Boundary Value Problems for Some Important Classes of Recurrent Relations with Two Independent Variables

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Abstract: It is shown that complex-valued boundary value problems for several classes of recurrent relations with two independent variables, of some considerable interest, are solvable on the following domain: $\mathcal{C} = \{(n,k) : 0 \le k \le n, k \in \mathbb{N}_0, n \in \mathbb{N}\}$, the so called combinatorial domain. The recurrent relations include some of the most important combinatorial ones, which, among other things, serve as a motivation for the investigation. The methods for solving the boundary value problems are presented and explained in detail.

Keywords: recurrent relations with two independent variables; boundary value problem; combinatorial domain; equation solvable in closed form; method of half-lines

MSC: 39A14; 05A10

1. Introduction

As usual, $\mathbb{N}$ stands for the set of all positive integers $\{1, 2, 3, \ldots\}$, whereas $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If m and n are two numbers from $\mathbb{N}_0$ such that $m \le n$, then for the set of all $i \in \mathbb{N}_0$ such that $m \le i \le n$, we use the following notation $i = \overline{m, n}$. Throughout the paper, as usual, we adopt the convention that:

$$\sum_{j=k}^{l} a_k = 0,$$

if $k, l \in \mathbb{N}_0$ are such that $l < k$ and that:

$$\prod_{j=k}^{k-1} a_k = 1,$$

for each $k \in \mathbb{N}_0$.

Let:

$$H_n = \sum_{j=1}^{n} \frac{1}{j}, \quad n \in \mathbb{N},$$

(the harmonic numbers).

By C_k^n, $0 \le k \le n$, are denoted the *Binomial coefficients*, which can be introduced, for example, as the coefficients of the polynomial $P_n(x) = (1 + x)^n$, $n \in \mathbb{N}_0$, that is, we have:

$$P_n(x) = C_0^n + C_1^n x + \cdots + C_n^n x^n.$$

Recall that, for the coefficients, the following formula holds:

$$C_k^n = \frac{n!}{k!(n-k)!},$$ (1)

where:

$$m! = \prod_{j=1}^{m} j \quad \text{and} \quad 0! := 1.$$

Note that:

$$C_0^n = C_n^n = 1, \quad n \in \mathbb{N}_0.$$ (2)

One of the basic relations involving the coefficients is the following two-dimensional recurrent relation:

$$C_k^n = C_k^{n-1} + C_{k-1}^{n-1},$$ (3)

for $k, n \in \mathbb{N}$, such that $1 \le k < n$, which is easily obtained from the following obvious identity $(1+x)^{n-1}(1+x) = (1+x)^n$. Many other equalities and relations involving the coefficients, can be found, for example, in the following books: [1–6]. More specifically, some basic relations can be found in [1,2], while some more complex ones can be found in [3]. Book [4] is of an encyclopedic character and presents a large list of various relations. More complex methods, among others, for investigating relations with the coefficients can be found in [5]. Use of the coefficients in combinatorics, as well as their combinatorial meaning can be found in many books, for example, in [2,6].

Formula (1), the equalities in (2), as well as the recurrent relation (3) were the main motivations in [7], in which the relation (3), together with (2), was considered in a new way.

One of the main things in [7], connected to the recurrent relation (3), is the solvability, which is an old topic (see the classics [1,3,5,8–10] for well-known methods for solving difference equations). The topic has reattracted some interest recently, especially after 2004 when Stevo Stević used a transformation for showing the solvability of the nonlinear difference equation:

$$x_{n+1} = \frac{x_{n-1}}{a + bx_{n-1}x_n}, \quad n \in \mathbb{N}_0,$$

(for some extensions of the results, see [11,12], while the corresponding results for some related systems of difference equations can be found in [13]). Since that time, related transformations have been frequently used on difference equations ([14–16]), as well as on close to symmetric systems (see [15,17,18] and numerous references therein), an area essentially initiated by Papaschinopoulos and Schinas (see [19–25]). Somewhat more complex methods can be found in [26]. For some related topics, such as finding invariants, special types of solutions and applications of solvable difference equations (see, for example, [21–23,25,27–29] and the references therein).

Quite frequently, the solvability was essentially shown by using some special cases of the following difference equation:

$$x_{n+1} = a_n x_n + b_n, \quad n \in \mathbb{N},$$ (4)

where coefficients $(a_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$, and the initial value x_0 are real or complex (see [11–15,17,18,26,30]), which shows how useful the equation is (for how Equation (4) is solved, consult, for example, [3,8]; the book [3] has a nice explanation of three methods for solving it). Recently, we have studied several classes of product-type equations and systems (see [31–34] and the references therein), which cannot be directly solved by Equation (4), but behind their solvability is hidden the equation. Namely, these papers on product-type equations and systems use the solvability of some special cases of the following product-type analog of Equation (4):

$$z_n = b_n z_{n-1}^{a_n}, \quad n \in \mathbb{N}_0$$

(note that, if a_n, b_n, z_n are positive, by taking the logarithm, the equation is transformed to Equation (4), which also shows the importance of the equation).

It has been noticed in [7] that Equation (4) can be useful not only in solving some ordinary difference equations, but also in solving some important equations with two independent variables, which has motivated us to conduct further investigations in this direction ([35,36]). In fact, Refs. [7,35,36] are devoted to the solvability of some boundary value problems. This paper is a continuation of our investigation of the solvability of difference equations, as well as the solvability of some boundary value problems for difference equations with two independent variables. It should be pointed out that not all of the coefficients of the equations studied here will be constant, which makes the study more complex. Note also that, for the special case $a_n = 1$ of Equation (4), which is used in [7], is a simpler case, but the behavior of the solutions to Equation (4) can be quite complex (see [37–39] for the case of metric spaces). Let $x^{(0)} := 1$ and:

$$x^{(n)} := \prod_{j=0}^{n-1} (x - j),$$

for $n \in \mathbb{N}$, the so called *falling factorial*.

It is easy to see that $x^{(n)}$ is a polynomial of degree n, so that the following equality holds

$$x^{(n)} = \sum_{k=0}^{n} S_k^n x^k, \tag{5}$$

for some integer numbers S_k^n, $k, n \in \mathbb{N}_0$. These numbers S_k^n are called (signed) *Stirling's numbers of the first kind* (see, for example, [3,9]).

