables. It is the basis of essential parts of the following considerations.

4. Geometry from Hamiltonian Motion

In the following we will demonstrate the *geometrical content* of the algebra of (co-)symplectics (Equation (35)) which emerges for *specific numbers of DOF* n . As shown above, pairs of canonical variables (DOFs) are a direct consequence of the abstract rules of our game. Though single DOFs are poor “objects”, it is remarkable to find physical structures emerging from our abstract rules *at all*. This suggests that there might be more structure to discover when n DOF are combined, for instance geometrical structures. The following considerations obey the rules of our game, since they are based purely on symmetry considerations like those that guided us towards Hamiltonian dynamics. The objects of interest in our algebraic interpretation of Hamiltonian dynamics are matrices. The first matrix (besides \mathcal{A}) with a specific form that we found, is γ_0 . It is a symplex:

$$\gamma_0^T = -\gamma_0 = \gamma_0 \gamma_0 \gamma_0 \quad (36)$$

According to Equation (17) there are $\nu_s = n(2n + 1)$ (*i.e.*, $\nu_s \geq 3$) symplectics. Hence it is nearby to ask if other symplectics with similar properties like γ_0 exist-and if so, what the relations between these matrices are. According to Equation (35) the commutator of two symplectics is again a symplex, while the anti-commutator is a cosymplex. As we are primarily interested in *observables* and components of

the Hamiltonian (*i.e.*, symplices), respectively, we look for further symplices that anti-commute with γ_0 and with each other. In this case, the product of two such matrices is also a symplex, *i.e.*, another potential contribution to the general Hamiltonian matrix \mathbf{F} .

Assumed we had a set of N mutually anti-commuting orthogonal symplices γ_0 and γ_k with $k \in [1 \dots N - 1]$, then a Hamiltonian matrix \mathbf{F} might look like

$$\mathbf{F} = \sum_{k=0}^{N-1} f_k \gamma_k + \dots \tag{37}$$

The γ_k are symplices *and* anti-commute with γ_0 :

$$\gamma_0 \gamma_k + \gamma_k \gamma_0 = 0 \tag{38}$$

Multiplication from the left with γ_0 gives:

$$-\gamma_k + \gamma_0 \gamma_k \gamma_0 = -\gamma_k + \gamma_k^T = 0 \tag{39}$$

so that all other possible symplices γ_k , which anticommute with γ_0 , are symmetric and square to $\mathbf{1}$. This is an important finding for what follows, as it can (within our game) be interpreted as a classical proof of the uniqueness of (observable) time-dimension: Time is one-dimensional as there is no other skew-symmetric symplex that anti-commutes with γ_0 . We can choose different forms for γ_0 , but the emerging algebra allows for no second “direction of time”.

The second order derivative of ψ is (for constant \mathbf{F}) given by $\ddot{\psi} = \mathbf{F}^2 \psi$ which yields:

$$\mathbf{F}^2 = \sum_{i=0}^{N-1} f_i^2 \gamma_i^2 + \sum_{i \neq j} f_i f_j (\gamma_i \gamma_j + \gamma_j \gamma_i) \tag{40}$$

Since the anti-commutator on the right vanishes by definition, we are left with:

$$\mathbf{F}^2 = \left(\sum_{k=1}^{N-1} f_k^2 - f_0^2 \right) \mathbf{1} \tag{41}$$

Thus-we find a set of (coupled) oscillators, if

$$f_0^2 > \sum_{k=1}^{N-1} f_k^2 \tag{42}$$

such that

$$\ddot{\psi} = -\omega^2 \psi \tag{43}$$

Given such matrix systems exist-then they generate a Minkowski type “metric” as in Equation (41) (Indeed it appears that Dirac derived his system of matrices from the this requirement [14]). The appearance of this metric shows how a Minkowski type geometry emerges from the driving terms of oscillatory motion. This is indeed possible- at least for symplices of certain dimensions as we will show below. The first thing needed is some kind of measure to define the length of a “vector”. Since the length is a measure that is invariant under certain transformations, specifically under rotations, we prefer to use a quantity with certain invariance properties to define a length. The only one we have at hand is given by Equation (29). Accordingly we define the (squared) length of a matrix representing a “vector” by

$$\|\mathbf{A}\|^2 \equiv \frac{1}{2n} \text{Tr}(\mathbf{A}^2) \tag{44}$$

The division by $2n$ is required to make the unit matrix have unit norm. Besides the norm we need a scalar product, *i.e.*, a definition of orthogonality. Consider the Pythagorean theorem which says that two vectors \vec{a} and \vec{b} are orthogonal iff

$$(\vec{a} + \vec{b})^2 = \vec{a}^2 + \vec{b}^2 \quad (45)$$

The general expression is

$$(\vec{a} + \vec{b})^2 = \vec{a}^2 + \vec{b}^2 + 2\vec{a} \cdot \vec{b} \quad (46)$$

The equations are equal, iff $\vec{a} \cdot \vec{b} = 0$. Hence the Pythagorean theorem yields a reasonable definition of orthogonality. However, we had no method yet to define vectors within our game. Using matrices \mathbf{A} and \mathbf{B} we may then write

$$\begin{aligned} \|\mathbf{A} + \mathbf{B}\|^2 &= \frac{1}{2n} \text{Tr}[(\mathbf{A} + \mathbf{B})^2] \\ &= \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 + \frac{1}{2n} \text{Tr}(\mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A}) \end{aligned} \quad (47)$$

If we compare this to Equations (45) and (46), respectively, then the obvious definition of the inner product is given by:

$$\mathbf{A} \cdot \mathbf{B} \equiv \frac{\mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A}}{2} \quad (48)$$

Since the anticommutator does in general not yield a scalar, we have to distinguish between inner product and scalar product:

$$(\mathbf{A} \cdot \mathbf{B})_S \equiv \frac{1}{4n} \text{Tr}(\mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A}) \quad (49)$$

where we indicate the scalar part by the subscript “S”. Accordingly we define the exterior product by the commutator

$$\mathbf{A} \wedge \mathbf{B} \equiv \frac{\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}}{2} \quad (50)$$

Now that we defined the products, we should come back to the unit vectors. The only “unit vector” that we explicitly defined so far is the symplectic unit matrix γ_0 . If it represents anything at all then it must be “the direction” of change, the direction of evolution in time as it was derived in this context and is the only “dimension” found so far. As we have already shown, all other unit vectors γ_k must be symmetric, if they are symplectic. And vice versa: If γ_k is symmetric and anti-commutes with γ_0 , then it is a symplectic. As only symplectic represent observables and are generators of symplectic transformations, we can have only a single “time” direction γ_0 and a yet unknown number of *symmetric* unit vectors (Thus we found a simple answer to the question, why only a single time direction is possible, a question also debated in Reference [15]). However, for $n > 1$, there might be different equivalent choices of γ_0 . Whatever the specific form of γ_0 is, we will show that in combination with some general requirements like completeness, normalizability and observability it determines the structure of the complete algebra. Though we don’t yet know how many symmetric and pairwise anti-commuting unit vectors γ_k exist—we have to interpret them as unit vectors in “spatial directions” (The meaning of what a spatial direction is, especially in contrast to the direction of time γ_0 , has to be derived from the form of the emerging equations, of course. As meaning follows form, we do not define space-time, but we identify structures that fit to the known concept of space-time). Of course unit vectors must have unit length, so that we have to demand that

$$\|\gamma_k\|^2 = \frac{1}{2n} \text{Tr}(\gamma_k^2) = \pm 1 \quad (51)$$

Note that (since our norm is not positive definite), we explicitly allow for unit vectors with negative “length” as we find it for γ_0 . Note furthermore that all skew-symmetric unit vectors square to -1 while the symmetric ones square to 1 [16].

Indeed systems of $N = p + q$ anti-commuting real matrices are known as real representations of Clifford algebras $Cl_{p,q}$. The index p is the number of unit elements (“vectors”) that square to $+1$ and q is the number of unit vectors that square to -1 . Clifford algebras are not necessarily connected to Hamiltonian motion, rather they can be regarded as purely mathematical “objects”. They can be defined without reference to matrices whatsoever. Hence in mathematics, sets of matrices are merely “representations” of Clifford algebras. But our game is about physics and due to the proven one-dimensionality of time we concentrate on Clifford algebras $Cl_{N-1,1}$ which link CHOs in the described way with the generators of a Minkowski type metric. Further below it will turn out that the representation by matrices is-within the game-indeed helpful, since it leads to an overlap of certain symmetry structures. The unit elements (or unit “vectors”) of a Clifford algebra, e_k , are called the *generators* of the Clifford algebra. They pairwise anticommute and they square to ± 1 (The role as *generator* of the Clifford algebra should not be confused with the role as generators of symplectic transformations (*i.e.*, symplexes). Though we are especially interested in Clifford algebras in which all generators are symplexes, not all symplexes are generators of the Clifford algebra. Bi-vectors for instance are symplexes, but not generators of the Clifford algebra). Since the inverse of the unit elements e_k of a Clifford algebra must be unique, the products of different unit vectors form new elements and all possible products including the unit matrix form a group. There are $\binom{N}{k}$ possible combinations (products without repetition) of k elements from a set of N generators. We therefore find $\binom{N}{2}$ *bi-vectors*, which are products of two generators, $\binom{N}{3}$ *trivectors*) and so on. The product of all N basic matrices is called *pseudoscalar*. The total number of all k -vectors then is (We identify $k = 0$ with the unit matrix $\mathbf{1}$):

$$\sum_{k=0}^N \binom{N}{k} = 2^N \tag{52}$$

If we desire to construct a *complete* system, then the number of variables of the Clifford algebra has to match the number of variables of the used matrix system:

$$2^N = (2n)^2 \tag{53}$$

Note that the root of this equation gives an even integer $2^{N/2} = 2n$ so that N must be even. Hence all Hamiltonian Clifford algebras have an even dimension. Of course not all elements of the Clifford algebra may be symplexes. The unit matrix (for instance) is a cosymplex. Consider the Clifford algebra $Cl_{1,1}$ with $N = 2$, which has two generators, say γ_0 with $\gamma_0^2 = -1$ and γ_1 with $\gamma_1^2 = 1$. Since these two anticommute (by definition of the Clifford algebra), so that we find (besides the unit matrix) a fourth matrix formed by the product $\gamma_0 \gamma_1$:

$$\begin{aligned} \gamma_0 \gamma_1 &= -\gamma_1 \gamma_0 \\ (\gamma_0 \gamma_1)^2 &= \gamma_0 \gamma_1 \gamma_0 \gamma_1 \\ &= -\gamma_0 \gamma_0 \gamma_1 \gamma_1 = \mathbf{1} \end{aligned} \tag{54}$$

The completeness of the Clifford algebras as we use them here implies that any $2n \times 2n$ -matrix \mathbf{M} with $(2n)^2 = 2^N$ can be written as a linear combination of all elements of the Clifford algebra:

$$\mathbf{M} = \sum_{k=0}^{4n^2-1} m_k \gamma_k \tag{55}$$

The coefficients can be computed from the scalar product of the unit vectors with the matrix \mathbf{M} :

$$m_k = (\gamma_k \cdot \mathbf{M})_S = \frac{S_k}{4n} \text{Tr}(\gamma_k \mathbf{M} + \mathbf{M} \gamma_k) \tag{56}$$

Recall that skew-symmetric γ_k have a negative length and therefore we included a factor s_k which represents the “signature” of γ_k , in order to get the correct sign of the coefficients m_k .

Can we derive more properties of the constructable space-times? One restriction results from representation theory: A theorem from the theory of Clifford algebras states that $Cl_{p,q}$ has a representation by real matrices if (and only if) [17]

$$p - q = 0 \text{ or } 2 \pmod{8} \quad (57)$$

The additional requirement that all generators must be symplectic so that $p = N - 1$ and $q = 1$ then restricts N to

$$N - 2 = 0 \text{ or } 2 \pmod{8} \quad (58)$$

Hence the only matrix systems that have the required symmetry properties within our modeling game are those that represent Clifford algebras with the dimensions $1 + 1, 3 + 1, 9 + 1, 11 + 1, 17 + 1, 19 + 1, 25 + 1, 27 + 1$ and so on. These correspond to matrix representations of size $2 \times 2, 4 \times 4, 32 \times 32, 64 \times 64, 512 \times 512$ and so on. The first of them is called *Pauli algebra*, the second one is the *Dirac algebra*. Do these two have special properties that the higher-dimensional algebras do not have? Yes, indeed.

