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Reference-Dependent Aggregation in Multi-Attribute Group Decision-Making

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Abstract: To characterize the influence of decision makers' psychological factors on the group decision process, this paper develops a new class of aggregation operators based on reference-dependent utility functions (RUs) in multi-attribute group decision analysis. We consider two types of RUs: *S*-shaped, representing decision makers who are risk-seeking for relative losses, and non-*S*-shaped, representing those that are risk-averse for relative losses. Based on these RUs, we establish two new classes of reference-dependent aggregation operators; we study their properties and show that their generality covers a number of existing aggregation operators. To determine the optimal weights for these aggregation operators, we construct an attribute deviation weight model and a decision maker (DM) deviation weight model. Furthermore, we develop a new multi-attribute group decision-making (MAGDM) approach based on these RU aggregation operators and weight models. Finally, numerical examples are given to illustrate the application of the approach.

Keywords: group decision-making; decision analysis; aggregation operator; reference-dependent; utility theory

1. Introduction

As an important part of modern decision-making science, multi-attribute group decision-making (MAGDM) has been applied with enormous success to various fields such as strategic planning, portfolio selection, medical diagnosis, and military system evaluation [1,2]. To better understand the procedure for solving MAGDM problems, we develop a general framework for the MAGDM aggregation procedure (see Figure 1) which contains two stages: (1) individual aggregation, which is a multi-attribute decision-making process for each decision maker (DM) composed by multiple attributes and multiple alternatives; and (2) group aggregation, which is a decision-making process composed of multiple experts and multiple alternatives. In both stages, developing efficient aggregation methods and determining the optimal weights for aggregation operators are two key steps.

Methods developed for aggregating information can be classified into three types. The first is the weighted-average method, which aggregates the information by using different importance degrees of the arguments [3,4]. The second is the probabilistic aggregation method, which unifies the ordered weighted averaging (OWA) operator and the corresponding probabilities by incorporating the importance degree of each case in the aggregation process [5,6]. The third is the deviation aggregation method, which minimizes the deviation between the aggregation result and evaluation information characterized by distance metrics or penalty functions [7,8].

Reference-dependent utility function to characterize psychological factors. There is a consensus that the psychological factors of the DM, such as reference wealth [9], cognitive elements [10] and the behavior towards risk [11], play important roles in decision analysis. Nevertheless, the aforementioned aggregation methods fail to capture the psychological character of DMs in the aggregation process. In this paper, we attempt to partially fill this gap by modeling psychological factors via reference-dependent utility functions (RUs). The most famous RU is the value function of Prospect theory [12,13], which involves a basic utility function, loss aversion coefficient and a reference outcome. This is the fundamental framework of RUs.

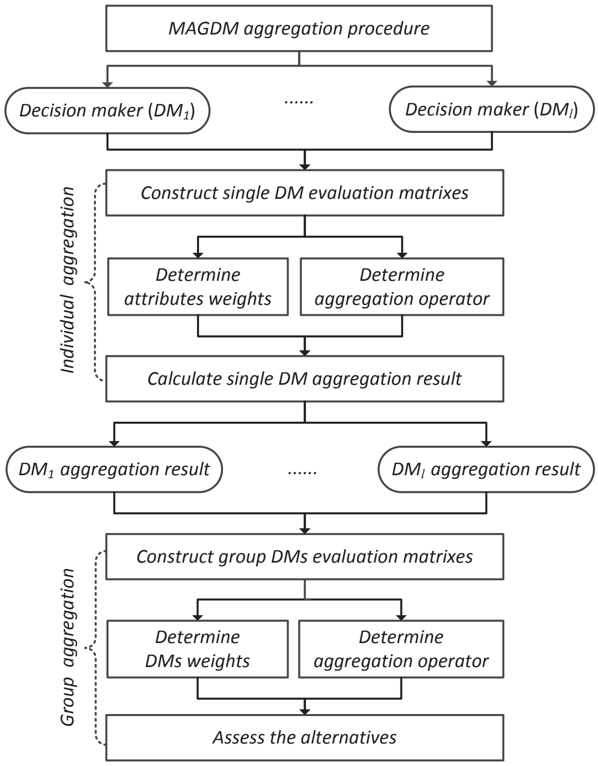

Figure 1. A general framework for the multi-attribute group decision-making (MAGDM) aggregation procedure. DM: decision maker.

We build the multi-attribute aggregation operators upon two types of RUs: an *S*-shaped RU [9,14] and a non-*S*-shaped RU [11,15]. The *S*-shaped RU describes the risk attitude of a DM who is risk-averse for relative gains (with a concave function above the reference point) and risk-seeking for relative losses (with a convex function below the reference point), as can be seen in Figure 2. On the other hand, the non-*S*-shaped RU maintains concavity regardless of the value of outcomes (either above or below the reference point), indicating risk-aversion for both gains and losses (see Figure 3). How to choose an RU depends on the DM's psychological standpoint: if the DM views relative losses as distorted positive (negative) outcomes, then his/her attitude tends to be risk-averse (risk-seeking) [16]. Although RUs have been widely applied to behavioral models of decision-making in economics and finance [17,18], to the best of our knowledge, there is little research of RU-based MAGDM aggregation methods.

Weight models of attributes and decision makers. Another crucial step in the application of aggregation operators to MAGDM is to determine the associated weights for both attributes and DMs (see, Figure 1). Relevant methods include the minimum variance method [19], minimum dispersion method [20], minimum chi-square method [21], minimum disparity method [22], and maximum Bayesian entropy method [23]. An unresolved issue in the aforementioned methods is how to factor the influence of input arguments in the process of determining weights. In practice, an attribute with similar attribute values across most alternatives is deemed less important, so it should be assigned a smaller weight; on the other hand, an attribute with values fluctuating across alternatives is considered more important, it then should be assigned a larger weight.

The weights of DMs also play an important role in the aggregation process. Many researchers directly apply attribute weight models to compute the weights of DMs, such as the minimum variance method [19], minimum chi-square method [21], minimizing distances from the extreme points (MDP) method [24], voting method [25] and improved minimax disparity method [26]. However, the common disadvantage of the above approaches is that the high subjectivity of DMs may cause inaccuracy, sometimes leading to biased decision results.

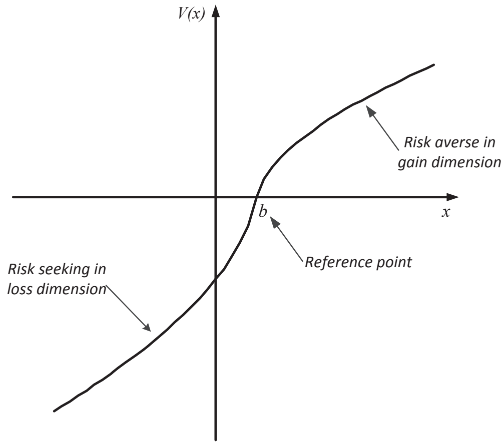

Figure 2. An S-Shaped Reference-Dependent Utility Function.

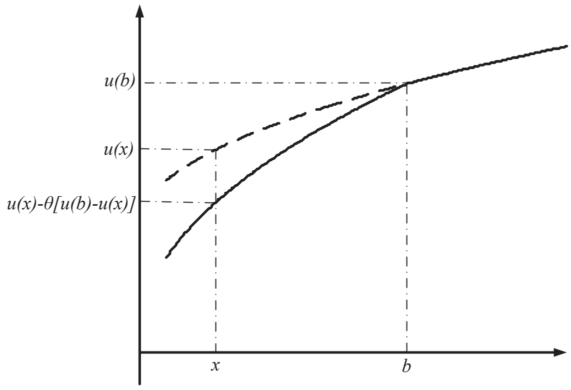

Figure 3. A non-S-shaped reference-dependent utility function (here $u(x) = \sqrt{x} + 1$ for $x \geq 0$ and $\theta = 1$).

We aim to resolve the above issues by developing two new optimization weight models. On the one hand, our model considers the impact of the attribute variation across the alternatives on choosing the optimal alternative; we will assign a relatively larger (smaller) weight to an attribute having a larger (smaller) variation across the alternatives. On the other hand, we try to achieve a fairly small deviation among the weights to maintain fairness.

Our contributions. We summarize our contributions in three directions.

(1) To investigate the impact of DMs psychological factors on the decision-making result, we propose for the first time two new operators based on RUs: the *S*-shaped and non-*S*-shaped operators. The DM can choose different RU operators to get the result according to his/her attitude toward the relative losses. To be specific, if the attitude of the DM is risk-seeking for relative losses, he/she can use the *S*-shaped operators (see Equation (11)) to select the optimal alternative. If the attitude of the DM is risk-averse for relative losses, he/she can apply the non-*S*-shaped operators (see, Equation (16)) in the decision-making process. If the attitude of a DM is risk-neutral, he/she can make decisions via the generalized ordered weighted multiple averaging (GOWMA) operator (see, Equation (18)) which is degenerated by the non-*S*-shaped operator. The main

advantage of the RU operators is that they not only reflect the psychological character of the DM while the aforementioned aggregation methods fail to capture in the aggregation process, but also generate a family of aggregation operators by taking different parameters. Specifically, the RU operators can degenerate to the existing aggregation operators (see Tables A1–A3 in Appendix A.3), which can be seen as the particular case of the RU operators.

(2) To determine the associated weights for the multiple attributes and DMs, we propose an attribute-deviation weight model and a DMs-deviation weight model (see, models (19) and (20)). Going beyond the framework of existing weight models which ignored the dependence on the attribute variation (deviation), our new weight models consider the variations impacts of the attribute values on the determination of the weight in aggregation process. In addition, the attributes weights and the DMs weights are calculated by using attribute-deviation and DM-deviation models respectively, while the most research uses the same model to determine the associated weights for both attributes and DMs, sometimes leading to biased decision results.

(3) We develop a new approach for MAGDM based on the RU operators and the weight models. In addition, the numerical examples are given to illustrate the application of the approach. Two novel findings emerge from the numerical analysis in Section 6. First, the optimal alternative will change to a relatively prudent alternative with the absolute risk aversion coefficient increasing. Second, the optimal alternative changes to a relatively risky one with the reference point (or, loss aversion coefficient) increasing.

The rest of the paper is organized as follows. Under the general frameworks of S-shaped RU, Section 2 derives the S-shaped operators for the DM whose attitude is risk-seeking for relative losses. As for the specific S-shaped RU, we focus on prospect value function and S-shaped hyperbolic absolute risk aversion function respectively and develop their corresponding aggregation operators. Under the general frameworks of non-S-shaped RU, Section 3 proposes the non-S-shaped operators for the DM whose attitude is risk-averse for relative losses. As for the specific non-S-shaped RU, we focus on non-S-shaped hyperbolic absolute risk aversion function and develop its corresponding aggregation operators. In Section 4, we construct two new nonlinear optimization weight models by applying the deviation measure method. The approach is summarized through six steps in Section 5 and its superiority is tested via numerical examples in Section 6. In the end, the conclusions are drawn in Section 7, and all proofs are given in the E-companion.

2. Aggregation Operators for Risk-Seeking DMs Regarding Relative Losses

2.1. General Framework

According to Prospect theory, the risk attitude of DMs for relative gains (outcomes above the reference point) is risk-averse whereas it tends to be risk-seeking for relative losses (outcomes below the reference point) [10,12,13]. Consequently, the utility function capturing DMs psychological effects is S-shaped; it is concave (convex) when the outcome is above (below) the reference point, as shown in Figure 2 (the curve is derived from utility Function (10) with $\beta = \beta_1 = 7$, $\eta = \eta_1 = 2$, $\gamma = \gamma_1 = 0.55$ and $\theta_1 = 1.5$). In this section, we study S-shaped RUs and develop their corresponding aggregation operators. In general, an S-shaped RU is given by:

$$v(x) = \begin{cases} v_1(x), & x \geq b \\ -v_2(x), & x < b \end{cases} \tag{1}$$

where $v_1(x) > 0$ and $v_2(x) > 0$. Let v_1 be the basic utility function, that is, $v(x) = v_1(x)$ if there is no reference point. The utility $v(x)$ satisfies the following conditions:

(1) v is strictly increasing;
(2) v is convex for $x < b$ and concave for $x > b$;
(3) v is asymmetry for $x > b$: $v(x - b) < -v(b - x)$;

(4) $v'(x-b) < v'(b-x)$ for all $x > b$, and $v(b) = 0$.

In the decision-making process, attribute values below (above) the reference point can be viewed as relative losses (gains). We next specify our S-shaped RUs. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be a collection of input arguments and $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ be a weight vector satisfying:

$$\sum_{i=1}^{n} w_i = 1 \quad \text{and} \quad w_i > 0 \quad \text{for} \quad 1 \leq i \leq n. \tag{2}$$

An n-dimensional S-shaped RU aggregation operator mapping f is determined by:

$$v_1(z) = f(v(x_1), v(x_2), \cdots, v(x_n)), \tag{3}$$

where $v(x)$ is an S-shaped RU and z is the aggregation result. Since the aggregation result z has no reference point, we use the basic utility function v_1 (rather than piecewise utility function v) to denote the utility of the aggregation result z. In order to obtain effective aggregation results, we aim to achieve the least variation in $v_1(z)$ across all outcomes, that is, we intend to minimize the weighted sum of deviations between the aggregation result $v_1(z)$ and the value of each input $v(x_i)$:

$$\min \quad \sum_{i=1}^{n} w_i d(v_1(z), v(x_i)) \tag{4}$$

where $d(v_1(z), v(x_i))$ is a deviation metric measuring the deviation between $v(x_i)$ and $v_1(z)$.

Based on the penalty function theory, there are three deviation measures [8,27], i.e.,

$$\begin{aligned} d_1(v_1(z), v(x_i)) &= \left(\frac{v(x_i)}{v_1(z)}\right)^{\lambda} + \left(\frac{v_1(z)}{v(x_i)}\right)^{\lambda} - 2, \\ d_2(v_1(z), v(x_i)) &= \left(1 - \left(\frac{v(x_i)}{v_1(z)}\right)^{\lambda}\right)^2, \\ d_3(v_1(z), v(x_i)) &= (v_1^{\lambda}(z) - v^{\lambda}(x_i))^2, \end{aligned}$$

where λ is a parameter satisfying $\lambda \in (-\infty, 0) \bigcup (0, +\infty)$. First-order condition (matching the derivative with respect to z to 0) of the objective function in model (4) yields the following three operators: (1) multiple-reference-dependent aggregation (MR) operator; (2) proportional-reference-dependent aggregation (PR) operator; and (3) subtract-reference-dependent aggregation (SR) operator. See Table 1 for the detailed formulas of these three operators.

Table 1. Reference-dependent aggregation operators based on three deviation metrics. MR: multiple-reference-dependent aggregation; PR: proportional-reference-dependent aggregation; SR: subtract-reference-dependent aggregation.

