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# Using the Supremum Form of Auxiliary Functions to Study the Common Coupled Coincidence Points in Fuzzy Semi-Metric Spaces

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**Abstract:** This paper investigates the common coupled coincidence points and common coupled fixed points in fuzzy semi-metric spaces. The symmetric condition is not necessarily satisfied in fuzzy semi-metric space. Therefore, four kinds of triangle inequalities are taken into account in order to study the Cauchy sequences. Inspired by the intuitive observations, the concepts of rational condition and distance condition are proposed for the purpose of simplifying the discussions.

**Keywords:** common coupled coincidence points; common coupled fixed points; distance condition; fuzzy semi-metric space; rational condition

**MSC:** 54E35; 54H25

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## 1. Introduction

The common coupled coincidence points and common coupled fixed points in conventional metric spaces and probabilistic metric spaces have been studied for a long time in which the symmetric condition is satisfied. In this paper, we shall consider the fuzzy semi-metric space in which the symmetric condition is not satisfied. In this case, the role of triangle inequality should be re-interpreted. Therefore, four kinds of triangle inequalities are considered, which can also refer to Wu [1].

Schweizer and Sklar [2–4] introduced probabilistic metric space, in which the (conventional) metric space is associated with the probability distribution functions. For more details on the theory of probabilistic metric space, we can refer to Hadžić and Pap [5] and Chang et al. [6]. An interesting special kind of probabilistic metric space is the so-called Menger space. Kramosil and Michalek [7] proposed the fuzzy metric space based on the idea of Menger space. The definition of fuzzy metric space is presented below. Let $X$ be a nonempty universal set associated with a t-norm $*$. Given a mapping $M$ from $X \times X \times [0,\infty)$ into $[0,1]$, the 3-tuple $(X, M, *)$ is called a fuzzy metric space when the following conditions are satisfied:

- for any $x, y \in X$, $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$;
- $M(x,y,0) = 0$ for all $x, y \in X$;
- $M(x,y,t) = M(y,x,t)$ for all $x, y \in X$ and $t \geq 0$; and,
- $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$ for all $x,y,z \in X$ and $s,t \geq 0$.

The mapping $M$ in the fuzzy metric space $(X, M, *)$ can be treated as a membership function of a fuzzy subset of the product space $X \times X \times [0,\infty)$. According to the first and second conditions of fuzzy metric space, the function value $M(x,y,t)$ means that the membership degree of the distance that is less than or equal to $t$ between $x$ and $y$.

In this paper, we are going to consider the semi-metric space that is completely different from the fuzzy metric space. The so-called fuzzy semi-metric space does not assume the symmetric condition $M(x,y,t) = M(y,x,t)$. Without this condition, the concept of triangle inequalities should be carefully treated. In this paper, there are four kinds of different triangle inequalities considered. It will be realized that, when the symmetric condition is satisfied, these four different kinds of triangle inequalities will be equivalent to the classical one. Being inspired by the intuitive observations, the concepts of rational condition and distance condition are proposed for the purpose of simplifying the discussions regarding the common coupled coincidence points and common coupled fixed points in a fuzzy semi-metric space.

Rakić et al. [8,9] studied the fixed points in b-fuzzy metric spaces. Mecheraoui et al. [10] obtained the sufficient condition for a G-Cauchy sequence to be an M-Cauchy sequence in fuzzy metric space. On the other hand, Gu and Shatanawi [11] used the concept of w-compatible mappings for studying the common coupled fixed points of two hybrid pairs of mappings in partial metric spaces. Petruel [12,13] studied the fixed point for graphic contractions and fixed point for multi-valued locally contractive operators. Hu et al. [14], Mohiuddine and Alotaibi [15], Qiu and Hong [16], and the references therein studied the common coupled coincidence points and common coupled fixed points in fuzzy metric spaces. Wu [17] also studied the common coincidence points in fuzzy semi-metric spaces. In this paper, the common coupled coincidence points and common coupled fixed points in fuzzy semi-metric spaces will be studied by considering four kinds of triangle inequalities. Although the common coupled fixed points are the common coupled coincidence points, the sufficient conditions will be completely different when considering the uniqueness.