Firstly, since dynamics is based on canonical pairs, the real Pauli algebra describes the motion of a single DOF and the Dirac algebra describes the *simplest system with interaction* between two DOF. This suggests the interpretation that within our game, objects (Dirac-particles) are not located “within space-time”, since we did not define space *at all* up to this point, but that space-time can be modeled as an emergent phenomenon. Space-time is in between particles.

Secondly, if we equate the number of fundamental variables ($2n$) of the oscillator phase space with the dimension of the Clifford space N , then Equation (53) leads to

$$2^N = N^2 \quad (59)$$

which allows for $N = 2$ and $N = 4$ only. But why should it be meaningful to assume $N = 2n$? The reason is quite simple: If $2n > N$ as for all higher-dimensional state vectors, there are less generators of the algebra than independent variables. This discrepancy increases with n . Hence the described objects can not be pure *vectors* anymore, but must contain tensor-type components (k -vectors) (For a deeper discussion of the dimensionality of space-time, see Reference [16] and references therein).

But before we describe a formal way to interpret Equation (59), let us first investigate the physical and geometrical implications of the game as described so far.

Matrix Exponentials

We said that the unit vectors γ_0 and γ_k are symplectic and therefore generators of symplectic transformations. All symplectic matrices are matrix exponentials of symplectic [12]. The computation of matrix exponentials is in the general case non-trivial. However, in the special case of matrices that square to ± 1 (e.g., along the “axis” γ_k of the coordinate system), the exponentials are readily evaluated:

$$\begin{aligned} \exp(\gamma_a \tau) &= \sum_{k=0}^{\infty} \frac{(\gamma_a \tau)^k}{k!} \\ \exp(\gamma_a \tau) &= \sum_{k=0}^{\infty} s^k \frac{\tau^{2k}}{(2k)!} + \gamma_a \sum_{k=0}^{\infty} s^k \frac{\tau^{2k+1}}{(2k+1)!} \end{aligned} \quad (60)$$

where $s = \pm 1$ is the sign of the matrix square of γ_a . For $s = -1$ ($\gamma_a^2 = -\mathbf{1}$), it follows that

$$\mathbf{R}_a(\tau) = \exp(\gamma_a \tau) = \cos(\tau) + \gamma_a \sin(\tau) \quad (61)$$

and for $s = 1$ ($\gamma_a^2 = \mathbf{1}$):

$$\mathbf{B}_a(\tau) = \exp(\gamma_a \tau) = \cosh(\tau) + \gamma_a \sinh(\tau) \quad (62)$$

We can identify skew-symmetric generators with rotations and (as we will show in more detail below) symmetric generators with boosts.

The (hyperbolic) sine/cosine structure of symplectic matrices are not limited to the generators but are a general property of the matrix exponentials of the symplectic \mathbf{F} (These properties are the main motivation to choose the nomenclature of “symplectic” and “cosymplectic”.):

$$\mathbf{M}(t) = \exp(\mathbf{F}t) = \mathbf{C} + \mathbf{S} \quad (63)$$

where the (co-) symplectic \mathbf{S} (\mathbf{C}) is given by:

$$\begin{aligned} \mathbf{S} &= \sinh(\mathbf{F}t) \\ \mathbf{C} &= \cosh(\mathbf{F}t) \end{aligned} \quad (64)$$

since (the linear combination of) all odd powers of a symplectic is again a symplectic and the sum of all even powers is a cosymplectic. The inverse transfer matrix $\mathbf{M}^{-1}(t)$ is given by:

$$\mathbf{M}^{-1}(t) = \mathbf{M}(-t) = \mathbf{C} - \mathbf{S} \quad (65)$$

The physical meaning of the matrix exponential results from Equation (13), which states that (for constant symplectics \mathbf{F}) the solutions are given by the matrix exponential of \mathbf{F} :

$$\psi(t) = \mathbf{M}(t) \psi(0) \quad (66)$$

A symplectic transformation can be regarded as the result of a *possible* evolution in time. There is no proof that non-symplectic processes are forbidden by nature, but that *only* symplectic transformations are *structure preserving*. Non-symplectic transformations are then *structure defining*. Both play a fundamental role in the physics of our model reality, *because* fundamental particles are—according to our model—represented by dynamical structures. Therefore symplectic transformations describe those processes and interactions, in which structure is preserved, *i.e.*, in which the type of the particle is not changed. The fundamental variables are just “carriers” of the dynamical structures. Non-symplectic transformations can be used to transform the structure. This could also be described by a rotation of the direction of time. Another interpretation is that of a gauge-transformation [18].

5. The Significance of (De-)Coupling

In physics it is a standard technique to reduce complexity of problems by a suitable change of variables. In case of linear systems, the change of variables is a linear canonical transformation. The goal of such transformations is usually to substitute the solution of a complicated problem by the solution of multiple simpler problems. This technique is known under various names, one of these names is *decoupling*, but it is also known as *principal component analysis* or (as we will later show) transformation into the “rest frame”. In other branches of science one might refer to it as *pattern recognition*.

In the following we investigate, how to transform a general oscillatory $2n \times 2n$ -dimensional symplectic to normal form. Certainly it would be preferable to find a “physical method”, *i.e.*, a method that matches to the concepts that we introduced so far and that has inherently physical significance. Or at least significance and explanatory power with respect to our modeling game. Let us start from the simplest systems, *i.e.*, with the Pauli and Dirac algebras which correspond to matrices of size 2×2 and 4×4 , respectively.

5.1. The Pauli Algebra

The fundamental significance of the Pauli algebra is based on the even dimensionality of (classical) phase space. The algebra of 2×2 matrices describes the motion of a single (isolated) DOF. Besides η_0 , the real Pauli algebra includes the following three matrices:

$$\begin{aligned}\eta_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \eta_2 &= \eta_0 \eta_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \eta_3 &= \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}\quad (67)$$

All except the unit matrix η_3 are symplectic. If η_0 and η_1 are chosen to represent the generators of the corresponding Clifford algebra $Cl_{1,1}$, then η_2 is the only possible bi-vector. A general symplectic has the form:

$$\begin{aligned}\mathbf{F} &= a \eta_0 + b \eta_1 + c \eta_2 \\ &= \begin{pmatrix} c & a+b \\ -a+b & -c \end{pmatrix}\end{aligned}\quad (68)$$

The characteristic equation is given by $\text{Det}(\mathbf{F} - \lambda \mathbf{1}) = 0$

$$\begin{aligned}0 &= (c - \lambda)(-c - \lambda) - (a + b)(-a + b) \\ \lambda &= \pm \sqrt{c^2 + b^2 - a^2}\end{aligned}\quad (69)$$

The eigenvalues λ_{\pm} are both either real for $a^2 < c^2 + b^2$ or both imaginary $a^2 > c^2 + b^2$ (or both zero). Systems in stable oscillation have purely imaginary eigenvalues. This case is most interesting for our modeling game.

Decoupling is usually understood in the more general sense to treat the interplay of several (at least two) DOF, but here we ask, whether all possible oscillating systems of $n = 1$ are isomorphic to normal form oscillators. Since there are 3 parameters in \mathbf{F} and only one COM, namely the frequency ω , we need at least two parameters in the transformation matrix. Let us see, if we can choose these two transformations along the axis of the Clifford algebra. In this case we apply subsequently two symplectic transformations along the axis η_0 and η_2 . Applying the symplectic transformation matrix $\exp(\eta_0 \tau/2)$ we obtain:

$$\begin{aligned}\mathbf{F}_1 &= \exp(\eta_0 \tau/2) \mathbf{F} \exp(-\eta_0 \tau/2) \\ &= a' \eta_0 + b' \eta_1 + c' \eta_2\end{aligned}\quad (70)$$

(The “half-angle” argument is for convenience). The transformed coefficients a' , b' and c' are given by

$$\begin{aligned}a' &= a \\ b' &= b \cos \tau - c \sin \tau \\ c' &= c \cos \tau + b \sin \tau\end{aligned}\quad (71)$$

so that—depending on the “duration of the pulse”, we can choose to transform into a coordinate system in which either $b' = 0$ or $c' = 0$. If we choose $t = \arctan(-c/b)$, then $c' = 0$, so that

$$\mathbf{F}' = a \eta_0 + \sqrt{b^2 + c^2} \eta_1 = a' \eta_0 + b' \eta_1 \quad (72)$$

If we choose the next generator to be η_2 , then:

$$\begin{aligned}a'' &= a' \cosh \tau - b' \sinh \tau \\ b'' &= b' \cosh \tau - a' \sinh \tau\end{aligned}\quad (73)$$

In this case we have to distinguish between the case, where $a' > b'$ and $a' < b'$. The former is the oscillatory system and in this case the transformation with $\tau = \text{artanh}(b'/a')$ leads to the normal form of a 1-dim. oscillator:

$$\begin{aligned} a'' &= \sqrt{a'^2 - b'^2 - c'^2} \\ b'' &= 0 \\ c'' &= 0 \end{aligned} \quad (74)$$

and the matrix \mathbf{F}'' has the form

$$\mathbf{F}'' = \sqrt{a'^2 - b'^2 - c'^2} \eta_0 \quad (75)$$

If the eigenvalues are imaginary, then $\lambda = \pm i \omega$ and hence

$$\mathbf{F}'' = \omega \eta_0 \quad (76)$$

so that the solution is-for constant frequency-given by the matrix exponential:

$$\begin{aligned} \psi(t) &= \exp(\omega \eta_0 t) \psi(0) \\ &= (\mathbf{1} \cos(\omega t) + \eta_0 \sin(\omega t)) \psi(0) \end{aligned} \quad (77)$$

This shows that in the context of stable oscillator algebras-the real Pauli algebra can be reduced to the complex number system: This becomes evident, if we consider possible representations of the complex numbers. Clearly we need two basic elements- the unit matrix and η_0 , *i.e.*, a matrix that commutes with the unit matrix and squares to -1 . If we write “ i ” instead of η_0 , then it is easily verified that (See also References [17,19] and Equation (34) in combination with Reference [20].):

$$\begin{aligned} z &= x + i y = \mathbf{Z} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \\ \bar{z} &= x - i y = \mathbf{Z}^T = x \mathbf{1} + \eta_0^T y \\ \exp(i \phi) &= \cos(\phi) + i \sin(\phi) \\ \|z\|^2 &= \mathbf{Z} \mathbf{Z}^T = z \bar{z} = x^2 + y^2 \end{aligned} \quad (78)$$

The theory of holomorphic functions is based on series expansions and can be equally well formulated with matrices. Viewed from our perspective the complex numbers are a special case of the real Pauli algebra- since we have shown above that any one-dimensional oscillator can be canonically transformed into a system of the form of Equation (76). Nevertheless we emphasize that the complex numbers interpreted this way can only represent the *normal form* of an oscillator. The normal form excludes a different scaling of coordinates and momenta as used in classical mechanics, *i.e.*, it avoids intrinsically the appearance of different “spring constants” and masses (There have been several attempts to explain the appearance of the complex numbers in quantum mechanics [21–27]. A general discussion of the use of complex numbers in physics is beyond the scope of this essay, therefore we add just a remark. Gary W. Gibbons wrote that “In particular there can be no evolution if ψ is real” [24]. We agree with Gibbons that the unit imaginary can be related to evolution in time as it implies oscillation, but we do not agree with his conclusion. Physics was able to describe evolution in time without imaginaries before quantum mechanics and it still is. The unconscious use of the unit imaginary did not prevent quantum mechanics from being experimentally successful. But it prevents physicists from understanding its structure).