Operator	$d(v_1(z), v(x_i))$	z
MR	$\frac{v^{\lambda}(x_i)}{v_1^{\lambda}(z)} + \frac{v_1^{\lambda}(z)}{v^{\lambda}(x_i)} - 2$	$v_1^{-1}\left(\left(\sum_{i=1}^{n} w_i v^{\lambda}(x_i) \Big/ \sum_{i=1}^{n} \left(w_i / v^{\lambda}(x_i)\right)\right)^{1/2\lambda}\right)$
PR	$\left(1 - \frac{v^{\lambda}(x_i)}{v_1^{\lambda}(z)}\right)^2$	$v_1^{-1}\left(\left(\sum_{i=1}^{n} w_i v^{2\lambda}(x_i) \Big/ \sum_{i=1}^{n} w_i v^{\lambda}(x_i)\right)^{1/\lambda}\right)$
SR	$(v_1^{\lambda}(z) - v^{\lambda}(x_i))^2$	$v_1^{-1}\left(\left(\sum_{i=1}^{n} w_i v^{\lambda}(x_i)\right)^{1/\lambda}\right)$

Without loss of generality, we focus on the MR operator throughout the rest of this paper; the other two operators in Table 1 can be derived by the similar method. For a given vector $\mathbf{x}$, let $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ be the ordered version of $\mathbf{x}$, that is, y_i is the ith largest of the argument of $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. In the following we propose a new S-shaped ordered multiple reference-dependent aggregation operator.

Definition 1. *An n-dimensional S-shaped ordered multiple reference-dependent aggregation (SOMR) operator is a mapping of SOMR: $R^{+n} \to R^{+}$ defined below:*

$$\mathrm{SOMR}(\mathbf{x}) = v_1^{-1}\left[\left(\left(\sum_{i\in Y_1} w_i v_1^{\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{\lambda}(y_i)\right) \Big/ \left(\sum_{i\in Y_1} w_i v_1^{-\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{-\lambda}(y_i)\right)\right)^{1/2\lambda}\right], \tag{5}$$

where the two sets are:

$$Y_1 = \{1 \le i \le n, y_i \ge b_i\}, \quad Y_2 = \{1 \le i \le n, y_i < b_i\}, \tag{6}$$

and b_i is a reference point of y_i (i.e., the reference point of the ith largest x_i), the weight vector $\mathbf{w}$ satisfies Expression (2), and $\lambda > 0$ is an odd number which ensures that the inverse function is well-defined (i.e., uniquely determined).

The following proposition shows that the SOMR operator is monotonic, bounded, commutative, idempotent, thus satisfying common properties of aggregation operators [6]. The proofs of these properties are given in Appendix A.1.

Proposition 1 (Properties of SOMR). *The SOMR operator given in Definition 1 satisfies the following properties:*

(1) (Monotonicity) For two vectors $\mathbf{x}$ and $\bar{\mathbf{x}}$ with $x_i \ge \bar{x}_i$ and the same reference points, then $\mathrm{SOMR}(\mathbf{x}) \ge \mathrm{SOMR}(\bar{\mathbf{x}})$.
(2) (Boundedness) If $b_1 \le y_1 = \max_i \{x_i\}$ and $b_n > y_n = \min_i \{x_i\}$, then $v_1^{-1}(v_2(y_n)) \le \mathrm{SOMR}(\mathbf{x}) \le y_1$.
(3) (Commutativity) If $\widehat{\mathbf{x}}$ is a permutation of $\mathbf{x}$, then $\mathrm{SOMR}(\mathbf{x}) = \mathrm{SOMR}(\widehat{\mathbf{x}})$.
(4) (Idempotency) If $x_i = x \ge x_0$ for all $1 \le i \le n$, then $\mathrm{SOMR}(\mathbf{x}) = x$.

2.2. Prospect Reference-Dependent Aggregation Operator

A commonly used RU in decision analysis is the prospect value function [10], which has the following form:

$$v(x) = \begin{cases} v_1(x) = (x-b)^{\alpha_0}, & x \ge b \\ -v_2(x) = -\theta_0 (b-x)^{\beta_0}, & x < b \end{cases} \tag{7}$$

where $0 < \alpha_0, \beta_0 < 1$, and $\theta_0 > 1$ is the loss aversion coefficient. Introducing prospect value Function (7) in Equation (5), we derive the following prospect ordered multiple reference-dependent aggregation operator.

Definition 2. *An n-dimensional prospect ordered multiple reference-dependent aggregation (POMR) operator is a mapping of POMR: $R^{+n} \to R^{+}$*

$$\mathrm{POMR}(\mathbf{x}) = \left(\left(\sum_{i\in Y_1} w_i v_1^{\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{\lambda}(y_i)\right) \Big/ \left(\sum_{i\in Y_1} w_i v_1^{-\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{-\lambda}(y_i)\right)\right)^{1/2\lambda\alpha_0}, \tag{8}$$

where $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ is the ordered version of $\mathbf{x}$, $v_1(y_i)$ and $v_2(y_i)$ are given in prospect value Function (7), and Y_1 and Y_2 are given by Set (6).

Taking special values of the parameters λ, α_0, β_0, θ_0 and b_i, the POMR operator degenerates to many different aggregation operators including ordered multiple reference-dependent (OMR) operator, GOWMA operator [8], prospect ordered multiple geometric (POMG) operator, ordered weighted multiple averaging (OWMA) operator [8], etc. (see, Table A1 in Appendix A.3).

We note that the power-utility structure of prospect value function is a function with a constant relative risk aversion (CRRA). However, empirical studies suggest that risk aversion among individuals is not an exact constant, which is referred to as constant absolute risk aversion (CARA) [28]. This motivates us to further consider a more general SOMR operator (extending the power-utility structure).

2.3. S-Shaped Hyperbolic Absolute Risk Aversion (HARA) Reference-Dependent Aggregation Operator

Grasselli [29] and Jung & Kim [30] introduced the hyperbolic absolute risk aversion utility function:

$$u(x) = \frac{1-\gamma}{\beta\gamma}\left(\frac{\beta}{1-\gamma}x+\eta\right)^{\gamma}, \tag{9}$$

where $\beta > 0$, $\eta > 0$ and $\gamma \in (-\infty, 0) \cup (0, 1)$ (for more discussions on the parameters, cf. [28,31]).

The general structure of HARA enables it to cover a rich class of operators by suitable adjustment of the parameters. For instance, special cases of HARA include the power utility function when $\beta = 1-\gamma$ and $\eta = 0$, exponential utility function when $\eta = 1$ and $\gamma \to -\infty$, and logarithm utility function when $\eta = \gamma = 0$ and $\beta = 1$. In this subsection we introduce an S-shaped HARA reference-dependent utility function and develop its corresponding aggregation operator.

We generalize the HARA utility Function (9) to the following S-shaped HARA RU:

$$v(x) = \begin{cases} v_1(x) = \dfrac{1-\gamma}{\beta\gamma}\left(\left(\dfrac{\beta}{1-\gamma}(x-b)+\eta\right)^{\gamma} - \eta^{\gamma}\right), & x \geq b \\ -v_2(x) = -\dfrac{\theta_1(1-\gamma_1)}{\beta_1\gamma_1}\left(\left(\dfrac{\beta_1}{1-\gamma_1}(b-x)+\eta_1\right)^{\gamma_1} - \eta_1^{\gamma_1}\right), & x < b \end{cases} \tag{10}$$

where $x-b$, the difference between outcome x and the reference point b, denotes relative gain (when $x \geq b$) and relative loss (when $x < b$), and $\theta_1 > 1$ is the loss aversion coefficient, $\beta, \beta_1, \eta, \eta_1 > 0$, and $\gamma, \gamma_1 \in (-\infty, 0) \cup (0, 1)$.

It is unequivocal to verify that the above S-shaped HARA RU satisfies all basic properties discussed in Section 2.1. Applying utility Function (10) to Equation (5) yields the following S-shaped HARA ordered multiple reference-dependent aggregation operator.

Definition 3. *An n-dimensional S-shaped HARA ordered multiple reference-dependent aggregation (SHOMR) operator is a mapping of SHOMR: $R^{+n} \to R^{+}$*

$$\mathrm{SHOMR}(\mathbf{x}) = \frac{1-\gamma}{\beta}\left\{\left[\frac{\beta\gamma}{1-\gamma}\left(\frac{\sum\limits_{i\in Y_1} w_i v_1^{\lambda}(y_i) - \sum\limits_{i\in Y_2} w_i v_2^{\lambda}(y_i)}{\sum\limits_{i\in Y_1} w_i v_1^{-\lambda}(y_i) - \sum\limits_{i\in Y_2} w_i v_2^{-\lambda}(y_i)}\right)^{1/2\lambda} + \eta^{\gamma}\right]^{1/\gamma} - \eta\right\}, \tag{11}$$

where $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ is the ordered version of $\mathbf{x}$, $v_1(y_i)$ and $v_2(y_i)$ are given in utility Function (10), and Y_1 and Y_2 are given by Set (6).

When $\beta = \beta_1, \gamma = \gamma_1, \eta = \eta_1$, the SHOMR operator degenerates into:

$$\mathrm{SHOMR}(\mathbf{x}) = \frac{1-\gamma}{\beta}\left\{\left[\left(\frac{\sum\limits_{i\in Y_1} w_i\mu_1^{\lambda}(y_i) - \sum\limits_{i\in Y_2} w_i\mu_2^{\lambda}(y_i)}{\sum\limits_{i\in Y_1} w_i\mu_1^{-\lambda}(y_i) - \sum\limits_{i\in Y_2} w_i\mu_2^{-\lambda}(y_i)}\right)^{1/2\lambda} + \eta^{\gamma}\right]^{1/\gamma} - \eta\right\}, \quad (12)$$

where $\mu_1(y_i) = (\beta(y_i - b_i)/(1-\gamma) + \eta)^{\gamma} - \eta^{\gamma}$ and $\mu_2(y_i) = \theta_1((\beta(b_i - y_i)/(1-\gamma) + \eta)^{\gamma} - \eta^{\gamma})$.

We remark that the family of the SHOMR operator is quite rich. Considering special values of λ, β, η, γ, θ_1 and b_i, we can show that the SHOMR operator covers a wide class of existing aggregation operators, including the *S*-shaped ordered multiple (SOM) operator, GOWMA operator [8], *S*-shaped ordered geometric (SOG) operator, ordered multiple geometric reference-dependent (OMGR) operator, OWMA operator [8], etc. See Table A2 for detailed parameters and formulations for these operators in Appendix A.3.

3. Aggregation Operators for Risk-Averse DMs Regarding Relative Losses

3.1. General Framework

Another interesting case is that a DM may have the risk-averse attitude for relative losses (rather than risk-seeking) if the DM views the relative losses as distorted positive outcomes [11]. In this case, the total utility curve (for both relative gains and losses) becomes a piecewise non-*S*-shaped RU with both pieces being concave. A non-*S*-shaped RU distinguishes from an *S*-shaped RU in two directions: (1) non-*S*-shaped RUs reflect a risk-averse attitude for relative losses whereas *S*-shaped RUs characterize a risk-seeking attitude for relative losses; (2) non-*S*-shaped RUs interpret a positive outcome below the reference point as a distorted positive gain, while *S*-shaped RUs view it as the real loss. Studies that investigate non-*S*-shaped RUs can be found in [15,16].

According to Shalev [11], utilities of a non-*S*-shaped RU below the reference point are scaled down by subtracting the loss multiplied by a loss aversion coefficient. That is,

$$v(x) = \begin{cases} u(x), & x \geq b \\ u_1(x) = u(x) - \theta(u(b) - u(x)), & x < b \end{cases} \quad (13)$$

where the function $v(x)$ is non-decreasing, b is the reference point, and $\theta \geq 0$ is the loss aversion coefficient. A larger value of θ indicates a higher degree of risk aversion.

The utility of the non-*S*-shaped RU for relative losses is concave (rather than convex for the *S*-shape RU). In the presence of reference points, utility values below the reference points represent that the gains do not meet the expectation. Hence, the utility of outcomes below the reference point is below the basic utility function, as depicted in Figure 3.

The non-*S*-shaped RU aggregation operator can be developed by using the penalty function method proposed in Section 2.

Definition 4. *An n-dimensional non-S-shaped ordered multiple reference-dependent aggregation (NOMR) operator is a mapping of NOMR: $R^{+n} \to R^{+}$*

$$\mathrm{NOMR}(\mathbf{x}) = u^{-1}\left[\left(\left(\sum_{i\in Y_1} w_i u^{\lambda}(y_i) + \sum_{i\in Y_2} w_i u_1^{\lambda}(y_i)\right)\Big/\left(\sum_{i\in Y_1} w_i u^{-\lambda}(y_i) + \sum_{i\in Y_2} w_i u_1^{-\lambda}(y_i)\right)\right)^{1/2\lambda}\right], \quad (14)$$

where $u(y_i)$ and $u_1(y_i)$ are given in utility Function (13), $\lambda \in (-\infty, 0) \cup (0, +\infty)$, $\mathbf{y}$ is the ordered sequence of $\mathbf{x}$, and Y_1 and Y_2 are given by Set (6).

Similar to SOMR, the NOMR operator satisfies desirable properties of aggregation operators, that is, it is monotonic, bounded, commutative, idempotent. See Proposition A2 in Appendix A.2.

3.2. Non-S-Shaped HARA Reference-Dependent Aggregation Operator

Introducing the HARA (9) in utility Function (13), we obtain,

$$v(x)=\begin{cases} u(x)=\dfrac{1-\gamma}{\beta\gamma}\left(\dfrac{\beta}{1-\gamma}x+\eta\right)^{\gamma}, & x\geq b\\ u_1(x)=\dfrac{1-\gamma}{\beta\gamma}\left[\left(\dfrac{\beta}{1-\gamma}x+\eta\right)^{\gamma}-\theta\left(\left(\dfrac{\beta}{1-\gamma}b+\eta\right)^{\gamma}-\left(\dfrac{\beta}{1-\gamma}x+\eta\right)^{\gamma}\right)\right], & x<b\end{cases} \tag{15}$$

where $\beta>0$, $\eta>0$, $\gamma\in(-\infty,0)\cup(0,1)$. Similar to Section 2.3, substituting the utility Function (15) into Equation (14), we can obtain a new non-*S*-shaped HARA ordered multiple reference-dependent aggregation operator.

Definition 5. *An n-dimensional non-S-shaped HARA ordered multiple reference-dependent aggregation (NHOMR) operator is a mapping of NHOMR: $R^{+n}\to R^{+}$*

$$\mathrm{NHOMR}(\mathbf{x})=\frac{1-\gamma}{\beta}\left[\left(\frac{\beta\gamma}{1-\gamma}\left(\frac{\sum\limits_{i\in Y_1}w_iu^{\lambda}(y_i)+\sum\limits_{i\in Y_2}w_i{u_1}^{\lambda}(y_i)}{\sum\limits_{i\in Y_1}w_iu^{-\lambda}(y_i)+\sum\limits_{i\in Y_2}w_i{u_1}^{-\lambda}(y_i)}\right)^{1/2\lambda}\right)^{1/\gamma}-\eta\right], \tag{16}$$

where $u(y_i)$ and $u_1(y_i)$ are given in utility Function (15), $\mathbf{y}$ is the ordered sequence of $\mathbf{x}$, and Y_1 and Y_2 are given by Set (6).