Now, we will present some basic facts on the Stirling numbers. First, from Equality (5), we immediately see that:

$$S_0^n = 0, \quad n \in \mathbb{N}, \tag{6}$$

$$S_n^n = 1, \quad n \in \mathbb{N}_0. \tag{7}$$

From the definition of $x^{(n)}$, Equality (5) and by using Viète formulas, it is easy to see that:

$$S_k^n = (-1)^{n-k} \sum_{1 \leq i_1 < i_2 < \ldots < i_{n-k} \leq n-1} i_1 i_2 \cdots i_{n-k}, \tag{8}$$

where $i_j \in \{1, 2, \ldots, n-1\}$, $j = \overline{1, n-k}$, and $1 \leq k \leq n-1$, from which it follows that:

$$S_1^n = (-1)^{n-1}(n-1)!, \quad n \in \mathbb{N}, \tag{9}$$

$$S_2^n = (-1)^{n-2}(n-1)! \sum_{j=1}^{n-1} \frac{1}{j} = (-1)^{n-2}(n-1)! H_{n-1}, \quad n \geq 2, \tag{10}$$

$$S_{n-1}^n = - \sum_{1 \leq j \leq n-1} j = -C_2^n, \quad n \geq 2. \tag{11}$$

From (5), since:

$$x^{(n+1)} = x^{(n)}(x - n),$$

and by some calculations, we also have that:

$$x^{(n+1)} = \sum_{k=0}^{n+1} S_k^{n+1} x^k = (x-n) \sum_{k=0}^{n} S_k^n x^k$$

$$= \sum_{k=0}^{n} S_k^n x^{k+1} - \sum_{k=0}^{n} n S_k^n x^k$$

$$= S_n^n x^{n+1} + \sum_{k=1}^{n} (S_{k-1}^n - n S_k^n) x^k - n S_0^n,$$

from which by comparing the coefficients of x^k are obtained the equations:

$$S_0^{n+1} = - n S_0^n, \quad n \in \mathbb{N}, \tag{12}$$

$$S_k^{n+1} = S_{k-1}^n - n S_k^n, \quad 1 \le k \le n, \ n \in \mathbb{N}, \tag{13}$$

$$S_{n+1}^{n+1} = S_n^n, \quad n \in \mathbb{N}. \tag{14}$$

From (12)–(14), Formulas (6), (7), (9)–(11) can also be obtained. Namely, from (12) and since $S_0^1 = 0$, we again obtain (6), whereas from (14) and since $S_1^1 = 1$, we again obtain (7). From (13) with $k = 1$ and by using (6),

$$S_1^n = -(n-1) S_1^{n-1},$$

is obtained, from which, along with $S_1^1 = 1$, it follows again that (9) holds. From (13) with $k = n$ and by using (7), is obtained:

$$S_{n-1}^n = S_{n-2}^{n-1} - (n-1),$$

from which by summing up this equality from 1 to n, and by using the fact $S_0^1 = 0$, it again easily follows that (11) holds. Formula (10) can also be found, essentially reducing recurrent relation (13) with $k = 2$, to a special case of Equation (4). Namely, using (9) into (13), we have that:

$$S_2^i = -(i-1) S_2^{i-1} + S_1^{i-1} = -(i-1) S_2^{i-1} + (-1)^{i-2}(i-2)!, \tag{15}$$

for $3 \le i \le n$.

Multiplying (15) by $\prod_{j=i}^{n-1}(-j)$, summing up such obtained equalities for $i = \overline{3, n}$, and using the fact that $S_2^2 = 1$, we obtain:

$$S_2^n = S_2^2 \prod_{j=2}^{n-1}(-j) + \sum_{i=3}^{n} \frac{1}{-(i-1)} \prod_{j=1}^{n-1}(-j)$$

$$= (-1)^{n-2}(n-1)! H_{n-1}.$$

The formula can be found in [9], but the author did not notice its connection with Equation (4).

This process can be continued for other values of k. It is a natural problem to try to find a general formula for solutions to Equation (13), including the (special) solution S_k^n. As far as we know, there are no formulas of this type in the literature and our task will be to present some.

If we use the change of variables:

$$d_{n,k} = (-1)^{n-k} S_k^n, \tag{16}$$

then the recurrent relation that is obtained from (13) when $n \to n-1$, is transformed to the following one:

$$d_{n,k} = d_{n-1,k-1} + (n-1) d_{n-1,k}, \tag{17}$$

where $1 \le k < n$, $k, n \in \mathbb{N}$, and, from (8) and (16), it is clear that sequence $|S_k^n|$ is a solution to Equation (17), which is a *partial difference equation* (for some classical results on the equations,

mostly connected to the their solvability, see, for example, [9,10], for some results up to 2003, see [40]; see also an interesting paper [41]). The numbers $|S_k^n|$, $0 \le k \le n$, are called the *unsigned Stirling's numbers of the first kind*. To reduce working with expressions with negative signs to a great extent, from now on, we will consider Equation (17) instead of the equation obtained from (13) when $n \to n-1$. Further details on the Stirling numbers can be found, for example, in [3,5,6,9].

From (17), we see that the Stirling numbers are a solution to a two-dimensional recurrent relation with given boundary conditions S_n^n and S_0^n, $n \in \mathbb{N}$. Since there is a closed form formula for it, that is, a formula for the solution to Equation (17) with the conditions:

$$d_{n,0} = 0, \quad d_{n,n} = 1,$$

$n \in \mathbb{N}$, it is expected that the equation can be also solved on the following domain:

$$\mathcal{C} = \{(n,k) : 0 \le k \le n, k \in \mathbb{N}_0, n \in \mathbb{N}\},$$

the so called *combinatorial domain* ([7]), for any values of $d_{n,0}$ and $d_{n,n}$, which is the case with the so called *Binomial partial difference equation* (see [7]). Our analysis has shown that the methods presented in [9,10] do not seem suitable for solving Equation (17) on domain $\mathcal{C}$, although we do not exclude a possibility that some modification of the methods could be suitable.

Using the idea and method in [7], we show, among other things, that there are closed form formulas for solutions to Equation (17) on domain $\mathcal{C}$ in terms of given boundary values $d_{n,0}$ and $d_{n,n}$, $n \in \mathbb{N}$.

