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Coincidences of the Concave Integral and the Pan-Integral

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Abstract: In this note, we discuss when the concave integral coincides with the pan- integral with respect to the standard arithmetic operations + and ·. The subadditivity of the underlying monotone measure is one sufficient condition for this equality. We show also another sufficient condition, which, in the case of finite spaces, is necessary, too.

Keywords: monotone measure; subadditivity; concave integral; pan-integral

1. Introduction

Integrals play a prominent role in almost any area dealing with quantitative information, varying from physics to sociology, including economics or engineering, but also many intelligent systems. The standard calculus is based on the Riemann integral [1]. Note that Riemann has generalized the earlier approaches known from antic Greece, and he has completed the ideas originated by Newton, Leibniz, Cauchy and others. Lebesgue [2] has further generalized this integral, working with σ-additive measures, and thus he has enabled the development of many other theories, first of all the Kolmogorovian probability theory [3]. Even in the Kolmogorov era, there were ideas of integrating some particular non-additive measures, especially outer and inner measures, see [4]. These efforts were completed by the introduction of the Choquet integral [5], which for σ-additive measures coincides with the Lebesgue integral. Further development of integrals based on monotone but non necessarily additive measures was initiated first of all by needs of economy, multicriteria decision support, psychology, sociology, etc., i.e., by needs of branches where the phenomenon of interaction is crucial. Among these new types of integrals (based on monotone measures) are the Sugeno integral [6], Shilkret integral [7], pan-integral [8], and the concave integral introduced by Lehrer [9,10]. Note that there are successful efforts how to axiomatize some types of integrals, see, e.g., the concept of universal integrals from [11], or how to construct integrals, recall the decomposition integrals introduced in [12]. As already mentioned, the Choquet integral generalizes the Lebesgue integral, i.e., for any σ-additive measure μ these integrals coincide. Similarly, when considering a σ-additive measure μ, the Lebesgue integral coincides with the pan-integral, as well as with the concave integral. Note that this is not the case of the Shilkret integral neither of the Sugeno integral. Recall also that all three earlier mentioned integrals (Choquet, pan and concave integrals) are decomposition integrals. Namely, the Choquet integral is based on finite chains, the pan-integral is based on finite partition while the concave integral is related to arbitrary finite set systems, for more details see [12]. The aim of this paper is a further discussion of the coincidence of integrals, whose starting point is the above-mentioned fact that, if a σ-additive measure μ is considered, the all four Lebesgue, Choquet, pan and concave integrals coincide. Obviously, for μ which is not σ-additive, the Lebesgue integral is not defined, and the remaining three integrals are different, in general. Nevertheless, for some particular monotone measure μ, some of these integrals may coincide.

Lehrer [9,13] discussed the relationship between the concave integral and the Choquet integral, and showed that these two integrals coincide if and only if the underlying capacity ν is convex (also known as supermodular). In [14] the order relationship between the pan-integral (with respect to the usual addition + and usual multiplication $\cdot$) and the Choquet integral was shown by using the subadditivity and superadditivity of monotone measures.

We have recently discussed the relationship between the concave integral and the pan-integral on finite spaces [15]. We have introduced the concept of *minimal atom* of a monotone measure. By means of two important structure characteristics related to minimal atoms: *minimal atoms disjoint property* and *subadditivity for minimal atoms*, we have shown a necessary and sufficient condition ensuring that the concave integral coincides with the pan-integral on finite spaces. A research on coincidences of the Choquet integral and the pan-integral on finite space was made by using the minimal atom of monotone measure (see [16]).

We pointed out that in the above-mentioned study we have only considered the case that the underlying space is finite. However, our approach based on minimal atoms does not apply to infinite spaces, see [15].

This paper will focus on the relationship between the concave integrals and pan-integrals on general spaces (not necessarily finite). We shall show that if the underlying monotone measure μ is subadditive, then the concave integral coincides with the pan-integral w.r.t. the usual addition + and usual multiplication $\cdot$.