Remark 1 (special cases of NHOMR operator)**.** *The NHOMR operator is quite general.*

(1) Considering the regular HARA utility without the reference point b and the loss aversion θ, NHOMR operator in Equation (16) degenerates to a HARA ordered multiple aggregation (HOM) operator:

$$\mathrm{HOM}(\mathbf{x})=\frac{1-\gamma}{\beta}\left(\left(\sum_{i=1}^{n}w_i\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\lambda\gamma}\Big/\sum_{i=1}^{n}w_i\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{-\lambda\gamma}\right)^{1/2\lambda\gamma}-\eta\right). \tag{17}$$

(2) In order to model a risk-neutral attitude of the DM, we can simply let $u(x)=x$ and $\theta=0$; in this case Equation (14) degenerates to,

$$\mathrm{GOWMA}(x)=\left(\sum_{i=1}^{n}w_i{y_i}^{\lambda}\Big/\sum_{i=1}^{n}w_i{y_i}^{-\lambda}\right)^{1/2\lambda}, \tag{18}$$

which is the well-known GOWMA operator [8]; (3) Taking special values of λ, γ, β, η and θ in NHOMR operator, we find that it covers many existing aggregation operators, such as the GOWMA operator [8], the ordered weighted geometric averaging (OWGA) operator [4], OWMA operator [8], the constant coefficient-OWGA (CC-OWGA) operator [32], etc. See Table A3 in Appendix A.3.

4. New Weight Models for Reference-Dependent Aggregation Operators

Because MAGDM problems have multiple alternatives, multiple attributes and multiple DMs, it becomes a crucial step to appropriately determine the weights of the attributes and the DMs. In this section, we propose new models to compute weights; our models capture realistic features in the decision-making process by taking into account the variations of the attribute values. In practice, if the values of an attribute have a small (large) variation across all alternatives, such an attribute should play a less (more) important role in choosing the optimal alternative, thus it deserves a smaller (larger) weight. We first compute weights for the attributes (Section 4.1) and next compute weights for the DMs (Section 4.2).

4.1. Weight Model for Attributes

We briefly review related weight methods in the literature. One popular idea is to obtain the desired OWA operator according to a given orness level (an attitudinal character of the DM; see [33] for details of orness) which is formulated as a constrained optimization problem. The objectives of the optimization problems include the minimum variance method [19], minimum dispersion method [20], minimum chi-square method [21], minimum disparity method [22], and maximum Bayesian entropy method [23], etc.

The methods mentioned above, however, fail to consider the influence of the input argument information on weighting the attribute. In other words, these methods assume that the determination of the weights for the attribute is independent of the distributions of the attribute values, which is evidently unreasonable [34].

It is a consensus in decision science that the larger variation the outcomes of an attribute have across the alternatives, the more important this attribute becomes (indicating a larger weight should be assigned) [34]. For instance, we consider the following two decision matrices $\mathbf{X}_1$ and $\mathbf{X}_2$; here $\mathbf{X}_2$ is derived from $\mathbf{X}_1$ (with the two columns in $\mathbf{X}_1$ swapped):

$$\mathbf{X}_1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} G_1 & G_2 \end{array} \\ \left[\begin{array}{cc} 0.2 & 0.1 \\ 0.3 & 0.9 \\ 0.2 & 0.5 \end{array}\right] \end{array} \quad \text{and} \quad \mathbf{X}_2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} G_1 & G_2 \end{array} \\ \left[\begin{array}{cc} 0.1 & 0.2 \\ 0.9 & 0.3 \\ 0.5 & 0.2 \end{array}\right] \end{array},$$

In the matrix $\mathbf{X}_1$, attribute G_1 should play a less significant role than G_2 in the decision-making process because outcomes of G_1 have similar values across all alternatives (thus a very small variation occurs) while the outcomes of G_2 have a much larger variation across A_1, A_2 and A_3. If outcomes of an attribute (such as G_1) are nearly identical, then such an attribute can perhaps be removed from the decision-making process (that is, a zero weight can be assigned to this attribute).

Nevertheless, the weight models reviewed above fail to take into account the impact of the attribute variation. In addition, these weight models will yield the same weights for the two decision matrices $\mathbf{X_1}$ and $\mathbf{X_2}$ (with the two columns in $\mathbf{X}_1$ swapped), which is unrealistic. To the best of our knowledge, there is little research considering the impact of the attributes variations on the determination of operators weights. To resolve this issue, we propose a new weight model for the attributes. In our model, a relatively larger (smaller) weight will be assigned to an attribute having a larger (smaller) variation across the alternatives. We name our model as the attribute-deviation weight model.

Generalizing the weight model in Wang & Parkan [20], we propose a new optimization model to determine the weights of the attributes. Here, $\mathbf{w}^{(k)} = (w_1^{(k)}, w_2^{(k)}, \cdots, w_n^{(k)})$ denotes the attribute weight vector for the kth DM.

$$\begin{aligned} \max \quad & P_1 \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{h=1}^{m} w_j^{(k)} d(x_{ij}^{(k)}, x_{hj}^{(k)}) - P_2 \sum_{j=1}^{n-1} d(w_j^{(k)}, w_{j+1}^{(k)}) \\ s.t. \quad & w_j^{(k)} \in H, \quad \sum_{j=1}^{n} w_j^{(k)} = 1, \quad w_j^{(k)} > 0. \end{aligned} \tag{19}$$

We now explain the objective function of our new model. To maintain fairness, all attributes should be considered equally important, thus suggesting equal weights. On the other hand, the impact of the attribute variations should also be considered in the determination of the weights. That is, an attribute with a larger variation should be assigned a larger weight. In our model, the first term in the objective function of model (19) accounts for the attribute deviation: $d(x_{ij}^{(k)}, x_{hj}^{(k)})$ denotes the deviation of x_{ij}^k and x_{hj}^k, and $\sum_{i=1}^{m} \sum_{h=1}^{m} w_j^{(k)} d(x_{ij}^{(k)}, x_{hj}^{(k)})$ represents the deviation of all alternatives for the attribute G_j under the DM_k. Next, the second term aims to minimize the total variation of the weights. The two factors P_1 and P_2 stand for the relative importance of the two terms, here, $P_1 + P_2 = 1$. The set H represents incomplete information regarding the weights (e.g., due to the lack of data and limited knowledge about the problem domain) [35,36]. The set H usually satisfies one or several of the following forms:

- A weak ranking: $\{w_i^{(k)} \geq w_j^{(k)}\}$;
- A strict ranking: $\{w_i^{(k)} - w_j^{(k)} \geq \alpha_i\}$;
- A ranking with multiples: $\{w_i^{(k)} \geq \alpha_i w_j^{(k)}\}$;
- An interval form: $\{\alpha_i \leq w_i^{(k)} \leq \alpha_i + \varepsilon_i\}$;
- A ranking of differences: $\{w_i^{(k)} - w_j^{(k)} \geq w_h^{(k)} - w_l^{(k)}\}$, for $j \neq h \neq l$, $\alpha_i, \varepsilon_i > 0$.

Remark 2 (Generality of the attribute-weight model (19))**.** *We advocate that model (19) not only captures the evaluation information of DM, but also maintains certain fairness. If $P_2 = 0$, the objective function of model (19) focuses on maximizing the total deviation for the attributes across all alternatives. On the other hand, if $P_1 = 0$, model (19) degenerates to several existing models, including the minimum chi-square model [21], minimum disparity model [22], etc.*

Similar to model (4), various forms for the deviation measure d can be used in model (19). Without loss of generality, this paper considers $d(p, q) = p^\lambda / q^\lambda + q^\lambda / p^\lambda - 2$.

4.2. Weight Model for Decision Makers

Once the individual aggregation is completed, there will exist a new matrix composed of multiple DMs and multiple alternatives. Due to the participation of multiple DMs, the final decision should be made collectively; it should reflect opinions of all DMs. A remaining question in the decision-making process is how weights of DMs should be determined.

Attribute weight models have been applied to compute the weights of DMs, such as the minimum variance method [19], minimum chi-square method [21], minimizing distances from the extreme points (MDP) method [24], voting method [25] and improved minimax disparity method [26]. However, the common disadvantage of the above approaches is that the high subjectivity of DMs may cause decision inaccuracy, sometimes leading to biased decision results.

In order to resolve this issue, we propose a new method. The idea of this method is to allocate larger weights to DMs that have smaller individual aggregation deviation. Specifically, if the kth DM's opinions are more agreeable to the optimal aggregation result, that is, the kth individual result $\mathbf{r}^{(k)} = \{r_{1k}, r_{2k}, \cdots, r_{mk}\}$ is closer to the optimal aggregation result $\mathbf{r}^* = \{r_1^*, r_2^*, \cdots, r_m^*\}$, a larger weight will be assigned to the kth DM.

We illustrate our idea by considering three DM matrices $\mathbf{D}^{(1)}$, $\mathbf{D}^{(2)}$ and $\mathbf{D}^{(3)}$, and their ideal matrix $\mathbf{D}^{(*)}$ (average of the three matrices). There are three alternatives and two attributes.

$$\mathbf{D}^{(1)} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} G_1 & G_2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 3 \end{bmatrix} \end{array}, \quad \mathbf{D}^{(2)} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} G_1 & G_2 \end{array} \\ \begin{bmatrix} 4 & 5 \\ 6 & 7 \\ 5 & 6 \end{bmatrix} \end{array},$$

$$\mathbf{D}^{(3)} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} G_1 & G_2 \end{array} \\ \begin{bmatrix} 6 & 7 \\ 8 & 9 \\ 7 & 8 \end{bmatrix} \end{array}, \quad \mathbf{D}^{*} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} G_1 & G_2 \end{array} \\ \begin{bmatrix} 3.7 & 4.7 \\ 5.7 & 6.7 \\ 4.7 & 5.7 \end{bmatrix} \end{array}$$

It is easy to see that $\mathbf{D}^{(2)}$ is closest to $\mathbf{D}^*$ while $\mathbf{D}^{(3)}$ is farthest from $\mathbf{D}^*$. Consequently, the weights should satisfy $w_2^{(DM)} > w_1^{(DM)} > w_3^{(DM)}$. Therefore, we propose the following new optimization model to compute the optimal weights for DMs.

$$\begin{array}{ll} \min & P_1 \sum_{k=1}^{l} \sum_{i=1}^{m} w_k^{(DM)} d(r_{ik}, r_i^*) + P_2 \sum_{k=1}^{l-1} d(w_k^{(DM)}, w_{k+1}^{(DM)}) \\ s.t. & w_k^{(DM)} \in H^{(DM)}, \quad \sum_{k=1}^{l} w_k^{(DM)} = 1, \quad w_k^{(DM)} > 0, \end{array} \tag{20}$$

where $\mathbf{w}^{(DM)} = (w_1^{(DM)}, w_2^{(DM)}, \cdots, w_l^{(DM)})$ represents the weight vector for the DMs, $d(r_{ik}, r_i^*)$ denotes the deviation r_{ik} and r_i^*, and $d(w_k^{(DM)}, w_{k+1}^{(DM)})$ denotes the deviation of $w_k^{(DM)}$ and $w_{k+1}^{(DM)}$.

Similar to the model (19), the second term of the objective function in (20) aims to maintain fairness for all DMs. $H^{(DM)}$ is a weighting set and has the similar condition in model (19). We call model (20) as the DM-deviation weight model.

We remark that the nonlinear optimization problems (19) and (20) (along with their corresponding optimal weight vectors $\mathbf{w}^{(k)}$ and $\mathbf{w}^{(DM)}$) can be quickly solved by numerical solvers, such as MATLAB and Lingo software. We conduct detailed numerical experiments in Section 6, where we use Lingo to solve models (19) and (20).

5. A New Reference-Dependent Aggregation Approach for MAGDM

In this section we develop a new approach for MAGDM based on the RU aggregation operators in Sections 2 and 3 and the new weight models in Section 4. The approach is summarized in a simple algorithm through six steps. We first describe the algorithm inputs.

Input data of our new MAGDM algorithm. Let $\mathcal{A} = \{A_1, A_2, \cdots, A_m\}$ be the set of m alternatives, $\mathcal{G} = \{G_1, G_2, \cdots, G_n\}$ be the set of n attributes, and $\mathcal{D} = \{D_1, D_2, \cdots, D_l\}$ be the set of l DMs. Assume that $\mathbf{w}^{(DM)} = (w_1^{(DM)}, w_2^{(DM)}, \cdots, w_l^{(DM)})$ is the weight vector for the DMs and $\mathbf{w}^{(k)} = (w_1^{(k)}, w_2^{(k)}, \cdots, w_n^{(k)})$ is the attribute weight vector for the kth DM such that $w_j^{(k)} \geq 0$, $\sum_{j=1}^{n} w_j^{(k)} = 1$, $w_k^{(DM)} \geq 0$ and $\sum_{k=1}^{l} w_k^{(DM)} = 1$. Opinions of DM D_k, $1 \leq k \leq l$, are characterized by the decision matrix $\mathbf{S}^{(k)} = (s_{ij}^{(k)})_{m \times n}$ and reference point vector $\mathbf{B}^{(k)} = (b_1^{(k)}, b_2^{(k)}, \cdots, b_n^{(k)})$, where $s_{ij}^{(k)}$ is the input argument of $D_k \in D$ for alternative $A_i \in \mathcal{A}$ and attribute $G_j \in \mathcal{G}$, and $b_j^{(k)}$ is the reference point of $D_k \in D$ for attribute $G_j \in \mathcal{G}$. We summarize all input data below,

$$\mathcal{I} = \left(\mathcal{A}, \mathcal{G}, \mathcal{D}, \mathbf{w}^{(k)}, \mathbf{w}^{(DM)}, \mathbf{S}^{(k)}, \mathbf{B}^{(k)}\right). \tag{21}$$

Given the input data $\mathcal{I}$ in (21), our objective is to determine the optimal alternative $A^* \in \mathcal{A}$. We give a new MAGDM algorithm below.

Step 1 Transform the decision matrixes $\mathbf{S}^{(k)}$ to the corresponding normalized version $\mathbf{R}^{(k)} = (r_{ij}^{(k)})_{m \times n}$ [8]:

$$r_{ij}^{(k)} = s_{ij}^{(k)} \Big/ \max_i s_{ij}^{(k)} \quad \text{for } j \in I_1 \quad \text{and} \quad r_{ij}^{(k)} = \min_i s_{ij}^{(k)} \Big/ s_{ij}^{(k)} \quad \text{for } j \in I_2 \tag{22}$$

where I_1 is a set of profit attributes and I_2 is a set of cost attribute.

Step 2 Calculate the attribute weight vector of the kth decision matrix $\mathbf{w}^{(k)}$ by solving the optimization problem in model (19) for $k = 1, 2, \cdots, l$.