2. Main Results

In this section, we show, among other things, how general solutions to the boundary-value problems for Equation (17) on domain $\mathcal{C}$ can be found by using the main idea and method in [7], which has been also recently employed in [35] for the case of another partial difference equation. Namely, domain $\mathcal{C}$ is divided into naturally chosen half-lines, and Equation (17) is regarded on each of these lines as an equation of type (4), that is, as a one-dimensional difference equation. Such one-dimensional equations are "solved" on the half lines, and, based on the obtained formulas, the general solution to the boundary-value problems for Equation (17) is obtained. In order to solve the equation, the method requires certain modifications, which are done/explained in the procedure that follows. There are also two natural ways of how to slice the domain $\mathcal{C}$ by lines. Based on the choice of slicing, some different looking solutions of the same problem are obtained. Besides Equation (17), an extension of it will be considered by using a different slicing, as well as another equation with a coefficient appearing in a different place.

2.1. Solving Equation (17)

To solve Equation (17), we will use the slicing of domain $\mathcal{C}$ by the lines:

$$k = c,$$

where $c \in \mathbb{N}$.

To be more suggestive, we will present the first three steps in detail and then give the general solution of the boundary-value problem on the domain. Let $k = 1$, and then (17) becomes:

$$d_{n,1} = (n-1)d_{n-1,1} + d_{n-1,0}, \tag{18}$$

for $n \ge 2$.

Multiplying the relation:

$$d_{i,1} = (i-1)d_{i-1,1} + d_{i-1,0},$$

by $\prod_{j=i}^{n-1} j$, summing up such obtained equalities for $2 \leq i \leq n$ and canceling the equal terms appearing on both sides of the summed equality, we get:

$$
\begin{aligned}
d_{n,1} &= (n-1)!d_{1,1} + \sum_{i=1}^{n-1} d_{i,0} \prod_{j=i+1}^{n-1} j \\
&= (n-1)! \left(d_{1,1} + \sum_{i=1}^{n-1} \frac{d_{i,0}}{i!} \right),
\end{aligned}
\tag{19}
$$

for $n \geq 2$. In fact, by using above-mentioned conventions, we see that (19) also holds for $n = 1$.

Letting $k = 2$, then (17) becomes:

$$
d_{n,2} = (n-1)d_{n-1,2} + d_{n-1,1},
\tag{20}
$$

for $n \geq 3$.

Multiplying the relation:

$$
d_{i,2} = (i-1)d_{i-1,2} + d_{i-1,1},
$$

by $\prod_{j=i}^{n-1} j$, summing up such obtained equalities for $3 \leq i \leq n$ and canceling the equal terms appearing on both sides of the summed equality, we get:

$$
\begin{aligned}
d_{n,2} &= \frac{(n-1)!}{1!} d_{2,2} + \sum_{i=2}^{n-1} d_{i,1} \prod_{j=i+1}^{n-1} j \\
&= (n-1)! \left(\frac{d_{2,2}}{1!} + \sum_{i=2}^{n-1} \frac{d_{i,1}}{i!} \right),
\end{aligned}
\tag{21}
$$

for $n \geq 3$. In fact, (21) also holds for $n = 2$.

By using (19) into (21), we get:

$$
\begin{aligned}
d_{n,2} &= (n-1)! \left(\frac{d_{2,2}}{1!} + \sum_{i=2}^{n-1} \frac{d_{i,1}}{i!} \right) \\
&= (n-1)! \left(\frac{d_{2,2}}{1!} + \sum_{i=2}^{n-1} \frac{1}{i!} \left((i-1)! \left(d_{1,1} + \sum_{j=1}^{i-1} \frac{d_{j,0}}{j!} \right) \right) \right) \\
&= (n-1)! \left(\frac{d_{2,2}}{1!} + d_{1,1} \sum_{i=2}^{n-1} \frac{1}{i} + \sum_{i=2}^{n-1} \frac{1}{i} \sum_{j=1}^{i-1} \frac{d_{j,0}}{j!} \right),
\end{aligned}
\tag{22}
$$

for $n \geq 3$.

Letting $k = 3$, then (17) becomes:

$$
d_{n,3} = (n-1)d_{n-1,3} + d_{n-1,2},
\tag{23}
$$

for $n \geq 4$.

Multiplying the relation:

$$
d_{i,3} = (i-1)d_{i-1,3} + d_{i-1,2},
$$

by $\prod_{j=i}^{n-1} j$, summing up such obtained equalities for all $4 \le i \le n$ and canceling the equal terms appearing on both sides of such obtained equality, it follows that:

$$d_{n,3} = \frac{(n-1)!}{2!} d_{3,3} + \sum_{i=3}^{n-1} d_{i,2} \prod_{j=i+1}^{n-1} j$$

$$= (n-1)! \left(\frac{d_{3,3}}{2!} + \sum_{i=3}^{n-1} \frac{d_{i,2}}{i!} \right), \tag{24}$$

for every $n \ge 4$ (in fact, it is immediately seen that the equality (24) holds also for $n = 3$).

By using (22) into (24), we get:

$$d_{n,3} = (n-1)! \left(\frac{d_{3,3}}{2!} + \sum_{i=3}^{n-1} \frac{d_{i,2}}{i!} \right)$$

$$= (n-1)! \left(\frac{d_{3,3}}{2!} + \sum_{i=3}^{n-1} \frac{1}{i!} \left((i-1)! \left(\frac{d_{2,2}}{1!} + d_{1,1} \sum_{j=2}^{i-1} \frac{1}{j} + \sum_{j=2}^{i-1} \frac{1}{j} \sum_{l=1}^{j-1} \frac{d_{l,0}}{l!} \right) \right) \right)$$

$$= (n-1)! \left(\frac{d_{3,3}}{2!} + \frac{d_{2,2}}{1!} \sum_{i=3}^{n-1} \frac{1}{i} + d_{1,1} \sum_{i=3}^{n-1} \frac{1}{i} \sum_{j=2}^{i-1} \frac{1}{j} + \sum_{i=3}^{n-1} \frac{1}{i} \sum_{j=2}^{i-1} \frac{1}{j} \sum_{l=1}^{j-1} \frac{d_{l,0}}{l!} \right), \tag{25}$$

for $n \ge 3$.

Assume that, for a fixed k such that $k \le n$, we have proved that:

$$d_{n,k} = (n-1)! \left(\frac{d_{k,k}}{(k-1)!} + \frac{d_{k-1,k-1}}{(k-2)!} \sum_{i_k=k}^{n-1} \frac{1}{i_k} + \cdots + d_{1,1} \sum_{i_k=k}^{n-1} \frac{1}{i_k} \cdots \sum_{i_3=3}^{i_4-1} \frac{1}{i_3} \sum_{i_2=2}^{i_3-1} \frac{1}{i_2} \right.$$

$$\left. + \sum_{i_k=k}^{n-1} \frac{1}{i_k} \cdots \sum_{i_2=2}^{i_3-1} \frac{1}{i_2} \sum_{i_1=1}^{i_2-1} \frac{d_{i_1,0}}{i_1!} \right), \tag{26}$$

for $n \ge k$.