5.2. The Dirac Algebra

In this subsection we consider the oscillator algebra for two coupled DOF, the algebra of 4×4 matrices. In contrast to the real Pauli algebra, where the parameters a , b and c did not suggest a specific physical meaning, the structure of the Dirac algebra bears geometrical significance

as has been pointed out by David Hestenes and others [28–30]. The (real) Dirac algebra is the simplest real algebra that enables for a description of two DOF and the interaction between them. Furthermore the eigenfrequencies of a Dirac simplex \mathbf{F} may be complex, while the spectrum of the Pauli matrices does not include complex numbers off the real and imaginary axis. The spectrum of general $2n \times 2n$ -simplices has a certain structure - since the coefficients of the characteristic polynomial are real: If λ is an eigenvalue of \mathbf{F} , then its complex conjugate $\bar{\lambda}$ as well as λ and $-\bar{\lambda}$ are also eigenvalues. As we will show, this is the spectrum of the Dirac algebra and therefore any $2n \times 2n$ -system can, at least in principle, be block-diagonalized using 4×4 -blocks. The Dirac algebra is therefore the simplest algebra that covers the general case.

The structure of Clifford algebras follows Pascal’s triangle. The Pauli algebra has the structure $1 - 2 - 1$ (scalar-vector-bivector), the Dirac algebra has the structure $1 - 4 - 6 - 4 - 1$, standing for unit element (scalar), vectors, bi-vectors, tri-vectors and pseudoscalar. The vector elements are by convention indexed with γ_μ with $\mu = 0 \dots 3$, i.e., the generators of the algebra (According to Pauli’s fundamental theorem of the Dirac algebra, all possible choices of the Dirac matrices are, as long as the “metric tensor” $g_{\mu\nu}$ remains unchanged, equivalent [31].):

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} & \gamma_1 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ \gamma_2 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \gamma_3 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{79}$$

We define the following numbering scheme for the remaining matrices (The specific choice of the matrices is not unique. A table of the different systems can be found in Reference ([32]).):

$$\begin{aligned} \gamma_{14} &= \gamma_0 \gamma_1 \gamma_2 \gamma_3; & \gamma_{15} &= \mathbf{1} \\ \gamma_4 &= \gamma_0 \gamma_1; & \gamma_7 &= \gamma_{14} \gamma_0 \gamma_1 = \gamma_2 \gamma_3 \\ \gamma_5 &= \gamma_0 \gamma_2; & \gamma_8 &= \gamma_{14} \gamma_0 \gamma_2 = \gamma_3 \gamma_1 \\ \gamma_6 &= \gamma_0 \gamma_3; & \gamma_9 &= \gamma_{14} \gamma_0 \gamma_3 = \gamma_1 \gamma_2 \\ \gamma_{10} &= \gamma_{14} \gamma_0 & &= \gamma_1 \gamma_2 \gamma_3 \\ \gamma_{11} &= \gamma_{14} \gamma_1 & &= \gamma_0 \gamma_2 \gamma_3 \\ \gamma_{12} &= \gamma_{14} \gamma_2 & &= \gamma_0 \gamma_3 \gamma_1 \\ \gamma_{13} &= \gamma_{14} \gamma_3 & &= \gamma_0 \gamma_1 \gamma_2 \end{aligned} \tag{80}$$

According to Equation (17) we expect 10 symplexes and since the 4 vectors and 6 bi-vectors are symplexes, all other elements are cosymplexes. With this ordering, the general 4×4 -symplex \mathbf{F} can be written as (instead of Equation (55)):

$$\mathbf{F} = \sum_{k=0}^9 f_k \gamma_k \tag{81}$$

In Reference [32] we presented a detailed survey of the Dirac algebra with respect to symplectic Hamiltonian motion. The essence of this survey is the insight that the real Dirac algebra describes Hamiltonian motion of an ensembles of two-dimensional oscillators, but as well the motion of a “point particle” in 3-dimensional space, i.e., that Equation (25) is, when expressed by the real Dirac algebra, *isomorphic to the Lorentz force equation* as we are going to show in Section 6.3. Or, in other words, the Dirac algebra allows to model a point particle and its interaction with the electromagnetic field in terms of the classical statistical ensemble of abstract oscillators.

6. Electromechanical Equivalence (EMEQ)

The number and type of symplexes within the Dirac algebra (80) suggests to use the following vector notation for the coefficients [32,33] of the observables:

$$\begin{aligned}\mathcal{E} &\equiv f_0 \\ \vec{P} &\equiv (f_1, f_2, f_3)^T \\ \vec{E} &\equiv (f_4, f_5, f_6)^T \\ \vec{B} &\equiv (f_7, f_8, f_9)^T\end{aligned}\quad (82)$$

where the “clustering” of the coefficients into 3-dimensional vectors will be explained in the following. The first four elements \mathcal{E} and \vec{P} are the coefficients of the generators of the Clifford algebra and the remaining symplexes are 3 symmetric bi-vectors \vec{E} and skew-symmetric bi-vectors \vec{B} . As explained above, the matrix exponentials of pure Clifford elements are readily evaluated (Equations (61) and (62)). The effect of a symplectic similarity transformation on a symplex

$$\begin{aligned}\tilde{\psi} &= \mathbf{R}(\tau/2) \psi \\ \tilde{\mathbf{F}} &= \mathbf{R}(\tau/2) \mathbf{F} \mathbf{R}^{-1}(\tau/2) \\ &= \mathbf{R}(\tau/2) \mathbf{F} \mathbf{R}(-\tau/2)\end{aligned}\quad (83)$$

can then be computed component-wise as in the following case of a rotation (using Equation (81)):

$$\begin{aligned}\tilde{\mathbf{F}} &= \sum_{k=0}^9 f_k \mathbf{R}_a \gamma_k \mathbf{R}_a^{-1} \\ \mathbf{R}_a \gamma_k \mathbf{R}_a^{-1} &= (\cos(\tau/2) + \gamma_a \sin(\tau/2)) \gamma_k (\cos(\tau/2) - \gamma_a \sin(\tau/2)) \\ &= \gamma_k \cos^2(\tau/2) - \gamma_a \gamma_k \gamma_a \sin^2(\tau/2) + (\gamma_a \gamma_k - \gamma_k \gamma_a) \cos(\tau/2) \sin(\tau/2)\end{aligned}\quad (84)$$

Since all Clifford elements either commute or anti-commute with each other, we have two possible solutions. The first (γ_k and γ_a commute) yields with $\gamma_a^2 = -\mathbf{1}$:

$$\mathbf{R}_a \gamma_k \mathbf{R}_a^{-1} = \gamma_k \cos^2(\tau/2) - \gamma_a^2 \gamma_k \sin^2(\tau/2) = \gamma_k \quad (85)$$

but if (γ_k and γ_a anti-commute) we obtain a rotation:

$$\begin{aligned}\mathbf{R}_a \gamma_k \mathbf{R}_a^{-1} &= \gamma_k (\cos^2(\tau/2) - \sin^2(\tau/2)) + \gamma_a \gamma_k 2 \cos(\tau/2) \sin(\tau/2) \\ &= \gamma_k \cos(\tau) + \gamma_a \gamma_k \sin(\tau)\end{aligned}\quad (86)$$

For $a = 9$ ($\gamma_a = \gamma_1 \gamma_2$) for instance we find:

$$\begin{aligned}\tilde{\gamma}_1 &= \gamma_1 \cos(\tau) + \gamma_1 \gamma_2 \gamma_1 \sin(\tau) = \gamma_1 \cos(\tau) - \gamma_2 \sin(\tau) \\ \tilde{\gamma}_2 &= \gamma_2 \cos(\tau) + \gamma_1 \gamma_2 \gamma_2 \sin(\tau) = \gamma_2 \cos(\tau) + \gamma_1 \sin(\tau) \\ \tilde{\gamma}_3 &= \gamma_3,\end{aligned}\quad (87)$$

which is formally equivalent to a rotation of \vec{P} about the “z-axis”. If the generator γ_a of the transformation is symmetric, we obtain:

$$\begin{aligned}\mathbf{R}_a \gamma_k \mathbf{R}_a^{-1} &= (\cosh(\tau/2) + \gamma_a \sinh(\tau/2)) \gamma_k (\cosh(\tau/2) - \gamma_a \sinh(\tau/2)) \\ &= \gamma_k \cosh^2(\tau/2) - \gamma_a \gamma_k \gamma_a \sinh^2(\tau/2) + (\gamma_a \gamma_k - \gamma_k \gamma_a) \cosh(\tau/2) \sinh(\tau/2)\end{aligned}\quad (88)$$

so that (if γ_a and γ_k commute):

$$\begin{aligned}\tilde{\gamma}_k &= \gamma_k \cosh^2(\tau/2) - \gamma_a^2 \gamma_k \sinh^2(\tau/2) \\ \tilde{\gamma}_k &= \gamma_k (\cosh^2(\tau/2) - \sinh^2(\tau/2)) = \gamma_k\end{aligned}\quad (89)$$

and if γ_a and γ_k anticommute:

$$\begin{aligned}\tilde{\gamma}_k &= \gamma_k (\cosh^2(\tau/2) + \sinh^2(\tau/2)) + 2 \gamma_a \gamma_k \cosh(\tau/2) \sinh(\tau/2) \\ &= \gamma_k \cosh(\tau) + \gamma_a \gamma_k \sinh(\tau),\end{aligned}\quad (90)$$

which is equivalent to a boost when the following parametrization of “rapidity” τ is used:

$$\begin{aligned}\tanh(\tau) &= \beta \\ \sinh(\tau) &= \beta \gamma \\ \cosh(\tau) &= \gamma \\ \gamma &= \frac{1}{\sqrt{1-\beta^2}}\end{aligned}\quad (91)$$

A complete survey of these transformations and the (anti-) commutator tables can be found in Reference [32] (This formalism corresponds exactly to the relativistic invariance of a Dirac spinor in QED as described for instance in Reference [34], although the Dirac theory uses complex numbers and a different sign-convention for the metric tensor). The “spatial” rotations are generated by the bi-vectors associated with \vec{B} and Lorentz boosts by the components associated with \vec{E} . The remaining 4 generators of symplectic transformations correspond to \mathcal{E} and \vec{P} . They were named *phase-rotation* (generated by γ_0) and *phase-boosts* (generated by $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$) and have been used for instance for symplectic decoupling as described in Reference [33].

It is nearby (and already suggested by our notation) to consider the possibility that the EMEQ (Equation (82)) allows to model a relativistic particle as represented by energy \mathcal{E} and momentum \mathbf{P} either in an external electromagnetic field given by \vec{E} and \vec{B} or-alternatively-in an accelerating and/or rotating reference frame, where the elements \vec{E} and \vec{B} correspond to the axis of acceleration and rotation, respectively. We assumed, that all components of the state vector ψ are equivalent in meaning and unit. Though we found that the state vector is formally composed of canonical pairs, the units are unchanged and identical for all elements of ψ . From Equation (13) we take, that the simplex \mathbf{F} (and also \mathcal{A}) have the unit of a frequency. If the Hamiltonian \mathcal{H} is supposed to represent energy, then the components of ψ have the unit of the square root of action.

If the coefficients are supposed to represent the electromagnetic field, then we need to express these fields in the unit of frequency. This can be done, but it requires to involve natural conversion factors like \hbar , charge e , velocity c and a mass, for instance the electron mass m_e . The magnetic field (for instance) is related to a “cyclotron frequency” ω_c by $\omega_c \propto \frac{e}{m_e} B$.

However, according to the rules of the game, the distinction between particle properties and “external” fields requires a reason, an explanation. Especially as it is physically meaningless for macroscopic coupled oscillators. In References [32,33] this nomenclature was used in a merely *formal* way, namely to find a descriptive scheme to order the symplectic generators, so to speak an *equivalent circuit* to describe the general possible coupling terms for two-dimensional coupled linear optics as required for the description of charged particles beams.