Step 3 Aggregate all individual decision matrixes $\mathbf{R}^{(k)}$ to obtain a collective decision matrix $\mathbf{R} = (r_{ik})_{m \times l}$ by using the attribute weight vector and the aggregation operator, and following Equations (11) and (16) for cases of risk-seeking and risk-averse attitude.
Step 4 Calculate the DMs weights $\mathbf{w}^{(DM)}$ by solving the optimization problem in model (20).
Step 5 Aggregate all attribute values r_{ik} to obtain an overall preference value t_i of the alternative A_i by using the DM weight vector $\mathbf{w}^{(DM)} = (w_1^{(DM)}, w_2^{(DM)}, \cdots, w_l^{(DM)})$ and the HOM operator given in Equation (17).
Step 6 Rank the collective overall preference values $t_1, t_2, \cdots, t_m$ in the descending order and consequently, select the optimal alternative(s) (e.g., the one(s) with the greatest value t_i).

6. Numerical Examples

This section conducts numerical experiments to evaluate the effectiveness of the new MAGDM approach developed in Section 5. First, in Section 6.1 we apply the new method to a multi-attribute investment selection problem introduced in Merigó & Casanovas [7], and show that the new method outperforms existing results in the presence of reference point and loss aversion coefficient. Second, to illustrate the influence of DMs' psychological factors (e.g., basic utility function, reference point and loss-aversion coefficient) on the decision-making results, Section 6.2 performs sensitivity analysis on β, η, γ, b and θ in the SHOMR operator and the NHOMR operator.

6.1. An Investment Selection Problem

We initially explain the framework of our experiments. Following Merigó & Casanovas [7], we consider an investment company that plans to invest in six possible alternatives $(A_1, A_2, \cdots, A_6)$. There are six attributes $(G_1, G_2, \cdots, G_6)$ in the group decision-making for each of the alternatives. Table 2 shows the detailed information of the alternatives and attributes. The group of company experts is constituted by three DMs (D_1, D_2, D_3), each DM provides his/her own opinions regarding all attributes of all alternatives. The results after standardization are given in Table 3.

Table 2. Detailed information of alternatives (**left**) and attributes (**right**).

Notations	Alternatives	Notations	Attributes
A_1	A chemical company	G_1	Benefits in the short term
A_2	A food company	G_2	Benefits in the mid-term
A_3	A computer company	G_3	Benefits in the long term
A_4	A car company	G_4	Risk of the investment
A_5	A furniture company	G_5	Difficulty of the investment
A_6	A pharmaceutical company	G_6	Other factors

Table 3. Evaluation of the investments.

	D_1						D_2						D_3					
	G_1	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6
A_1	0.7	0.8	0.6	0.7	0.5	0.9	0.6	0.8	0.5	0.6	0.4	0.8	0.7	0.6	0.6	0.6	0.4	0.7
A_2	0.8	0.6	0.9	0.7	0.6	0.7	0.7	0.6	0.8	0.6	0.7	0.7	0.7	0.6	0.7	0.6	0.6	0.7
A_3	0.5	0.4	0.8	0.3	0.8	0.8	0.7	0.6	0.8	0.7	0.8	0.8	0.6	0.5	0.8	0.5	0.8	0.8
A_4	0.6	0.7	0.6	0.7	0.8	0.6	0.6	0.7	0.5	0.6	0.8	0.7	0.6	0.7	0.7	0.5	0.8	0.6
A_5	0.9	0.8	0.4	0.7	0.7	0.8	0.7	0.8	0.7	0.7	0.6	0.8	0.7	0.8	0.6	0.7	0.6	0.8
A_6	0.8	0.3	0.7	0.7	0.6	0.7	0.6	0.4	0.8	0.7	0.6	0.7	0.4	0.5	0.9	0.7	0.6	0.6

We consider three DMs with different risk attitudes for losses: (1) D_1 is risk-seeking; (2) D_2 is risk-averse; and (3) D_3 is risk-neutral for completeness. Their corresponding aggregation operators are given by Equations (12), (16) and (18) respectively. Similar to Ahn [35], weight vectors $\mathbf{w}^{(k)} \in H$ and $\mathbf{w}^{(DM)} \in H^{(DM)}$ satisfy,

$$H = \left\{ \begin{array}{l} 0.05 \leq w_1^{(k)} \leq 0.2, \quad w_2^{(k)} \geq 0.1, \quad w_3^{(k)} - w_2^{(k)} \geq 0.2, \quad 0.2 \leq w_3^{(k)} \leq 0.3, \\ 0.05 \leq w_4^{(k)} \leq 0.15, \quad 0.1 \leq w_5^{(k)} - w_4^{(k)} \leq 0.2, \quad 0.1 \leq w_6^{(k)} \leq 0.3 \end{array} \right\}$$

for $k = 1, 2, 3$ and,

$$H^{(DM)} \in \{w_2^{(DM)} \leq w_1^{(DM)}, 0.1 \leq w_2^{(DM)} \leq 0.2, 0.1 \leq w_3^{(DM)} - w_2^{(DM)} \leq 0.2\}.$$

Throughout the numerical analysis, for simplicity, we assume that $\lambda = 1, \theta = \theta_1$ and $P_1 = P_2 = 0.5$. Following Cox & Huang [31], we let $\beta = \beta_1 = 5$, $\eta = \eta_1 = 0.5$ and $\gamma = \gamma_1 = -7$ for the basic HARA utility. By applying the new MAGDM algorithm, we derive different optimal alternatives in the following three cases.

Case 1. Let the reference points be the averages of attributes values, i.e., $\mathbf{B}^{(1)} = (0.72, 0.60, 0.67, 0.63, 0.67, 0.75)$, $\mathbf{B}^{(2)} = (0.65, 0.65, 0.68, 0.65, 0.65, 0.75)$ and loss aversion coefficient $\theta = 1.5$. Our MAGDM algorithm in Section 5 yields an aggregation result and the ranked alternatives:

$$(t_1, t_2, t_3, t_4, t_5, t_6) = (0.461, 0.558, 0.485, 0.484, 0.534, 0.441), A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6.$$

Therefore, the optimal investment alterative is the food company A_2.

Case 2. Assume that the reference points are $\mathbf{B}^{(1)} = (0.62, 0.50, 0.57, 0.53, 0.57, 0.65)$ and $\mathbf{B}^{(2)} = (0.55, 0.55, 0.58, 0.55, 0.55, 0.65)$, and the loss aversion coefficient is $\theta = 1$. Applying the MAGDM algorithm presented in Section 5, we derive that:

$$(t_1, t_2, t_3, t_4, t_5, t_6) = (0.529, 0.576, 0.585, 0.547, 0.592, 0.483), A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_6.$$

This implies that the optimal investment alterative is the furniture company A_5. We remark that our result in this case coincides with Merigó & Casanovas [7].

Case 3. Setting $\mathbf{B}^{(1)} = (0.77, 0.65, 0.72, 0.68, 0.72, 0.80)$, $\mathbf{B}^{(2)} = (0.7, 0.7, 0.73, 0.7, 0.7, 0.8)$ and $\theta = 3$ yields that:

$$(t_1, t_2, t_3, t_4, t_5, t_6) = (0.292, 0.316, 0.325, 0.287, 0.304, 0.238), A_3 \succ A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_6.$$

In this case, the optimal investment alterative is the computer company A_3.

Remark 3 (Characteristics of the new approach)**.** *Compared with the approach of Merigó & Casanovas [7], the main characteristics of our approach can be concluded as follows:*

(1) The SHOMR operators (see, Equation (11)) and NHOMR operators (see, Equation (16)) can capture the psychological preference of the DM with regard to the input argument information, while the aggregation operators in Merigó & Casanovas [7] fail to consider in the decision-making process. Specifically, the above three cases clearly show that the optimal alternative highly depends on the reference point **B** *and the loss aversion coefficient θ; this confirms the significance of capturing psychological factors of DMs in the aggregation process. The DMs can choose different* **B** *and θ based on their risk preference to select the optimal alternative.*

(2) The attribute-deviation weight model (see, model (19)) and DM-deviation weight model (see, model (20)) are constructed to determine the associated weights of the attributes and DMs, while the weights of the attributes and DMs are completely known in Merigó & Casanovas [7]. In fact, owing to the complexity and uncertainty of things in reality, the weights of the attributes and DMs are generally incomplete known.

(3) *The new aggregation operators can degenerate to the traditional aggregation operators including the OWGA operator [4], OWMA operator [8], CC-OWGA operator [32] and GOWMA operator [8], etc. (see, Table A1–A3 in Appendix A.3). In this way, the new aggregation approach can consider a wide range of scenarios according to the interests of the DM and select the alternative which is closest to his/her real interests.*

6.2. Sensitive Analysis of Reference-Dependent Aggregation Operators

We note that introducing the SHOMR and NHOMR operators has an impact on the DM to aggregate information and finally influences the selection of the optimal alternative(s) (see, Equations (12) and (16)). However, it is difficult to find what role these parameters of the RU aggregation operators play in the selection process. Therefore, we conduct sensitivity analysis about: (1) the parameters of the basic utility function in Section 6.2.1; (2) the reference points in Section 6.2.2; and (3) the loss aversion coefficient in Section 6.2.3.

6.2.1. Sensitive Analysis of Parameters in the Basic Utility Function

Based on the parameters given in Section 6.1, Figures 4 and 5 and Table 4 illustrate the change of the choice on the optimal alternative with the variations of one parameter, two parameters and three parameters, respectively.

From Figure 4 we find that the optimal alternative will change with the parameter β increasing to a certain point (see, Figure 4a) or with η (or, γ) decreasing to a certain point (see, Figure 4b,c). For example, the optimal alternative switches from A_2 to A_5 when β increases from 4 to about 7, or when η decreases from 1 to about 0.1, or when γ decreases from -6.5 to about -8.3.

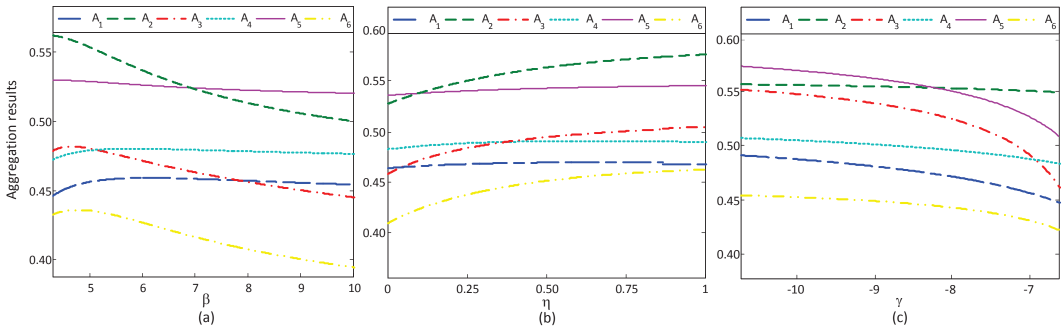

Figure 4. Sensitivity analysis of the optimal alternatives with respect to one parameter: (**a**) β; (**b**) η; and (**c**) γ.

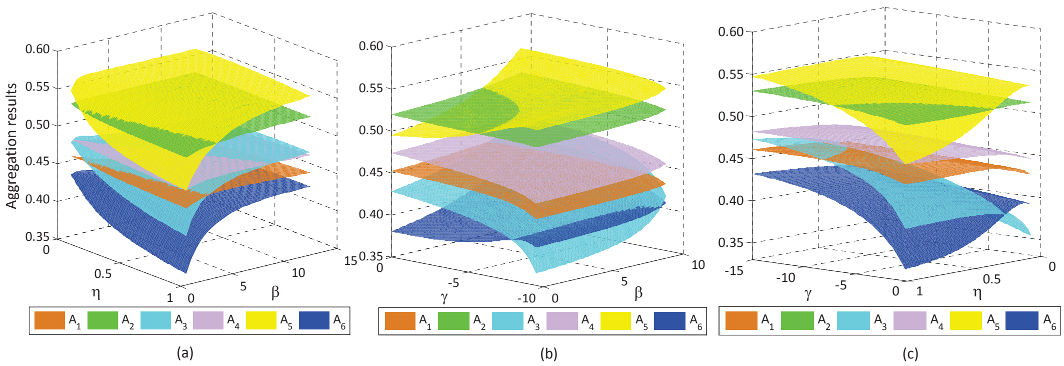

Figure 5. Sensitivity analysis of the optimal alternatives with respect to two parameters: (**a**) β and η; (**b**) β and γ; and (**c**) η and γ.

Table 4. Sensitivity analysis of the optimal alternatives with respect to parameters β, η and γ.

β	η	γ	$r(x)$	Overall Preference Value t_i						Ordering
				t_1	t_2	t_3	t_4	t_5	t_6	
1	12	0.5	0.0746	0.4502	0.5226	0.3793	0.4783	0.4539	0.3361	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3 \succ A_6$
3	5	-3	0.5430	0.4557	0.5235	0.4110	0.4804	0.4709	0.3630	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3 \succ A_6$
4	2	-5	1.6216	0.4598	0.5252	0.4395	0.4825	0.4907	0.3881	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3 \succ A_6$
5	0.5	-7	5.3333	0.4613	0.5384	0.4854	0.4839	0.5336	0.4406	$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_6$
6	0.45	-6	5.7143	0.4597	0.5327	0.4815	0.4801	0.5329	0.4371	$A_5 \succ A_2 \succ A_3 \succ A_4 \succ A_1 \succ A_6$
7	0.3	-9	8.8608	0.4578	0.5298	0.4795	0.4769	0.5307	0.4329	$A_5 \succ A_2 \succ A_3 \succ A_4 \succ A_1 \succ A_6$
9	0.1	-11	14.400	0.4418	0.5241	0.4725	0.4711	0.5279	0.4290	$A_5 \succ A_2 \succ A_3 \succ A_4 \succ A_1 \succ A_6$
12	0.05	-12	17.238	0.4388	0.5125	0.4688	0.4695	0.5210	0.4258	$A_5 \succ A_2 \succ A_4 \succ A_3 \succ A_1 \succ A_6$

Figure 5 and Table 4 show that the changing of two or three parameters will also affect the selection of the optimal alternative. For instance, Figure 5a illustrates that when $0 < \beta < 5.5$ and $\eta > 0.3$, the optimal investment alternative is A_2; when $\beta > 5.5$ and $0 < \eta < 0.3$, the optimal investment alternative is A_5.

In particular, Table 4 shows that the optimal alternative will change from A_2 to A_5 with the absolute risk aversion coefficient changing to a certain point. This change tendency is also suitable for one-parameter and two-parameters situations. For example, in Table 4, the optimal alternative is still A_2 if the value of absolute risk aversion coefficient $r(x,\gamma,\beta,\eta) = \beta(1-\gamma)/(\beta x + \eta(1-\gamma)) \le 5.33$ (here, we assume $x = 0.7$), while it will become A_5 when the value of $r(x,\gamma,\beta,\eta) > 5.71$. We further investigate the sensitivity of $r(x,\gamma,\beta,\eta)$ in Figures 6 and 7. We see that $r(x,\gamma,\beta,\eta)$ increases in β (Figure 6a) and decreases in both η and γ (Figure 6b,c). Figure 7 shows how $r(x,\gamma,\beta,\eta)$ depends on two parameters with the third parameter fixed.