This paper is organized, as follows. In Section 2, the concept of fuzzy semi-metric spaces will be introduced. Because the symmetric condition is not satisfied, four different kinds of triangle inequalities will be taken into account to study the common coupled fixed points. In Section 3, in order to study the Cauchy sequence in fuzzy semi-metric space, the auxiliary functions that are based on the supremun are proposed. In Section 4, while using the auxiliary functions proposed in Section 3, the desired property regarding the Cauchy sequence in fuzzy semi-metric space will be presented. In Section 5, many kinds of common coupled coincidence points in fuzzy semi-metric spaces will be investigated by considering the four different kinds of triangle inequalities. Finally, in Section 6, the common coupled fixed points shown in fuzzy semi-metric spaces will also be studied based on the four different kinds of triangle inequalities.

## 2. Fuzzy Semi-Metric Spaces

The concept of fuzzy semi-metric space is based on the concept of t-norm (triangular norm), which will be introduced below. Let $* : [0,1] \times [0,1] \to [0,1]$ be a function that is defined on the product set $[0,1] \times [0,1]$. We say that $*$ is a t-norm when the following conditions are satisfied:

- $a * 1 = a$.
- $a * b = b * a$.
- $b < c$ implies $a * b \leq a * c$.
- $(a * b) * c = a * (b * c)$.

The following properties regarding t-norm will be used in the further study.

**Proposition 1.** *We have the following properties.*

(i) *Suppose that the t-norm $*$ is left-continuous at 1 with respect to the first or second component. For any $a, b \in (0,1)$ with $a > b$, there exists $r \in (0,1)$ that satisfies $a * r \geq b$.*

(ii) *Suppose that the t-norm $*$ is left-continuous at 1 with respect to the first or second component. For any $a \in (0,1)$ and any $p \in \mathbb{N}$, there exists $r \in (0,1)$ satisfying $\overbrace{r * r * \cdots * r}^{p \text{ times}} > a$.*

(iii) *Given any fixed $a, b \in [0,1]$, suppose that the t-norm $*$ is continuous at a and b with respect the first or second component, and that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences in $[0,1]$ satisfying $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Subsequently, we have $a_n * b_n \to a * b$ as $n \to \infty$.*

(iv) *Given any fixed $a, b \in (0,1]$, suppose that the t-norm $*$ is left-continuous at a and b with respect to the first or second component, and that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences in*

$[0,1]$ *satisfying* $a_n \to a-$ *and* $b_n \to b-$ *as* $n \to \infty$. *Afterwards, we have* $a_n * b_n \to a * b$ *as* $n \to \infty$.

(v) *Given any fixed* $a, b \in [0,1)$, *suppose that the t-norm* $*$ *is right-continuous at* $a$ *and* $b$ *with respect to the first or second component, and that* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are two sequences in* $[0,1]$ *satisfying* $a_n \to a+$ *and* $b_n \to b+$ *as* $n \to \infty$. *Subsequently, we have* $a_n * b_n \to a * b$ *as* $n \to \infty$.

Wu [1,17,18] proposed the concept of fuzzy semi-metric space. The formal definition is given below.

**Definition 1.** *Let* $X$ *be a nonempty set and let* $M$ *be a mapping from* $X \times X \times [0,\infty)$ *into* $[0,1]$. *We say that* $(X, M)$ *is fuzzy semi-metric space when the following conditions are satisfied:*

- *for any* $x, y \in X$, $M(x,y,t) = 1$ *for all* $t \geq 0$ *if and only if* $x = y$;
- $M(x,y,0) = 0$ *for all* $x, y \in X$ *with* $x \neq y$;

*The mapping* $M$ *is said to satisfy the symmetric condition when* $M(x,y,t) = M(y,x,t)$ *for any* $x, y \in X$ *and* $t \geq 0$.