Here we play the reversed modeling game: Instead of using the EMEQ as an equivalent circuit to describe ensembles of oscillators, we now use ensembles of oscillators as an equivalent circuit to describe point particles. The motivation for Equation (82) is nevertheless similar, *i.e.*, it follows from the formal structure of the Dirac Clifford algebra. The grouping of the coefficients comes along with the number of vector- and bi-vector-elements, 4 and 6, respectively. The second criterium is to distinguish between generators of rotations and boost, *i.e.*, between symmetric and skew-symmetric symplexes, which separates energy from momentum and electric from magnetic elements. Third of all, we note that even (*Even k-vectors* are those with even $k = 2m$, where m is a natural number) elements (scalar, bi-vectors, 4-vectors *etc.*) of even-dimensional Clifford algebras form a sub-algebra. This means that we can generate the complete Clifford algebra from the vector-elements by matrix multiplication (this is why we call them generators), but we can not generate vectors from bi-vectors by multiplication. And therefore the vectors are the particles (which are understood as the sources of fields) and the

bi-vectors are the fields, which are generated by the objects and influence their motion. The full Dirac symplex-algebra includes the description of a particle (vector) in a field (bi-vector). But why would the field be *external*? Simply, because it is impossible to generate bi-vectors from a single vector-type object, since any single vector-type object written as $\mathcal{E} \gamma_0 + \vec{P} \cdot \vec{\gamma}$ squares to a scalar. Therefore, the fields must be the result of interaction with other particles and hence we call them “external”. This is in some way a “first-order” approach, since there might be higher order processes that we did not consider yet. But in the linear approach (*i.e.*, for second-order Hamiltonians), this distinction is reasonable and hence a legitimate move in the game.

Besides the Hamiltonian structure (symplices *vs.* co-symplices) and the Clifford algebraic structure (distinguishing vectors, bi-vectors, tri-vectors *etc.*) there is a third essential symmetry, which is connected to the real matrix representation of the Dirac algebra and to the fact that it describes the general Hamiltonian motion of coupled oscillators: To distinguish the even from the odd elements with respect to the block-diagonal matrix structure. We used this property in Reference [33] to develop a general geometrical decoupling algorithm (see also Section 6.2).

Now it may appear that we are cheating somehow, as relativity is usually “derived” from the constancy of the speed of light, while in our modeling game, we did neither introduce spatial notions nor light at all. Instead we directly arrive at notions of quantum electrodynamics (QED). How can this be? The definition of “velocity” within wave mechanics usually involves the dispersion relation of waves, *i.e.*, the velocity of a wave packet is given by the group velocity \vec{v}_{gr} defined by

$$\vec{v}_{gr} \equiv \vec{\nabla}_{\vec{k}} \omega(\vec{k}) \quad (92)$$

and the so-called phase velocity v_{ph} defined by

$$v_{ph} = \frac{\omega}{k} \quad (93)$$

It is then typically mentioned that the product of these two velocities is a constant $v_{gr} v_{ph} = c^2$. By the use of the EMEQ and Equation (29), the eigenvalues of \mathbf{F} can be written as:

$$\begin{aligned} K_1 &= -\text{Tr}(\mathbf{F}^2)/4 \\ K_2 &= \text{Tr}(\mathbf{F}^4)/16 - K_1^2/4 \\ \omega_1 &= \sqrt{K_1 + 2\sqrt{K_2}} \\ \omega_2 &= \sqrt{K_1 - 2\sqrt{K_2}} \\ \omega_1^2 \omega_2^2 &= K_1^2 - 4K_2 = \text{Det}(\mathbf{F}) \\ K_1 &= \mathcal{E}^2 + \vec{B}^2 - \vec{E}^2 - \vec{P}^2 \\ K_2 &= (\mathcal{E} \vec{B} + \vec{E} \times \vec{P})^2 - (\vec{E} \cdot \vec{B})^2 - (\vec{P} \cdot \vec{B})^2 \end{aligned} \quad (94)$$

Since symplectic transformations are similarity transformations, they do not alter the eigenvalues of the matrix \mathbf{F} and since all possible evolutions in time (which can be described by the Hamiltonian) are symplectic transformations, the eigenvalues (of closed systems) are conserved. If we consider a “free particle”, then we obtain from Equation (94):

$$\omega_{1,2} = \pm \sqrt{\mathcal{E}^2 - \vec{P}^2} \quad (95)$$

As we mentioned before both, energy and momentum, have (within this game) the unit of frequencies. If we take into account that $\omega_{1,2} \equiv m$ is fixed, then the dispersion relation for “the energy” $\mathcal{E} = \omega$ is

$$\mathcal{E} = \omega = \sqrt{m^2 + \vec{P}^2} \quad (96)$$

which is indeed the correct relativistic dispersion. But how do we make the step from pure oscillations to *waves*? (The question if Quantum theory requires Planck's constant \hbar , has been answered negative by John P. Ralston [35]).

6.1. Moments and The Fourier Transform

In case of "classical" probability distribution functions (PDFs) $\phi(x)$ we may use the Taylor terms of the *characteristic function* $\tilde{\phi}_x(t) = \langle \exp i t x \rangle_x$, which is the Fourier transform of $\phi(x)$, at the origin. The k -th moment is then given by

$$\langle x^k \rangle = i^k \tilde{\phi}^{(k)}(0) \quad (97)$$

where $\phi^{(k)}$ is the k -th derivative of $\tilde{\phi}_x(t)$.

A similar method would be of interest for our modeling game. Since a (phase space-) density is positive definite, we can always take the square root of the density instead of the density itself: $\phi = \sqrt{\rho}$. The square root can also be defined to be a complex function, so that the density is $\rho = \phi\phi^* = \|\phi\|^2$ and, if mathematically well-defined (convergent), we can also define the Fourier transform of the complex root, *i.e.*,

$$\tilde{\phi}(\omega, \vec{k}) = N \int \phi(t, \vec{x}) \exp(i\omega t - i\vec{k}\vec{x}) dt d^3x \quad (98)$$

and vice versa:

$$\tilde{\phi}(t, \vec{x}) = \tilde{N} \int \phi(\omega, \vec{k}) \exp(-i\omega t + i\vec{k}\vec{x}) d\omega d^3k \quad (99)$$

In principle, we may *define* the density not only by real and imaginary part, but by an arbitrary number of components. Thus, if we consider a four-component spinor, we may of course mathematically define its Fourier transform. But in order to see, why this might be more than a mathematical "trick", but *physically meaningful*, we need to go back to the notions of classical statistical mechanics. Consider that we replace the single state vector by an "ensemble", where we leave the question open, if the ensemble should be understood as a single phase space trajectory, averaged over time, or as some (presumably large) number of different trajectories. It is well-known, that the phase space density $\rho(\psi)$ is stationary, if it depends only on constants of motion, for instance if it depends only on the Hamiltonian itself. With the Hamiltonian of Equation (12), the density could for example have the form

$$\rho(\mathcal{H}) \propto \exp(-\beta \mathcal{H}) = \exp(-\beta \psi \mathcal{A} \psi / 2) \quad (100)$$

which corresponds to a multivariate Gaussian. But more important is the insight, that the density exclusively depends on the second moments of the phase space variables as given by the Hamiltonian, *i.e.*, in case of a "free particle" it depends on \mathcal{E} and \vec{P} . And therefore we should be able to use energy and momentum as frequency ω and wave-vector \vec{k} .

But there are more indications in our modeling game that suggest the use of a Fourier transform as we will show in the next section.

6.2. The Geometry of (De-)Coupling

In the following we give a (very) brief summary of Reference [33]. As already mentioned, decoupling is meant-despite the use of the EMEQ-first of all purely technical-mathematical. Let us delay the question, if the notions that we define in the following have any physical relevance. Here we

refer first of all to block-diagonalization, *i.e.*, we treat the symplectic \mathbf{F} just as a “Hamiltonian” matrix. From the definition of the real Dirac matrices we obtain \mathbf{F} in explicit 4×4 matrix form:

$$\mathbf{F} = \begin{pmatrix} -E_x & E_z + B_y & E_y - B_z & B_x \\ E_z - B_y & E_x & -B_x & -E_y - B_z \\ E_y + B_z & B_x & E_x & E_z - B_y \\ -B_x & -E_y + B_z & E_z + B_y & -E_x \end{pmatrix} + \begin{pmatrix} -P_z & \mathcal{E} - P_x & 0 & P_y \\ -\mathcal{E} - P_x & P_z & P_y & 0 \\ 0 & P_y & -P_z & \mathcal{E} + P_x \\ P_y & 0 & -\mathcal{E} + P_x & P_z \end{pmatrix} \quad (101)$$

If we find a (sequence of) symplectic similarity transformations that would allow to reduce the 4×4 -form to a block-diagonal form, then we would obtain two separate systems of size 2×2 and we could continue with the transformations of Section 5.1.

Inspection of Equation (101) unveils that \mathbf{F} is block-diagonal, if the coefficients E_y, P_y, B_x and B_z vanish. Obviously this implies that $\vec{E} \cdot \vec{B} = 0$ and $\vec{P} \cdot \vec{B} = 0$. Or vice versa, if we find a symplectic method that transforms into a system in which $\vec{E} \cdot \vec{B} = 0$ and $\vec{P} \cdot \vec{B} = 0$, then we only need to apply appropriate rotations to achieve block-diagonal form. As shown in Reference [33] this can be done in different ways, but in general it requires the use of the “phase rotation” γ_0 and “phase boosts” $\vec{\gamma}$. Within the conceptual framework of our game, the application of these transformations equals the use of “matter fields”. But furthermore, this shows that block-diagonalization has also geometric significance within the Dirac algebra and, with respect to the Fourier transformation, the requirement $\vec{P} \cdot \vec{B} = 0$ indicates a divergence free magnetic field, as the replacement of \vec{P} by $\vec{\nabla}$ yields $\vec{\nabla} \cdot \vec{B} = 0$. The additional requirement $\vec{E} \cdot \vec{B} = 0$ also fits well to our physical picture of e.m. waves. Note furthermore, that there is no analogous requirement to make $\vec{P} \cdot \vec{E}$ equal to zero. Thus (within this analogy) we can accept $\vec{\nabla} \cdot \vec{E} \neq 0$.

But this is not everything to be taken from this method. If we analyze in more detail, which expressions are *required* to vanish and which may remain, then it appears that $\vec{P} \cdot \vec{B}$ is explicitly given by

$$\begin{aligned} P_x B_x \gamma_1 \gamma_2 \gamma_3 + P_y B_y \gamma_2 \gamma_3 \gamma_1 + P_z B_z \gamma_3 \gamma_1 \gamma_2 &= (\vec{P} \cdot \vec{B}) \gamma_{10} \\ E_x B_x \gamma_4 \gamma_2 \gamma_3 + E_y B_y \gamma_5 \gamma_3 \gamma_1 + E_z B_z \gamma_6 \gamma_1 \gamma_2 &= (\vec{E} \cdot \vec{B}) \gamma_{14} \\ P_x E_x \gamma_1 \gamma_4 \gamma_3 + P_y E_y \gamma_2 \gamma_5 \gamma_1 + P_z E_z \gamma_3 \gamma_6 \gamma_2 &= -(\vec{P} \cdot \vec{E}) \gamma_0 \end{aligned} \quad (102)$$

That means that exactly those products have to vanish which yield *cosymplices*. This can be interpreted via the structure preserving properties of symplectic motion. Since within our game, the particle *type* can only be represented by the structure of the dynamics, and since electromagnetic processes do not change the type of a particle, then they are quite obviously *structure preserving* which then implies the non-appearance of co-symplices. Or-in other words-electromagnetism is of Hamiltonian nature. We will come back to this point in Section 6.4.

