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# Symmetry Analysis and Conservation Laws of the Zoomeron Equation

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**Abstract:** In this work, we study the (2 + 1)-dimensional Zoomeron equation which is an extension of the famous (1 + 1)-dimensional Zoomeron equation that has been studied extensively in the literature. Using classical Lie point symmetries admitted by the equation, for the first time we develop an optimal system of one-dimensional subalgebras. Based on this optimal system, we obtain symmetry reductions and new group-invariant solutions. Again for the first time, we construct the conservation laws of the underlying equation using the multiplier method.

**Keywords:** (2 + 1)-dimensional Zoomeron equation; Lie point symmetries; optimal system; exact solutions; conservation laws; multiplier method

## 1. Introduction

Many physical phenomena of the real world are governed by nonlinear partial differential equations (NLPDEs). It is therefore absolutely necessary to analyse these equations from the point of view of their integrability and finding exact closed form solutions. Although this is not an easy task, many researchers have developed various methods to find exact solutions of NLPDEs. These methods include the sine-cosine method [1], the extended tanh method [2], the inverse scattering transform method [3], the Hirota's bilinear method [4], the multiple exp-function method [5], the simplest equation method [6,7], non-classical method [8], method of generalized conditional symmetries [9], and the Lie symmetry method [10,11].

This paper aims to study one NLPDE; namely, the (2 + 1)-dimensional Zoomeron equation [12]

$$\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2(u^2)_{tx} = 0, \tag{1}$$

which has attracted some attention in recent years. Many authors have found closed-form solutions of this equation. For example, the $(G'/G)-$expansion method [12,13], the extended tanh method [14], the tanh-coth method [15], the sine-cosine method [16,17], and the modified simple equation method [18] have been used to find closed-form solutions of (1). The (2 + 1)-dimensional Zoomeron equation with power-law nonlinearity was studied in [19] from a Lie point symmetries point of view and symmetry reductions, and some solutions were obtained. Additionally, in [19], the authors have given a brief history of the (1 + 1)-dimensional Zoomeron equation. See also [20–22].

In this paper we first use the classical Lie point symmetries admitted by Equation (1) to find an optimal system of one-dimensional subalgebras. These are then used to perform symmetry reductions and determine new group-invariant solutions of (1). It should be noted that such approach was previously used for examination of a wide range of nonlinear PDEs [23–31]. Furthermore, we derive the conservation laws of (1) using the multiplier method [32,33].

The paper is organized as follows: in Section 2, we compute the Lie point symmetries of (1) and use them to construct the optimal system of one-dimensional subalgebras. These are then used to perform symmetry reductions and determine new group-invariant solutions of (1). In Section 3, we derive conservation laws of (1) by employing the multiplier method. Finally, concluding remarks are presented in Section 4.

## 2. Symmetry Reductions and Exact Solutions of (1) Based on Optimal System

In this section, firstly we use the Lie point symmetries admitted by (1) to construct an optimal system of one-dimensional subalgebras. Thereafter, we obtain symmetry reductions and group-invariant solutions based on the optimal system of one-dimensional subalgebras [23,24].

### *2.1. Lie Point Symmetries of (1)*

The Lie point symmetries of the Zoomeron Equation (1) are given by [19]

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},\ X_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},\ X_5 = 2y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u},$$

which generate a five-dimensional Lie algebra $L_5$.

### *2.2. Optimal System of One-Dimensional Subalgebras*

In this subsection, we use the Lie point symmetries of (1) to compute an optimal system of one-dimensional subalgebras. We employ the method given in [23,24], which takes a general element from the Lie algebra and reduces it to its simplest equivalent form by using the chosen adjoint transformations

$$\mathrm{Ad}(\exp(\varepsilon X_i))X_j = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(\mathrm{ad}X_i)^n(X_j) = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2!}[X_i, [X_i, X_j]] - \cdots,$$

where $\varepsilon$ is a real number, and $[X_i, X_j]$ denotes the commutator defined by

$$[X_i, X_j] = X_iX_j - X_jX_i.$$

The table of commutators of the Lie point symmetries of Equation (1) and the adjoint representations of the symmetry group of (1) on its Lie algebra are given in Tables 1 and 2, respectively. Then, Tables 1 and 2 are used to construct the optimal system of one-dimensional subalgebras for Equation (1).