**Definition 2.** *Let* $(X, M)$ *be a fuzzy semi-metric space. We say that* $M$ *satisfies the distance condition when, for any* $x, y \in X$ *with* $x \neq y$, *there exists* $t_0 > 0$, *such that* $M(x,y,t_0) \neq 0$.

Because the symmetric condition is not necessarily be satisfied in fuzzy semi-metric space $(X, M)$, by referring to Wu [1,17,18], four kinds of triangle inequalities are proposed below.

**Definition 3.** *Let* $X$ *be a nonempty set, let* $*$ *be a t-norm, and let* $M$ *be a mapping that is defined on* $X \times X \times [0,\infty)$ *into* $[0,1]$.

- *We say that* $M$ *satisfies the* $\bowtie$*-triangle inequality when the following inequality is satisfied:*

$$M(x,y,t) * M(y,z,s) \leq M(x,z,t+s) \text{ for all } x,y,z \in X \text{ and } s,t > 0.$$

- *We say that* $M$ *satisfies the* $\triangleright$*-triangle inequality when the following inequality is satisfied:*

$$M(x,y,t) * M(z,y,s) \leq M(x,z,t+s) \text{ for all } x,y,z \in X \text{ and } s,t > 0.$$

- *We say that* $M$ *satisfies the* $\triangleleft$*-triangle inequality when the following inequality is satisfied:*

$$M(y,x,t) * M(y,z,s) \leq M(x,z,t+s) \text{ for all } x,y,z \in X \text{ and } s,t > 0.$$

- *We say that* $M$ *satisfies the* $\diamond$*-triangle inequality when the following inequality is satisfied:*

$$M(y,x,t) * M(z,y,s) \leq M(x,z,t+s) \text{ for all } x,y,z \in X \text{ and } s,t > 0.$$

**Remark 1.** *Suppose that the mapping* $M$ *satisfies the* $\bowtie$*-triangle inequality. Subsequently, we have*

$$M(a,b,t_1) * M(b,c,t_2) * M(c,d,t_3) \leq M(a,c,t_1+t_2) * M(c,d,t_3) \leq M(a,d,t_1+t_2+t_3)$$

*and*

$$M(b,a,t_1) * M(c,b,t_2) = M(c,b,t_2) * M(b,a,t_1) \leq M(c,a,t_1+t_2),$$

*which implies*

$$M(b,a,t_1) * M(c,b,t_2) * M(d,c,t_3) \leq M(d,a,t_1+t_2+t_3).$$

*In general, we have*

$$M(x_1, x_2, t_1) * M(x_2, x_3, t_2) * \cdots * M(x_p, x_{p+1}, t_p) \leq M(x_1, x_{p+1}, t_1 + t_2 + \cdots + t_p)$$

*and*

$$M(x_2, x_1, t_1) * M(x_3, x_2, t_2) * \cdots * M(x_{p+1}, x_p, t_{p+1}) \leq M(x_{p+1}, x_1, t_1 + t_2 + \cdots + t_p).$$

*For the case of satisfying the $\triangleright$-triangle inequality, $\triangleleft$-triangle inequality and $\diamond$-triangle inequality, we can refer to Wu [17].*

**Proposition 2** (Wu [1])**.** *Let $(X, M)$ be a fuzzy semi-metric space. Then we have the following properties.*

(i) *Suppose that the mapping $M$ satisfies the $\bowtie$-triangle inequality. Subsequently, $M$ is non-decreasing in the sense of $M(x, y, t_1) \geq M(x, y, t_2)$ for any fixed $x, y \in X$ and $t_1 > t_2$.*
(ii) *Suppose that the mapping $M$ satisfies the $\diamond$-triangle inequality. Subsequently, $M$ is symmetrically non-decreasing in the sense of $M(x, y, t_1) \geq M(y, x, t_2)$ for any fixed $x, y \in X$ and $t_1 > t_2$.*
(iii) *Suppose that the mapping $M$ satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality. Afterwards, $M$ is both non-decreasing and symmetrically non-decreasing.*

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in the fuzzy semi-metric space $(X, M)$.