2. Preliminaries

Let X be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of X. $\mathbf{F}_+$ denotes the class of all finite nonnegative real-valued measurable functions on the measurable space $(X, \mathcal{A})$. Unless stated otherwise all the subsets mentioned are supposed to belong to $\mathcal{A}$, and all the functions mentioned are supposed to belong to $\mathbf{F}_+$.

Definition 1. ([14]) A *monotone measure* on $\mathcal{A}$ is an extended real valued set function $\mu : \mathcal{A} \to [0, +\infty]$ satisfying the following conditions:

(1) $\mu(\emptyset) = 0$; (vanishing at $\emptyset$)

(2) $\mu(A) \leq \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{F}$. (monotonicity)

When μ is a monotone measure, the triple $(X, \mathcal{A}, \mu)$ is called a monotone measure space ([14,17,18]). In some literature, such a monotone measure μ constrained by the boundary condition $\mu(X) = 1$ is also called a capacity or a fuzzy measure or a nonadditive probability, etc.

Let μ be a monotone measure on $(X, \mathcal{A})$. μ is said to be

(i) *subadditive* if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$;

(ii) *superadditive* if $\mu(A \cup B) \geq \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$ such that $A \cap B = \emptyset$ [19];

(iii) *supermodular* if $\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$ [19];

(iv) *continuous from below* (resp. *from above*), if $\lim_{n\to\infty} \mu(E_n) = \mu(E)$ whenever $E_n \nearrow E$ (resp. whenever $E_n \searrow E$ and $\mu(E_1) < \infty$) ([20]).

In our discussions we concern three types of nonlinear integrals, the Choquet integral, the concave integral and the pan-integral. We recall their definitions.

We consider a given monotone measure space $(X, \mathcal{A}, \mu)$, and let $f \in \mathbf{F}_+$, χ_A denote the indicator function of measurable set A.

The *Choquet integral* [5] (see also [18,19]) of f on X with respect to μ, is defined by

$$\int^{Cho} f\,d\mu = \int_0^{\infty} \mu(\{x : f(x) \geq t\})\,dt,$$

where the right side integral is the Riemann integral.

Lehrer [13] introduced a new integral known as concave integral (see also [9,21]), as follows: The *concave integral* of f on X is defined by

$$\int^{cav} f d\mu = \sup\Big\{ \sum_{i=1}^{n} \lambda_i \mu(A_i) : \sum_{i=1}^{n} \lambda_i \chi_{A_i} \leq f, \\ \{A_i\}_{i=1}^{n} \subset \mathcal{A}, \lambda_i \geq 0, n \in \mathbb{N} \Big\}.$$

The concept of a pan-integral [8,14] involves two binary operations, the pan-addition $\oplus$ and pan-multiplication $\otimes$ of real numbers (see also [14,18,22–26]). In this paper we only consider the pan-integrals with respect to the usual addition + and usual multiplication $\cdot$. Note that the general case of pan-integrals is discussed in Concluding Remarks.

The pan-integral of f on X w.r.t. the usual addition + and usual multiplication $\cdot$ (in short, pan-integral), is given by

$$\int^{pan} f d\mu = \sup\Big\{ \sum_{i=1}^{n} \lambda_i \mu(A_i) : \sum_{i=1}^{n} \lambda_i \chi_{A_i} \leq f, \\ \{A_i\}_{i=1}^{n} \subset \mathcal{A} \text{ is a partition of } X, \lambda_i \geq 0, n \in \mathbb{N} \Big\}.$$

All these integrals are covered by a recent concept of decomposition integrals by Even and Lehrer [12].

Note that the pan-integral is related to finite partitions of X, the concave integral to any finite set systems of measurable subsets of X. The Choquet integral is based on chains of sets, it can be expressed in the following form:

$$\int^{Cho} f\,d\mu = \sup\Big\{ \sum_{i=1}^{n} \lambda_i \mu(A_i) : \sum_{i=1}^{n} \lambda_i \chi_{A_i} \leq f, \\ \{A_i\}_{i=1}^{n} \subset \mathcal{A} \text{ is a chain}, \lambda_i \geq 0, n \in \mathbb{N} \Big\}.$$

Comparing above three definitions, it is obvious that for each $f \in \mathbf{F}_+$,

$$\int^{cav} f d\mu \geq \int^{pan} f d\mu \tag{1}$$

and

$$\int^{cav} f d\mu \geq \int^{Cho} f d\mu. \tag{2}$$

In general, $\int^{cav} f d\mu \neq \int^{pan} f d\mu$, $\int^{cav} f d\mu \neq \int^{Cho} f d\mu$.

