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New Generalized Ekeland's Variational Principle, Critical Point Theorems and Common Fuzzy Fixed Point Theorems Induced by Lin-Du's Abstract Maximal Element Principle

Junjian Zhao [1] and Wei-Shih Du [2,*]

[1] School of Mathematical Sciences, Tiangong University, Tianjin 300387, China; zhaojunjian@tiangong.edu.cn
[2] Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan
* Correspondence: wsdu@mail.nknu.edu.tw

Abstract: In this paper, by applying the abstract maximal element principle of Lin and Du, we present some new existence theorems related with critical point theorem, maximal element theorem, generalized Ekeland's variational principle and common (fuzzy) fixed point theorem for essential distances.

Keywords: maximal element; fixed point; sizing-up function; μ-bounded quasi-ordered set; critical point; fuzzy mapping; Ekeland's variational principle; Caristi's fixed point theorem; Takahashi's nonconvex minimization theorem; essential distance

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1. Introduction

Maximal element principle (MEP, for short) is a fascinating theory that has a wide range of applications in many fields of mathematics. Various generalizations in different directions of maximal element principle have been investigated by several authors, see [1–8] and references therein. Lin and Du [3,4,7] introduced the concepts of the sizing-up function and μ-bounded quasi-ordered set to define sufficient conditions for a nondecreasing sequence on a quasi-ordered set to have an upper bound and used them to establish an abstract MEP.

Definition 1 (see [3,4,7]). *Let E be a nonempty set. A function $\mu : 2^E \to [0, +\infty]$ defined on the power set 2^E of E is called sizing-up if it satisfies the following properties*

(μ1) $\mu(\emptyset) = 0$;
(μ2) $\mu(C) \le \mu(D)$ *if* $C \subseteq D$.

Definition 2 (see [3,4,7]). *Let E be a nonempty set and $\mu : 2^E \to [0, +\infty]$ a sizing-up function. A multivalued map $T : E \to 2^E$ with nonempty values is said to be of type (μ) if for each $x \in E$ and $\epsilon > 0$, there exists a $y = y(x, \epsilon) \in T(x)$ such that $\mu(T(y)) \le \epsilon$.*

Definition 3 (see [3,4,7]). *A quasi-ordered set $(E, \lesssim)$ with a sizing-up function $\mu : 2^E \to [0, +\infty]$, in short $(E, \lesssim, \mu)$, is said to be μ-bounded if every $\lesssim$-nondecreasing sequence $z_1 \lesssim z_2 \lesssim \cdots \lesssim z_n \lesssim z_{n+1} \lesssim \cdots$ in E satisfying*

$$\lim_{n \to +\infty} \mu(\{z_n, z_{n+1}, \cdots\}) = 0$$

has an upper bound.

The following abstract maximal element principle of Lin and Du is established in [3,4,7].

Theorem 1. *Let $(E, \lesssim, \mu)$ be a μ-bounded quasi-ordered set with a sizing-up function $\mu : 2^E \to [0, +\infty]$. For each $x \in E$, let $S : E \to 2^E$ be defined by $S(x) = \{y \in E : x \lesssim y\}$. If S is of type (μ), then for each $z_0 \in E$, there exists a nondecreasing sequence $z_0 \lesssim z_1 \lesssim z_2 \lesssim \cdots$ in E and $v \in E$ such that*

(i) v *is an upper bound of* $\{z_n\}_{n=0}^{\infty}$;
(ii) $S(v) \subseteq \bigcap_{n=1}^{+\infty} S(z_n)$;
(iii) $\mu(\bigcap_{n=1}^{+\infty} S(z_n)) = \mu(S(v)) = 0$.

Ekeland's variational principle [9,10] is a very important tool for the study of approximate solutions approximate solutions of nonconvex minimization problems.

Theorem 2. *(Ekeland's variational principle) Let (M, d) be a complete metric space and $f : M \to (-\infty, +\infty]$ be a proper lower semicontinuous and bounded below function. Let $\varepsilon > 0$ and $u \in M$ with $f(u) < +\infty$. Then there exists $v \in M$ such that*

(a) $f(v) + \varepsilon d(u, v) \leq f(u)$;
(b) $f(z) + \varepsilon d(v, z) > f(v)$ *for all* $z \in M$ *with* $z \neq v$.