Using Tables 1 and 2, we can construct an optimal system of one-dimensional subalgebras, which is given by $\{X_3, X_4, X_5, X_1 + X_3, X_2 + X_3, X_1 + X_5, X_2 + X_5, X_4 + X_5, X_1 + X_2 + X_3, X_1 + X_2 + X_5\}$.

**Table 1.** Lie brackets for Equation (1).

| $[,]$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ |
|---|---|---|---|---|---|
| $X_1$ | 0 | 0 | 0 | $X_1$ | 0 |
| $X_2$ | 0 | 0 | 0 | $X_2$ | 0 |
| $X_3$ | 0 | 0 | 0 | $-X_3$ | $2X_3$ |
| $X_4$ | $-X_1$ | $-X_2$ | $X_3$ | 0 | 0 |
| $X_5$ | 0 | 0 | $-2X_3$ | 0 | 0 |

**Table 2.** Adjoint representation of subalgebras.

| **Ad** | $\mathbf{X_1}$ | $\mathbf{X_2}$ | $\mathbf{X_3}$ | $\mathbf{X_4}$ | $\mathbf{X_5}$ |
|---|---|---|---|---|---|
| $X_1$ | $X_1$ | $X_2$ | $X_3$ | $X_4 - \varepsilon X_1$ | $X_5$ |
| $X_2$ | $X_1$ | $X_2$ | $X_3$ | $X_4 - \varepsilon X_2$ | $X_5$ |
| $X_3$ | $X_1$ | $X_2$ | $X_3$ | $X_4 + \varepsilon X_3$ | $X_5 - 2\varepsilon X_3$ |
| $X_4$ | $e^{\varepsilon} X_1$ | $e^{\varepsilon} X_2$ | $e^{-\varepsilon} X_3$ | $X_4$ | $X_5$ |
| $X_5$ | $X_1$ | $X_2$ | $e^{2\varepsilon} X_3$ | $X_4$ | $X_5$ |

### *2.3. Symmetry Reductions*

In this subsection, we use the optimal system of one-dimensional subalgebras computed in the previous subsection, and present symmetry reductions of (1) to two-dimensional partial differential equations.

For the first operator $X_3$ of the optimal system, we have the three invariants $s = t,\ r = x,\ f = u$, and using these invariants, (1) reduces to

$$(f^2)_{sr} = 0.$$

Likewise for $X_4$, the invariants $s = ty,\ r = xy,\ f = u$ transforms (1) to

$$\left(\frac{f_{rr}}{f^2} - \frac{f_{ss}}{f^2} + \frac{2f_s^2}{f^3} - \frac{2f_r^2}{f^3}\right)(sf_s + rf_r)_r + \frac{2f_r}{f^2}(sf_s + rf_r)_{rr} - \frac{2f_s}{f^2}(sf_s + rf_r)_{sr}$$
$$+ \frac{1}{f}(sf_s + rf_r)_{ssr} - \frac{1}{f}(sf_s + rf_r)_{rrr} + 2\left(f^2\right)_{sr} = 0.$$

The invariants $s = t,\ r = x,\ f = u\sqrt{y}$ of $X_5$ reduces (1) to

$$\left(\frac{f_r}{2f}\right)_{rr} - \left(\frac{f_r}{2f}\right)_{ss} + 2\left(f^2\right)_{sr} = 0.$$

Using the invariants $s = x,\ r = y - t,\ f = u$ of $X_1 + X_3$, (1) reduces to

$$\left(\frac{f_{sr}}{f}\right)_{rr} - \left(\frac{f_{sr}}{f}\right)_{ss} - 2\left(f^2\right)_{sr} = 0.$$

Similarly, the invariants $s = t,\ r = y - x,\ f = u$ of $X_2 + X_3$ reduces (1) to

$$\left(\frac{f_{rr}}{f}\right)_{rr} - \left(\frac{f_{rr}}{f}\right)_{ss} - 2\left(f^2\right)_{sr} = 0.$$