Example 2. Let $X = \mathbb{N}$ (the set of all positive integers). The monotone measure $\mu\colon 2^{\mathbb{N}} \to [0,1]$ is defined by

$$\mu(E) = \begin{cases} 1 & \text{if } |E| = \infty \text{ and } 1 \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We take

$$f(x) = \begin{cases} 2, & \text{if } x = 1; \\ 1, & \text{if } x = 2, 3, \ldots. \end{cases}$$

Then $\int^{cav} f d\mu = 2$, and $\int^{pan} f d\mu = \int^{Cho} f d\mu = 1$. Thus, $\int^{cav} f d\mu \neq \int^{pan} f d\mu$, $\int^{cav} f d\mu \neq \int^{Cho} f d\mu$.

Observe that the Choquet integral and the pan-integral are not comparable.

Example 3. Let $X = \{1,2\}, \mathcal{A} = 2^X$, and the monotone measure μ be defined as $\mu(X) = 3$, $\mu(\{1\}) = \mu(\{2\}) = 1, \mu(\emptyset) = 0$. Let $f(x) = x$. Then

$$\int^{Cho} f d\mu = \mu(X) + \mu(\{2\}) = 4$$

and

$$\int^{pan} f d\mu = \max\Big(\mu(X), \mu(\{1\}) + 2\mu(\{2\})\Big) = 3.$$

Thus, we have $\int^{Cho} f d\mu > \int^{pan} f d\mu$.

Example 4. Let $X = \{1,2\}, \mathcal{A} = 2^X$, and the monotone measure μ be defined as $\mu(A) = 1$ if $A \neq \emptyset$ and $\mu(\emptyset) = 0$. Let $f(x) = x$. Then

$$\int^{Cho} f d\mu = \mu(X) + \mu(\{2\}) = 2$$

and

$$\int^{pan} f d\mu = \max\Big(\mu(X), \mu(\{1\}) + 2\mu(\{2\})\Big) = 3.$$

Thus, $\int^{Cho} f d\mu < \int^{pan} f d\mu$.

The above examples indicate that any two of the three integrals do not coincide, in general. They are significantly different from each other.

3. The Main Results

We consider a given measurable space $(X, \mathcal{A})$, and let $\mathcal{M}$ be the class of all monotone measures defined on $(X, \mathcal{A})$.

For the convenience of our discussion, we denote $\mathbf{Ch}_\mu(f) = \int^{Cho} f\, d\mu$, $\mathbf{Cav}_\mu(f) = \int^{cav} f\, d\mu$ and $\mathbf{Pan}_\mu(f) = \int^{pan} f\, d\mu$.

In [13] (see also [9,27,28]) the relationship between the the concave integral and the Choquet integral was discussed, as follows:

Theorem 5. *Let $\mu \in \mathcal{M}$. Then $\mathbf{Cav}_\mu \equiv \mathbf{Ch}_\mu$, i.e., for each $f \in \mathbf{F}_+$,*

$$\int^{cav} f d\mu = \int^{Cho} f d\mu$$

if and only if μ is supermodular, i.e., for any $A, B \in \mathcal{A}$

$$\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B).$$

The following results were shown in [14] (Theorems 10.7 and 10.8 in [14]).

Theorem 6. *Let $\mu \in \mathcal{M}$. Then*

(i) *if μ is superadditive, then $\mathbf{Pan}_\mu \le \mathbf{Ch}_\mu$, i.e., for each $f \in \mathbf{F}_+$, $\mathbf{Pan}_\mu(f) \le \mathbf{Ch}_\mu(f)$;*

(ii) *if μ is subadditive, then $\mathbf{Pan}_\mu \ge \mathbf{Ch}_\mu$.*

Moreover, we have the following result (see also Mesiar et al. [16]):

Theorem 7. *Let $\mu \in \mathcal{M}$. If $\mathbf{Pan}_\mu \equiv \mathbf{Ch}_\mu$, i.e., for each $f \in \mathbf{F}_+$,*

$$\int^{pan} f d\mu = \int^{Cho} f d\mu,$$

then μ is superadditive.