6.3. The Lorentz Force

In the previous section we constituted the distinction between the “mechanical” elements $\mathbf{P} = \mathcal{E} \gamma_0 + \vec{\gamma} \cdot \vec{P}$ of the general matrix \mathbf{F} and the electrodynamical elements $\mathbf{F} = \gamma_0 \vec{\gamma} \cdot \mathbf{E} + \gamma_{14} \gamma_0 \vec{\gamma} \cdot \vec{B}$. Since the matrix $\mathbf{S} = \Sigma \gamma_0$ is a symplectic, let us assume to be equal to \mathbf{P} and apply Equation (25). We then find (with the appropriate relative scaling between \mathbf{P} and \mathbf{F} as explained above):

$$\frac{d\mathbf{P}}{d\tau} = \dot{\mathbf{P}} = \frac{q}{2m} (\mathbf{F}\mathbf{P} - \mathbf{P}\mathbf{F}) \quad (103)$$

which yields written with the coefficients of the real Dirac matrices:

$$\begin{aligned}\frac{d\mathcal{E}}{d\tau} &= \frac{q}{m} \vec{P} \cdot \vec{E} \\ \frac{d\vec{P}}{d\tau} &= \frac{q}{m} (\mathcal{E} \vec{E} + \vec{P} \times \vec{B})\end{aligned}\quad (104)$$

where τ is the proper time. If we convert to the lab frame time t using $dt = \frac{d\tau}{\gamma}$ Equation (103) yields (setting $c = 1$):

$$\begin{aligned}\gamma \frac{d\mathcal{E}}{dt} &= q \gamma \vec{v} \cdot \vec{E} \\ \gamma \frac{d\vec{P}}{dt} &= \frac{q}{m} (m \gamma \vec{E} + m \gamma \vec{v} \times \vec{B}) \\ \frac{dE}{dt} &= q \vec{v} \cdot \vec{E} \\ \frac{d\vec{P}}{dt} &= q (\vec{E} + \vec{v} \times \vec{B})\end{aligned}\quad (105)$$

which is the Lorentz force. Therefore the Lorentz force acting on a charged particle in 3 spatial dimensions can be modeled by an ensemble of 2-dimensional CHOs. The isomorphism between the observables of the perceived 3-dimensional world and the second moments of density distributions in the phase space of 2-dimensional oscillators is remarkable.

In any case, Equation (103) clarifies two things within the game. Firstly, that both, energy \mathcal{E} and momentum \vec{p} , have to be interpreted as mechanical energy and momentum (and not canonical), secondly the relative normalization between fields and mechanical momentum is fixed and last, but not least, it clarifies the relation between the time related to mass (proper time) and the time related to γ_0 and energy, which appears to be the laboratory time.

6.4. The Maxwell Equations

As we already pointed out, waves are (within this game) the result of a Fourier transformation (FT). But there are different ways to argue this. In Reference [16] we argued that Maxwell's equations can be derived within our framework by (a) the postulate that space-time emerges from interaction, *i.e.*, that the fields \vec{E} and \vec{B} have to be constructed from the 4-vectors. $\mathbf{X} = t \gamma_0 + \vec{x} \cdot \vec{\gamma}$, $\mathbf{J} = \rho \gamma_0 + \vec{j} \cdot \vec{\gamma}$ and $\mathbf{A} = \Phi \gamma_0 + \vec{A} \cdot \vec{\gamma}$ with (b) the requirement that no co-symplices emerge. But we can also argue with the FT of the density (see Section 6.1).

If we introduce the 4-derivative

$$\partial \equiv -\partial_t \gamma_0 + \partial_x \gamma_1 + \partial_y \gamma_2 + \partial_z \gamma_3 \quad (106)$$

The non-abelian nature of matrix multiplication requires to distinguish differential operators acting to the right and to the left, *i.e.*, we have ∂ as defined in Equation (106), $\overrightarrow{\partial}$ and $\overleftarrow{\partial}$ which is written to the right of the operand (thus indicating the order of the matrix multiplication) so that

$$\begin{aligned}\overleftarrow{\mathbf{H}} \partial &\equiv -\partial_t \mathbf{H} \gamma_0 + \partial_x \mathbf{H} \gamma_1 + \partial_y \mathbf{H} \gamma_2 + \partial_z \mathbf{H} \gamma_3 \\ \overrightarrow{\partial} \mathbf{H} &\equiv -\gamma_0 \partial_t \mathbf{H} + \gamma_1 \partial_x \mathbf{H} + \gamma_2 \partial_y \mathbf{H} + \gamma_3 \partial_z \mathbf{H}\end{aligned}\quad (107)$$

The we find the following general rules (see Equation (35)) that prevent from non-zero cosymplices:

$$\begin{aligned}\frac{1}{2} \left(\overrightarrow{\partial} \text{vector} - \text{vector} \overleftarrow{\partial} \right) &\Rightarrow \text{bi-vector} \\ \frac{1}{2} \left(\overrightarrow{\partial} \text{bi-vector} - \text{bi-vector} \overleftarrow{\partial} \right) &\Rightarrow \text{vector} \\ \frac{1}{2} \left(\overrightarrow{\partial} \text{bi-vector} + \text{bi-vector} \overleftarrow{\partial} \right) &\Rightarrow \text{axial vector} = 0 \\ \frac{1}{2} \left(\overrightarrow{\partial} \text{vector} + \text{vector} \overleftarrow{\partial} \right) &\Rightarrow \text{scalar} = 0\end{aligned}\quad (108)$$

Application of these derivatives yields:

$$\begin{aligned}
 \mathbf{F} &= \frac{1}{2} \left(\vec{\partial} \mathbf{A} - \mathbf{A} \overleftarrow{\partial} \right) \\
 4 \pi \mathbf{J} &= \frac{1}{2} \left(\vec{\partial} \mathbf{F} - \mathbf{F} \overleftarrow{\partial} \right) \\
 0 &= \vec{\partial} \mathbf{F} + \mathbf{F} \overleftarrow{\partial} \\
 0 &= \frac{1}{2} \left(\vec{\partial} \mathbf{A} + \mathbf{A} \overleftarrow{\partial} \right) \\
 0 &= \frac{1}{2} \left(\vec{\partial} \mathbf{J} + \mathbf{J} \overleftarrow{\partial} \right)
 \end{aligned} \tag{109}$$

The first row of Equation (109) corresponds to the usual definition of the bi-vector fields from a vector potential \mathbf{A} and is (written by components) given by

$$\begin{aligned}
 \vec{E} &= -\vec{\nabla} \phi - \partial_t \vec{A} \\
 \vec{B} &= \vec{\nabla} \times \vec{A}
 \end{aligned} \tag{110}$$

The second row of Equation (109) corresponds to the usual definition of the 4-current \mathbf{J} as sources of the fields and the last three rows just express the impossibility of the appearance of cosymplices. They explicitly represent the homogenous Maxwell equations

$$\begin{aligned}
 \vec{\nabla} \cdot \vec{B} &= 0 \\
 \vec{\nabla} \times \vec{E} + \partial_t \vec{B} &= 0
 \end{aligned} \tag{111}$$

the continuity equation

$$\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0 \tag{112}$$

and the so-called ‘‘Lorentz gauge’’

$$\partial_t \Phi + \vec{\nabla} \cdot \vec{A} = 0 \tag{113}$$

The simplest idea about the 4-current within QED is to assume that it is proportional to the ‘‘probability current’’, which is within our game given by the vector components of $\mathbf{S} = \Sigma \gamma_0$.

7. The Phase Space

Up to now, our modeling game referred to the second moments and the elements of \mathbf{S} are second moments such that the observables are given by (averages over) the following quadratic forms:

$$\begin{aligned}
 \mathcal{E} &\propto \psi^T \psi = q_1^2 + p_1^2 + q_2^2 + p_2^2 \\
 p_x &\propto -q_1^2 + p_1^2 + q_2^2 - p_2^2 \\
 p_y &\propto 2(q_1 q_2 - p_1 p_2) \\
 p_z &\propto 2(q_1 p_1 + q_2 p_2) \\
 E_x &\propto 2(q_1 p_1 - q_2 p_2) \\
 E_y &\propto -2(q_1 p_2 + q_2 p_1) \\
 E_z &\propto q_1^2 - p_1^2 + q_2^2 - p_2^2 \\
 B_x &\propto 2(q_1 q_2 + p_1 p_2) \\
 B_y &\propto q_1^2 + p_1^2 - q_2^2 - p_2^2 \\
 B_z &\propto 2(q_1 p_2 - p_1 q_2)
 \end{aligned} \tag{114}$$

If we analyze the real Dirac matrix coefficients of $\mathbf{S} = \psi \psi^T \gamma_0$ in terms of the EMEQ and evaluate the quadratic relations between those coefficients, then we obtain:

$$\begin{aligned}
\vec{P}^2 &= \vec{E}^2 = \vec{B}^2 = \mathcal{E}^2 \\
0 &= \vec{E}^2 - \vec{B}^2 \\
\mathcal{E}^2 &= \frac{1}{2} (\vec{E}^2 + \vec{B}^2) \\
\mathcal{E} \vec{P} &= \vec{E} \times \vec{B} \\
\mathcal{E}^3 &= \vec{P} \cdot (\vec{E} \times \vec{B}) \\
m^2 &\propto \mathcal{E}^2 - \vec{P}^2 = 0 \\
\vec{P} \cdot \vec{E} &= \vec{E} \cdot \vec{B} = \vec{P} \cdot \vec{B} = 0
\end{aligned} \tag{115}$$

Besides a missing renormalization these equations describe an object without mass but with the geometric properties of light as described by electrodynamics, e.g., by the electrodynamic description of electromagnetic waves, which are $\vec{E} \cdot \vec{B} = 0$, $\vec{P} \propto \vec{E} \times \vec{B}$, $\vec{E}^2 = \vec{B}^2$ and so on. Hence single spinors are *light-like* and can not represent massive particles.

Consider the spinor as a vector in a four-dimensional Euclidean space. We write the symmetric matrix \mathcal{A} (or Σ , respectively) as a product in the form of a Gramian:

$$\mathcal{A} = \mathcal{B}^T \mathcal{B} \tag{116}$$

or-componentwise:

$$\begin{aligned}
\mathcal{A}_{ij} &= \sum_k (\mathcal{B}^T)_{ik} \mathcal{B}_{kj} \\
&= \sum_k \mathcal{B}_{ki} \mathcal{B}_{kj}
\end{aligned} \tag{117}$$

The last line can be read such that matrix element \mathcal{A}_{ij} is the conventional 4-dimensional scalar product of column vector \mathcal{B}_i with column vector \mathcal{B}_j .

From linear algebra we know that Equation (116) yields a non-singular matrix \mathcal{A} , iff the column-vectors of the matrix \mathcal{B} are linearly independent. In the orthonormal case, the matrix \mathcal{A} simply is the pure form of a non-singular matrix, *i.e.*, the unit matrix. Hence, if we want to construct a massive object from spinors, we need several spinors to fill the columns of \mathcal{B} . The simplest case is the orthogonal case: the combination of four mutual orthogonal vectors. Given a general 4-component Hamiltonian spinor $\psi = (q_1, p_1, q_2, p_2)$, how do we find a spinor that is orthogonal to this one? In 3 (*i.e.*, odd) space dimensions, we know that there are two vectors that are perpendicular to any vector $(x, y, z)^T$, but without fixing the first vector, we can't define the others. In even dimensions this is different: it suffices to find a non-singular skew-symmetric matrix like γ_0 to generate a vector that is orthogonal to ψ , namely $\gamma_0 \psi$. As in Equation (3), it is the skew-symmetry of the matrix that ensures the orthogonality. A third vector $\gamma_k \psi$ must then be orthogonal to ψ and to $\gamma_0 \psi$. It must be skew-symmetric and it must hold $\psi^T \gamma_k^T \gamma_0 \psi = 0$. This means that the product $\gamma_k^T \gamma_0$ must also be skew-symmetric and hence that γ_k must anti-commute with γ_0 :

$$\begin{aligned}
(\gamma_k^T \gamma_0)^T &= \gamma_0^T \gamma_k = -\gamma_k^T \gamma_0 \\
\Rightarrow \gamma_0^T \gamma_k + \gamma_k^T \gamma_0 &= 0 \\
0 &= \gamma_0 \gamma_k + \gamma_k \gamma_0
\end{aligned} \tag{118}$$

Now let us for a moment return to the question of dimensionality. There are in general $2n(2n-1)/2$ non-zero independent elements in a skew-symmetric square $2n \times 2n$ matrix. But how many matrices are there in the considered phase space dimensions, *i.e.*, in $1+1$, $3+1$ and $9+1$ (*etc.*) dimensions which anti-commute with γ_0 ? We need at least $2n-1$ skew-symmetric anti-commuting elements to obtain a diagonal \mathcal{A} . However, this implies at least $N-1$ anticommuting elements of the Clifford algebra that square to -1 . Hence the ideal case is $2n = N$, which is only true for the Pauli and Dirac algebra. For the Pauli algebra, there is one skew-symmetric element, namely η_0 . In the Dirac algebra there are 6 skew-symmetric generators that contain two sets of mutually commuting skew-symmetric

matrices: γ_0, γ_{10} and γ_{14} on the one hand and γ_7, γ_8 and γ_9 on the other hand. The next considered Clifford algebra with $N = 9 + 1$ dimensions requires a representation by $2n = 32 = \sqrt{2^{10}}$ -dimensional real matrices. Hence this algebra may not represent a Clifford algebra with more than 10 unit elements—certainly not $2n$. Hence, we can not use the algebra to generate purely massive objects (e.g., diagonal matrices) without further restrictions (i.e., projections) of the spinor ψ .