The symmetry $X_1 + X_5$ has invariants $s = x,\ r = ye^{-2t},\ f = uy^{1/2}$, and these reduce (1) to

$$\frac{8r^2 f_r^2 f_{sr}}{f^3} + \frac{f_s^3}{rf^3} - \frac{2f_s^2 f_{sr}}{f^3} - \frac{4rf_r^2 f_s}{f^3} - \frac{4r^2 f_{rr} f_{sr}}{f^2} - \frac{8r^2 f_r f_{srr}}{f^2} + \frac{2f_r f_s}{f^2} + \frac{2rf_{rr} f_s}{f^2} - \frac{3f_{ss} f_s}{2rf^2}$$
$$+ \frac{2f_s f_{ssr}}{f^2} - \frac{8rf_r f_{sr}}{f^2} + \frac{f_{ss} f_{sr}}{f^2} + \frac{2f_{sr}}{f} + \frac{10rf_{srr}}{f} + \frac{4r^2 f_{srrr}}{f} + \frac{f_{sss}}{2rf} - \frac{f_{sssr}}{f} - 4\left(f^2\right)_{sr} = 0.$$

The invariants $s = t,\ r = ye^{-2x},\ f = uy^{1/2}$ of $X_2 + X_5$ transform (1) to

$$\frac{16r^3 f_r^2 f_{rr}}{f^3} + \frac{8r^2 f_r^3}{f^3} - \frac{4r f_{rr} f_s^2}{f^3} - \frac{2 f_r f_s^2}{f^3} - \frac{8r^3 f_{rr}^2}{f^2} - \frac{16r^3 f_r f_{rrr}}{f^2} - \frac{52r^2 f_r f_{rr}}{f^2} + \frac{4r f_s f_{srr}}{f^2} - \frac{12 r f_r^2}{f^2} + \frac{2r f_{rr} f_{ss}}{f^2} + \frac{2 f_s f_{sr}}{f^2} + \frac{f_r f_{ss}}{f^2} - \frac{2r f_{ssrr}}{f} + \frac{44 r^2 f_{rrr}}{f} + \frac{44 r f_{rr}}{f} + \frac{8r^3 f_{rrrr}}{f} + \frac{4 f_r}{f} - \frac{f_{ssr}}{f} - 8 f_{sr} - 8 f_r f_s = 0.$$

Using the invariants $s = x/t,\ r = y/t,\ f = tu$ of $X_4 + X_5$, (1) reduces to

$$\frac{2r^2 f_r^2 f_{sr}}{f^3} - \frac{r^2 f_{rr} f_{sr}}{f^2} - \frac{2r^2 f_r f_{srr}}{f^2} + \frac{r^2 f_{srrr}}{f} + \frac{2s^2 f_s^2 f_{sr}}{f^3} - \frac{s^2 f_{ss} f_{sr}}{f^2} - \frac{2s^2 f_s f_{ssr}}{f^2} + \frac{s^2 f_{sssr}}{f} - \frac{2rs f_{sr}^2}{f^2} - 4r f f_{sr} - \frac{6r f_r f_{sr}}{f^2} + \frac{4rs f_r f_s f_{sr}}{f^3} - \frac{2rs f_s f_{srr}}{f^2} + \frac{6r f_{srr}}{f} - \frac{2rs f_r f_{ssr}}{f^2} + \frac{2rs f_{ssrr}}{f} - 12 f_s f - \frac{6s f_s f_{sr}}{f^2} + \frac{6 f_{sr}}{f} - 4s f_{ss} f + \frac{6s f_{ssr}}{f} - 4r f_r f_s - 4s f_s^2 + \left(\frac{f_{sr}}{f}\right)_{ss} = 0.$$

The operator $X_1 + X_2 + X_3$ has invariants $s = x - t,\ r = y - t,\ f = u$, and with the use of these invariants, (1) reduces to

$$\left(\frac{(f_s + f_r)_r}{f}\right)_{rr} - \left(\frac{(f_s + f_r)_r}{f}\right)_{ss} + 2\left(f^2\right)_{sr} = 0.$$

Finally, $X_1 + X_2 + X_5$ has invariants $s = x - t,\ r = ye^{-2t},\ f = ue^t$, and their use reduces (1) to

$$\frac{8r^2 f_r^2 f_{sr}}{f^3} + \frac{8r f_r f_s f_{sr}}{f^3} - \frac{4r^2 f_{rr} f_{sr}}{f^2} - \frac{8r^2 f_r f_{srr}}{f^2} - \frac{4r f_{sr}^2}{f^2} - \frac{12 r f_r f_{sr}}{f^2} - \frac{4r f_s f_{srr}}{f^2} - \frac{4r f_r f_{ssr}}{f^2} - \frac{4 f_s f_{sr}}{f^2} + \frac{4r f_{ssrr}}{f} + \frac{12 r f_{srr}}{f} + \frac{4 f_{sr}}{f} + \frac{4 f_{ssr}}{f} + \frac{4r^2 f_{srrr}}{f} - 8 f_s f - 8r f_r f_s - 8r f f_{sr} - 4 f_s^2 - 4 f_{ss} f = 0.$$

### *2.4. Group-Invariant Solutions*

We now obtain group-invariant solutions based on the optimal system of one-dimensional subalgebras. However, in this paper we are looking only at some interesting cases.