If, in (17), we replace k by $k+1$, we get:

$$d_{n,k+1} = (n-1)d_{n-1,k+1} + d_{n-1,k}, \tag{27}$$

for $n \ge k+2$.

Multiplying the relation:

$$d_{i,k+1} = (i-1)d_{i-1,k+1} + d_{i-1,k},$$

by $\prod_{j=i}^{n-1} j$, summing up such obtained equalities for all $k+2 \le i \le n$ and canceling the equal terms appearing on both sides of the such obtained equality, it follows that:

$$d_{n,k+1} = \frac{(n-1)!}{k!} d_{k+1,k+1} + \sum_{i_{k+1}=k+1}^{n-1} d_{i_{k+1},k} \prod_{j=i_{k+1}+1}^{n-1} j$$

$$= (n-1)! \left(\frac{d_{k+1,k+1}}{k!} + \sum_{i_{k+1}=k+1}^{n-1} \frac{d_{i_{k+1},k}}{i_{k+1}!} \right), \tag{28}$$

for $n \ge k+2$ (in fact, for $n \ge k+1$).

Using (26) into (28), we get:

$$
\begin{aligned}
d_{n,k+1} =&(n-1)!\left(\frac{d_{k+1,k+1}}{k!} + \sum_{i_{k+1}=k+1}^{n-1}\frac{d_{i_{k+1},k}}{i_{k+1}!}\right)\\
=&(n-1)!\left(\frac{d_{k+1,k+1}}{k!} + \sum_{i_{k+1}=k+1}^{n-1}\frac{1}{i_{k+1}}\left(\frac{d_{k,k}}{(k-1)!} + \frac{d_{k-1,k-1}}{(k-2)!}\sum_{i_k=k}^{i_{k+1}-1}\frac{1}{i_k}+\cdots\right.\right.\\
&+ d_{1,1}\sum_{i_k=k}^{i_{k+1}-1}\frac{1}{i_k}\cdots\sum_{i_2=2}^{i_3-1}\frac{1}{i_2} + \sum_{i_k=k}^{i_{k+1}-1}\frac{1}{i_k}\cdots\sum_{i_2=2}^{i_3-1}\frac{1}{i_2}\sum_{i_1=1}^{i_2-1}\frac{d_{i_1,0}}{i_1!}\bigg)\bigg)\\
=&(n-1)!\left(\frac{d_{k+1,k+1}}{k!} + \frac{d_{k,k}}{(k-1)!}\sum_{i_{k+1}=k+1}^{n-1}\frac{1}{i_{k+1}}\right.\\
&+ \frac{d_{k-1,k-1}}{(k-2)!}\sum_{i_{k+1}=k+1}^{n-1}\frac{1}{i_{k+1}}\sum_{i_k=k}^{i_{k+1}-1}\frac{1}{i_k}+\cdots\\
&+ d_{1,1}\sum_{i_{k+1}=k+1}^{n-1}\frac{1}{i_{k+1}}\sum_{i_k=k}^{i_{k+1}-1}\frac{1}{i_k}\cdots\sum_{i_2=2}^{i_3-1}\frac{1}{i_2}\\
&+ \sum_{i_{k+1}=k+1}^{n-1}\frac{1}{i_{k+1}}\sum_{i_k=k}^{i_{k+1}-1}\frac{1}{i_k}\cdots\sum_{i_2=2}^{i_3-1}\frac{1}{i_2}\sum_{i_1=1}^{i_2-1}\frac{d_{i_1,0}}{i_1!}\bigg),
\end{aligned}
\tag{29}
$$

for $n \geq k+2$. Note that (29) also holds for $n = k+1$.

From (19), (29) and the method of induction, it follows that Formula (26) holds for $n \geq k$.

From the above conducted detailed analysis and consideration, we see that the following result holds.

Theorem 1. *If* $(a_k)_{k\in\mathbb{N}}$*,* $(b_k)_{k\in\mathbb{N}}$ *are given sequences of complex numbers. Then, the solution to the partial difference Equation* (17) *on the domain* $\mathcal{C}$*, with the boundary value conditions,*

$$
d_{k,0} = a_k \quad and \quad d_{k,k} = b_k, \quad k \in \mathbb{N},
\tag{30}
$$

is given by:

$$
d_{n,k} =(n-1)!\left(\sum_{j=1}^{k}\frac{b_j}{(j-1)!}\sum_{i_k=k}^{n-1}\frac{1}{i_k}\cdots\sum_{i_{j+1}=2}^{i_{j+2}-1}\frac{1}{i_{j+1}} + \sum_{i_k=k}^{n-1}\frac{1}{i_k}\cdots\sum_{i_2=2}^{i_3-1}\frac{1}{i_2}\sum_{i_1=1}^{i_2-1}\frac{a_{i_1}}{i_1!}\right).
\tag{31}
$$

Proof. Using all the conditions given in (30) in Formula (26), Formula (31) is obtained. □

From Theorem 1 and (16), the following corollary follows.

Corollary 1. *If* $(a_k)_{k\in\mathbb{N}}$*,* $(b_k)_{k\in\mathbb{N}}$ *are given sequences of real numbers, then the solution to the partial difference equation:*

$$
s_{n,k} = s_{n-1,k-1} - (n-1)s_{n-1,k},
\tag{32}
$$

on the domain $\mathcal{C}$*, with the boundary value conditions:*

$$
s_{k,0} = a_k \quad and \quad s_{k,k} = b_k, \quad k \in \mathbb{N},
\tag{33}
$$

is given by:

$$s_{n,k} = (-1)^{n-k}(n-1)! \left(\sum_{j=1}^{k} \frac{b_j}{(j-1)!} \sum_{i_k=k}^{n-1} \frac{1}{i_k} \cdots \sum_{i_{j+1}=2}^{i_{j+2}-1} \frac{1}{i_{j+1}} + \sum_{i_k=k}^{n-1} \frac{1}{i_k} \cdots \sum_{i_2=2}^{i_3-1} \frac{1}{i_2} \sum_{i_1=1}^{i_2-1} \frac{a_{i_1}}{i_1!} \right). \tag{34}$$

Remark 1. *Note that, in the first three steps above, as well as in the main part of the inductive argument, we essentially used a method for solving Equation (4) (see, how we "solve" Equations (18), (20), (23) and (27)).*

2.2. Partial Difference Equations with an Interchanged Coefficient

In this section, we consider the following partial difference equation:

$$d_{n,k} = (n-1)d_{n-1,k-1} + d_{n-1,k}, \tag{35}$$

on domain $\mathcal{C}$.