- We write $x_n \xrightarrow{M^{\triangleright}} x$ as $n \to \infty$ when $M(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$.
- We write $x_n \xrightarrow{M^{\triangleleft}} x$ as $n \to \infty$ when $M(x, x_n, t) \to 1$ as $n \to \infty$ for all $t > 0$.
- We write $x_n \xrightarrow{M} x$ as $n \to \infty$ when $x_n \xrightarrow{M^{\triangleright}} x$ and $x_n \xrightarrow{M^{\triangleleft}} x$ as $n \to \infty$.

**Proposition 3** (Wu [17])**.** *Let $(X, M)$ be a fuzzy semi-metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $X$. Suppose that the t-norm $*$ is left-continuous at 1 with respect to the first or second component. Afterwards, we have the following results.*

(i) *Assume that the mapping $M$ satisfies the $\bowtie$-triangle inequality or $\diamond$-triangle inequality. Subsequently, we have the following properties.*
   - *If $x_n \xrightarrow{M^{\triangleleft}} x$ and $x_n \xrightarrow{M^{\triangleright}} y$ as $n \to \infty$, then $x = y$.*
   - *If $x_n \xrightarrow{M^{\triangleright}} x$ and $x_n \xrightarrow{M^{\triangleleft}} y$ as $n \to \infty$, then $x = y$.*

(ii) *Assume that $M$ satisfies the $\triangleleft$-triangle inequality. If $x_n \xrightarrow{M^{\triangleright}} x$ and $x_n \xrightarrow{M^{\triangleright}} y$ as $n \to \infty$, then $x = y$.*
(iii) *Assume that $M$ satisfies the $\triangleright$-triangle inequality. If $x_n \xrightarrow{M^{\triangleleft}} x$ and $x_n \xrightarrow{M^{\triangleleft}} y$ as $n \to \infty$, then $x = y$.*

**Proposition 4** (Wu [18])**.** *Let $(X, M)$ be a fuzzy semi-metric space, and let $\{(x_n, y_n, t_n)\}_{n=1}^{\infty}$ be a sequence in $X \times X \times (0, \infty)$. Assume that the t-norm $*$ is left-continuous with respect to the first or second component. For any sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ in $[0, 1]$, we also assume that the following inequality is satisfied*

$$\sup_n (a_n * b_n) \geq \left(\sup_n a_n\right) * \left(\sup_n b_n\right).$$

(i) *Suppose that $M$ satisfies the $\bowtie$-triangle inequality, and that $t_n \to t^{\circ}$, $x_n \xrightarrow{M} x^{\circ}$ and $y_n \xrightarrow{M} y^{\circ}$ as $n \to \infty$. Subsequently, the following statements hold true.*
   - *If $M$ is continuous with respect to the distance at $t^{\circ}$, then $M(x_n, y_n, t_n) \to M(x^{\circ}, y^{\circ}, t^{\circ})$ as $n \to \infty$.*

- *If $M$ is symmetrically continuous with respect to the distance at $t^\circ$, then $M(x_n, y_n, t_n) \to M(y^\circ, x^\circ, t^\circ)$ as $n \to \infty$.*

(ii) *Suppose that $M$ satisfies the $\circ$-triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$, and that $t_n \to t^\circ$, $x_n \xrightarrow{M} x^\circ$ and $y_n \xrightarrow{M} y^\circ$ as $n \to \infty$. If $M$ is continuous or symmetrically continuous with respect to the distance at $t^\circ$, then $M(x_n, y_n, t_n) \to M(x^\circ, y^\circ, t^\circ) = M(y^\circ, x^\circ, t^\circ)$ as $n \to \infty$.*