But what exactly does this mean? Of course we can easily find 32 linearly independent spinors to generate an orthogonal matrix \mathcal{B} . So what exactly is special in the Pauli- and Dirac algebra? To see this, we need to understand, what it means that we can use the matrix \mathcal{B} of mutually orthogonal column-spinors

$$\mathcal{B} = (\psi, \gamma_0 \psi, \gamma_{10} \psi, \gamma_{14} \psi) \tag{119}$$

This form implies that we can define the *mass* of the “particle” algebraically, and since we have $N - 1 = 3$ anticommuting skew-symmetric matrices in the Dirac algebra, we can find a multispinor \mathcal{B} for any arbitrary point in phase space. This does not seem to be sensational at first sight, since this appears to be a property of any Euclidean space. The importance comes from the fact that ψ is a “point” in a very special space—a point in phase space. In fact, we will argue in the following that this possibility to factorize ψ and the density ρ is everything but self-evident.

If we want to simulate a phase space distribution, we can either define a phase space density $\rho(\psi)$ or we use the technique of Monte-Carlo simulations and represent the phase space by (a huge number of random) samples. If we generate a random sample and we like to implement a certain exact symmetry of the density in phase space, then we would (for instance) form a symmetric sample by appending not only a column-vector to \mathcal{B} , but also its negative $-\psi$. In this way we obtain a sample with an exact symmetry. In a more general sense: If a phase space symmetry can be represented by a matrix γ_s that allows to associate to an arbitrary phase space point ψ a second point $\gamma_s \psi$ where γ_s is skew-symmetric, then we have a certain continuous linear rotational symmetry in this phase space. As we have shown, phase-spaces are intrinsically structured by γ_0 and insofar much more restricted than Euclidean spaces. This is due to the distinction of symplectic from non-symplectic transformations and due to the intrinsic relation to Clifford algebras: *Phase spaces are spaces structured by time*. Within our game, the phase space is the only possible fundamental space.

We may imprint the mentioned symmetry to an arbitrary phase space density ρ by taking all phase space samples that we have so far and adding the same number of samples, each column multiplied by γ_s . Thus, we have a single rotation in the Pauli algebra and two of them in the Dirac algebra:

$$\begin{aligned} & \mathcal{B}_0 = \psi \\ \gamma_0 & \rightarrow \mathcal{B}_1 = (\psi, \gamma_0 \psi) \\ \gamma_{14} & \rightarrow \mathcal{B}_2 = (\psi, \gamma_0 \psi, \gamma_{14} \psi, \gamma_{14} \gamma_0 \psi) \\ & = (\psi, \gamma_0 \psi, \gamma_{14} \psi, \gamma_{10} \psi) \end{aligned} \tag{120}$$

or:

$$\begin{aligned} & \mathcal{B}_0 = \psi \\ \gamma_7 & \rightarrow \mathcal{B}_1 = (\psi, \gamma_7 \psi) \\ \gamma_8 & \rightarrow \mathcal{B}_2 = (\psi, \gamma_7 \psi, \gamma_8 \psi, \gamma_8 \gamma_7 \psi) \\ & = (\psi, \gamma_7 \psi, \gamma_8 \psi, -\gamma_9 \psi) \end{aligned} \tag{121}$$

Note that order and sign of the column-vectors in \mathcal{B} are irrelevant—at least with respect to the autocorrelation matrix $\mathcal{B} \mathcal{B}^T$. Thus we find that there are two fundamental ways to represent a positive mass in the Dirac algebra and one in the Pauli-algebra. The 4-dimensional phase space of the Dirac algebra is in two independent ways self-matched.

Our starting point was the statement that $2n$ linear independent vectors are needed to generate mass. If we can't find $2n$ vectors in the way described above for the Pauli and Dirac algebra, then this does (of course) not automatically imply that there are not $2n$ linear independent vectors.

But what does it mean that the dimension of the Clifford algebra of observables (N) does not match the dimension of the phase space ($2n$) in higher dimensions? There are different physical descriptions given. Classically we would say that a positive definite $2n$ -component spinor describes a system of n (potentially) coupled oscillators with n frequencies. If \mathcal{B} is orthogonal, then all oscillators have the same frequency, *i.e.*, the system is degenerate. But for $n > 2$ we find that not all eigenmodes can involve the complete $2n$ -dimensional phase space. This phenomenon is already known in 3 dimensions: The trajectory of the isotropic three-dimensional oscillator always happens in a 2-dimensional plane, *i.e.*, in a subspace. If it did not, then the angular momentum would not be conserved. In this case the isotropy of space would be broken. Hence one may say in some sense that the *isotropy of space* is the reason for a 4-dimensional phase-space and hence the reason for the $3 + 1$ -dimensional observable space-time of objects. Or in other words: higher-dimensional spaces are incompatible with isotropy, *i.e.*, with the conservation of angular momentum. There is an intimate connection of these findings to the impossibility of Clifford algebras $Cl_{p,1}$ with $p > 3$ to create a homogeneous "Euclidean" space: Let γ_0 represent time and γ_k with $k \in [1, \dots, N - 1]$ the spatial coordinates. The spatial rotators are products of two spatial basis vectors. The generator of rotations in the $(1, 2)$ -plane is $\gamma_1 \gamma_2$. Then we have 6 rotators in 4 "spatial" dimensions:

$$\gamma_1 \gamma_2, \gamma_1 \gamma_3, \gamma_1 \gamma_4, \gamma_2 \gamma_3, \gamma_2 \gamma_4, \gamma_3 \gamma_4 \tag{122}$$

However, we find that some generators commute and while others anticommute and it can be taken from combinatorics that only sets of 3 mutual anti-commuting rotators can be formed from a set of symmetric anti-commuting γ_k . The 3 rotators

$$\gamma_1 \gamma_2, \gamma_2 \gamma_3, \gamma_1 \gamma_3 \tag{123}$$

mutually anticommute, but $\gamma_1 \gamma_2$ and $\gamma_3 \gamma_4$ commute. Furthermore, in $9 + 1$ dimensions, the spinors are either projections into 4-dimensional subspaces or there are non-zero off-diagonal terms in \mathcal{A} , *i.e.*, there is "internal interaction".

Another way to express the above considerations is the following: Only in 4 phase space dimensions we may construct a massive object from a matrix \mathcal{B} that represents a multispinor Ψ of exactly $N = 2n$ single spinors and construct a wave-function according to

$$\Psi = \phi \mathcal{B} \tag{124}$$

where $\rho = \phi^2$ is the phase space density.

It is easy to prove and has been shown in Reference [16] that the elements γ_0, γ_{10} and γ_{14} represent parity, time reversal and charge conjugation. The combination of these operators to form a multispinor, may lead (with normalization) to the construction of symplectic matrices \mathbf{M} . Some examples are:

$$\begin{aligned} \mathbf{M} &= (\mathbf{1} \psi, \gamma_0 \psi, -\gamma_{14} \psi, -\gamma_{10} \psi) / \sqrt{\psi^T \psi} \\ \mathbf{M} \gamma_0 \mathbf{M}^T &= \gamma_0 \\ \mathbf{M} &= (\mathbf{1} \psi, -\gamma_{14} \psi, -\gamma_{10} \psi, \gamma_0 \psi) / \sqrt{\psi^T \psi} \\ \mathbf{M} \gamma_{10} \mathbf{M}^T &= \gamma_{10} \\ \mathbf{M} &= (\gamma_{10} \psi, -\mathbf{1} \psi, -\gamma_{14} \psi, \gamma_0 \psi) / \sqrt{\psi^T \psi} \\ \mathbf{M} \gamma_{14} \mathbf{M}^T &= \gamma_{14} \end{aligned} \tag{125}$$

Hence the combination of the identity and CPT-operators can be arranged such that the multispinor \mathbf{M} is symplectic with respect to the directions of time γ_0, γ_{10} and γ_{14} , but not with respect to γ_7, γ_8 or γ_9 . As we tried to explain, the specific choice of the skew-symmetric matrix γ_0 is determined by a structure defining transformation. Since particles are nothing but dynamical structures in this game, the 6 possible SUMs should stand for 6 different particle types. However, for each direction of time, there are also two choices of the spatial axes. For γ_0 we have chosen γ_1, γ_2 and γ_3 , but we could have used $\gamma_4 = \gamma_0 \gamma_1, \gamma_5 = \gamma_0 \gamma_2$ and $\gamma_6 = \gamma_0 \gamma_3$ as well.

Thus, there should be either 6 or 12 different types of structures (types of fermions) that can be constructed within the Dirac algebra. The above construction allows for three different types corresponding to three different forms of the symplectic unit matrix, further three types are expected to be related to γ_7, γ_8 and γ_9 :

$$\begin{aligned} \mathbf{M} &= (\mathbf{1} \psi, -\gamma_9 \psi, -\gamma_8 \psi, -\gamma_7 \psi) / \sqrt{\psi^T \psi} \\ \mathbf{M} \gamma_7 \mathbf{M}^T &= \gamma_7 \\ \\ \mathbf{M} &= (\mathbf{1} \psi, -\gamma_8 \psi, -\gamma_7 \psi, -\gamma_9 \psi) / \sqrt{\psi^T \psi} \\ \mathbf{M} \gamma_8 \mathbf{M}^T &= \gamma_8 \\ \\ \mathbf{M} &= (\gamma_7 \psi, -\mathbf{1} \psi, -\gamma_8 \psi, -\gamma_9 \psi) / \sqrt{\psi^T \psi} \\ \mathbf{M} \gamma_9 \mathbf{M}^T &= \gamma_9 \end{aligned} \quad (126)$$

These matrices describe specific symmetries of the 4-dimensional phase space, *i.e.*, geometric objects in phase space. Therefore massive multispinors can be described as volumes in phase space. If we deform the figure by stretching parameters a, b, c, d such that

$$\tilde{\mathbf{M}} = (a \mathbf{1} \psi, -b \gamma_0 \psi, -c \gamma_{14} \psi, -d \gamma_{10} \psi) / \sqrt{\psi^T \psi} \quad (127)$$

then one obtains with f_k taken from Equation (114):

$$\begin{aligned} \tilde{\mathbf{M}} \tilde{\mathbf{M}}^T \gamma_0 &= \sum_{k=0}^9 g_k f_k \gamma_k / \sqrt{\psi^T \psi} \\ g_0 &= a^2 + b^2 + c^2 + d^2 \\ g_1 &= -g_2 = g_3 = a^2 - b^2 + c^2 - d^2 \\ g_4 &= -g_5 = g_6 = a^2 - b^2 - c^2 + d^2 \\ g_7 &= g_8 = g_9 = a^2 + b^2 - c^2 - d^2 \end{aligned} \quad (128)$$

This result reproduces the quadratic forms f_k of Equation (114), but furthermore the phase space radii a, b, c and d reproduce the structure of the Clifford algebra, *i.e.*, the classification into the 4 types of observables $\mathcal{E}, \vec{P}, \vec{E}$ and \vec{B} . This means that a deformation of the phase space “unit cell” represents momenta and fields, *i.e.*, the dimensions of the phase space unit cell are related to the appearance of certain symplectics:

$$\begin{aligned} (a = b) \text{ AND } (c = d) &\Rightarrow \vec{P} = \vec{E} = 0 \\ (a = c) \text{ AND } (b = d) &\Rightarrow \vec{E} = \vec{B} = 0 \\ (a = d) \text{ AND } (b = c) &\Rightarrow \vec{P} = \vec{B} = 0 \end{aligned} \quad (129)$$

while for $a = b = c = d$ all vectors but \mathcal{E} vanish. Only in this latter case, the matrix \mathbf{M} is symplectic for $a = b = c = d = 1$. These relations confirm the intrinsic connection between a classical 4-dimensional Hamiltonian phase space and Clifford algebras in dimension 3+1.