**Case 1.** $X_5 = 2y\partial/\partial y - u\partial/\partial u$

The associated Lagrange system to the operator $X_4$ yields three invariants

$$s = t, \quad r = x, \quad u = y^{-1/2} U(r, s),$$

which give group-invariant solution $u = y^{-1/2} U(s, r)$ and transforms (1) to

$$\left(\frac{U_r}{U}\right)_{ss} - \left(\frac{U_r}{U}\right)_{rr} - 4\left(U^2\right)_{rs} = 0. \tag{2}$$

This equation has three Lie point symmetries, viz.,

$$\Gamma_1 = \frac{\partial}{\partial s},\ \Gamma_2 = \frac{\partial}{\partial r},\ \Gamma_3 = 2s\frac{\partial}{\partial s} + 2r\frac{\partial}{\partial r} - U\frac{\partial}{\partial U}.$$

The symmetry $\Gamma_1 - \nu\Gamma_2$ gives the two invariants $z = r + \nu s$ and $F = U$. Using these invariants, (2) transforms to the nonlinear third-order ordinary differential equation

$$\left(\frac{F'}{F}\right)'' + \frac{4\nu}{1-\nu^2}\left(F^2\right)'' = 0. \tag{3}$$

Integrating (3) twice with respect to $z$, we obtain

$$F'(z) + \frac{4\nu}{1-\nu^2}(F(z))^3 - k_1 z F(z) - k_2 F(z) = 0, \tag{4}$$

where $k_1$ and $k_2$ are constants of integration. The solutions of this equation are given by

$$F(z) = \pm\sqrt{\frac{\sqrt{k_1}(1-\nu^2)\exp\{(k_1 z + k_2)^2/k_1\}}{k_3\sqrt{k_1}(1-\nu^2)\exp\{k_2^2/k_1\} + 4\nu\sqrt{\pi}\,\mathrm{erfi}\left((k_1 z + k_2)/\sqrt{k_1}\right)}},$$

where $k_3$ is a constant of integration and $\mathrm{erfi}(z)$ is the imaginary error function [34]. Thus, solutions of (1) are

$$u(t,x,y) = \pm y^{-1/2}\sqrt{\frac{\sqrt{k_1}(1-\nu^2)\exp\{(k_1(x+\nu t) + k_2)^2/k_1\}}{k_3\sqrt{k_1}(1-\nu^2)\exp\{k_2^2/k_1\} + 4\nu\sqrt{\pi}\,\mathrm{erfi}\left((k_1(x+\nu t) + k_2)/\sqrt{k_1}\right)}}.$$

**Case 2.** $X_1 + X_5 = \partial/\partial t + 2y\partial/\partial y - u\partial/\partial u$

The associated Lagrange system to this operator yields the three invariants

$$s = x, \quad r = ye^{-2t}, \quad u = e^{-t}U,$$

which give group-invariant solution $u = e^{-t}U(s,r)$ and transforms (1) to

$$\begin{aligned} &U\left(U_{sr}\left(4r^2U_{rr} - U_{ss}\right) + 4rU_r\left(3U_{sr} + 2rU_{srr}\right) - 2U_sU_{ssr}\right) + 8U^4\left(rU_{sr} + U_s\right) + 8rU_rU_sU^3 \\ &+U^2\left(U_{sssr} - 4\left(U_{sr} + r\left(3U_{srr} + rU_{srrr}\right)\right)\right) + 2\left(U_s^2 - 4r^2U_r^2\right)U_{sr} = 0. \end{aligned} \tag{5}$$

The Lie point symmetries of the above equation are

$$\Gamma_1 = \frac{\partial}{\partial s}, \ \Gamma_2 = 2r\frac{\partial}{\partial r} - U\frac{\partial}{\partial U}.$$