In 1976, Caristi [11] established the following famous fixed point theorem:

Theorem 3. *(Caristi's fixed point theorem) Let (M, d) be a complete metric space and $f : M \to (-\infty, +\infty]$ be a proper lower semicontinuous and bounded below function. Suppose that $T : M \to M$ is selfmapping, satisfying*

$$f(Tz) + d(z, Tz) \leq f(z)$$

for each $z \in M$. Then there exists $w \in M$ such that $Tw = w$.

In 1991, Takahashi [12] proved the following nonconvex minimization theorem:

Theorem 4. *(Takahashi's nonconvex minimization theorem) Let (M, d) be a complete metric space and $f : M \to (-\infty, +\infty]$ be a proper lower semicontinuous and bounded below function. Suppose that for any $x \in M$ with $f(x) > \inf_{z \in M} f(z)$, there exists $y_x \in M$ with $y_x \neq x$ such that*

$$f(y_x) + d(x, y_x) \leq f(x).$$

Then there exists $w \in M$ such that $f(w) = \inf_{z \in M} f(z)$.

It is well known that Caristi's fixed point theorem, Takahashi's nonconvex minimization theorem and Ekeland's variational principle are logically equivalent; for detail, one can refer to [3,6–8,13–24]. Many authors have devoted their attention to investigating generalizations and applications in various different directions of the well-known fixed point theorems (see, e.g., [3–8,12–31] and references therein). By using Theorem 1, Du proved several versions of generalized Ekeland's variational principle and maximal element principle and established their equivalent formulations in complete metric spaces, for detail, see [3,4].

In this paper, we present some new existence theorems related with critical point theorem, generalized Ekeland's variational principle, maximal element principle, and common (fuzzy) fixed point theorem for essential distances by applying Theorem 1.

2. Preliminaries

Let E be a nonempty set. A fuzzy set in E is a function of E into $[0, 1]$. Let $\mathcal{F}(E)$ be the family of all fuzzy sets in E. A fuzzy mapping on E is a mapping from E into $\mathcal{F}(E)$. This enables us to regard each fuzzy map as a two variable function of $E \times E$ into $[0, 1]$. Let F be a fuzzy mapping on E. An element a of E is said to be a fuzzy fixed point of F if $F(a, a) = 1$ (see, e.g., [4]). Let $\Gamma : E \to 2^E$ be a multivalued mapping. A point $x \in E$ is called to be a *critical point* (or *stationary point* or *strict fixed point*) [4] of Γ if $\Gamma(v) = \{v\}$.

Let E be a nonempty set and "$\lesssim$" a quasi-order (preorder or pseudo-order; that is, a reflexive and transitive relation) on E. Then $(E, \lesssim)$ is called a quasi-ordered set. An element v in E is called a *maximal element* of E if there is no element x of E, different from v, such that $v \lesssim x$; that is, $v \lesssim w$ for some $w \in E$ implies that $v = w$. Let $(E, \lesssim)$ be a quasi-ordered set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called $\lesssim$-*nondecreasing* (resp. $\lesssim$-*nonincreasing*) if $x_n \lesssim x_{n+1}$ (resp. $x_{n+1} \lesssim x_n$) for each $n \in \mathbb{N}$.

Let (X, d) be a metric space. A real valued function $\varphi : X \to \mathbb{R}$ is *lower semicontinuous* (in short *l.s.c*) (resp. *upper semicontinuous*, in short *u.s.c*) if $\{x \in X : \varphi(x) \leq r\}$ (resp. $\{x \in X : \varphi(x) \geq r\}$) is *closed* for each $r \in \mathbb{R}$. A real-valued function $f : X \to (-\infty, +\infty]$ is said to be proper if $f \not\equiv +\infty$. Recall that a function $p : X \times X \to [0, +\infty)$ is called a *w-distance* [17,23], if the following are satisfied

- $(w1)$ $p(x,z) \leq p(x,y) + p(y,z)$ for any $x, y, z \in X$;
- $(w2)$ For any $x \in X$, $p(x, \cdot) : X \to [0, +\infty)$ is l.s.c.;
- $(w3)$ For any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