We will solve the equation also by slicing domain $\mathcal{C}$ by the lines $k = c$, $c \in \mathbb{N}$ and use the above procedure for solving the corresponding one-dimensional difference equations on such obtained half-lines that will lead to the general solution to the boundary-value problem for Equation (35) on the domain.

Assuming first that $k = 1$, then (35) becomes:

$$d_{n,1} = (n-1)d_{n-1,0} + d_{n-1,1}, \tag{36}$$

for $n \geq 2$.

Summing up the following relations:

$$d_{i,1} = (i-1)d_{i-1,0} + d_{i-1,1},$$

for $2 \leq i \leq n$, we get:

$$d_{n,1} = d_{1,1} + \sum_{i=1}^{n-1} i d_{i,0}, \tag{37}$$

for $n \geq 2$. Note that (37) also holds for $n = 1$.

Letting $k = 2$, then (35) becomes:

$$d_{n,2} = (n-1)d_{n-1,1} + d_{n-1,2}, \tag{38}$$

for $n \geq 3$.

Summing up the relations:

$$d_{i,2} = (i-1)d_{i-1,1} + d_{i-1,2},$$

for $3 \leq i \leq n$, canceling the equal terms and using Formula (37), we get:

$$
\begin{aligned}
d_{n,2} &= d_{2,2} + \sum_{j=2}^{n-1} j d_{j,1} \\
&= d_{2,2} + \sum_{j=2}^{n-1} j \left(d_{1,1} + \sum_{i=1}^{j-1} i d_{i,0} \right) \\
&= d_{2,2} + d_{1,1} \sum_{j=2}^{n-1} j + \sum_{j=2}^{n-1} j \sum_{i=1}^{j-1} i d_{i,0},
\end{aligned} \tag{39}
$$

for $n \geq 3$. Note that (39) also holds for $n = 2$.

Assume that the following formula has been proved for a $k \in \mathbb{N}$ and $n \geq k$,

$$
\begin{aligned}
d_{n,k} = {} & d_{k,k} + d_{k-1,k-1} \sum_{i_k=k}^{n-1} i_k + d_{k-2,k-2} \sum_{i_k=k}^{n-1} i_k \sum_{i_{k-1}=k-1}^{i_k-1} i_{k-1} + \cdots \\
& + d_{1,1} \sum_{i_k=k}^{n-1} i_k \cdots \sum_{i_3=3}^{i_4-1} i_3 \sum_{i_2=2}^{i_3-1} i_2 + \sum_{i_k=k}^{n-1} i_k \cdots \sum_{i_3=3}^{i_4-1} i_3 \sum_{i_2=2}^{i_3-1} i_2 \sum_{i_1=1}^{i_2-1} i_1 d_{i_1,0}.
\end{aligned}
\tag{40}
$$

Then, by using (35) with $k \to k+1$, we get:

$$
d_{n,k+1} = (n-1)d_{n-1,k} + d_{n-1,k+1},
\tag{41}
$$

for $n \geq k+2$.

Summing up the relations:

$$
d_{i,k+1} = (i-1)d_{i-1,k} + d_{i-1,k+1},
$$

for $k+2 \leq i \leq n$, and using (40), it follows that:

$$
\begin{aligned}
d_{n,k+1} = {} & d_{k+1,k+1} + \sum_{i_{k+1}=k+1}^{n-1} i_{k+1} d_{i_{k+1},k} \\
= {} & d_{k+1,k+1} + \\
& \sum_{i_{k+1}=k+1}^{n-1} i_{k+1} \Bigg(d_{k,k} + d_{k-1,k-1} \sum_{i_k=k}^{i_{k+1}-1} i_k + d_{k-2,k-2} \sum_{i_k=k}^{i_{k+1}-1} i_k \sum_{i_{k-1}=k-1}^{i_k-1} i_{k-1} + \\
& \cdots + d_{1,1} \sum_{i_k=k}^{i_{k+1}-1} i_k \cdots \sum_{i_3=3}^{i_4-1} i_3 \sum_{i_2=2}^{i_3-1} i_2 + \sum_{i_k=k}^{i_{k+1}-1} i_k \cdots \sum_{i_3=3}^{i_4-1} i_3 \sum_{i_2=2}^{i_3-1} i_2 \sum_{i_1=1}^{i_2-1} i_1 d_{i_1,0} \Bigg) \\
= {} & d_{k+1,k+1} + d_{k,k} \sum_{i_{k+1}=k+1}^{n-1} i_{k+1} + d_{k-1,k-1} \sum_{i_{k+1}=k+1}^{n-1} i_{k+1} \sum_{i_k=k}^{i_{k+1}-1} i_k \\
& + \cdots + d_{1,1} \sum_{i_{k+1}=k+1}^{n-1} i_{k+1} \sum_{i_k=k}^{i_{k+1}-1} i_k \cdots \sum_{i_3=3}^{i_4-1} i_3 \sum_{i_2=2}^{i_3-1} i_2 \\
& + \sum_{i_{k+1}=k+1}^{n-1} i_{k+1} \sum_{i_k=k}^{i_{k+1}-1} i_k \cdots \sum_{i_3=3}^{i_4-1} i_3 \sum_{i_2=2}^{i_3-1} i_2 \sum_{i_1=1}^{i_2-1} i_1 d_{i_1,0},
\end{aligned}
\tag{42}
$$

for $n \geq k+2$ ((42) obviously holds for $n = k+1$).

From (37), (42) and the method of mathematical induction, it follows that Formula (40) really holds for $n \geq k$.

Now, note that the procedure described above can be applied to every partial difference equation in the following form:

$$
d_{n,k} = f(n-1)d_{n-1,k-1} + d_{n-1,k},
\tag{43}
$$

on domain $\mathcal{C}$, where $(f(m))_{m \in \mathbb{N}}$ is a complex sequence. The following theorem holds (the proof is omitted since it is similar to the one given in the case $f(m) = m$).