The symmetry $\Gamma_2$ gives the two invariants $z = s$ and $U = r^{-1/2}F$, and using these invariants, (2) transforms to the nonlinear third-order ordinary differential equation

$$\left(\frac{F'}{F}\right)'' = 0. \tag{6}$$

Integrating (6) twice with respect to $z$, we obtain

$$F'(z) = k_1 z F(z) + k_2 F(z), \tag{7}$$

where $k_1$ and $k_2$ are constants of integration. The solution of this equation is given by

$$F(z) = k_3 \exp\left(\frac{k_1}{2}z^2 + k_2 z\right),$$

where $k_3$ is a constant of integration. Thus, a solution of (1) is

$$u(t,x,y) = k_3 y^{-1/2} \exp\left(\frac{k_1}{2}x^2 + k_2 x\right),$$

which is a steady-state solution.

**Case 3.** $X_1 + X_2 + X_3$

The associated Lagrange system to this symmetry operator gives three invariants, viz.,

$$s = x - t, \quad r = y - t, \quad U = u,$$

which give group-invariant solution $u = U(s,r)$ and reduces (1) to

$$\begin{aligned} &U^2 \left(U_{srrr} + 2U_{ssrr}\right) - 4U^4 \left(U_{sr} + U_{ss}\right) - 4U_s U^3 \left(U_r + U_s\right) + 2U_r \left(U_r + 2U_s\right) U_{sr} \\ &-U \left(U_{rr}U_{sr} + 2\left({U_{sr}}^2 + U_s U_{srr} + U_r \left(U_{srr} + U_{ssr}\right)\right)\right) = 0. \end{aligned} \tag{8}$$

The Lie point symmetries of the above equation are

$$\Gamma_1 = \frac{\partial}{\partial s}, \ \Gamma_2 = \frac{\partial}{\partial r}, \ \Gamma_3 = s\frac{\partial}{\partial s} + r\frac{\partial}{\partial r} - U\frac{\partial}{\partial U}.$$

The symmetry $\Gamma_1 - \nu\Gamma_2$ gives the two invariants $z = r + \nu s$ and $F = U$. Using these invariants, (8) transforms to the nonlinear fourth-order ordinary differential equation

$$\left(\frac{F''}{F}\right)'' - \frac{2(\nu+1)}{2\nu+1}\left(F^2\right)'' = 0. \tag{9}$$

Integrating (9) twice with respect to $z$, we obtain

$$F'' - \frac{2(\nu+1)}{2\nu+1}F^3 - k_1 z F - k_2 F = 0, \tag{10}$$

where $k_1$ and $k_2$ are constants of integration. This equation can not be integrated in the closed form. However, by taking $k_1 = 0$, one can obtain its solution in the closed form in the following manner. Multiplying (10) with $k_1 = 0$ by $F'$ and integrating, we obtain

$$F'^2 = \frac{\nu+1}{2\nu+1}F^4 + k_2F^2 + k_3, \tag{11}$$

where $k_3$ is a constant of integration. The solution of this equation is given by

$$F(z) = \sqrt{\frac{2k_3(2\nu+1)}{C}}\,\mathrm{sn}\left(\sqrt{\frac{C}{2(2\nu+1)}}\,z + k_4, 2\sqrt{\frac{-k_3(\nu+1)}{Ck_2 + 4k_3 + 4k_3\nu}}\right),$$

where $k_4$ is a constant of integration, $C = \sqrt{4k_2^2\nu^2 + 4k_2^2\nu + k_2^2 - 16k_3\nu^2 - 24k_3\nu - 8k_3} - 2k_2\nu - k_2 \neq 0$ and sn is the Jacobi elliptic sine function [35]. Thus, a solution of (1) is

$$u(t,x,y) = \sqrt{\frac{2k_3(2\nu+1)}{C}}\,\mathrm{sn}\left(\sqrt{\frac{C}{2(2\nu+1)}}\,(y + \nu x - (\nu+1)t) + k_4, 2\sqrt{\frac{-k_3(\nu+1)}{Ck_2 + 4k_3 + 4k_3\nu}}\right).$$

For $k_3 = 0$ we have the solution given by

$$u(t,x,y) = \frac{2k_2(2\nu+1)\exp[\sqrt{k_2}\{\pm(\nu x + y - (\nu+1)t)\}]}{2\nu+1-k_2(\nu+1)\exp[2\sqrt{k_2}\{\pm(\nu x + y - (\nu+1)t)\}]}$$

and when $C = 0$ we have

$$u(t,x,y) = \left\{\sqrt{\frac{\nu+1}{2\nu+1}}(\nu x + y - (\nu+1)t)\right\}^{-1}.$$

Likewise, one may obtain more group-invariant solutions using the other symmetry operators of the optimal system of one-dimensional subalgebras. For example, the symmetry operator $X_2 + X_3$ of the optimal system gives us the group-invariant solution (2.9) of [19] in terms of the Airy functions.