The concept of τ-function was introduced and studied by Lin and Du as follows. A function $p : X \times X \to [0, \infty)$ is said to be a *τ-function* [4,13,15,20,22,24,25], if the following conditions hold

- $(\tau1)$ $p(x,z) \leq p(x,y) + p(y,z)$ for all $x, y, z \in X$;
- $(\tau2)$ If $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ such that $p(x, y_n) \leq c$ for some $c = c(x) > 0$, then $p(x,y) \leq c$;
- $(\tau3)$ For any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to+\infty} p(x_n, y_n) = 0$, then $\lim_{n\to+\infty} d(x_n, y_n) = 0$;
- $(\tau4)$ For $x, y, z \in X$, $p(x,y) = 0$ and $p(x,z) = 0$ imply $y = z$.

It is worth mentioning that a τ-function is nonsymmetric in general. It is known that any metric d is a w-distance and any w-distance is a τ-function, but the converse is not true, see [24] for more detail.

Lemma 1 (see [15,16,26]). *If condition $(\tau4)$ is weakened to the following condition $(\tau4)'$:*

$(\tau4)'$ for any $x \in X$ with $p(x,x) = 0$, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$,

then $(\tau3)$ implies $(\tau4)'$.

The concept of essential distance was introduced by Du [15] in 2016.

Definition 4 (see [15]). *Let (X,d) be a metric space. A function $p : X \times X \to [0, +\infty)$ is called an essential distance if conditions $(\tau1)$, $(\tau2)$, and $(\tau3)$ hold.*

Remark 1. *It is obvious that any τ-function is an essential distance. By Lemma 1, we know that if p is an essential distance, then condition $(\tau4)'$ holds.*

The following known result is very crucial in our proofs.

Lemma 2 (see [4]). *Let (X,d) be a metric space and $p : X \times X \to [0, +\infty)$ be a function. Assume that p satisfies the condition $(\tau3)$. If a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence in X.*

3. Main Results

Lemma 3. *Let (M, d) be a metric space and $p : M \times M \to [0, +\infty)$ be a function satisfying $p(x,x) = 0$ for all $x \in M$ and $p(x,z) \leq p(x,y) + p(y,z)$ for any $x, y, z \in M$. Suppose that the extended real-valued function $L : M \times M \to (-\infty, +\infty]$ satisfies the following assumptions*

- *(i)* $L(x,x) \leq 0$ *for all* $x \in M$;
- *(ii)* $L(x,z) \leq L(x,y) + L(y,z)$ *for all* $x, y, z \in M$;

(iii) *For each $x \in M$, $y \to L(x,y)$ is l.s.c.;*
(iv) $\{x \in X : \inf_{y \in M} L(x,y) > -\infty\} \neq \emptyset$.

Define a binary relation $\lesssim$ on M by

$$x \lesssim y \iff L(x,y) + p(x,y) \leq 0.$$

Then $\lesssim$ is a quasi-order.

Proof. Clearly, $x \lesssim x$ for all $x \in M$. If $x \lesssim y$ and $y \lesssim z$, then

$$L(x,y) + p(x,y) \leq 0$$

and

$$L(y,z) + p(y,z) \leq 0.$$

By (ii), we get

$$L(x,z) + p(x,z) \leq L(x,y) + L(y,z) + p(x,y) + p(y,z) \leq 0,$$

which shows that $x \lesssim z$. Hence $\lesssim$ is a quasi-order. □

Lemma 4. *Let (M,d), p, L, and $\lesssim$ be the same as in Lemma 3. Assume that for each $x \in M$, the function $y \to p(x,y)$ is l.s.c. Define $G : M \to 2^M$ by*

$$G(x) = \{y \in M : x \lesssim y\} \quad \text{for } x \in M.$$

Then the following hold

(a) *$G(x)$ is nonempty and closed for each $x \in M$;*
(b) *$G(y) \subseteq G(x)$ for each $y \in G(x)$.*

Proof. Obviously, the conclusion (a) holds. To see (b), let $y \in G(x)$. Then $x \lesssim y$. We claim that $G(y) \subseteq G(x)$. Given $z \in G(y)$. Thus $y \lesssim z$. By the transitive relation, we get $x \lesssim z$ which means $z \in G(x)$. Hence $G(y) \subseteq G(x)$. □

The following theorem is one of the main results of this paper.