(iii) *Suppose that $M$ satisfies the $\diamond$-triangle inequality, and that $t_n \to t^\circ$ as $n \to \infty$, $x_n \xrightarrow{M^\triangleright} x^\circ$, and $y_n \xrightarrow{M^\triangleright} y^\circ$ as $n \to \infty$ simultaneously, or $x_n \xrightarrow{M^\triangleleft} x^\circ$ and $y_n \xrightarrow{M^\triangleleft} y^\circ$ as $n \to \infty$ simultaneously. If $M$ is continuous or symmetrically continuous with respect to the distance at $t^\circ$, then $M(x_n, y_n, t_n) \to M(y^\circ, x^\circ, t^\circ) = M(x^\circ, y^\circ, t^\circ)$ as $n \to \infty$.*

**Definition 4.** *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in the fuzzy semi-metric space $(X, M)$.*

- *We say that $\{x_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence when, given any pair $(r, t)$ with $t > 0$ and $0 < r < 1$, there exists $n_{r,t} \in \mathbb{N}$ satisfying $M(x_m, x_n, t) > 1 - r$ for all pairs $(m, n)$ of integers $m$ and $n$ with $m > n \geq n_{r,t}$.*
- *We say that $\{x_n\}_{n=1}^{\infty}$ is a $<$-Cauchy sequence when, given any pair $(r, t)$ with $t > 0$ and $0 < r < 1$, there exists $n_{r,t} \in \mathbb{N}$ satisfying $M(x_n, x_m, t) > 1 - r$ for all pairs $(m, n)$ of integers $m$ and $n$ with $m > n \geq n_{r,t}$.*
- *We say that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence when, given any pair $(r, t)$ with $t > 0$ and $0 < r < 1$, there exists $n_{r,t} \in \mathbb{N}$ satisfying $M(x_m, x_n, t) > 1 - r$ and $M(x_n, x_m, t) > 1 - r$ for all pairs $(m, n)$ of integers $m$ and $n$ with $m, n \geq n_{r,t}$ and $m \neq n$.*
- *We say that $(X, M)$ is $(>, \triangleright)$-complete when each $>$-Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ is convergent in the sense of $x_n \xrightarrow{M^\triangleright} x$.*
- *We say that $(X, M)$ is $(>, \triangleleft)$-complete when each $>$-Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ is convergent in the sense of $x_n \xrightarrow{M^\triangleleft} x$.*
- *We say that $(X, M)$ is $(<, \triangleright)$-complete when each $<$-Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ is convergent in the sense of $x_n \xrightarrow{M^\triangleright} x$.*
- *We say that $(X, M)$ is $(<, \triangleleft)$-complete when each $<$-Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ is convergent in the sense of $x_n \xrightarrow{M^\triangleleft} x$.*

**Definition 5.** *Let $(X, M)$ be a fuzzy semi-metric space. Four types of continuities are defined below.*

- *We say that the function $f : X \to X$ is $(\triangleright, \triangleright)$-continuous with respect to $M$ when, given any sequence $\{x_n\}_{n=1}^{\infty}$ in $X$, $x_n \xrightarrow{M^\triangleright} x$, as $n \to \infty$ implies $f(x_n) \xrightarrow{M^\triangleright} f(x)$ as $n \to \infty$.*
- *We say that the function $f : X \to X$ is $(\triangleright, \triangleleft)$-continuous with respect to $M$ when, given any sequence $\{x_n\}_{n=1}^{\infty}$ in $X$, $x_n \xrightarrow{M^\triangleright} x$, as $n \to \infty$ implies $f(x_n) \xrightarrow{M^\triangleleft} f(x)$ as $n \to \infty$.*
- *We say that the function $f : X \to X$ is $(\triangleleft, \triangleright)$-continuous with respect to $M$ when, given any sequence $\{x_n\}_{n=1}^{\infty}$ in $X$, $x_n \xrightarrow{M^\triangleleft} x$, as $n \to \infty$ implies $f(x_n) \xrightarrow{M^\triangleright} f(x)$ as $n \to \infty$.*
- *We say that the function $f : X \to X$ is $(\triangleleft, \triangleleft)$-continuous with respect to $M$ when, given any sequence $\{x_n\}_{n=1}^{\infty}$ in $X$, $x_n \xrightarrow{M^\triangleleft} x$, as $n \to \infty$ implies $f(x_n) \xrightarrow{M^\triangleleft} f(x)$ as $n \to \infty$.*