The above changes in Figures 4 and 5 and Table 4 can be understood by the implication of the absolute risk aversion coefficient $r(x,\gamma,\beta,\eta)$. Note that as the absolute risk aversion coefficient increases, the risk attitude of the DM becomes more prudent [37]. We find that from Table 3 the alternative A_5 dominates alternative A_2 with respect to the attributes G_1, G_2 and G_4. Notice that G_1, G_2 and G_4 denote the short term benefits, the mid-term benefits, and the risk of the investment, respectively. In the decision-making process, these attributes generally draw higher attention from DMs than those of G_3 (benefits in the long term), G_5 (difficulty of the investment) and G_6 (other factors). Therefore, from the perspective of investment prudency, A_5 dominates A_2 as $r(x,\gamma,\beta,\eta)$ increases. In other words, we can conclude that the optimal alternative will change to a relatively prudent alternative with the absolute risk aversion coefficient increasing. We remark that the new MAGDM method has practical values because the results in Figures 4 and 5 or Table 4 can quickly provide useful advice to DMs.

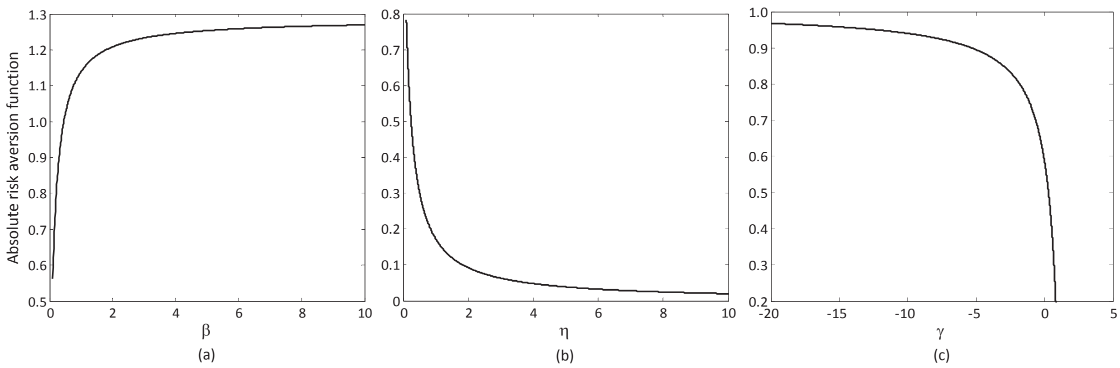

Figure 6. Variations of the absolute risk aversion function for one parameter: (**a**) β; (**b**) η; and (**c**) γ.

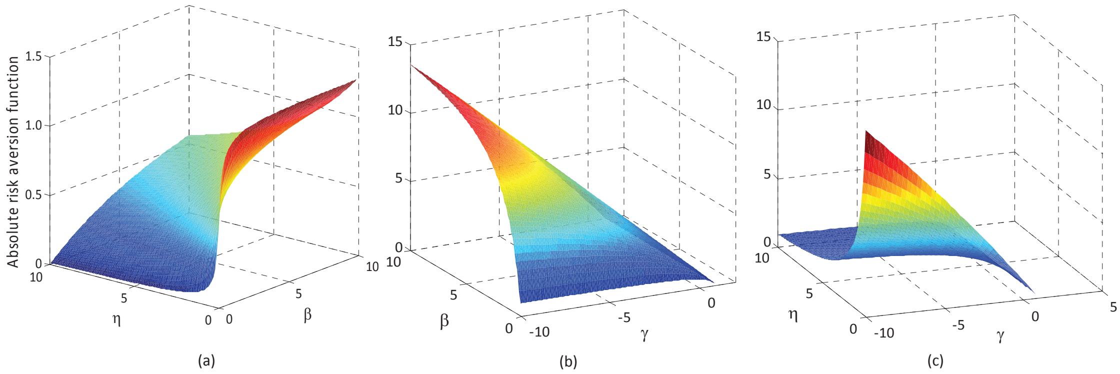

Figure 7. Variations of the absolute risk aversion function for two parameters: (**a**) β and η; (**b**) β and γ; and (**c**) η and γ.

6.2.2. Sensitive Analysis of Reference Points

With all other parameters specified in Section 6.1, Figure 8 shows that the change of the reference points can affect the choice of the optimal investment alternative.

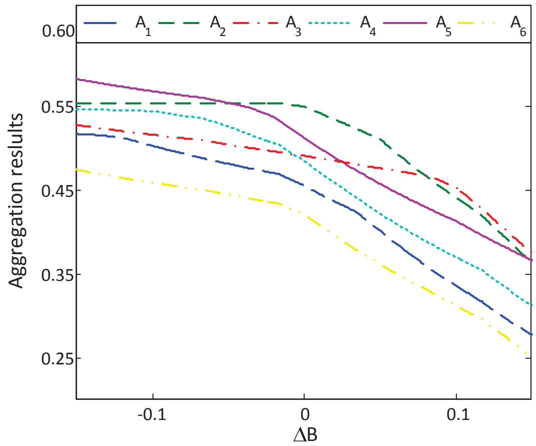

Figure 8. Sensitivity analysis of the optimal alternatives with respect to reference point **B**.

From Figure 8, we see that the aggregation results of all alternatives decrease as the reference point **B** increases, and that the optimal alternative changes with the reference point **B** increasing to a certain point. More precisely, the optimal alternative changes to a relatively risky one with the reference point **B** increasing to a certain point. For example, with the reference point **B** changing from -0.15 to -0.05, the aggregation result of A_5 decreases from 0.57 to 0.555. Consequently, the optimal alternative switches from A_5 to A_2. When the reference point **B** continuously increases to about 0.07, the optimal alternative shifts from A_2 to A_3.

The above changing tendency in Figure 8 can be explained by the implication of the reference point **B** in RUs. On the one hand, the input arguments below the reference point **B** are regarded as relative losses for *S*-shaped RUs, and the utility curve for relative losses is convex which reflects the risk-seeking attitude of the DM. As the reference point **B** increases, the relative loss becomes larger, which further makes the DM more risk-seeking. On the other hand, although outcomes below the reference point **B** are viewed as distorted positive gains for non-*S*-shaped RUs, the utility curve below this reference point is steeper than in the case without any reference point, which means that larger reference points imply smaller degrees of risk-aversion.

In addition, Table 3 shows that alternatives $A_3 \prec A_2 \prec A_5$ for attributes G_1 (benefits in the short term), G_2 (benefits in the mid-term) and G_4 (risk of the investment). In the view of investment prudency, these attributes usually seem more important than the other three attributes: G_3 (benefits in the long term), G_5 (difficulty of the investment) and G_6 (other factors). Therefore, larger reference points imply less risk-averse attitudes (DMs pay more attention to the risky alternative). As a result, the optimal alternative changes $A_5 \rightarrow A_2 \rightarrow A_3$ as the reference points increase.

6.2.3. Sensitive Analysis of the Loss Aversion Coefficient

Finally, we study the impact of the loss aversion coefficient on the ranking of alternatives in the decision-making process. Using the example in Section 6.1, we plot the aggregation results for six alternatives as functions of the loss aversion coefficient θ in Figure 9.

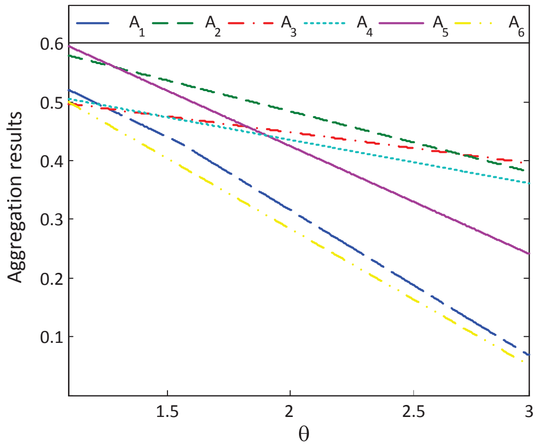

Figure 9. Sensitivity analysis of the optimal alternatives with respect to loss aversion coefficient θ.

We find that from Figure 9 the aggregation results of each alternative decrease as θ increases, and that the optimal alternative changes to a relatively risky one with the loss aversion coefficient increasing to a certain point. For instance, the aggregation result of A_5 decreases from 0.6 to 0.24 when θ increases from 1 to about 3; the optimal alternative switches from A_5 to A_2 when θ increases from 1 to about 1.3, and the optimal alternative will change from A_2 to A_3 with the loss aversion coefficient θ continuously increasing to about 2.7.

We provide some intuitive insights for the above observations. On the one hand, for S-shaped RUs, a larger value for θ implies a steeper utility curve for relative losses (i.e., a larger negative utility value for losses), which means that the risk attitude of the DM is more risk-seeking. On the other hand, for non-S-shaped RUs, a larger loss aversion coefficient indicates a smaller utility value for relative losses, and this case illustrates a steeper utility curve for relative losses. In other words, larger loss aversion coefficients imply smaller degrees of risk-aversion for relative losses. From Table 3, we find that alternatives $A_3 \prec A_2 \prec A_5$ for attributes G_1, G_2 and G_4. In the view of investment prudency, these tattributes are more attractive to DMs than other attributes. Note that the attitude of the DM towards risk becomes more adventurous with the loss aversion coefficient increasing, therefore, the optimal alternative changes $A_5 \rightarrow A_2 \rightarrow A_3$ as θ increases.

7. Conclusions

This paper introduced new reference-dependent utility functions in the aggregation process. To better model the psychological factors in MAGDM, we proposed S-shaped RU and non-S-shaped RU aggregation operators to characterize two attitudes of DMs for relative losses: risk-seeking and risk-averse. The S-shaped operator represented one type of aggregation functions where the attitude of the DM is risk-seeking for relative losses, and the non-S-shaped operator indicated another type of aggregation functions where the attitude of the DM is risk-averse for relative losses. In addition, the non-S-shaped operator can degenerate into the GOWMA operator, which implied the attitude of the DM is risk-neutral. Specifically, we developed an SOMR aggregation operator and an NOMR aggregation operator under S-shaped HARA and non-S-shaped HARA utility framework; we found that they are commutative, monotonic, bounded and idempotent.

In addition, we proposed an attribute-deviation weight model and a DMs-deviation weight model to determine the weights of attributes and DMs, which overcame the shortcomings of the existing aggregation operator weight models. We summarized the new MAGDM approach based on the RU operators and the weight models. In the end, we tested its effectiveness and demonstrated how to choose the optimal alternatives via numerical examples. The approach can be used in many fields such as strategic planning, portfolio selection, medical diagnosis, and military system evaluation. We believe that our proposed approach leaves space for further study of many interesting questions regarding (1) how to obtain accurate information about psychological preference from the DM; and (2) how to choose the utilities that will most efficiently guide the optimization problem to an optimal decision-making.

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Appendix A

This appendix has three sections, presenting the supporting materials of the main paper. In Appendix A.1 we give the proof of Proposition 1 in Section 2. In Appendix A.2 we give the proof for Properties of NOMR operator in Section 3. In Appendix A.3 we provide the families of the reference-dependent aggregation operators and the corresponding proofs.

Appendix A.1. Proof for Properties of SOMR Operator

Proposition A1. *The SOMR operator given in Definition 1 satisfies the following properties:*

(1) *(Monotonicity) For two vectors $\mathbf{x}$ and $\bar{\mathbf{x}}$ with $x_i \geq \bar{x}_i$ and the same reference points, then* $\mathrm{SOMR}(\mathbf{x}) \geq \mathrm{SOMR}(\bar{\mathbf{x}})$.
(2) *(Boundedness) If $b_1 \leq y_1 = \max_i \{x_i\}$ and $b_n > y_n = \min_i \{x_i\}$, then* $v_1^{-1}(v_2(y_n)) \leq \mathrm{SOMR}(\mathbf{x}) \leq y_1$.
(3) *(Commutativity) If $\hat{\mathbf{x}}$ is a permutation of $\mathbf{x}$, then* $\mathrm{SOMR}(\mathbf{x}) = \mathrm{SOMR}(\hat{\mathbf{x}})$.
(4) *(Idempotency) If $x_i = x \geq x_0$ for all $1 \leq i \leq n$, then* $\mathrm{SOMR}(\mathbf{x}) = x$.

Proof of Proposition A1. (1) *Monotonicity.* For convenience, let S denote:

$$\mathrm{SOMR}(\mathbf{x}) = v_1^{-1}\left[\left(\left(\sum_{i\in Y_1} w_i v_1^{\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{\lambda}(y_i)\right) \Big/ \left(\sum_{i\in Y_1} w_i v_1^{-\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{-\lambda}(y_i)\right)\right)^{1/2\lambda}\right].$$

- For $i \in Y_1$, by taking the first-order condition of S with respect to y_i, we have that,

$$\begin{aligned}\frac{\partial S}{\partial y_i} = \; & \frac{1}{2v_1'(S)}\left[\left(\left(\sum_{i\in Y_1} w_i v_1^{\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{\lambda}(y_i)\right) \Big/ \left(\sum_{i\in Y_1} w_i v_1^{-\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{-\lambda}(y_i)\right)\right)^{1/2\lambda-1}\right] \times \\ & \left(\sum_{i\in Y_1} w_i v_1^{-\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{-\lambda}(y_i)\right)^{-2}\left[\left(w_i v_1^{\lambda-1}(y_i) v_1'(y_i)\right)\left(\sum_{i\in Y_1} w_i v_1^{-\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{-\lambda}(y_i)\right) + \right. \\ & \left.\left(\sum_{i\in Y_1} w_i v_1^{\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{\lambda}(y_i)\right)\left(w_i v_1^{-\lambda-1}(y_i) v_1'(y_i)\right)\right].\end{aligned}$$

- For $i \in Y_2$, by taking the first-order condition of S with respect to y_i, we have that,

$$\begin{aligned}\frac{\partial S}{\partial y_i} = & \frac{1}{2v_1'(S)}\left[\left(\left(\sum_{i\in Y_1} w_i v_1^{\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{\lambda}(y_i)\right)\Big/\left(\sum_{i\in Y_1} w_i v_1^{-\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{-\lambda}(y_i)\right)\right)^{1/2\lambda-1}\right]\times \\ & \left(\sum_{i\in Y_1} w_i v_1^{-\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{-\lambda}(y_i)\right)^{-2}\left[\left(-w_i v_2^{\lambda-1}(y_i)(v_2(y_i))'\right)\left(\sum_{i\in Y_1} w_i v_1^{-\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{-\lambda}(y_i)\right)+\right. \\ & \left.\left(\sum_{i\in Y_1} w_i v_1^{\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{\lambda}(y_i)\right)\left(-w_i v_2^{-\lambda-1}(y_i)(v_2(y_i))'\right)\right].\end{aligned}$$

Since $v_1(y_i) > 0$, $v_1'(y_i) > 0$, $v_2(y_i) > 0$ and $(v_2(\cdot))' = (v_2(b_i - y_i))' = -v_2'(b_i - y_i) < 0$, we obtain that $\partial S/\partial y_i > 0$, which implies that S increases monotonically with respect to y_i. Note that $x_i \geq \overline{x}_i$ and they have the same reference point b_i $(i = 1, 2, \cdots, n)$, we then get $\mathrm{SOMR}(\mathbf{x}) \geq \mathrm{SOMR}(\overline{\mathbf{x}})$.