Theorem 5. *Let (M,d) be a metric space and p be an essential distance on M with $p(x,\cdot)$ is l.s.c. for each $x \in M$ and $p(a,a) = 0$ for all $a \in M$. Suppose that L, $\lesssim$ and G be the same as in Lemmas 3 and 4. If,*

$$p(y,x) \leq p(x,y) \quad \text{for all } y \in G(x),$$

then the following hold:

(a) *G is of type (μ_p) where $\mu_p(D) := \sup\{p(x,y) : x,y \in D\}$ for $D \subseteq M$;*
(b) *If M is $\lesssim$-complete, then $(M, \lesssim, \mu_p)$ is a μ_p-bounded quasi-ordered set.*

Proof. We first show that G *is of type* (μ_p). Let $x \in M$ and $\epsilon > 0$ be given. Then there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$, such that $2^{-n_0} < \frac{\epsilon}{2}$. Define a function $\kappa : M \to [-\infty, +\infty]$ by

$$\kappa(x) = \inf_{y \in G(x)} L(x,y).$$

Let $y \in G(x)$. If $\kappa(x) = -\infty$, then $0 \leq p(x,y) < -\kappa(x)$. Otherwise, if $\kappa(x) > -\infty$, then

$$p(x,y) \leq -L(x,y) \leq -\kappa(x).$$

Hence we conclude

$$0 \leq p(x,y) \leq -\kappa(x) \quad \text{for all } y \in G(x). \tag{1}$$

Set $x_1 := x \in M$. Thus one can choose $x_2 \in G(x_1) \subseteq M$, such that

$$L(x_1, x_2) \leq \kappa(x_1) + \frac{1}{2}.$$

Let $k \in \mathbb{N}$ and assume that $x_k \in M$ is already known. Then, one can choose $x_{k+1} \in G(x_k)$ such that

$$L(x_k, x_{k+1}) \leq \kappa(x_k) + \frac{1}{2^k}.$$

Hence, by induction, we obtain a nondecreasing sequence $x_1 \lesssim x_2 \lesssim \cdots$ in M such that $x_{n+1} \in G(x_n)$ and

$$L(x_n, x_{n+1}) \leq \kappa(x_n) + \frac{1}{2^n} \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

By Lemma 4, we have $G(x_{n+1}) \subseteq G(x_n)$ for all $n \in \mathbb{N}$. So it follows that

$$\begin{aligned} \kappa(x_{n+1}) &= \inf_{y \in G(x_{n+1})} L(x_{n+1}, y) \\ &\geq \inf_{y \in G(x_n)} L(x_{n+1}, y) \\ &\geq \inf_{y \in G(x_n)} [L(x_n, y) - L(x_n, x_{n+1})] \\ &= \kappa(x_n) - L(x_n, x_{n+1}). \end{aligned} \tag{3}$$

Combining (2) with (3), we obtain

$$\kappa(x_{n+1}) + \frac{1}{2^n} \geq 0,$$

and hence

$$0 \leq -\kappa(x_{n+1}) \leq \frac{1}{2^n} < \frac{\epsilon}{2} \quad \text{for all } n \geq n_0.$$

Put $w = x_{n_0+1}$. Thus $w \in G(x)$ and

$$0 \leq -\kappa(w) < \frac{\epsilon}{2}.$$

If $G(w)$ is a singleton set, then $\mu_p(G(w)) = 0 \leq \epsilon$. Assume that $G(w)$ is not a singleton set. Let $u, v \in G(w)$. By our hypothesis, we have $p(u, w) \leq p(w, u)$. So, by (1), we obtain

$$\begin{aligned} p(u, v) &\leq p(u, w) + p(w, v) \\ &\leq -2\kappa(w) \\ &< \epsilon \end{aligned}$$

which implies

$$\mu_p(G(w)) = \sup\{p(u, v) : u, v \in G(w)\} \leq \epsilon.$$

Therefore G is of type (μ_p). Finally, we prove (b). Let $\alpha_1 \lesssim \alpha_2 \lesssim \cdots$ be a $\lesssim$-nondecreasing sequence in M satisfying $\lim_{n \to +\infty} \mu_p(\{\alpha_n, \alpha_{n+1}, \cdots\}) = 0$. Since