Proof. Observe that $\mathbf{Ch}_\mu(\chi_E) = \mu(E)$ for any $E \subseteq X$ and, thus for any $A, B \subseteq X,\ A \cap B = \emptyset$, we have

$$\begin{aligned} & \mu(A \cup B) = \mathbf{Ch}_\mu(\chi_{A\cup B}) = \mathbf{Pan}_\mu(\chi_{A\cup B}) \\ = \quad & \sup\left\{ \sum_{i=1}^{k} \lambda_i \cdot \mu(D_i) \mid (D_i)_{i=1}^k \text{ is a disjoint system,} \right. \\ & \qquad\qquad \left. \lambda_1, \lambda_2, \cdots, \lambda_k \ge 0 \text{ and } \sum_{i=1}^{k} \lambda_i \chi_{A_i} \le \chi_{A\cup B} \right\} \\ \ge \quad & \mu(A) + \mu(B), \end{aligned}$$

i.e., μ is superadditive. □

Remark 8. The converse of Theorem 7 may not be true. Observe that in Example 2, the monotone measure μ is superadditive, but $\int^{Cho} f d\mu > \int^{pan} f d\mu$.

Now we present our main result.

Theorem 9. *Let $\mu \in \mathcal{M}$. If μ is subadditive, then $\mathbf{Cav}_\mu \equiv \mathbf{Pan}_\mu$, i.e., for each $f \in \mathbf{F}_+$,*

$$\int^{pan} f d\mu = \int^{cav} f d\mu.$$

Proof. It suffices to prove that $\int^{pan} f d\mu \ge \int^{cav} f d\mu$ holds for any $f \in \mathbf{F}_+$. To prove this fact, it suffices to prove that for any $\{A_i\}_{i=1}^N \subset \mathcal{A}$ and $\lambda_i \ge 0, i = 1, 2, \ldots, N$, there is a sequence of pairwise disjoint subsets $\{B_j\}_{j=1}^M \subset \mathcal{A}$ and a sequence of nonnegative numbers $l_j, j = 1, 2, \ldots, M$ such that

$$\sum_{i=1}^{N} \lambda_i \chi_{A_i} = \sum_{j=1}^{M} l_j \chi_{B_j} \tag{3}$$

and

$$\sum_{i=1}^{N} \lambda_i \mu(A_i) \le \sum_{j=1}^{M} l_j \mu(B_j). \tag{4}$$

For $N = 2$, observe that

$$\lambda_1 \chi_{A_1} + \lambda_2 \chi_{A_2} = \lambda_1 \chi_{A_1 - (A_1 \cap A_2)} + \lambda_2 \chi_{A_2 - (A_1 \cap A_2)} + (\lambda_1 + \lambda_2) \chi_{A_1 \cap A_2}.$$

If we let
$$l_1 = \lambda_1,\ l_2 = \lambda_2,\ l_3 = \lambda_1 + \lambda_2$$
and
$$B_1 = A_1 - (A_1 \cap A_2),\ B_2 = A_2 - (A_1 \cap A_2),\ B_3 = A_1 \cap A_2,$$
then
$$\sum_{i=1}^{2} \lambda_i \chi_{A_i} = \sum_{j=1}^{3} l_j \chi_{B_j}.$$
Moreover, thanks to the subadditivity of μ, we have
$$\begin{aligned} & \lambda_1 \mu(A_1) + \lambda_2 \mu(A_2) \\ \leq\ & \lambda_1 (\mu(B_1) + \mu(B_3)) + \lambda_2 (\mu(B_2) + \mu(B_3)) \\ =\ & l_1 \mu(B_1) + l_2 \mu(B_2) + l_3 \mu(B_3). \end{aligned}$$