(2) *Boundedness.* If $b_1 \leq y_1 = \max\limits_{1\leq i\leq n}\{x_i\}$, according to the above proof, we have that,

$$\mathrm{SOMR}(\mathbf{x}) \leq \mathrm{SOMR}(y_1, y_1, \cdots, y_1) = v_1^{-1}\left[\left(\left(\sum_{i=1}^{n} w_i v_1^{\lambda}(y_1)\right)\Big/\left(\sum_{i=1}^{n} w_i v_1^{-\lambda}(y_1)\right)\right)^{1/2\lambda}\right] = y_1.$$

If $b_n > y_n = \min\limits_{1\leq i\leq n}\{x_i\}$, then by monotonicity, we have that,

$$\mathrm{SOMR}(\mathbf{x}) \geq \mathrm{SOMR}(y_n, y_n, \cdots, y_n) = v_1^{-1}\left[\left(\left(\sum_{i=1}^{n} w_i v_2^{\lambda}(y_n)\right)\Big/\left(\sum_{i=1}^{n} w_i v_2^{-\lambda}(y_n)\right)\right)^{1/2\lambda}\right] = v_1^{-1}\left(v_2(y_n)\right).$$

Thus, $v_1^{-1}\left(v_2(y_n)\right) \leq \mathrm{SOMR}(\mathbf{x}) \leq y_1$.

(3) *Commutativity.* Let,

$$\mathrm{SOMR}(\widehat{\mathbf{x}}) = v_1^{-1}\left[\left(\left(\sum_{i\in T_1} w_i v_1^{\lambda}(t_i) - \sum_{i\in T_2} w_i v_2^{\lambda}(t_i)\right)\Big/\left(\sum_{i\in T_1} w_i v_1^{-\lambda}(t_i) - \sum_{i\in T_2} w_i v_2^{-\lambda}(t_i)\right)\right)^{1/2\lambda}\right].$$

Since $(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n)$ is any permutation of the arguments $(x_1, x_2, \cdots, x_n)$, we can get $y_i = t_i$ for all i. We then obtain that $\mathrm{SOMR}(\mathbf{x}) = \mathrm{SOMR}(\widehat{\mathbf{x}})$.

(4) *Idempotency.* Since $x_i = x \geq x_0$, then we have,

$$\mathrm{SOMR}(\mathbf{x}) = v_1^{-1}\left[\left(\left(\sum_{i=1}^{n} w_i v_1^{\lambda}(x)\right)\Big/\left(\sum_{i=1}^{n} w_i v_1^{-\lambda}(x)\right)\right)^{1/2\lambda}\right] = x. \quad \square$$

Appendix A.2. Proof for Properties of NOMR Operator

Proposition A2. *The NOMR operator given in Definition 4 satisfies:*

(1) *(Monotonicity) For two vectors* $\mathbf{x}$ *and* $\overline{\mathbf{x}}$ *with* $x_i \geq \overline{x}_i$ *and the same reference points, then* $\mathrm{NOMR}(\mathbf{x}) \geq \mathrm{NOMR}(\overline{\mathbf{x}})$.
(2) *(Boundedness) If* $b_1 \leq y_1 = \max\limits_i \{x_i\}$ *and* $b_n > y_n = \min\limits_i \{x_i\}$, *then* $u^{-1}(u_1(y_n)) \leq \mathrm{NOMR}(\mathbf{x}) \leq y_1$. *Especially,* $y_n \leq \mathrm{NOMR}(\mathbf{x}) \leq y_1$ *while* $y_n = b_n$.
(3) *(Commutativity) If* $\widehat{\mathbf{x}}$ *is a permutation of* $\mathbf{x}$, *then* $\mathrm{NOMR}(\mathbf{x}) = \mathrm{NOMR}(\widehat{\mathbf{x}})$.
(4) *(Idempotency). If* $x_i = x \geq x_0$ *for all* $1 \leq i \leq n$, *then* $\mathrm{NOMR}(\mathbf{x}) = x$.

(1) *Monotonicity.* For convenience, let N denote,

$$\mathrm{NOMR}(\mathbf{x}) = u^{-1}\left[\left(\left(\sum_{i\in Y_1} w_i u^{\lambda}(y_i) + \sum_{i\in Y_2} w_i u_1^{\lambda}(y_i)\right)\Big/\left(\sum_{i\in Y_1} w_i u^{-\lambda}(y_i) + \sum_{i\in Y_2} w_i u_1^{-\lambda}(y_i)\right)\right)^{1/2\lambda}\right],$$

where $u_1(y_i) = u(y_i) - \theta(u(b_i) - u(y_i))$.

- For $i \in Y_1$, the first-order condition of N with respect to y_i implies that,

$$\begin{aligned}\frac{\partial N}{\partial y_i} = \; & \frac{1}{2u'(N)}\left[\left(\left(\sum_{i\in Y_1} w_i u^{\lambda}(y_i) + \sum_{i\in Y_2} w_i u_1^{\lambda}(y_i)\right)\Big/\left(\sum_{i\in Y_1} w_i u^{-\lambda}(y_i) + \sum_{i\in Y_2} w_i u_1^{-\lambda}(y_i)\right)\right)^{1/2\lambda-1}\right]\times \\ & \left[\left(w_i u^{\lambda-1}(y_i)u'(y_i)\right)\left(\sum_{i\in Y_1} w_i u^{-\lambda}(y_i) + \sum_{i\in Y_2} w_i u_1^{-\lambda}(y_i)\right) + \left(w_i u^{-\lambda-1}(y_i)u'(y_i)\right)\times\right. \\ & \left.\left(\sum_{i\in Y_1} w_i u^{\lambda}(y_i) + \sum_{i\in Y_2} w_i u_1^{\lambda}(y_i)\right)\right]\left(\sum_{i\in Y_1} w_i u^{-\lambda}(y_i) + \sum_{i\in Y_2} w_i u_1^{-\lambda}(y_i)\right)^{-2}.\end{aligned}$$

- For $i \in Y_2$, the first-order condition of N with respect to y_i implies that,

$$\begin{aligned}\frac{\partial N}{\partial y_i} = \; & \frac{1}{2u'(N)}\left[\left(\left(\sum_{i\in Y_1} w_i u^{\lambda}(y_i) + \sum_{i\in Y_2} w_i u_1^{\lambda}(y_i)\right)\Big/\left(\sum_{i\in Y_1} w_i u^{-\lambda}(y_i) + \sum_{i\in Y_2} w_i u_1^{-\lambda}(y_i)\right)\right)^{1/2\lambda-1}\right]\times \\ & \left[\left(w_i u_1^{\lambda-1}(y_i)u_1'(y_i)\right)\left(\sum_{i\in Y_1} w_i u^{-\lambda}(y_i) + \sum_{i\in Y_2} w_i u_1^{-\lambda}(y_i)\right) + \left(w_i u_1^{-\lambda-1}(y_i)u_1'(y_i)\right)\times\right. \\ & \left.\left(\sum_{i\in Y_1} w_i u^{\lambda}(y_i) + \sum_{i\in Y_2} w_i u_1^{\lambda}(y_i)\right)\right]\left(\sum_{i\in Y_1} w_i u^{-\lambda}(y_i) + \sum_{i\in Y_2} w_i u_1^{-\lambda}(y_i)\right)^{-2}.\end{aligned}$$

Since $u(y_i) > 0$, $u'(y_i) > 0$, $u_1(y_i) > 0$ and $u_1{}'(y_i) > 0$, we obtain that $\partial N/\partial y_i > 0$ for $i \in Y_1$ or $i \in Y_2$, meaning that N increases monotonically with respect to y_i. Note that $x_i \geq \overline{x}_i$ and they have the same reference point b_i $(i = 1, 2, \cdots, n)$, we then get $\mathrm{NOMR}(\mathbf{x}) \geq \mathrm{NOMR}(\overline{\mathbf{x}})$.

(2) *Boundedness.* Similar to Proposition 1, if $b_1 \leq y_1 = \max_{1\leq i\leq n}\{x_i\}$, we then have that,

$$\mathrm{NOMR}(\mathbf{x}) \leq \mathrm{NOMR}(y_1, \cdots, y_1) = u^{-1}\left[\left(\sum_{i=1}^{n} w_i u^{\lambda}(y_1)\Big/\sum_{i=1}^{n} w_i u^{-\lambda}(y_1)\right)^{1/2\lambda}\right] = y_1.$$

If $b_n > y_n = \min_{1\leq i\leq n}\{x_i\}$, then from monotonicity we derive,

$$\mathrm{NOMR}(\mathbf{x}) \geq \mathrm{NOMR}(y_n, \cdots, y_n) = u^{-1}\left[\left(\sum_{i=1}^{n} w_i u_1^{\lambda}(y_n)\Big/\sum_{i=1}^{n} w_i u_1^{-\lambda}(y_n)\right)^{1/2\lambda}\right] = u^{-1}(u_1(y_n)).$$

Thus, $u^{-1}(u_1(y_n)) \leq \mathrm{NOMR}(\mathbf{x}) \leq y_1$. Especially, $y_n \leq \mathrm{NOMR}(\mathbf{x}) \leq y_1$ while $y_n = b_n$.

(3) *Commutativity.* Let,

$$\mathrm{NOMR}(\widehat{\mathbf{x}}) = u^{-1}\left[\left(\left(\sum_{i\in T_1} w_i u^{\lambda}(t_i) + \sum_{i\in T_2} w_i u_1^{\lambda}(t_i)\right)\Big/\left(\sum_{i\in T_1} w_i u^{-\lambda}(t_i) + \sum_{i\in T_2} w_i u_1^{-\lambda}(t_i)\right)\right)^{1/2\lambda}\right],$$

Since $(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n)$ is any permutation of the arguments $(x_1, x_2, \cdots, x_n)$, we can get $y_i = t_i$ for all i. We then obtain that $\mathrm{NOMR}(\mathbf{x}) = \mathrm{NOMR}(\widehat{\mathbf{x}})$.

(4) *Idempotency.* Since $x_i = x \geq x_0$, then we have,

$$\mathrm{NOMR}(\mathbf{x}) = u^{-1}\left[\left(\left(\sum_{i=1}^{n} w_i u^{\lambda}(x)\right)\Big/\left(\sum_{i=1}^{n} w_i u^{-\lambda}(x)\right)\right)^{1/2\lambda}\right] = x.$$

Appendix A.3. Families of the Reference-Dependent Aggregation Operators

Table A1. Families of the POMR operator.

λ	$\alpha_0, \beta_0, \theta_0$	b_i	Formulation	The Name of Aggregation Operator
λ is odd and $\lambda > 0$	$0 < \alpha_0 < 1$, $0 < \beta_0 < 1$, $\theta_0 > 1$	$y_i \geq b_i$, $b_i \neq 0$	$\left(\left(\sum_{i=1}^{n} w_i(y_i - b_i)^{\alpha_0\lambda}\right) \Big/ \left(\sum_{i=1}^{n} w_i(y_i - b_i)^{-\alpha_0\lambda}\right)\right)^{1/2\lambda\alpha_0}$	Prospect gain ordered multiple reference-dependent operator (PGOMR)
		$y_i \geq b_i$, $b_i = 0$	$\left(\left(\sum_{i=1}^{n} w_i y_i^{\alpha_0\lambda}\right) \Big/ \left(\sum_{i=1}^{n} w_i y_i^{-\alpha_0\lambda}\right)\right)^{1/2\lambda\alpha_0}$	Prospect gain ordered multiple operator (PGOM)
		$y_i < b_i$, $b_i \neq 0$	$\left(\sum_{i=1}^{n} w_i(-\theta_0(b_i - y_i)^{\beta_0})^{\lambda} \Big/ \sum_{i=1}^{n} w_i(-\theta_0(b_i - y_i)^{\beta_0})^{-\lambda}\right)^{1/2\lambda\alpha_0}$	Prospect loss ordered multiple reference-dependent operator (PLOMR)
		$y_i < b_i$, $b_i = 0$	$\left(\sum_{i=1}^{n} w_i(-\theta_0(-y_i)^{\beta_0})^{\lambda} \Big/ \sum_{i=1}^{n} w_i(-\theta_0(-y_i)^{\beta_0})^{-\lambda}\right)^{1/2\lambda\alpha_0}$	Prospect loss ordered multiple operator (PLOM)
	$\alpha_0 \to 1$, $\beta_0 \to 1$, $\theta_0 \to 1$	$b_i \neq 0$	$\left(\sum_{i=1}^{n} w_i(y_i - b_i)^{\lambda} \Big/ \sum_{i=1}^{n} w_i(y_i - b_i)^{-\lambda}\right)^{1/2\lambda}$	Ordered multiple reference-dependent operator (OMR)
		$b_i = 0$	$\left(\sum_{i=1}^{n} w_i {y_i}^{\lambda} \Big/ \sum_{i=1}^{n} w_i {y_i}^{-\lambda}\right)^{1/2\lambda}$	GOWMA operator [8]
$\lambda \to 0$	$0 < \alpha_0 < 1$, $0 < \beta_0 < 1$, $\theta_0 > 1$	$b_i \neq 0$	$\prod_{i \in Y_1} (y_i - b_i)^{w_i} \Big/ \prod_{i \in Y_2} (\theta_0(b_i - y_i)^{\beta_0})^{w_i/\alpha_0}$	Prospect ordered multiple geometric reference-dependent operator (POMGR)
		$b_i = 0$	$\prod_{i \in Y_1} y_i^{w_i} \Big/ \prod_{i \in Y_2} (\theta_0(-y_i)^{\beta_0})^{w_i/\alpha_0}$	Prospect ordered multiple geometric operator (POMG)
	$\alpha_0 \to 1$, $\beta_0 \to 1$, $\theta_0 \to 1$	$b_i \neq 0$	$\prod_{i \in Y_1} (y_i - b_i)^{w_i} \Big/ \prod_{i \in Y_2} (b_i - y_i)^{w_i}$	Ordered multiple geometric reference-dependent aggregation operator (OMGR)
		$b_i = 0$	$\prod_{i \in Y_1} {y_i}^{w_i} \Big/ \prod_{i \in Y_2} (-y_i)^{w_i}$	Ordered multiple geometric aggregation operator (OMG)
$\lambda = 1$	$0 < \alpha_0 < 1$, $0 < \beta_0 < 1$, $\theta_0 > 1$	$b_i \neq 0$	$\left(\frac{\sum_{i \in Y_1} w_i(y_i - b_i)^{\alpha_0} - \sum_{i \in Y_2} w_i\theta_0(b_i - y_i)^{\beta_0}}{\sum_{i \in Y_1} w_i(y_i - b_i)^{-\alpha_0} - \sum_{i \in Y_2} w_i(\theta_0(b_i - y_i)^{\beta_0})^{-1}}\right)^{1/2\alpha_0}$	Constant prospect ordered multiple reference-dependent operator (CPOMR)
		$b_i = 0$	$\left(\frac{\sum_{i \in Y_1} w_i y_i^{\alpha_0} - \sum_{i \in Y_2} w_i\theta_0(-y_i)^{\beta_0}}{\sum_{i \in Y_1} w_i y_i^{-\alpha_0} - \sum_{i \in Y_2} w_i(\theta_0(-y_i)^{\beta_0})^{-1}}\right)^{1/2\alpha_0}$	Constant prospect ordered multiple dependent operator (CPOM)
	$\alpha_0 \to 1$, $\beta_0 \to 1$, $\theta_0 \to 1$	$b_i \neq 0$	$\sqrt{\sum_{i=1}^{n} w_i(y_i - b_i) \Big/ \sum_{i=1}^{n} w_i(y_i - b_i)^{-1}}$	Constant ordered multiple reference-dependent operator (COMR)
		$b_i = 0$	$\sqrt{\sum_{i=1}^{n} w_i y_i \Big/ \sum_{i=1}^{n} w_i {y_i}^{-1}}$	OWMA operator [8]
	$\alpha_0 \to 0$, $\beta_0 \to 0$, $\theta_0 \to 1$	$b_i \neq 0$	$\left(\prod_{i \in Y_1} (y_i - b_i)^{w_i} \Big/ \prod_{i \in Y_2} (y_i - b_i)^{w_i}\right)^{\left(\sum_{i \in Y_1} w_i - \sum_{i \in Y_2} w_i\right)}.$	Constant prospect ordered multiple geometric reference-dependent operator (CPOMGR)
		$b_i = 0$	$\left(\prod_{i \in Y_1} {y_i}^{w_i} \Big/ \prod_{i \in Y_2} {y_i}^{w_i}\right)^{\left(\sum_{i \in Y_1} w_i - \sum_{i \in Y_2} w_i\right)}.$	Constant prospect ordered multiple geometric operator (CPOMG)

Table A2. Families of the SHOMR operator ($\beta = \beta_1, \gamma = \gamma_1, \eta = \eta_1$).