$$\begin{aligned} 0 &= \lim_{n \to +\infty} \mu_p(\{\alpha_n, \alpha_{n+1}, \cdots\}) \\ &= \lim_{n \to +\infty} \sup\{p(u, v) : u, v \in \{\alpha_n, \alpha_{n+1}, \cdots\}\}, \end{aligned}$$

we get

$$\lim_{n \to +\infty} \sup\{p(\alpha_n, \alpha_m) : m > n\} = 0.$$

So, by applying Lemma 2, we show that $\{\alpha_n\}$ is a nondecreasing Cauchy sequence in M. By the $\lesssim$-completeness of M, there exists $\beta \in M$ such that $\alpha_n \to \beta$ as $n \to +\infty$. We claim that β is an upper bound of $\{\alpha_n\}_{n=1}^{+\infty}$. For each $n \in \mathbb{N}$, since $\alpha_m \in G(\alpha_n)$ for all $m \geq n$ and $\alpha_n \to \beta$, by the closedness of $G(\alpha_n)$, we have $\beta \in G(\alpha_n)$ or $\alpha_n \lesssim \beta$ for all $n \in \mathbb{N}$. Therefore β is an upper bound of $\{\alpha_n\}$ and hence $(M, \lesssim, \mu_p)$ is a μ_p-bounded quasi-ordered set. The proof is completed. □

The following result is immediate from Theorem 5 and Lemmas 3 and 4.

Corollary 1. *Let (M, d) be a metric space and p be an essential distance on M with $p(x, \cdot)$ a l.s.c. for each $x \in M$ and $p(a, a) = 0$ for all $a \in M$. Suppose that the extended real-valued function $f : M \to (-\infty, +\infty]$ is proper, l.s.c. and bounded below. Let $\varepsilon > 0$. Define a binary relation $\lesssim_{(\varepsilon,f,p)}$ on M by*

$$x \lesssim_{(\varepsilon,f,p)} y \iff \varepsilon p(x, y) \leq f(x) - f(y).$$

Let $\Gamma : M \to 2^M$ be defined by

$$\Gamma(x) = \{y \in M : x \lesssim_{(\varepsilon,f,p)} y\} \text{ for } x \in M.$$

Then the following hold:

(a) *$\lesssim_{(\varepsilon,f,p)}$ is a quasi-order;*
(b) *For each $x \in M$, $\Gamma(x)$ is closed;*
(c) *Γ is of type (μ_p) where $\mu_p(D) := \sup\{p(x, y) : x, y \in D\}$ for $D \subset M$;*
(d) *If M is complete, then $(M, \lesssim_{(\varepsilon,f,p)}, \mu_p)$ is a μ_p-bounded quasi-ordered set.*

Proof. Define $L : M \times M \to (-\infty, \infty]$ by

$$L(x, y) = \frac{1}{\varepsilon}(f(y) - f(x)).$$

Then the following hold

- $x \lesssim_{(\varepsilon,f,p)} y \iff L(x, y) + p(x, y) \leq 0$;
- $L(x, x) = 0$ for all $x \in M$;
- $L(x, z) = L(x, y) + L(y, z)$ for all $x, y, z \in M$;
- For each $x \in M$, $y \to L(x, y)$ is l.s.c.;
- $\{x \in X : \inf_{y \in M} L(x, y) > -\infty\} \neq \emptyset$.

Therefore, applying Theorem 5 and Lemmas 3 and 4, we show the desired conclusions. □

By applying Theorem 5, we obtain a new result related to common fuzzy fixed point theorem, critical point theorem, maximal element principle and generalized Ekeland's variational principle for essential distances.