8. Summary and Discussion

Based on three fundamental principles, which describe the form of physics, we have shown that the algebraic structure of coupled classical degrees of freedom is (depending on the number of the DOFs) isomorph to certain Clifford algebras that allow to explain the dimensionality of space-time, to model Lorentz-transformations, the relativistic energy-momentum relation and even Maxwell's equations.

It is usually assumed that we have to define the properties of space-time in the first place: "In Einstein's theory of gravitation matter and its dynamical interaction are based on the notion of an intrinsic geometric structure of the space-time continuum" [36]. However, as we have shown within this "game", it has far more explanatory power to derive and explain space-time from the principles of interaction. Hence we propose to reverse the above statement: The intrinsic geometric structure of the space-time continuum is based on the dynamical interaction of matter. A rigorous consequence of this reversal of perspective is that "space-time" does not need to have a fixed and unique dimensionality at all. It appears that the dimensionality is a property of the type of interaction. However, supposed higher-dimensional space-times (see Reference [16]) would emerge in analogy to the method presented here, for instance in nuclear interaction, then these space-times would not simply be Euclidean spaces of higher dimension. Clifford algebras, especially if they are restricted by symplectic conditions by a Hamiltonian function, have a surprisingly complicated intrinsic structure. As we pointed out, if all generators of a Clifford algebra are symplectic, then in $9 + 1$ dimensions, we find k -vectors with $k \in [0..10]$ but k -vectors generated from symplectic are themselves symplectic only for $k \in [1, 2, 5, 6, 9, 10, \dots]$. However, if space-time is constraint by Hamiltonian motion, then ensembles of oscillators may also clump together to form "objects" with $9 + 1$ or $25 + 1$ -dimensional interactions, despite the fact that we gave strong arguments for the fundamentality of the $3 + 1$ -dimensional Hamiltonian algebra.

There is no a priori reason to exclude higher order terms-whenver they include constants of motion. However, as the Hamiltonian then involves terms of higher order, we might then need to consider higher order moments of the phase space distribution. In this case we would have to invent an action constant in order to scale ψ .

Our game is based a few general rules and symmetry considerations. The math used in our derivation-taken the results of representation theory for granted-is simple and can be understood on an undergraduate level. And though we never intended to find a connection to string theory, we found-besides the $3 + 1$ -dimensional interactions a list of possible higher-dimensional candidates, two of which are also in the focus of string theories, namely $9 + 1 = 10$ -dimensional and $25 + 1 = 26$ -dimensional theories [37].

We understand this modeling game as a contribution to the demystification (and unification) of our understanding of space-time, relativity, electrodynamics and quantum mechanics. Despite the fact that it has become tradition to write all equations of motion of QED and QM in a way that requires the use of the unit imaginary, our model seems to indicate that it does not have to be that way. Though it is frequently postulated that evolution in time has to be unitary within QM, it appears that symplectic motion does not only suffice, but is superior as it yields the correct number of relevant operators. While in the unitary case, one should expect 16 (15) unitary (traceless) operators for a 4-component spinor, but the natural number of generators in the corresponding symplectic treatment is 10 as found by Dirac himself in QED [2,38]. If a theory contains things which are *not required*, then we have added something arbitrary and artificial. The theory as we described it indicates that in momentum space, which is used here, there is no immediate need for the use of the unit imaginary and no need for more than 10 fundamental generators. The use of the unit imaginary however appears unavoidable when we switch via Fourier transform to the "real space".

There is a dichotomy in physics. On the one hand all *causes* are considered to inhabit space-time (*local causality*), but on the other hand the *physical reasoning* mostly happens in energy-momentum space: There are no Feynman-graphs, no scattering amplitudes, no fundamental physical relations, that

do not refer in some way to energy or momentum (-conservation). We treat problems in solid state physics as well as in high energy physics mostly in Fourier space (reciprocal lattice).

We are aware that the rules of the game are, due to their rigour, difficult to accept. However, maybe it does not suffice to speculate that the world might be a hologram (As t'Hooft suggested [39] and Leonard Susskind sketched in his celebrated paper, Reference [40])-we really should play modeling games that might help to decide, if and *how* it could be like that.

Conflicts of Interest: "The author declares no conflict of interest."

Appendix A. Microcanonical Ensemble

Einstein once wrote that "A theory is the more impressive the greater the simplicity of its premises, the more different kinds of things it relates, and the more extended its area of applicability. Hence the deep impression that classical thermodynamics made upon me. It is the only physical theory of universal content concerning which I am convinced that, within the framework of the applicability of its basic concepts, it will never be overthrown [...]" [10]. We agree with him and we will try to show in the following that this holds also for the branch of thermodynamics that is called statistical mechanics.

By the use of the EMEQ it has been shown, that the expectation values

$$f_k = \frac{\text{Tr}(\gamma_k^2)}{16} \bar{\psi} \gamma_k \psi \quad (\text{A1})$$

can be associated with energy \mathcal{E} and momentum \vec{p} of and with the electric (magnetic) field \vec{E} and \vec{B} as *seen by* a relativistic charged particle. It has also been shown that stable systems can always be transformed in such a way as to bring \mathcal{H} into a diagonal form:

$$\mathbf{F} = \begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{pmatrix} \quad (\text{A2})$$

In the following we will use the classical model of the microcanonical ensemble to compute some phase space averages. Let the constant value of the Hamiltonian be $\mathcal{H} = U$ where U is some energy, the volume in phase space Φ^* that is limited by the surface of constant energy U is given by [41]:

$$\Phi^* = \int_{\mathcal{H} < U} dq_1 dp_1 dq_2 dp_2 \quad (\text{A3})$$

and the partition function ω^* is the derivative

$$\omega^* = \frac{d\Phi^*}{dU} \quad (\text{A4})$$

which is the phase space integral over all states of constant energy U . The average value of any phase space function $f(p, q)$ is then given by

$$\overline{f(p, q)} = \frac{1}{\omega^*} \frac{d}{dU} \int_{\mathcal{H} < U} f(p, q) dq_1 dp_1 dq_2 dp_2 \quad (\text{A5})$$

In case of a 2-dimensional harmonic oscillator, for instance, we may take the following parametrization of the phase space:

$$\begin{aligned} q_1 &= r \cos(\alpha) \cos(\beta) \\ p_1 &= r \cos(\alpha) \sin(\beta) \\ q_2 &= r \sin(\alpha) \cos(\gamma) \\ p_2 &= r \sin(\alpha) \sin(\gamma) \end{aligned} \quad (\text{A6})$$

Note that Equation (A6) describes a solution of the equations of motion

$$\dot{\psi} = \mathbf{F} \psi \quad (\text{A7})$$

when we replace

$$\begin{aligned} \beta &\rightarrow -\omega_1 t \\ \gamma &\rightarrow -\omega_2 t \end{aligned} \quad (\text{A8})$$

This means that the (normalized) integration over β and γ is mathematically identical to an integration over all times (time average). From Equation (A5) one would directly conclude

$$\overline{f(p, q)} = \frac{1}{\omega^*} \frac{d}{dU} \int_{\mathcal{H} < U} f(p, q) \sqrt{g} dr d\alpha d\beta d\gamma \quad (\text{A9})$$

where g is the Gramian determinant. However the relative amplitude controlled by the parameter α can not be changed by symplectic transformations and hence remains constant in a closed system. Therefore the phase space trajectory of the oscillator can not cover the complete 3-dim. energy surface, but only a 2-dim. subset thereof. This is known very well in accelerator physics as the emittance preservation of decoupled DOF. And we have shown in Reference [33] that all stable harmonic oscillators are symplectically similar to a decoupled system. Consequently α has to be excluded from the integration of a “single particle” average and has to be treated instead as an additional parameter or “boundary condition”:

$$\overline{f(p, q)} = \frac{1}{\omega^*} \frac{d}{dU} \int_{\mathcal{H} < U} f(p, q) \sqrt{g} dr d\beta d\gamma \quad (\text{A10})$$

The Gramian determinant hence is given by:

$$\begin{aligned} g &= \text{Det}(\mathbf{G}^T \mathbf{G}) \\ \mathbf{G} &= \begin{pmatrix} \frac{\partial q_1}{\partial r} & \frac{\partial q_1}{\partial \beta} & \frac{\partial q_1}{\partial \gamma} \\ \frac{\partial p_1}{\partial r} & \frac{\partial p_1}{\partial \beta} & \frac{\partial p_1}{\partial \gamma} \\ \frac{\partial q_2}{\partial r} & \frac{\partial q_2}{\partial \beta} & \frac{\partial q_2}{\partial \gamma} \\ \frac{\partial p_2}{\partial r} & \frac{\partial p_2}{\partial \beta} & \frac{\partial p_2}{\partial \gamma} \end{pmatrix} \end{aligned} \quad (\text{A11})$$

so that one finds

$$\sqrt{g} = r^2 \cos(\alpha) \sin(\alpha) \quad (\text{A12})$$

Accordingly, Equation (A3) has to be written as

$$\Phi^* = \int_{\mathcal{H} < U} \sqrt{g} dr d\beta d\gamma \quad (\text{A13})$$

We use the abbreviations

$$\begin{aligned}\bar{\omega} &= \frac{\omega_1 + \omega_2}{2} \\ \Delta\omega &= \frac{\omega_1 - \omega_2}{2} \\ \Omega &= \bar{\omega} + \Delta\omega \cos(2\alpha)\end{aligned}\quad (\text{A14})$$

The Hamilton function is given in the new coordinates by

$$\mathcal{H} = \frac{r^2 \Omega}{2} \quad (\text{A15})$$

so that the condition $\mathcal{H} < U$ translates into

$$r \leq \sqrt{2\varepsilon} \quad (\text{A16})$$

where $\varepsilon = U/\Omega$. The integration over β and γ is taken from 0 to 2π . The complete integration results in

$$\begin{aligned}\Phi^* &= \frac{2\pi^2}{3} \sin(2\alpha) \left(\frac{2U}{\Omega}\right)^{3/2} \\ \omega^* &= \frac{3}{2U} \Phi^*\end{aligned}\quad (\text{A17})$$

The following average values are computed from Equation (A10):

$$\begin{aligned}\overline{\mathcal{H}} &= U \\ \overline{\mathcal{H}^2} &= U^2 \\ \overline{f_k} &= \begin{cases} \varepsilon & \text{for } k = 0 \\ \varepsilon \cos(2\alpha) & \text{for } k = 8 \\ 0 & \text{for } k \in \{1-7, 9\} \end{cases} \\ \overline{f_k^2} &= \begin{cases} \overline{f_k}^2 & \text{for } k = 0, 8 \\ \frac{1}{4}\varepsilon^2(1 + \cos^2(2\alpha)) & \text{for } k \in \{1, 3, 4, 6\} \\ \frac{1}{2}\varepsilon^2 \sin^2(2\alpha) & \text{for } k \in \{2, 5, 7, 9\} \end{cases}\end{aligned}\quad (\text{A18})$$

Hence we find that f_0 (energy), f_8 (one spin component) and \mathcal{H} (mass) are “sharp” (*i.e.*, operators with an eigenvalue), while the other “expectation values” have a non-vanishing variance. The fact that spin always has a “direction of quantization”, *i.e.*, that only *one single* “sharp” component, can therefore be modelled within our game. It is a consequence of symplectic motion. Note also that the squared expectation values of all even ($\gamma_1, \gamma_3, \gamma_4$ and γ_6 , except γ_0 and γ_8) and all odd ($\gamma_2, \gamma_5, \gamma_7$ and γ_9) operators are equal (The *even* Dirac matrices are block-diagonal, the *odd* ones not. There are six even symplectics and four odd ($\gamma_2, \gamma_5, \gamma_7$ and γ_9) ones [32]. Obviously this pattern is the reason for the grouping in Equation (A18)).