Now suppose that (3) and (4) hold for $N = k$, we need to verify that they are also true for $N = k+1$. For $\sum_{i=1}^{k+1} \lambda_i \chi_{A_i}$, we have
$$\begin{aligned} \sum_{i=1}^{k+1} \lambda_i \chi_{A_i} &= \sum_{i=1}^{k} \lambda_i \chi_{A_i} + \lambda_{k+1} \chi_{A_{k+1}} \\ &= \sum_{j=1}^{N'} \alpha_j \chi_{C_j} + \lambda_{k+1} \chi_{A_{k+1}}, \end{aligned}$$
where $C_j, j = 1, 2, \ldots, N'$ are pairwise disjoint subsets of X, $\alpha_j \geq 0$ with $\sum_{i=1}^{k} \lambda_i \mu(A_i) \leq \sum_{j=1}^{N'} \alpha_j \mu(C_j)$. Observe the facts that
$$C_j = \left(C_j - (C_j \cap A_{k+1})\right) \bigcup (C_j \cap A_{k+1})$$
and
$$A_{k+1} = \left(A_{k+1} - \bigcup_{j=1}^{N'} (A_{k+1} \cap C_j)\right) \bigcup \left(\bigcup_{j=1}^{N'} (A_{k+1} \cap C_j)\right).$$
If we let
$$\begin{aligned} B_j &= C_j - (C_j \cap A_{k+1}),\ \ j = 1, 2, \ldots, N' \\ B_{N'+j} &= C_j \cap A_{k+1},\ j = 1, 2, \ldots, N', \\ B_{2N'+1} &= A_{k+1} - \bigcup_{j=1}^{N'} (A_{k+1} \cap C_j) \end{aligned}$$
and let
$$l_j = \alpha_j,\ l_{N'+j} = \alpha_j + \lambda_{k+1}, j = 1, 2, \ldots, N',\ l_{2N'+1} = \lambda_{k+1},$$
then
$$\sum_{i=1}^{k+1} \lambda_i \chi_{A_i} = \sum_{j=1}^{2N'+1} l_j \chi_{B_j}$$

and

$$\begin{aligned}
& \sum_{i=1}^{k+1} \lambda_i \mu(A_i) \\
\leq \quad & \sum_{j=1}^{N'} \alpha_j \mu(C_j) + \lambda_{k+1} \mu(A_{k+1}) \\
\leq \quad & \sum_{j=1}^{N'} \alpha_j \left(\mu(B_j) + \mu(B_{N'+j}) \right) \\
& \quad + \lambda_{k+1} \left(\mu(B_{2N'+1}) + \sum_{j=1}^{N'} \mu(B_{N'+j}) \right) \\
= \quad & \sum_{j=1}^{N'} \alpha_j \mu(B_j) + \sum_{j=1}^{N'} (\alpha_j + \lambda_{k+1}) \mu(B_{N'+j}) + \lambda_{k+1} \mu(B_{2N'+1}) \\
= \quad & \sum_{j=1}^{2N'+1} l_j \mu(B_j).
\end{aligned}$$

□

The following example shows that the subadditivity in Theorem 9 is not a necessary condition.

Example 10. Let $X = [0,1]$ and $\mathcal{A} = \mathcal{B}(X)$ (the Borel σ-algebra over X). Let a monotone measure μ be defined as

$$\mu(E) = \begin{cases} 1 & \text{if } E = X, \\ 0 & \text{if } E \neq X. \end{cases}$$

Then, for all $f \in \mathbf{F}_+$

$$\int^{cav} f d\mu = \int^{pan} f d\mu = \int^{Cho} f d\mu = \inf\{f(x) | x \in X\}.$$

But μ is not subadditive. Indeed, for any Borel measurable proper subset E of $\mathcal{A}$, we have $\mu(E \cup E^c) = \mu(X) = 1 > 0 = \mu(E) + \mu(E^c)$.

The next theorem gives another sufficient condition ensuring the coincidence of the pan-integral and concave integral, now covering Example 10, too.