## 3. Auxiliary Functions Based on the Supremum

The concept of auxiliary function based on $X^2$ was proposed by Wu [17] to study the common coincidence point. In this paper, we are going to consider the auxiliary function that is based on $X^4$ to study the common coupled coincidence point.

**Definition 6.** *Let $(X, M)$ be a fuzzy semi-metric space. We say that the mapping $M$ satisfies the rational condition when $M(x, y, t) \to 0$, as $t \to 0+$ for any fixed $x, y \in X$.*

Let $(X, M)$ be a fuzzy semi-metric space along with a t-norm $*$. We define the mapping $\eta : X^4 \times [0, \infty) \to [0, 1]$ on the product space $X^4 \times [0, \infty)$, as follows

$$\eta(x, y, u, v, t) = M(x, y, t) * M(u, v, t).$$

Subsequently, we have the following interesting result that will be used to define the auxiliary functions.

**Proposition 5.** *Let $(X, M)$ be a fuzzy semi-metric space, such that the mapping $M$ satisfies the rational condition. Suppose that the t-norm $*$ is right-continuous at $0$ with respect to the first or second component. Subsequently, we have*

$$\lim_{t \to 0+} \eta(x, y, u, v, t) = 0. \tag{1}$$

The following definition of auxiliary functions are based on $X^4$. This new concept extends the auxiliary functions based on $X^2$, as proposed by Wu [17].

**Definition 7.** *Let $(X, M)$ be a fuzzy semi-metric space, such that $M$ satisfies the rational condition in which the t-norm $*$ is also right-continuous at $0$ with respect to the first or second component. For any fixed $x, y, u, v \in X$ and $\lambda \in [0, 1)$ with $x \neq y$ or $u \neq v$, we define a function $\Phi : X^4 \to [0, \infty)$ on the product space $X^4$ by*

$$\Phi(\lambda, x, y, u, v) = \sup\{t > 0 : \eta(x, y, u, v, t) \leq 1 - \lambda\}$$

*and $\Phi(\lambda, x, x, u, u) = 0$ for $\lambda \in [0, 1)$.*

For $x \neq y$ or $u \neq v$, we need to claim that the set $\{t > 0 : \eta(x, y, u, v, t) \leq 1 - \lambda\}$ is not empty. Suppose that $\{t > 0 : \eta(x, y, u, v, t) \leq 1 - \lambda\} = \emptyset$. By definition, we must have $\eta(x, y, u, v, t) > 1 - \lambda$ for all $t > 0$. This says that

$$\lim_{t \to 0+} \eta(x, y, u, v, t) \geq 1 - \lambda,$$

which contradicts (1). Therefore, we indeed have $\{t > 0 : \eta(x, y, u, v, t) \leq 1 - \lambda\} \neq \emptyset$, which says that the function $\Phi$ is well-defined.

**Proposition 6.** *Let $(X, M)$ be a fuzzy semi-metric space such that the mapping $M$ satisfies the rational condition in which the t-norm $*$ is right-continuous at $0$ with respect to the first or second component. Given any fixed $x, y, u, v \in X$ and $\lambda \in (0, 1)$, we have the following properties.*