λ	$\beta, \eta, \gamma, \theta_1$	b_i	Formulation	The Name of Aggregation Operator
λ is odd and $\lambda > 0$	$\beta = 1-\gamma$, $\eta \to 0$, $\gamma \to 1$ $\theta_1 > 1$	$b_i \neq 0$	$\left(\dfrac{\sum\limits_{i\in Y_1} w_i(y_i-b_i)^{\lambda} - \sum\limits_{i\in Y_2} w_i\theta_1^{\lambda}(b_i-y_i)^{\lambda}}{\sum\limits_{i\in Y_1} w_i(y_i-b_i)^{-\lambda} - \sum\limits_{i\in Y_2} w_i\theta_1^{-\lambda}(b_i-y_i)^{-\lambda}}\right)^{1/2\lambda}$	*S*-shaped ordered multiple reference-dependent operator (SOMR)
		$b_i = 0$	$\left(\dfrac{\sum\limits_{i\in Y_1} w_i y_i^{\lambda} + \sum\limits_{i\in Y_2} w_i\theta_1^{\lambda} y_i^{\lambda}}{\sum\limits_{i\in Y_1} w_i y_i^{-\lambda} + \sum\limits_{i\in Y_2} w_i\theta_1^{-\lambda} y_i^{-\lambda}}\right)^{1/2\lambda}$	*S*-shaped ordered multiple operator (SOM)
		$y_i \geq b_i or$ $y_i < b_i$, $b_i \neq 0$	$\left(\sum\limits_{i=1}^{n} w_i(y_i-b_i)^{\lambda} \Big/ \sum\limits_{i=1}^{n} w_i(y_i-b_i)^{-\lambda}\right)^{1/2\lambda}$	Ordered multiple reference-dependent operator (OMR)
		$y_i \geq b_i or$ $y_i < b_i$, $b_i = 0$	$\left(\sum\limits_{i=1}^{n} w_i y_i^{\lambda} \Big/ \sum\limits_{i=1}^{n} w_i y_i^{-\lambda}\right)^{1/2\lambda}$	GOWMA operator [8]
$\lambda \to 0$	$\beta, \eta > 0$, $\gamma \in R^{-} \cup (0,1)$ $\theta_1 > 1$	$b_i \neq 0$	$\dfrac{1-\gamma}{\beta}\left(\left(\dfrac{\prod\limits_{i\in Y_1}\left(\left(\frac{\beta}{1-\gamma}(y_i-b_i)+\eta\right)^{\gamma}-\eta^{\gamma}\right)^{w_i}}{\prod\limits_{i\in Y_2}\theta_1^{w_i}\left(\left(\frac{\beta}{1-\gamma}(b_i-y_i)+\eta\right)^{\gamma}-\eta^{\gamma}\right)^{w_i}}+\eta^{\gamma}\right)^{1/\gamma}-\eta\right)$	*S*-shaped HARA ordered geometric reference-dependent operator (SHOGR)
		$b_i = 0$	$\dfrac{1-\gamma}{\beta}\left(\left(\dfrac{\prod\limits_{i\in Y_1}\left(\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\gamma}-\eta^{\gamma}\right)^{w_i}}{\prod\limits_{i\in Y_2}\theta_1^{w_i}\left(\left(\frac{\beta}{1-\gamma}(-y_i)+\eta\right)^{\gamma}-\eta^{\gamma}\right)^{w_i}}+\eta^{\gamma}\right)^{1/\gamma}-\eta\right)$	*S*-shaped HARA ordered geometric operator (SHOG)
	$\beta = 1-\gamma$, $\eta \to 0$, $\theta_1 > 1$	$b_i \neq 0$	$\prod\limits_{i\in Y_1}(y_i-b_i)^{w_i} \Big/ \prod\limits_{i\in Y_2}\theta_1^{w_i/\gamma}(b_i-y_i)^{w_i}$	*S*-shaped ordered geometric reference-dependent operator (SOGR)
		$b_i = 0$	$\prod\limits_{i\in Y_1} y_i^{w_i} \Big/ \prod\limits_{i\in Y_2}\theta_1^{w_i/\gamma}(-y_i)^{w_i}$	*S*-shaped ordered geometric operator (SOG)
	$\beta = 1-\gamma$, $\eta \to 0, \gamma \to 1$ $\theta_1 \to 1$	$b_i \neq 0$	$\prod\limits_{i\in Y_1}(y_i-b_i)^{w_i} \Big/ \prod\limits_{i\in Y_2}(b_i-y_i)^{w_i}$	Ordered multiple geometric reference-dependent operator (OMGR)
		$b_i = 0$	$\prod\limits_{i\in Y_1} y_i^{w_i} \Big/ \prod\limits_{i\in Y_2}(-y_i)^{w_i}$	Ordered multiple geometric operator (OMG)
$\lambda = 1$	$\beta, \eta > 0$, $\gamma \in R^{-} \cup (0,1)$ $\theta_1 > 1$	$b_i \neq 0$	$\dfrac{1-\gamma}{\beta}\left(\left(\sqrt{\dfrac{\sum\limits_{i\in Y_1} w_i\mu_1(y_i) - \sum\limits_{i\in Y_2} w_i\mu_2(y_i)}{\sum\limits_{i\in Y_1} w_i\mu_1^{-1}(y_i) - \sum\limits_{i\in Y_2} w_i{\mu_2}^{-1}(y_i)}}+\eta^{\gamma}\right)^{1/\gamma}-\eta\right)$	Constant HARA ordered multiple reference-dependent operator(CHOMR)
		$b_i = 0$	$\dfrac{1-\gamma}{\beta}\left(\left(\sqrt{\dfrac{\sum\limits_{i\in Y_1} w_iA - \sum\limits_{i\in Y_2} w_iB}{\sum\limits_{i\in Y_1} w_iA^{-1} - \sum\limits_{i\in Y_2} w_iB^{-1}}}+\eta^{\gamma}\right)^{1/\gamma}-\eta\right)$, $A = \left(\dfrac{\beta y_i}{1-\gamma}+\eta\right)^{\gamma}-\eta^{\gamma}, B = \theta_1\left(\left(-\dfrac{\beta y_i}{1-\gamma}+\eta\right)^{\gamma}-\eta^{\gamma}\right)$	Constant HARA ordered multiple operator (CHOM)
	$\beta > 0, \eta \to 0$, $\gamma \in R^{-} \cup (0,1)$ $\theta_1 > 1$	$b_i \neq 0$	$\left(\dfrac{\sum\limits_{i\in Y_1} w_i(y_i-b_i)^{\gamma} - \sum\limits_{i\in Y_2} w_i\theta_1(b_i-y_i)^{\gamma}}{\sum\limits_{i\in Y_1} w_i(y_i-b_i)^{-\gamma} - \sum\limits_{i\in Y_2} w_i\left(\theta_1(b_i-y_i)^{\gamma}\right)^{-1}}\right)^{1/2\gamma}$	Constant ordered multiple reference-dependent operator (COMR)
		$b_i = 0$	$\left(\dfrac{\sum\limits_{i\in Y_1} w_i y_i^{\gamma} - \sum\limits_{i\in Y_2} w_i\theta_1(-y_i)^{\gamma}}{\sum\limits_{i\in Y_1} w_i y_i^{-\gamma} - \sum\limits_{i\in Y_2} w_i\left(\theta_1(-y_i)^{\gamma}\right)^{-1}}\right)^{1/2\gamma}$	Constant ordered multiple operator (COM)
	$\beta > 0, \eta \to 0$, $\gamma, \theta_1 \to 1$	$b_i \neq 0$	$\sqrt{\sum\limits_{i=1}^{n} w_i(y_i-b_i) \Big/ \sum\limits_{i=1}^{n} w_i(y_i-b_i)^{-1}}$	Constant ordered multiple reference-dependent operator (COMR)
		$b_i = 0$	$\sqrt{\sum\limits_{i=1}^{n} w_i y_i \Big/ \sum\limits_{i=1}^{n} w_i {y_i}^{-1}}$	OWMA operator [8]

Table A3. Families of the NHOMR operator.

λ	β, η, γ	θ	Formulation	The Name of Aggregation Operator
$\lambda \in R$, $\lambda \neq 0$	$\beta = 1 - \gamma$, $\eta \to 0, \gamma \to 1$	$\theta \neq 0$	$\left(\frac{\sum_{i \in Y_1} w_i y_i^{\lambda} + \sum_{i \in Y_2} w_i \left((1+\theta) y_i - \theta b_i \right)^{\lambda}}{\sum_{i \in Y_1} w_i y_i^{-\lambda} + \sum_{i \in Y_2} w_i \left((1+\theta) y_i - \theta b_i \right)^{-\lambda}} \right)^{1/2\lambda}$	Non-*S*-shaped ordered multiple reference-dependent basic operator (NOMRB)
		$\theta = 0$	$\left(\sum_{i=1}^{n} w_i y_i^{\lambda} \Big/ \sum_{i=1}^{n} w_i y_i^{-\lambda} \right)^{1/2\lambda}$	GOWMA operator [8]
$\lambda \to 0$	$\beta > 0, \eta > 0$, $\gamma \in R^- \cup (0,1)$	$\theta \neq 0$	$\frac{1-\gamma}{\beta} \left(\left(\prod_{i \in Y_1} C^{w_i} \prod_{i \in Y_2} \left((1+\theta) C^{\gamma} - \theta D^{\gamma} \right)^{w_i/\gamma} \right) - \eta \right)$, $C = \frac{\beta}{1-\gamma} y_i + \eta, D = \frac{\beta}{1-\gamma} b_i + \eta$	Non-*S*-shaped HARA ordered reference-dependent geometric operator (NHORG)
		$\theta = 0$	$\frac{1-\gamma}{\beta} \left(\left(\prod_{i=1}^{n} \left(\frac{\beta}{1-\gamma} y_i + \eta \right)^{w_i} \right) - \eta \right)$	Ordered weighted utility geometric averaging-HARA operator (OWUGA -HARA) [32]
	$\beta > 0, \eta \to 0$, $\gamma \to 1$	$\theta \neq 0$	$\prod_{i \in Y_1} y_i^{w_i} \prod_{i \in Y_2} \left((1+\theta) y_i - \theta b_i \right)^{w_i}$	Non-*S*-shaped ordered geometric operator (NOG)
		$\theta = 0$	$\prod_{i=1}^{n} y_i^{w_i}$	OWGA operator [4]
$\lambda = 1$ or $\lambda = -1$	$\beta, \eta > 0$, $\gamma \in R^- \cup (0,1)$	$\theta \neq 0$	$\frac{1-\gamma}{\beta} \left(\left(\frac{\sum_{i \in Y_1} w_i C^{\gamma} + \sum_{i \in Y_2} w_i \left((1+\theta) C^{\gamma} - \theta D^{\gamma} \right)}{\sum_{i \in Y_1} w_i C^{-\gamma} + \sum_{i \in Y_2} w_i \left((1+\theta) C^{\gamma} - \theta D^{\gamma} \right)^{-1}} \right)^{1/2\gamma} - \eta \right)$	Constant non-*S*-shaped HARA ordered multiple reference-dependent operator (CNHOMR)
		$\theta = 0$	$\frac{1-\gamma}{\beta} \left(\left(\sum_{i=1}^{n} w_i \left(\frac{\beta}{1-\gamma} y_i + \eta \right)^{\gamma} \Big/ \sum_{i=1}^{n} w_i \left(\frac{\beta}{1-\gamma} y_i + \eta \right)^{-\gamma} \right)^{1/2\gamma} - \eta \right)$	Constant HARA ordered multiple operator (CHOM)
	$\beta > 0, \eta \to 0$, $\gamma \in R^- \cup (0,1)$	$\theta \neq 0$	$\left(\frac{\sum_{i \in Y_1} w_i y_i^{\gamma} + \sum_{i \in Y_2} w_i \left((1+\theta) y_i^{\gamma} - \theta b_i^{\gamma} \right)}{\sum_{i \in Y_1} w_i y_i^{-\gamma} + \sum_{i \in Y_2} w_i \left((1+\theta) y_i^{\gamma} - \theta b_i^{\gamma} \right)^{-1}} \right)^{1/2\gamma}$	Constant non-*S*-shaped HARA ordered reference-dependent power operator (CNHORP)
		$\theta = 0$	$\left(\sum_{i=1}^{n} w_i y_i^{\gamma} \Big/ \sum_{i=1}^{n} w_i y_i^{-\gamma} \right)^{1/2\gamma}$	Constant ordered power operator (COP)
	$\beta > 0, \eta \to 0$, $\gamma \to 1$	$\theta \neq 0$	$\sqrt{\frac{\sum_{i \in Y_1} w_i y_i + \sum_{i \in Y_2} w_i \left((1+\theta) y_i - \theta b_i \right)}{\sum_{i \in Y_1} w_i y_i^{-1} + \sum_{i \in Y_2} w_i \left((1+\theta) y_i - \theta b_i \right)^{-1}}}$	Constant non-*S*-shaped ordered reference-dependent operator (CNOR)
		$\theta = 0$	$\sqrt{\sum_{i=1}^{n} w_i y_i \Big/ \sum_{i=1}^{n} w_i y_i^{-1}}$	OWMA operator [8]
	$\beta, \eta > 0$, $\gamma \to 0$	$\theta \neq 0$	$\frac{1}{\beta} \left(\left(\prod_{i \in Y_1} (\beta y_i + \eta)^{w_i} \prod_{i \in Y_2} \left((\beta y_i + \eta)^{(1+\theta) w_i} \Big/ (\beta b_i + \eta)^{\theta w_i} \right) \right) - \eta \right)$	Constant non-*S*-shaped HARA ordered geometric operator (CNHOG)
		$\theta = 0$	$\frac{1}{\beta} \left(\left(\prod_{i=1}^{n} (\beta y_i + \eta)^{w_i} \right) - \eta \right)$	CC-OWGA operator [32]
	$\beta > 0$, $\eta, \gamma \to 0$	$\theta \neq 0$	$\prod_{i \in Y_1} y_i^{w_i} \prod_{i \in Y_2} \left(y_i^{(1+\theta) w_i} \Big/ b_i^{\theta w_i} \right)$	Constant non-*S*-shaped ordered geometric operator (CNOG)
		$\theta = 0$	$\prod_{i=1}^{n} y_i^{w_i}$	OWGA operator [4]