## 3. Conservation Laws of (1)

Conservation laws describe physical conserved quantities, such as mass, energy, momentum and angular momentum, electric charge, and other constants of motion [32]. They are very important in the study of differential equations. Conservation laws can be used in investigating the existence, uniqueness, and stability of the solutions of nonlinear partial differential equations. They have also been used in the development of numerical methods and in obtaining exact solutions for some partial differential equations.

A local conservation law for the (2 + 1)-dimensional Zoomeron Equation (1) is a continuity equation

$$D_t T + D_x X + D_y Y = 0 \tag{12}$$

holding for all solutions of Equation (1), where the conserved density $T$ and the spatial fluxes $X$ and $Y$ are functions of $t, x, y, u$. The results in [11] show that all non-trivial conservation laws arise from multipliers. Specifically, when we move off of the set of solutions of Equation (1), every non-trivial local conservation law (12) is equivalent to one that can be expressed in the characteristic form

$$D_t\tilde{T} + D_x\tilde{X} + D_y\tilde{Y} = \left(\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2(u^2)_{tx}\right)Q \tag{13}$$

holding off of the set of solutions of Equation (1) where $Q(x,y,t,u\ldots)$ is the multiplier, and where $(\tilde{T},\tilde{X},\tilde{Y})$ differs from $(T,X,Y)$ by a trivial conserved current. On the set of solutions $u(x,y,t)$ of Equation (1), the characteristic form (13) reduces to the conservation law (12).

In general, a function $Q(x,t,u\ldots)$ is a multiplier if it is non-singular on the set of solutions $u(x,y,t)$ of Equation (1), and if its product with Equation (1) is a divergence expression with respect to $t,x,y$. There is a one-to-one correspondence between non-trivial multipliers and non-trivial conservation laws in characteristic form.

The determining equation to obtain all multipliers is

$$\frac{\delta}{\delta u}\left(\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2(u^2)_{tx}\right)Q = 0, \tag{14}$$

where $\delta/\delta u$ is the Euler–Lagrange operator given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u_{i_1 i_2\cdots i_s}}.$$

Equation (14) must hold off of the set of solutions of Equation (1). Once the multipliers are found, the corresponding non-trivial conservation laws are obtained by integrating the characteristic Equation (13) [11].

We will now find all multipliers $Q(x,y,t,u)$ and obtain corresponding non-trivial (new) conservation laws. The determining Equation (14) splits with respect to the variables $u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xy}, u_{yy}, u_{ttt}, u_{ttx}, u_{txy}, u_{tyy}, u_{xyy}, u_{tttx}, u_{txyy}$. This yields a linear determining system for $Q(x,y,t,u)$ which can be solved by the same algorithmic method used to solve the determining equation for infinitesimal symmetries. By applying this method, for Equation (1), we obtain the following linear determining equations for the multipliers:

$$Q_u\left(t,x,y,u\right) = 0, \tag{15}$$

$$Q_{ty}\left(t,x,y,u\right) = 0, \tag{16}$$

$$Q_{yyy}\left(t,x,y,u\right) = 0, \tag{17}$$

$$Q_{tt}\left(t,x,y,u\right) - Q_{yy}\left(t,x,y,u\right) = 0. \tag{18}$$

It is straightforward using Maple to set up and solve this determining system (15)–(18), and we get the four multipliers given by

$$Q_1 = \frac{1}{2}\left(t^2 + y^2\right) f_1(x), \tag{19}$$

$$Q_2 = f_2(x)y, \tag{20}$$

$$Q_3 = f_3(x)t, \tag{21}$$

$$Q_4 = f_4(x). \tag{22}$$

For each solution $Q$, a corresponding conserved density and flux can be derived (up to local equivalence) by integration of the divergence identity (13) [11,36]. We obtain the following results.