Consider the coordinates are given by the fields $(\vec{E} \propto \vec{x}) \vec{Q} = (f_4, f_5, f_6)^T$ and the momenta as usual by $\vec{P} = (f_1, f_2, f_3)^T$, then the angular momentum \vec{L} should be given by $\vec{L} = \vec{Q} \times \vec{P}$. We obtain the following expectation values from the microcanonical ensemble:

$$\begin{aligned}\overline{L_x} &= \overline{L_z} = 0 \\ \overline{L_x^2} &= \overline{L_z^2} = \frac{1}{2}\varepsilon^4 \sin^2(2\alpha) \\ \overline{L_y} &= \varepsilon^2 \cos(2\alpha) \\ \overline{L_y^2} &= \varepsilon^4 \cos^2(2\alpha)\end{aligned}\quad (\text{A19})$$

That is-up to a common scale factor of ε (or ε^2 , respectively)-we have the same results as in Equation (A18). Consider now the quantum mechanical postulate that the spin component of a fermion is $s_z = \pm s = \pm \frac{1}{2}$ and $|\vec{s}|^2 = s_x^2 + s_y^2 + s_z^2 = s(s+1) = \frac{3}{4}$. We can “derive” this result (up to a factor) from an isotropy requirement for the 4th order moments, *i.e.*, from the condition that $\langle P_x^2 \rangle = \langle P_y^2 \rangle = \langle P_z^2 \rangle$:

$$\sin^2(2\alpha) = 2 \cos^2(2\alpha) \quad (\text{A20})$$

so that

$$\alpha = \frac{1}{2} \arctan \sqrt{2} = 27.3678^\circ \quad (\text{A21})$$

or equivalently with

$$1 + 3 \cos^2(2\alpha) = 0 \quad (\text{A22})$$

we obtain

$$\cos(2\alpha) = \pm \frac{1}{\sqrt{3}} \quad (\text{A23})$$

With respect to the simplex \mathbf{F} as defined in Equation (A2), we have

$$\mathbf{F} = \bar{\omega} \gamma_0 + \Delta\omega \gamma_8 \quad (\text{A24})$$

so that with Equation (A18) one finds

$$f_8/f_0 = \bar{f}_8/\bar{f}_0 = \cos(2\alpha) \quad (\text{A25})$$

The total spin would then be given by the 4th order moments as $\vec{S}^2 = \varepsilon^2$, so that for a spin- $\frac{1}{2}$ -particle we would have to normalize to $\varepsilon^2 = (s(s+1))\hbar^2 = \frac{3}{4}\hbar^2$ and hence $f_8 = \sqrt{\frac{3}{4}}\hbar \frac{1}{\sqrt{3}} = \frac{\hbar}{2}$. However, the mass formula (Equation (94)), refers to the second moments, so that in linear theory we would have

$$\mathcal{H} = m = \omega_0 \sqrt{f_0^2 + f_8^2} = \omega_0 \sqrt{\varepsilon^2 + \frac{1}{3}\varepsilon^2} = \omega_0 \sqrt{\frac{4}{3}}\varepsilon = \omega_0 \hbar \quad (\text{A26})$$

In order to relate this to a frequency difference, we use Equation (A25):

$$\begin{aligned} \frac{f_8}{f_0} &= \frac{\Delta\omega}{\bar{\omega}} = \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2} = \cos(2\alpha) = \frac{1}{\sqrt{3}} \\ \Rightarrow \\ \frac{\omega_1}{\omega_2} &= 2 + \sqrt{3} \\ \Omega &= 4\Delta\omega = \frac{4}{\sqrt{3}}\bar{\omega} \end{aligned} \quad (\text{A27})$$

Then from $r = \sqrt{2}\varepsilon$ and Equation (A15) we find

$$\begin{aligned} \mathcal{H} &= \frac{r^2\Omega}{2} = \varepsilon\Omega = \sqrt{\frac{3}{4}}\hbar\Omega = \sqrt{\frac{3}{4}}\hbar \frac{4}{\sqrt{3}}\bar{\omega} \\ &= 2\hbar\bar{\omega} = \hbar(\omega_1 + \omega_2) \end{aligned} \quad (\text{A28})$$

To conclude, classical statistical mechanics allows for a description of spin, if the rules of symplectic motion are taken into account. This alone is remarkable. Secondly, assumed that the microcanonical ensemble is the right approach, then the isotropy of the emergent 3+1-dimensional space-time (with respect to 4th-order moments) apparently requires a certain ratio between the frequencies and amplitudes of the two coupled oscillators, *i.e.*, an asymmetry on the fundamental level.

Appendix A.1. Entropy and Heat Capacity

The entropy S of the microcanonical ensemble can be written as [41]:

$$S = k \log \Phi^* \quad (\text{A29})$$

The temperature T of the system is given by

$$\begin{aligned} \frac{\partial S}{\partial U} &= \frac{1}{T} = \frac{\partial(k \log \Phi^*)}{\partial U} = k \frac{\omega^*}{\Phi^*} \\ &= \frac{3k}{2U} \end{aligned} \quad (\text{A30})$$

so that energy as a function of temperature is

$$U = \frac{3}{2} k T \quad (\text{A31})$$

and the heat capacity $C_V = \frac{\partial U}{\partial T}$ is (per particle)

$$C_V = \frac{3}{2} k \quad (\text{A32})$$

This important result demonstrates—according to statistical mechanics—that the 3-dimensionality of the “particle” as the energy per DOF is $\frac{kT}{2}$: a two-dimensional harmonic oscillator of fundamental variables is equivalent to an free 3-dimensional “point particle”. To our knowledge this is the first *real physical* model of a relativistic point particle.

References

- Hestenes, D. Modeling games in the Newtonian World. *Am. J. of Phys.* **1992**, *60*, 732–748.
- Dirac, P.A.M. A remarkable representation of the 3 + 2 de Sitter group. *J. Math. Phys.* **1963**, *4*, 901–909.
- Kim, Y.S.; Noz, M.E. Symplectic formulation of relativistic quantum mechanics. *J. Math. Phys.* **1981**, *22*, 2289–2293.
- Kim, Y.S.; Noz, M.E. Dirac’s light—Cone coordinate system. *Am. J. Phys.* **1982**, *50*, 721–724.
- Kim, Y.S.; Noz, M.E. Coupled oscillators, entangled oscillators, and Lorentz-covariant harmonic oscillators. *J. Opt. B Quantum Semiclass. Opt.* **2005**, *7*, S458–S467.
- Briggs, J.; Einfeld, A. Equivalence of quantum and classical coherence in electronic energy transfer. *Phys. Rev. E* **2011**, *83*, 051911, doi:10.1103/PhysRevE.83.051911.
- Briggs, J.; Einfeld, A. Coherent quantum states from classical oscillator amplitudes. *Phys. Rev. A* **2012**, *85*, 052111, doi:10.1103/PhysRevA.85.052111.
- Briggs, J.; Einfeld, A. Quantum dynamics simulations with classical oscillators. *Phys. Rev. A* **2013**, *88*, 062104, doi:10.1103/PhysRevA.88.062104.
- Jaynes, E.T. *The Electron*; Hestenes, D., Weingartshofer, A., Eds; Kluwer Academic Publishers: Dordrecht, Holland, 1991.
- Schilpp, P.A. *Albert Einstein: Autobiographical Notes*; Open Court Publishing Company: Chicago, IL, USA, 1979.
- Lieb, E.H. The stability of matter, *Rev. Mod. Phys.* **1976**, *48*, 553–569.
- Meyer, K.R.; Hall, G.R.; Offin, D. *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, 2nd ed.; Springer, New York, NY, USA, 2000.
- Lax, P.D. Integrals of Nonlinear Equations of Evolution and Solitary Waves. *Comm. Pure Appl. Math.* **1968**, *21*, 467–490.
- Dirac, P.A.M. The Quantum Theory of the Electron. *Proc. R. Soc. A* **1928**, *117*, 610–624.
- Borstnik, N.M.; Nielson, H.B.; Why odd-space and odd-time dimensions in even-dimensional spaces? *Phys. Lett. B* **2000**, *486*, 314–321.
- Baumgarten, C. Minkowski Spacetime and QED from Ontology of Time. Kastner, R.E., Jeknic-Dugic, J., Jaroszkiewicz, G., Eds; **2014/2015**. arXiv:1409.5338v5; to appear in *Quantum Structural Studies*, forthcoming from World Scientific.
- Lounesto, P. *Clifford Algebras and Spinors*, 2nd ed.; Cambridge University Press: New York, NY, USA, 2001.
- Baumgarten, C. A new look at linear (non-?) symplectic ion beam optics in magnets. *Nucl. Instr. Meth. A* **2014**, *735*, 546–551.
- Dyson, F.J. The Threefold Way. Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics. *J. Math. Phys.* **1962**, *3*, 1199–1215.
- Ralston, J.P. Berrys phase and the symplectic character of quantum time evolution. *Phys. Rev. A* **1989**, *40*, 4872–4884.
- Hestenes, D. Observables, Operators, and Complex Numbers in the Dirac Theory. *J. Math. Phys.* **1975**, *16*, 556–572.

22. Baylis, W.E.; Huschlit, J.; Wei, J. Why i ? *Am. J. Phys.* **1992**, *60*, 788–796.
23. Horn, M.E. Living in a world without imaginaries. *J. Phys.: Conf. Ser.* **2012**, *380*, 012006, doi:10.1088/1742-6596/380/1/012006.
24. Gibbons, G.W. The Emergent Nature of Time and the Complex Numbers in Quantum Cosmology. In *The Arrows of Time: A debate in Cosmology*; Mersini-Houghton, L., Vass, R., Eds; Springer Berlin Heidelberg: Berlin, Germany, 2012; pp. 109–148.
25. Schiff, J.; Poirier, B. Communication: Quantum mechanics without wavefunctions. *J. Chem. Phys.* **2012**, *136*, 031102, doi:10.1063/1.3680558.
26. Strocchi, F. Complex coordinates and quantum mechanics. *Rev. Mod. Phys.* **1966**, *38*, 36–40.
27. Stückelberg, E.C.G. Quantum theory in real Hilbert space. *Helv. Phys. Acta* **1960**, *33*, 727–752.
28. Hestenes, D. *Space-Time-Algebra*; Gordon and Breach: New York, NY, USA, 1966.
29. Hestenes, D. Spacetime Physics with Geometric Algebra. *Am. J. Phys.* **2003**, *71*, 691–714.
30. Gull, S.; Lasenby, A.; Doran, C. Imaginary numbers are not real—The geometric algebra of spacetime. *Found. Phys.* **1993**, *23*, 1175–1201.
31. Pauli, W. Elementary Particles and Their Interactions: Concepts and Phenomena. *Ann. de l'Inst. Henry Poincaré* **1936**, *6*, 109–136.
32. Baumgarten, C. Use of real Dirac matrices in two-dimensional coupled linear optics. *Phys. Rev. ST Accel. Beams* **2011**, *14*, 114002, doi:10.1103/PhysRevSTAB.15.124001.
33. Baumgarten, C. Geometrical method of decoupling. *Phys. Rev. ST Accel. Beams* **2012**, *15*, 124001, doi:10.1103/PhysRevSTAB.15.124001.
34. Schmüser, P. *Feynman-Graphen und Eichtheorien für Experimentalphysiker* (Lecture Notes in Physics No. 295); Springer-Verlag: Heidelberg, Germany, 1988. (in German)
35. Ralston, J.P. Quantum Theory without Planck's Constant. **2012**, arXiv:1203.5557.
36. Schrödinger, E. *Space-Time Structure*; Cambridge University Press: New York, NY, USA, 1950, ISBN 0-521-31520-4.
37. Blumenhagen, R.; Lüst, D.; Theissen, S. *Basic Concepts of String Theory*; Springer-Verlag Berlin Heidelberg: Berlin, Germany, 2013.
38. Dirac, P.A.M. Forms of Relativistic Dynamics. *Rev. Mod. Phys.* **1949**, *21*, 392–399.
39. 't Hooft, G. Dimensional Reduction in Quantum Gravity. **2009**, arxiv:gr-qc/9310026.
40. Susskind, L. The world as a hologram. *J. Math. Phys.* **1995**, *36*, 6377–6396.
41. Becker, R. *Theory of Heat*, 2nd ed.; Springer-Verlag Berlin Heidelberg: Berlin, Germany, 1967.



© 2016 by the author; licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC-BY) license (<http://creativecommons.org/licenses/by/4.0/>).