Theorem 6. *Let (M, d) be a complete metric space. Suppose that p, L, $\lesssim$, and G be the same as in Theorem 5. Let I be any index set. For each $i \in I$, let F_i be a fuzzy mapping on M. Assume that for each $(i, x) \in I \times M$, there exists $y_{(i,x)} \in G(x)$ such that $F_i(x, y_{(i,x)}) = 1$. Then for every $\varepsilon > 0$ and for every $u \in M$, there exists $v \in M$ such that*

(a) *v is a maximal element of $(M, \lesssim)$;*
(b) *$G(v) = \{v\}$;*
(c) *$L(u, v) + p(u, v) \leq 0$;*
(d) *$L(v, x) + p(v, x) > 0$ for all $x \in M$ with $x \neq v$;*
(e) *$F_i(v, v) = 1$ for all $i \in I$.*

Proof. By applying Theorem 5, G is of type (μ_p) and $(M, \lesssim, \mu_p)$ is a μ_p-bounded quasi-ordered set, where

$$\mu_p(D) := \sup\{p(x, y) : x, y \in D\} \quad \text{for } D \subseteq M.$$

Let $u \in M$ be given. Put $u_0 := u$. Since G is of type (μ_p), by Theorem 1, there exists a $\lesssim$-nondecreasing sequence $u_0 \lesssim u_1 \lesssim u_2 \lesssim \cdots$ in M and $v \in M$ such that

(i) v is an upper bound of $\{u_n\}_{n=0}^{+\infty}$;
(ii) $\mu_p(G(v)) = 0$.

From (i), we prove (c). Next, we claim that $G(v) = \{v\}$. Let $z \in G(v)$. By $(\mu 2)$ and (ii), we have

$$p(v,z) = \mu_p(\{v,z\}) \leq \mu_p(G(v)) = 0,$$

which deduces $p(v,z) = 0$. Since $p(v,v) = 0$, by Lemma 1, we get $z = v$. Therefore $G(v) = \{v\}$ and, equivalency, (d) holds. For each $(i,v) \in I \times M$, due to $G(v) = \{v\}$ and our hypothesis, there exists $y_{(i,v)} := v \in G(v)$ such that $F_i(v,v) = F_i(v, y_{(i,v)}) = 1$. So (e) is true. Finally, we verify (a). If $v \lesssim w$ for some $w \in W$, then $w \in G(v) = \{v\}$, which implies $v = w$. Hence v is a maximal element of $(M, \lesssim)$. The proof is completed. □

Corollary 2. *Let (M,d) be a complete metric space and $\varepsilon > 0$. Suppose that f, p, $\lesssim_{(\varepsilon,f,p)}$, and Γ be the same as in Corollary 1. Let I be any index set. For each $i \in I$, let F_i be a fuzzy mapping on M. Assume that for each $(i,x) \in I \times M$, there exists $y_{(i,x)} \in \Gamma(x)$ such that $F_i(x, y_{(i,x)}) = 1$. Then for every $u \in M$, there exists $v \in M$ such that*

(a) *v is a maximal element of $(M, \lesssim_{(\varepsilon,f,p)})$;*
(b) *$\Gamma(v) = \{v\}$;*
(c) *$f(v) + \varepsilon p(u,v) \leq f(u)$;*
(d) *$f(z) + \varepsilon p(v,z) > f(v)$ for all $z \in M$ with $z \neq v$;*
(e) *$F_i(v,v) = 1$ for all $i \in I$.*

Proof. Define $L : M \times M \to (-\infty, +\infty]$ by

$$L(x,y) = \frac{1}{\varepsilon}(f(y) - f(x)).$$

Then,

$$x \lesssim_{(\varepsilon,f,p)} y \iff L(x,y) + p(x,y) \leq 0.$$

So the desired conclusions follow from Theorem 6 immediately. □

Let (M,d) *be a metric space and* $T : M \to 2^M$ be a multivalued mapping with nonempty values. Then we can define a fuzzy mapping K on M by

$$K(x,y) = \chi_{T(x)}(y),$$

where χ_A is the characteristic function for an arbitrary set $A \subset M$. Note that

$$K(x,y) = 1 \iff y \in T(x).$$

The following new result related to critical point theorem, generalized Ekeland's variational principle, maximal element principle, and common fixed point theorem for essential distances can be established by Theorem 6 immediately.