Theorem 11. *Let μ be a monotone measure on $(X, \mathcal{A})$. If there is a countable partition $\{E_t \mid t \in T\} \subset \mathcal{A}$ of X, so that $e_t = \mu(E_t), t \in T$, and*

$$\mu(E) \leq \sum_{t \in T, E_t \subset E} e_t, \ \ \forall E \in \mathcal{A},$$

then the concave integral coincides with the pan-integral with respect to the usual arithmetic operation " $+$ " and " $\cdot$ ".

Proof. It is not difficult to check that under the above constraints on μ, for any $f \in \mathbf{F}_+$ it holds

$$\int^{cav} f d\mu = \int^{pan} f d\mu = \sum_{t \in T} e_t \cdot \inf\{f(x) | x \in E_t\}.$$

□

Observe that if X is a finite space, then the constraints on μ given in Theorem 11 are also necessary, see [15]. Moreover, consider a lower probability μ on a finite set $X = \{1, 2, \ldots, n\}$ in the sense of de Finetti [29], i.e., there is partition $\{E_1, E_2, \cdots, E_r\}$ of X such that

$$\mu(E_1) = e_1, \mu(E_2) = e_2, \cdots, \mu(E_r) = e_r, \;\; e_1 + e_2, \cdots, e_r = 1,$$

and for any $E \subset X$ it holds

$$\mu(E) = \sum_{1 \le i \le r, E_i \subset E} e_i.$$

Note that μ is then a belief measure [14] which is k-additive [30]. Clearly, μ satisfies the constraints of Theorem 11, and thus $\mathbf{Cav}_\mu = \mathbf{Pan}_\mu$. Moreover, both these integrals coincide in this case also with the Choquet integral, i.e., $\mathbf{Ch}_\mu = \mathbf{Cav}_\mu = \mathbf{Pan}_\mu$. Note that the case when μ is σ-additive (i.e., a discrete probability measure on X) is a particular subcase of the mentioned class of lower probabilities related to the finest partition of X into the singletons, i.e., when $E_1 = \{1\}, E_2 = \{2\}, \cdots, E_n = \{n\}$. Another particular subclass of de Finetti's lower probabilities, known from the game theory, is formed by the unanimity games. In that case, for a non-empty subset E of X, we define a monotone measure μ_E on X as

$$\mu_E(A) = \begin{cases} 1 & \text{if } \; E \subset A, \\ 0 & \text{otherwise.} \end{cases}$$

and then for all three considered integrals their equal output is $\min\{f(i) \mid i \in E\}$.

4. Concluding Remarks

We have proved the coincidence of the concave integral and the pan-integral w.r.t. the usual addition + and usual multiplication · on general spaces (not necessarily finite spaces) by considering the subadditivity of related monotone measures. However, the subadditivity condition is only sufficient, but not necessary (see Example 10). We have shown also some other sufficient conditions ensuring the discussed coincidence $\mathbf{Cav}_\mu = \mathbf{Pan}_\mu$, including Theorem 11 which in the case of a finite universe X gives also a necessary condition. In general, a complete characterization of capacities μ ensuring the coincidence $\mathbf{Cav}_\mu = \mathbf{Pan}_\mu$ is a challenging open problem.

Note that the pan-integral [8,14] was established based on a special type of commutative isotonic semiring $(\overline{R}_+, \oplus, \otimes)$. A related concept of generalizing Lebesgue integral based on a *generalized ring* $(\overline{R}_+, \oplus, \otimes)$ (the commutativity of $\otimes$ is not required) was proposed and discussed in [31]. On the other hand, Mesiar et al. introduced pseudo-concave integrals [32] (see also [33]) and pseudo-concave Benvenuti integrals [34] by means of the pseudo-addition $\oplus$ and pseudo-multiplication $\otimes$ of reals based on a generalized ring $(\overline{R}_+, \oplus, \otimes)$. Similarly, Choquet-like integrals [35] are based on a particular ring $(\overline{R}_+, \oplus, \otimes)$.

In further research, we shall investigate the relationships among these four integrals on a fixed generalized ring $(\overline{R}_+, \oplus, \otimes)$.

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