Corresponding to these multipliers, we obtain four conservation laws. Thus, the multiplier (19) gives the conservation law with the following conserved vector:

$$\begin{aligned}
T_1 &= f_1(x)\left\{\frac{1}{2}(t^2+y^2)\left(\frac{{u_t}^2 u_x}{u^3} - \frac{u_x u_{tt}}{u^2}\right) + \frac{t u_x u_t}{u^2} - 2yu^2\right\} \\
&\quad + f_1'(x)\left\{\frac{1}{2}(t^2+y^2)\left(\frac{u_{tt}}{u} - \frac{1}{2}\frac{{u_t}^2}{u^2}\right) - \frac{t u_t}{u}\right\}, \\
X_1 &= f_1(x)\left\{\frac{1}{2}(t^2+y^2)\left(\frac{2u_t u_{tt}}{u^2} - \frac{u_{ttt}}{u} - \frac{{u_t}^3}{u^3}\right) - \frac{1}{2}\frac{{u_t}^2 t}{u^2} + \frac{u_t}{u}\right\}, \\
Y_1 &= f_1(x)\left\{\frac{1}{2}(t^2+y^2)\left(4uu_t + \frac{u_{txy}}{u} - \frac{u_y u_{tx}}{u^2}\right) - \frac{y u_{tx}}{u}\right\}.
\end{aligned}$$

Likewise, the multiplier (20) yields

$$\begin{aligned}
T_2 &= f_2(x)y\left(4\,uu_y - \frac{u_x u_{tt}}{u^2} + \frac{{u_t}^2 u_x}{u^3}\right) + f_2'(x)y\left(\frac{u_{tt}}{u} - \frac{1}{2}\frac{{u_t}^2}{u^2}\right), \\
X_2 &= f_2(x)y\left(\frac{2u_t u_{tt}}{u^2} - \frac{u_{ttt}}{u} - \frac{{u_t}^3}{u^3}\right), \\
Y_2 &= f_2(x)\left(\frac{y u_{txy}}{u} - \frac{y u_y u_{tx}}{u^2} - \frac{u_{tx}}{u}\right)
\end{aligned}$$

as conserved vector.

Similarly, the multiplier (21) results in the following conserved vector

$$
\begin{aligned}
T_3 &= f_3(x)\left(4\,tuu_y - \frac{tu_x u_{tt}}{u^2} + \frac{tu_t{}^2 u_x}{u^3} + \frac{u_x u_t}{u^2}\right) + f_3'(x)\left(\frac{tu_{tt}}{u} - \frac{1}{2}\frac{tu_t{}^2}{u^2} - \frac{u_t}{u}\right),\\
X_3 &= f_3(x)\left(\frac{2tu_t u_{tt}}{u^2} - \frac{tu_t{}^3}{u^3} - \frac{1}{2}\frac{u_t{}^2}{u^2} - \frac{tu_{ttt}}{u}\right),\\
Y_3 &= f_3(x)\frac{(tuu_{txy} - 2\,u^4 - tu_y u_{tx})}{u^2}.
\end{aligned}
$$

Lastly, the multiplier (22) gives the conserved vector whose components are

$$
\begin{aligned}
T_4 &= f_4(x)\left(4uu_y - \frac{u_x u_{tt}}{u^2} + \frac{u_t{}^2 u_x}{u^3}\right) + f_4'(x)\left(\frac{u_{tt}}{u} - \frac{1}{2}\frac{u_t{}^2}{u^2}\right),\\
X_4 &= f_4(x)\left(\frac{2u_t u_{tt}}{u^2} - \frac{u_{ttt}}{u} - \frac{u_t{}^3}{u^3}\right),\\
Y_4 &= f_4(x)\left(\frac{u_{txy}}{u} - \frac{u_y u_{tx}}{u^2}\right).
\end{aligned}
$$

## 4. Concluding Remarks

In this paper, we studied the (2 + 1)-dimensional Zoomeron Equation (1). For the first time, the classical Lie point symmetries were used to construct an optimal system of one-dimensional subalgebras. This system was then used to obtain symmetry reductions and new group-invariant solutions of (1). Again for the first time, we derived the conservation laws for (1) by employing the multiplier method. We note that since we had arbitrary functions in the multipliers, we obtained infinitely many conservation laws for Equation (1).

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