Theorem 7. *Let (M,d) be a complete metric space. Suppose that p, L, $\lesssim$, and G are the same as in Theorem 5. Let I be any index set. For each $i \in I$, let $T_i : M \to 2^M$ be a multivalued mapping with nonempty values such that for each $(i,x) \in I \times M$, there exists $y_{(i,x)} \in T_i(x) \bigcap G(x)$. Then for every $\varepsilon > 0$ and for every $u \in M$, there exists $v \in M$ such that*

(a) *v is a maximal element of $(M, \lesssim)$;*
(b) *$G(v) = \{v\}$;*
(c) *$L(u,v) + p(u,v) \leq 0$;*
(d) *$L(v,x) + p(v_i, x_i) > 0$ for all $x \in M$ with $x \neq v$;*

(e) v is a common fixed point for the family $\{T_i\}_{i\in I}$.

Corollary 3. *Let (M,d) be a complete metric space and $\varepsilon > 0$. Suppose that f, p, $\precsim_{(\varepsilon,f,p)}$, and Γ be the same as in Corollary 1. Let I be any index set. For each $i \in I$, let $T_i : M \to 2^M$ be a multivalued mapping with nonempty values such that for each $(i,x) \in I \times M$, there exists $y_{(i,x)} \in T_i(x) \bigcap \Gamma(x)$. Then for every $u \in M$, there exists $v \in M$ such that*

(a) v is a maximal element of $(M, \precsim_{(\varepsilon,f,p)})$;
(b) $\Gamma(v) = \{v\}$;
(c) $f(v) + \varepsilon p(u,v) \le f(u)$;
(d) $f(z) + \varepsilon p(v,z) > f(v)$ for all $z \in M$ with $z \neq v$;
(e) v is a common fixed point for the family $\{T_i\}_{i\in I}$.

Finally, the following simple example is given to illustrate Corollary 3.

Example 1. *Let $M = [-1,1]$ with the metric $d(x,y) = |x-y|$ for $x,y \in M$. Then (M,d) is a complete metric space. Let $T_1, T_2 : M \to 2^M$ be defined by $T_1 x = \left\{\frac{1}{2}x\right\}$ and $T_2 x = \left\{\frac{1}{3}x\right\}$ for $x \in M$. Clearly, 0 is the unique common fixed point of T_1 and T_2. Let $f : M \to \mathbb{R}$ by $f(x) = |x|$ for $x \in M$. Define a binary relation $\precsim_{(1,f,d)}$ on M by*

$$x \precsim_{(1,f,d)} y \iff d(x,y) \le f(x) - f(y).$$

Then $\precsim_{(1,f,d)}$ is a quasi-order and

$$\Gamma(x) = \{y \in M : x \precsim_{(1,f,d)} y\} = \{y \in M : d(x,y) \le f(x) - f(y)\} \neq \varnothing.$$

It is easy to see that for each $x \in M$, we have

$$d\left(x, \frac{1}{2}x\right) = f(x) - f\left(\frac{1}{2}x\right)$$

and

$$d\left(x, \frac{1}{3}x\right) = f(x) - f\left(\frac{1}{3}x\right).$$

Hence $\frac{1}{2}x \in T_1 x \cap \Gamma(x)$ and $\frac{1}{3}x \in T_2 x \cap \Gamma(x)$ for any $x \in M$. Therefore, all the assumptions of Corollary 3 are satisfied. By applying Corollary 3, for every $u \in M$, we can obtain $v \in M$ (in fact, $v = 0$) such that

(a) 0 is a common fixed point for T_1 and T_2;
(b) 0 is a maximal element of $(M, \precsim_{(1,f,d)})$;
(c) $\Gamma(0) = \{0\}$;
(d) $f(0) + d(u,0) \le f(u)$;
(e) $f(z) + d(0,z) > f(0)$ for all $z \in M$ with $z \neq 0$.

Remark 2.

(a) *Theorems 5–7 and Corollaries 1–3 improve and generalize some of the existence results on the topic in the literature, see, e.g., [3,4,8,17,23,24] and references therein;*
(b) *Following the same argument as in the proof of [16], one can establish the equivalence of Ekeland's variational principle Caristi's fixed point theorem and Takahashi's nonconvex minimization theorem for essential distances.*

4. Conclusions

Maximal element principle is a significant theory and has already been proposed and investigated its potential applications in several areas of mathematics. In this paper, by applying the abstract maximal element principle of Lin and Du, we present some new existence theorems related with common (fuzzy) fixed point theorem, maximal element theorem, critical point theorem and generalized Ekeland's variational principle for essential distances.

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