Proofs of Operators in Tables A1–A3.
Proof of the POMGR operator. Let $\mathrm{P}(\mathbf{x})$ denote,

$$\mathrm{POMR}(\mathbf{x}) = \left(\left(\sum_{i\in Y_1} w_i v_1^{\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{\lambda}(y_i)\right)\bigg/\left(\sum_{i\in Y_1} w_i v_1^{-\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{-\lambda}(y_i)\right)\right)^{1/2\lambda\alpha_0},$$

where $v_1(y_i) = (y_i - b_i)^{\alpha_0}$, $v_2(y_i) = \theta_0(b_i - y_i)^{\beta_0}$. By the L'Hôpital's rule, we have that

$$\begin{aligned}
\lim_{\lambda\to 0}\mathrm{P}(\mathbf{x}) = & \lim_{\lambda\to 0}\exp\left[\frac{1}{2\alpha_0\lambda}\log\left(\left(\sum_{i\in Y_1} w_i v_1^{\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{\lambda}(y_i)\right)\bigg/\left(\sum_{i\in Y_1} w_i v_1^{-\lambda}(y_i) - \sum_{i\in Y_2} w_i v_2^{-\lambda}(y_i)\right)\right)\right] \\
= & \exp\left\{\frac{1}{\alpha_0}\left(\sum_{i\in Y_1} w_i\alpha_0\log(y_i - b_i) - \sum_{i\in Y_2} w_i\log(\theta_0(b_i - y_i)^{\beta_0})\right)\right\} \\
= & \prod_{i\in Y_1}(y_i - b_i)^{w_i}\bigg/\prod_{i\in Y_2}(\theta_0(b_i - y_i)^{\beta_0})^{w_i/\alpha_0}.
\end{aligned}$$

Thus,

$$\lim_{\lambda\to 0}\mathrm{P}(\mathbf{x}) = \prod_{i\in Y_1}(y_i - b_i)^{w_i}\bigg/\prod_{i\in Y_2}(\theta_0(b_i - y_i)^{\beta_0})^{w_i/\alpha_0}.$$

Especially, if $b_i = 0$, then we have,

$$\lim_{\lambda\to 0}\mathrm{P}(\mathbf{x}) = \prod_{i\in Y_1} y_i^{w_i}\bigg/\prod_{i\in Y_2}(\theta_0(-y_i)^{\beta_0})^{w_i/\alpha_0}. \quad \square$$

Proof of the CPOMGR operator. By the L'Hôpital's rule, we get that,

$$\begin{aligned}
\lim_{\alpha_0,\,\beta_0\to 0,\theta_0\to 1}\mathrm{P}(\mathbf{x}) = & \lim_{\alpha_0,\,\beta_0\to 0,\theta_0\to 1}\exp\left\{\frac{1}{2\alpha_0}\log\left(\frac{\sum_{i\in Y_1} w_i v_1(y_i) - \sum_{i\in Y_2} w_i v_2(y_i)}{\sum_{i\in Y_1} w_i v_1^{-1}(y_i) - \sum_{i\in Y_2} w_i v_2^{-1}(y_i)}\right)\right\} \\
= & \exp\left\{\left(\sum_{i\in Y_1} w_i\log(y_i - b_i) - \sum_{i\in Y_2} w_i\log(y_i - b_i)\right)\left(\sum_{i\in Y_1} w_i - \sum_{i\in Y_2} w_i\right)\right\} \\
= & \left(\prod_{i\in Y_1}(y_i - b_i)^{w_i}\bigg/\prod_{i\in Y_2}(y_i - b_i)^{w_i}\right)^{\left(\sum_{i\in Y_1} w_i - \sum_{i\in Y_2} w_i\right)}.
\end{aligned}$$

Thus,

$$\lim_{\alpha_0,\,\beta_0\to 0,\theta_0\to 1}\mathrm{P}(\mathbf{x}) = \left(\prod_{i\in Y_1}(y_i - b_i)^{w_i}\bigg/\prod_{i\in Y_2}(y_i - b_i)^{w_i}\right)^{\left(\sum_{i\in Y_1} w_i - \sum_{i\in Y_2} w_i\right)}.$$

Especially, if $b_i = 0$, then we have,

$$\lim_{\alpha_0,\,\beta_0\to 0,\theta_0\to 1}\mathrm{P}(\mathbf{x}) = \left(\prod_{i\in Y_1} y_i^{w_i}\bigg/\prod_{i\in Y_2} y_i^{w_i}\right)^{\left(\sum_{i\in Y_1} w_i - \sum_{i\in Y_2} w_i\right)}. \quad \square$$

Proof of the SHOGR operator. Let $\mathrm{S}(\mathbf{x})$ denote

$$\mathrm{SHOMR}(\mathbf{x})=\frac{1-\gamma}{\beta}\left\{\left[\left(\frac{\sum\limits_{i\in Y_1}w_i\mu_1^{\lambda}(y_i)-\sum\limits_{i\in Y_2}w_i\mu_2^{\lambda}(y_i)}{\sum\limits_{i\in Y_1}w_i\mu_1^{-\lambda}(y_i)-\sum\limits_{i\in Y_2}w_i\mu_2^{-\lambda}(y_i)}\right)^{1/2\lambda}+\eta^{\gamma}\right]^{1/\gamma}-\eta\right\},$$

where $\mu_1(y_i)=(\beta(y_i-b_i)/(1-\gamma)+\eta)^{\gamma}-\eta^{\gamma}$, $\mu_2(y_i)=\theta_1\left((\beta(b_i-y_i)/(1-\gamma)+\eta)^{\gamma}-\eta^{\gamma}\right)$. By the L'Hôpital's rule, we get that,

$$\begin{aligned}
&\lim_{\lambda\to 0}\left(\left(\sum_{i\in Y_1}w_i\mu_1^{\lambda}(y_i)-\sum_{i\in Y_2}w_i\mu_2^{\lambda}(y_i)\right)\Big/\left(\sum_{i\in Y_1}w_i\mu_1^{-\lambda}(y_i)-\sum_{i\in Y_2}w_i\mu_2^{-\lambda}(y_i)\right)\right)^{1/2\lambda}\\
=\;&\lim_{\lambda\to 0}\exp\left\{\frac{1}{2\lambda}\log\left(\left(\sum_{i\in Y_1}w_i\mu_1^{\lambda}(y_i)-\sum_{i\in Y_2}w_i\mu_2^{\lambda}(y_i)\right)\Big/\left(\sum_{i\in Y_1}w_i\mu_1^{-\lambda}(y_i)-\sum_{i\in Y_2}w_i\mu_2^{-\lambda}(y_i)\right)\right)\right\}\\
=\;&\prod_{i\in Y_1}\mu_1^{w_i}(y_i)\Big/\prod_{i\in Y_2}\mu_2^{w_i}(y_i).
\end{aligned}$$

Thus,

$$\lim_{\lambda\to 0}\mathrm{S}(\mathbf{x})=\frac{1-\gamma}{\beta}\left\{\left[\left(\frac{\prod\limits_{i\in Y_1}\left(\left(\frac{\beta}{1-\gamma}(y_i-b_i)+\eta\right)^{\gamma}-\eta^{\gamma}\right)^{w_i}}{\prod\limits_{i\in Y_2}\theta_1^{w_i}\left(\left(\frac{\beta}{1-\gamma}(b_i-y_i)+\eta\right)^{\gamma}-\eta^{\gamma}\right)^{w_i}}\right)+\eta^{\gamma}\right]^{1/\gamma}-\eta\right\}.$$

Especially, if $b_i=0$, then we have,

$$\lim_{\lambda\to 0}\mathrm{S}(\mathbf{x})=\frac{1-\gamma}{\beta}\left\{\left[\left(\frac{\prod\limits_{i\in Y_1}\left(\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\gamma}-\eta^{\gamma}\right)^{w_i}}{\prod\limits_{i\in Y_2}\theta_1^{w_i}\left(\left(\frac{\beta}{1-\gamma}(-y_i)+\eta\right)^{\gamma}-\eta^{\gamma}\right)^{w_i}}\right)+\eta^{\gamma}\right]^{1/\gamma}-\eta\right\}.\quad\square$$

Proof of the NOMRB operator. Let $\mathrm{N}(\mathbf{x})$ denote,

$$\mathrm{NHOMR}(\mathbf{x})=\frac{1-\gamma}{\beta}\left\{\left(\left(\sum_{i\in Y_1}w_iu^{\lambda}(y_i)+\sum_{i\in Y_2}w_iu_1^{\lambda}(y_i)\right)\Big/\left(\sum_{i\in Y_1}w_iu^{-\lambda}(y_i)+\sum_{i\in Y_2}w_iu_1^{-\lambda}(y_i)\right)\right)^{1/2\lambda\gamma}-\eta\right\},$$

where $u(y_i)=(\beta y_i/(1-\gamma)+\eta)^{\gamma}$, $u_1(y_i)=(1+\theta)\left(\beta y_i/(1-\gamma)+\eta\right)^{\gamma}-\theta\left(\beta b_i/(1-\gamma)+\eta\right)^{\gamma}$. Thus, we get that

$$\begin{aligned}
&\lim_{\beta=1-\gamma,\,\eta\to 0,\,\gamma\to 1}\mathrm{N}(\mathbf{x})\\
=\;&\lim_{\beta=1-\gamma,\,\eta\to 0,\,\gamma\to 1}\frac{1-\gamma}{\beta}\left[\left(\left(\sum_{i\in Y_1}w_iu^{\lambda}(y_i)+\sum_{i\in Y_2}w_iu_1^{\lambda}(y_i)\right)\Big/\left(\sum_{i\in Y_1}w_iu^{-\lambda}(y_i)+\sum_{i\in Y_2}w_iu_1^{-\lambda}(y_i)\right)\right)^{1/2\lambda\gamma}-\eta\right]\\
=\;&\left(\left(\sum_{i\in Y_1}w_iy_i^{\lambda}+\sum_{i\in Y_2}w_i\left((1+\theta)\,y_i-\theta b_i\right)^{\lambda}\right)\Big/\left(\sum_{i\in Y_1}w_iy_i^{-\lambda}+\sum_{i\in Y_2}w_i\left((1+\theta)\,y_i-\theta b_i\right)^{-\lambda}\right)\right)^{1/2\lambda}.
\end{aligned}$$

Noting that if $\theta=0$, we then obtain,

$$\lim_{\beta=1-\gamma,\,\eta\to 0,\,\gamma\to 1}\mathrm{N}(\mathbf{x})=\left(\sum_{i=1}^{n}w_iy_i^{\lambda}\Big/\sum_{i=1}^{n}w_iy_i^{-\lambda}\right)^{1/2\lambda}.\quad\square$$

Proof of the NHORG operator. By the L'Hôpital's rule, we have that,

$$\begin{aligned}
&\lim_{\lambda\to 0}\left(\left(\sum_{i\in Y_1} w_i u^{\lambda}(y_i)+\sum_{i\in Y_2} w_i u_1^{\lambda}(y_i)\right)\Big/\left(\sum_{i\in Y_1} w_i u^{-\lambda}(y_i)+\sum_{i\in Y_2} w_i u_1^{-\lambda}(y_i)\right)\right)^{1/2\lambda\gamma}\\
=\;&\lim_{\lambda\to 0}\exp\left\{\frac{1}{2\lambda\gamma}\log\left(\left(\sum_{i\in Y_1} w_i u^{\lambda}(y_i)+\sum_{i\in Y_2} w_i u_1^{\lambda}(y_i)\right)\Big/\left(\sum_{i\in Y_1} w_i u^{\lambda}(y_i)+\sum_{i\in Y_2} w_i u_1^{-\lambda}(y_i)\right)\right)\right\}\\
=\;&\left(\prod_{i\in Y_1}(u(y_i))^{w_i/\gamma}\right)\left(\prod_{i\in Y_2}(u_1(y_i))^{w_i/\gamma}\right).
\end{aligned}$$

Thus,

$$\lim_{\lambda\to 0}\mathrm{N}(\mathbf{x})=\frac{1-\gamma}{\beta}\left\{\left[\prod_{i\in Y_1}\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{w_i}\prod_{i\in Y_2}\left((1+\theta)\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{\gamma}-\theta\left(\frac{\beta}{1-\gamma}b_i+\eta\right)^{\gamma}\right)^{w_i/\gamma}\right]-\eta\right\}.$$

In addition, if $\theta=0$, we derive,

$$\lim_{\lambda\to 0}\mathrm{N}(\mathbf{x})=\frac{1-\gamma}{\beta}\left\{\left[\prod_{i=1}^{n}\left(\frac{\beta}{1-\gamma}y_i+\eta\right)^{w_i}\right]-\eta\right\}.\quad\square$$

Proof of the CNHOG operator. By the L'Hôpital's rule, we have that,

$$\begin{aligned}
&\lim_{\gamma\to 0}\left(\left(\sum_{i\in Y_1} w_i u(y_i)+\sum_{i\in Y_2} w_i u_1(y_i)\right)\Big/\left(\sum_{i\in Y_1} w_i u^{-1}(y_i)+\sum_{i\in Y_2} w_i u_1^{-1}(y_i)\right)\right)^{1/2\gamma}\\
=\;&\lim_{\gamma\to 0}\exp\left\{\frac{1}{2\gamma}\log\left(\left(\sum_{i\in Y_1} w_i u(y_i)+\sum_{i\in Y_2} w_i u_1(y_i)\right)\Big/\left(\sum_{i\in Y_1} w_i u^{-1}(y_i)+\sum_{i\in Y_2} w_i u_1^{-1}(y_i)\right)\right)\right\}\\
=\;&\left(\prod_{i\in Y_1}(\beta y_i+\eta)^{w_i}\right)\left(\prod_{i\in Y_2}\left((\beta y_i+\eta)^{(1+\theta)}\Big/(\beta b_i+\eta)^{\theta}\right)^{w_i}\right).
\end{aligned}$$

Thus,

$$\lim_{\gamma\to 0}\mathrm{N}(\mathbf{x})=\frac{1}{\beta}\left(\left(\prod_{i\in Y_1}(\beta y_i+\eta)^{w_i}\right)\left(\prod_{i\in Y_2}\left((\beta y_i+\eta)^{(1+\theta)}\Big/(\beta b_i+\eta)^{\theta}\right)^{w_i}\right)-\eta\right).$$

This completes the proof. $\square$

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