Theorem 2. *Let $(u_j)_{j \in \mathbb{N}}$, $(v_j)_{j \in \mathbb{N}}$ be given sequences of complex numbers. Then, the solution to Equation (43) on the domain $\mathcal{C}$, with the following conditions*

$$
d_{j,0} = u_j \quad \text{and} \quad d_{j,j} = v_j, \quad j \in \mathbb{N},
$$

is given by:

$$
\begin{aligned}
d_{n,k} = {}& v_k + v_{k-1} \sum_{i_k=k}^{n-1} f(i_k) + v_{k-2} \sum_{i_k=k}^{n-1} f(i_k) \sum_{i_{k-1}=k-1}^{i_k-1} f(i_{k-1}) + \cdots \\
& + v_1 \sum_{i_k=k}^{n-1} f(i_k) \cdots \sum_{i_3=3}^{i_4-1} f(i_3) \sum_{i_2=2}^{i_3-1} f(i_2) \\
& + \sum_{i_k=k}^{n-1} f(i_k) \cdots \sum_{i_3=3}^{i_4-1} f(i_3) \sum_{i_2=2}^{i_3-1} f(i_2) \sum_{i_1=1}^{i_2-1} f(i_1) u_{i_1},
\end{aligned}
\tag{44}
$$

for $n \geq k$.

Theorem 2 shows that the calculation of iterated sums plays an important role in solving Equation (43).

2.3. An Extension to Equation (17)

Here, we consider the following class of partial difference equations:

$$
d_{n,k} = d_{n-1,k-1} + f(n,k)d_{n-1,k},
\tag{45}
$$

where $(n,k) \in \mathcal{C}$, which is an extension to Equation (17), and whose special cases appear in the literature.

In contrast to the previous two equations, this time, we will find a solution to the equation by slicing domain $\mathcal{C}$ in a different way. Namely, the domain will be sliced now by the lines:

$$
y_l(k) = k + l, \quad k \in \mathbb{N},
$$

where $l \in \mathbb{N}$.

For $n = k + 1$, Equation (45) becomes:

$$
d_{k+1,k} = d_{k,k-1} + f(k+1,k)d_{k,k},
\tag{46}
$$

for $k \in \mathbb{N}$.

Summing up equalities (46) from 1 to k,

$$
d_{k+1,k} = d_{1,0} + \sum_{j=1}^{k} f(j+1,j)d_{j,j},
\tag{47}
$$

is obtained for $k \in \mathbb{N}$. Note that (47) holds also for $k = 0$.

For $n = k + 2$, Equation (45) becomes:

$$
d_{k+2,k} = d_{k+1,k-1} + f(k+2,k)d_{k+1,k},
\tag{48}
$$

for $k \in \mathbb{N}$.

Summing up equalities (48) from 1 to k, and using (47) in such obtained equality, we get:

$$d_{k+2,k} = d_{2,0} + \sum_{j=1}^{k} f(j+2,j)d_{j+1,j}$$

$$= d_{2,0} + \sum_{j=1}^{k} f(j+2,j)\left(d_{1,0} + \sum_{i=1}^{j} f(i+1,i)d_{i,i}\right)$$

$$= d_{2,0} + d_{1,0}\sum_{j=1}^{k} f(j+2,j) + \sum_{j=1}^{k} f(j+2,j)\sum_{i=1}^{j} f(i+1,i)d_{i,i}, \tag{49}$$

for $k \in \mathbb{N}$. Note that (49) holds also for $k = 0$.

Assume that, for an $l \in \mathbb{N}$,

$$d_{k+l,k} = d_{l,0} + d_{l-1,0}\sum_{i_l=1}^{k} f(i_l+l,i_l) + d_{l-2,0}\sum_{i_l=1}^{k} f(i_l+l,i_l)\sum_{i_{l-1}=1}^{i_l} f(i_{l-1}+l-1,i_{l-1})$$

$$+ \cdots + d_{1,0}\sum_{i_l=1}^{k} f(i_l+l,i_l)\sum_{i_{l-1}=1}^{i_l} f(i_{l-1}+l-1,i_{l-1})\cdots\sum_{i_2=1}^{i_3} f(i_2+2,i_2)$$

$$+ \sum_{i_l=1}^{k} f(i_l+l,i_l)\sum_{i_{l-1}=1}^{i_l} f(i_{l-1}+l-1,i_{l-1})\cdots\sum_{i_1=1}^{i_2} f(i_1+1,i_1)d_{i_1,i_1}, \tag{50}$$

for $k \in \mathbb{N}_0$.

For $n = k + l + 1$, Equation (45) becomes:

$$d_{k+l+1,k} = d_{k+l,k-1} + f(k+l+1,k)d_{k+l,k}, \tag{51}$$

for $k \in \mathbb{N}$.

Summing up equalities (51) from 1 to k, and using (50) in such obtained equalities, we get:

$$d_{k+l+1,k} = d_{l+1,0} + \sum_{i_{l+1}=1}^{k} f(i_{l+1}+l+1,i_{l+1})d_{i_{l+1}+l,i_{l+1}}$$

$$= d_{l+1,0} + \sum_{i_{l+1}=1}^{k} f(i_{l+1}+l+1,i_{l+1})\left(d_{l,0} + d_{l-1,0}\sum_{i_l=1}^{i_{l+1}} f(i_l+l,i_l)\right.$$

$$+ \cdots + d_{1,0}\sum_{i_l=1}^{i_{l+1}} f(i_l+l,i_l)\sum_{i_{l-1}=1}^{i_l} f(i_{l-1}+l-1,i_{l-1})\cdots\sum_{i_2=1}^{i_3} f(i_2+2,i_2)$$

$$+ \left.\sum_{i_l=1}^{i_{l+1}} f(i_l+l,i_l)\sum_{i_{l-1}=1}^{i_l} f(i_{l-1}+l-1,i_{l-1})\cdots\sum_{i_1=1}^{i_2} f(i_1+1,i_1)d_{i_1,i_1}\right)$$

$$= d_{l+1,0} + d_{l,0}\sum_{i_{l+1}=1}^{k} f(i_{l+1}+l+1,i_{l+1})$$

$$+ d_{l-1,0}\sum_{i_{l+1}=1}^{k} f(i_{l+1}+l+1,i_{l+1})\sum_{i_l=1}^{i_{l+1}} f(i_l+l,i_l)$$

$$+ \cdots + d_{1,0}\sum_{i_{l+1}=1}^{k} f(i_{l+1}+l+1,i_{l+1})\sum_{i_l=1}^{i_{l+1}} f(i_l+l,i_l)\cdots\sum_{i_2=1}^{i_3} f(i_2+2,i_2)$$

$$+ \sum_{i_{l+1}=1}^{k} f(i_{l+1}+l+1,i_{l+1})\sum_{i_l=1}^{i_{l+1}} f(i_l+l,i_l)\cdots\sum_{i_1=1}^{i_2} f(i_1+1,i_1)d_{i_1,i_1}, \tag{52}$$

for $k \in \mathbb{N}$. Since (52) obviously holds for $k = 0$, it holds for every $k \in \mathbb{N}_0$.

From (47), (52) and the method of induction, we see that (50) holds for every $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$.

From the above considerations, we have the following results.

Theorem 3. *Let* $(u_j)_{j \in \mathbb{N}}$, $(v_j)_{j \in \mathbb{N}}$ *be given sequences of complex numbers. Then, the solution to Equation* (45) *on the domain* $\mathcal{C}$, *with the following boundary-value conditions:*

$$d_{j,0} = u_j \quad \text{and} \quad d_{j,j} = v_j, \quad j \in \mathbb{N}, \tag{53}$$

is given by:

$$
\begin{aligned}
d_{n,k} =\,& u_{n-k} + u_{n-k-1} \sum_{i_{n-k}=1}^{k} f(i_{n-k} + n - k, i_{n-k}) \\
& + u_{n-k-2} \sum_{i_{n-k}=1}^{k} f(i_{n-k} + n - k, i_{n-k}) \sum_{i_{n-k-1}=1}^{i_{n-k}} f(i_{n-k-1} + n - k - 1, i_{n-k-1}) \\
& + \cdots + u_1 \sum_{i_{n-k}=1}^{k} f(i_{n-k} + n - k, i_{n-k}) \cdots \sum_{i_3=1}^{i_4} f(i_3 + 3, i_3) \sum_{i_2=1}^{i_3} f(i_2 + 2, i_2) \\
& + \sum_{i_{n-k}=1}^{k} f(i_{n-k} + n - k, i_{n-k}) \cdots \sum_{i_2=1}^{i_3} f(i_2 + 2, i_2) \sum_{i_1=1}^{i_2} f(i_1 + 1, i_1) v_{i_1},
\end{aligned} \tag{54}
$$

for $k, n \in \mathbb{N}$ *such that* $n \geq k + 1$.

Proof. If we apply the change of variables $l = n - k$ in Formula (50) and then use the boundary value conditions (53), we obtain (54). □

If we apply Theorem 3 for the case $f(n, k) = n - 1$, we obtain another formula for the solution to the boundary value problem in Theorem 1. Namely, the following result holds.

Corollary 2. *Let* $(u_j)_{j \in \mathbb{N}}$, $(v_j)_{j \in \mathbb{N}}$ *be given sequences of complex numbers. Then the solution to Equation* (17) *on the domain* $\mathcal{C}$, *with the following boundary-value conditions:*

$$d_{j,0} = u_j \quad \text{and} \quad d_{j,j} = v_j, \quad j \in \mathbb{N},$$

is given by:

$$
\begin{aligned}
d_{n,k} =\,& u_{n-k} + u_{n-k-1} \sum_{i_{n-k}=1}^{k} (i_{n-k} + n - k - 1) \\
& + u_{n-k-2} \sum_{i_{n-k}=1}^{k} (i_{n-k} + n - k - 1) \sum_{i_{n-k-1}=1}^{i_{n-k}} (i_{n-k} + n - k - 2) \\
& + \cdots + u_1 \sum_{i_{n-k}=1}^{k} (i_{n-k} + n - k - 1) \cdots \sum_{i_3=1}^{i_4} (i_3 + 2) \sum_{i_2=1}^{i_3} (i_2 + 1) \\
& + \sum_{i_{n-k}=1}^{k} (i_{n-k} + n - k - 1) \cdots \sum_{i_2=1}^{i_3} (i_2 + 1) \sum_{i_1=1}^{i_2} i_1 v_{i_1},
\end{aligned} \tag{55}
$$

for $k, n \in \mathbb{N}$ *such that* $n \geq k + 1$.

If, in Corollary 2, we choose $d_{j,0} = 0$ and $d_{j,j} = 1$, for $j \in \mathbb{N}$, and use (16), we get the following formula for the Stirling numbers of the first kind:

$$S_k^n = (-1)^{n-k} \sum_{i_{n-k}=1}^{k} (i_{n-k} + n - k - 1) \cdots \sum_{i_2=1}^{i_3} (i_2 + 1) \sum_{i_1=1}^{i_2} i_1,$$

or by changing the order of the summation:

$$S_k^n = (-1)^{n-k} \sum_{i_1=1}^{k} i_1 \sum_{i_2=i_1}^{k} (i_2 + 1) \cdots \sum_{i_{n-k}=i_{n-k-1}}^{k} (i_{n-k} + n - k - 1),$$

for $k, n \in \mathbb{N}$, such that $n \geq k + 1$.

3. Conclusions

Here, we have presented another influence and application of linear first-order difference equations, this time on the solvability of some boundary value problems for some important classes of recurrent relations with two independent variables on the so-called combinatorial domain. Our method of half-lines is used and suitably modified to show the solvability of the problems. To a certain extent, we have been motivated by some recurrent relations in combinatorics, particularly by the recurrent relation generating Stirling's numbers of the first kind. Using this fact, as well as the fact that the recurrent relation for the binomial coefficients can be solved by the method of half-lines, it is highly expected that some suitable modifications of the method can be successfully used in considering the other related recurrent relations that stem from combinatorics, as well as the other areas of mathematics and science, which will be one of our topics for further investigation.

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References

1. Krechmar, V.A. *A Problem Book in Algebra*; Mir Publishers: Moscow, Russia, 1974.
2. Mitrinović, D.S. *Mathematical Induction, Binomial Formula, Combinatorics*; Gradjevinska Knjiga: Beograd, Serbia, 1980. (In Serbian)
3. Mitrinović, D.S.; Kečkić, J.D. *Methods for Calculating Finite Sums*; Naučna Knjiga: Beograd, Serbia, 1984. (In Serbian)
4. Prudnikov, A.P.; Brychkov, Y.A.; Marichev, O.I. *Integrals and Series*; Nauka: Moscow, Russian, 1981. (In Russian)
5. Riordan, J. *Combinatorial Identities*; John Wiley & Sons Inc.: New York, NY, USA; London, UK; Sydney, Australia, 1968.
6. Yablonskiy, S.V. *Introduction to Discrete Mathematics*; Mir Publishers: Moscow, Russia, 1989.
7. Stević, S. Note on the binomial partial difference equation. *Electron. J. Qual. Theory Differ. Equ.* **2015**, *2015*, Article 96.
8. Brand, L. *Differential and Difference Equations*; John Wiley & Sons, Inc.: New York, NY, USA, 1966.
9. Jordan, C. *Calculus of Finite Differences*; Chelsea Publishing Company: New York, NY, USA, 1956.
10. Levy, H.; Lessman, F. *Finite Difference Equations*; Dover Publications: New York, NY, USA, 1992.
11. Stević, S. On the difference equation $x_n = x_{n-k}/(b + cx_{n-1} \cdots x_{n-k})$, *Appl. Math. Comput.* **2012**, *218*, 6291–6296.

12. Stević, S.; Diblik, J.; Iričanin, B.; Šmarda, Z. On the difference equation $x_n = a_n x_{n-k}/(b_n + c_n x_{n-1} \cdots x_{n-k})$. *Abstr. Appl. Anal.* **2012**, *2012*, Article 409237.

13. Stević, S.; Diblik, J.; Iričanin. B.; Šmarda, Z. On a third-order system of difference equations with variable coefficients. *Abstr. Appl. Anal.* **2012**, *2012*, Article 508523.

14. Papaschinopoulos, G.; Stefanidou, G. Asymptotic behavior of the solutions of a class of rational difference equations. *Int. J. Differ. Equ.* **2010**, *5*, 233–249.

15. Stević, S.; Diblik, J.; Iričanin, B.; Šmarda, Z. On some solvable difference equations and systems of difference equations. *Abstr. Appl. Anal.* **2012**, *2012*, Article 541761.

16. Stević, S.; Diblik, J.; Iričanin, B.; Šmarda, Z. Solvability of nonlinear difference equations of fourth order. *Electron. J. Differ. Equ.* **2014**, *2014*, Article 264.

17. Berg, L.; Stević, S. On some systems of difference equations. *Appl. Math. Comput.* **2011**, *218*, 1713–1718.

18. Stević, S.; Iričanin, B.; Šmarda, Z. On a close to symmetric system of difference equations of second order. *Adv. Differ. Equ.* **2015**, *2015*, Article 264.

19. Fotiades, N.; Papaschinopoulos, G. On a system of difference equations with maximum, *Appl. Math. Comput.* **2013**, *221*, 684–690.

20. Papaschinopoulos, G.; Schinas, C.J. On a system of two nonlinear difference equations. *J. Math. Anal. Appl.* **1998**, *219*, 415–426.

21. Papaschinopoulos, G.; Schinas, C.J. On the behavior of the solutions of a system of two nonlinear difference equations. *Commun. Appl. Nonlinear Anal.* **1998**, *5*, 47–59.

22. Papaschinopoulos, G.; Schinas, C.J. Invariants for systems of two nonlinear difference equations. *Differ. Equ. Dyn. Syst.* **1999**, *7*, 181–196.

23. Papaschinopoulos, G.; Schinas, C.J. Invariants and oscillation for systems of two nonlinear difference equations. *Nonlinear Anal. Theory Methods Appl.* **2001**, *46*, 967–978.

24. Papaschinopoulos, G.; Schinas, C.J. On the dynamics of two exponential type systems of difference equations. *Comput. Math. Appl.* **2012**, *64*, 2326–2334.

25. Papaschinopoulos, G.; Schinas, C.J.; Stefanidou, G. On a *k*-order system of Lyness-type difference equations. *Adv. Differ. Equ.* **2007**, *2007*, Article 31272.

26. Stević, S.; Diblik, J.; Iričanin, B.; Šmarda, Z. On a solvable system of rational difference equations. *J. Differ. Equ. Appl.* **2014**, *20*, 811–825.

27. Berezansky, L.; Braverman, E. On impulsive Beverton-Holt difference equations and their applications. *J. Differ. Equ. Appl.* **2004**, *10*, 851–868.

28. Diblik, J.; Halfarová, H. Explicit general solution of planar linear discrete systems with constant coefficients and weak delays. *Adv. Differ. Equ.* **2013**, *2013*, Article 50.

29. Iričanin, B.; Stević, S. Eventually constant solutions of a rational difference equation. *Appl. Math. Comput.* **2009**, *215*, 854–856.

30. Stević, S.; Diblik, J.; Iričanin, B.; Šmarda, Z. On the difference equation $x_{n+1} = x_n x_{n-k}/(x_{n-k+1}(a + bx_n x_{n-k}))$. *Abstr. Appl. Anal.* **2012**, *2012*, Article 108047.

31. Stević, S. First-order product-type systems of difference equations solvable in closed form. *Electron. J. Differ. Equ.* **2015**, *2015*, Article 308.

32. Stević, S.; Iričanin, B.; Šmarda, Z. On a product-type system of difference equations of second order solvable in closed form. *J. Inequal. Appl.* **2015**, *2015*, Article 327.

33. Stević, S.; Iričanin, B.; Šmarda, Z. Solvability of a close to symmetric system of difference equations. *Electron. J. Differ. Equ.* **2016**, *2016*, Article 159.

34. Stević, S.; Iričanin, B.; Šmarda, Z. Two-dimensional product-type system of difference equations solvable in closed form. *Adv. Differ. Equ.* **2016**, *2016*, Article 253.

35. Stević, S. Solvability of boundary value problems for a class of partial difference equations on the combinatorial domain. *Adv. Differ. Equ.* **2016**, *2016*, Article 262.

36. Stević, S. Solvability of boundary-value problems for a linear partial difference equation. *Electron. J. Differ. Equ.* **2017**, *2017*, Article 17.

37. Mitrinović, D.S.; Adamović, D.D. *Sequences and Series*; Naučna Knjiga: Beograd, Serbia, 1980. (In Serbian)

38. Polya, G.; Szegö, G. *Aufgaben Und Lehrsätze Aus Der Analysis*; Verlag von Julius Springer: Berlin, Germany, 1925.

39. Ašić, M.D.; Adamović, D.D. Limit points of sequences in metric spaces. *Am. Math. Mon.* **1970**, *77*, 613–616.

40. Cheng, S.S. *Partial Difference Equations*; Taylor & Francis: London, UK; New York, NY, USA, 2003.
41. Musielak, A.; Popenda, J. On the hyperbolic partial difference equations and their oscillatory properties. *Glas. Mat.* **1998**, *33*, 209–221.