(i) *Suppose that $\Phi(\lambda, x, y, u, v) < \infty$. For any $\epsilon > 0$, we have*

$$\eta(x, y, u, v, \Phi(\lambda, x, y, u, v) + \epsilon) > 1 - \lambda$$

(ii) *Assume that $\epsilon > 0$ is sufficiently small satisfying $\Phi(\lambda, x, y, u, v) > \epsilon$. Subsequently, we have the following properties.*
- *If the mapping $M$ satisfies the $\bowtie$-triangle inequality or the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality, then*

$$\eta(x, y, u, v, \Phi(\lambda, x, y, u, v) - \epsilon) \leq 1 - \lambda.$$

- *If the mapping $M$ satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality, then*

$$\eta(y, x, u, v, \Phi(\lambda, x, y, u, v) - \epsilon) \leq 1 - \lambda \text{ and } \eta(x, y, v, u, \Phi(\lambda, x, y, u, v) - \epsilon) \leq 1 - \lambda.$$

- *If the mapping $M$ satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality or the $\diamond$-triangle inequality, then*

$$\eta(y, x, v, u, \Phi(\lambda, x, y, u, v) - \epsilon) \leq 1 - \lambda.$$

**Proof.** The proof is similar to the argument in Wu [17] by considering $X^4$ instead of $X^2$. □

**Proposition 7.** *Let $(X, M)$ be a fuzzy semi-metric space, such that the mapping $M$ satisfies the rational condition in which the t-norm $*$ is right-continuous at 0 with respect to the first or second component. Given any fixed $x, y, u, v \in X$ and $\lambda \in (0, 1)$, we have the following properties.*

(i) *Assume that $\eta(x, y, u, v, t) \leq 1 - \lambda$. Then, we have the following results.*
- *If the mapping $M$ satisfies the $\bowtie$-triangle inequality or the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality, then $t \leq \Phi(\lambda, x, y, u, v)$.*
- *If the mapping $M$ satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality, then $t \leq \Phi(\lambda, y, x, u, v)$ and $t \leq \Phi(\lambda, x, y, v, u)$.*
- *If the mapping $M$ satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality or the $\diamond$-triangle inequality, then $t \leq \Phi(\lambda, y, x, v, u)$.*

(ii) *We have the following results.*
- *Suppose that the mapping $M$ satisfies the $\bowtie$-triangle inequality or the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality. If $\eta(x, y, u, v, t) > 1 - \lambda$, then $\Phi(\lambda, x, y, u, v) < \infty$ and $t \geq \Phi(\lambda, x, y, u, v)$.*
- *Suppose that the mapping $M$ satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality.*
  - *If $\eta(x, y, u, v, t) > 1 - \lambda$, then $\Phi(\lambda, y, x, u, v) < \infty$ and $\Phi(\lambda, x, y, v, u) < \infty$.*
  - *If $\eta(x, y, u, v, t) > 1 - \lambda$ and $\Phi(\lambda, x, y, u, v) < \infty$, then $t \geq \Phi(\lambda, x, y, u, v)$.*
- *Suppose that the mapping $M$ satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality or the $\diamond$-triangle inequality.*
  - *If $\eta(x, y, u, v, t) > 1 - \lambda$, then $\Phi(\lambda, y, x, v, u) < \infty$.*
  - *If $\eta(x, y, u, v, t) > 1 - \lambda$ and $\Phi(\lambda, x, y, u, v) < \infty$, then $t \geq \Phi(\lambda, x, y, u, v)$.*

**Proof.** The proof is similar to the argument in Wu [17] by considering $X^4$ instead of $X^2$. □

**Proposition 8.** *Let $(X, M)$ be a fuzzy semi-metric space, such that $M$ satisfies the rational condition, in which the t-norm $*$ is right-continuous at 0 and left-continuous at 1 with respect to the first or second component.*

(i) *Suppose that $M$ satisfies the $\bowtie$-triangle inequality. Given any fixed $x_1, x_2, \cdots, x_p, y_1, y_2, \cdots, y_p \in X$ and any fixed $\mu \in (0, 1]$, there exists $\lambda \in (0, 1)$, such that*

$$\begin{aligned}
\Phi(\mu, x_1, x_p, y_1, y_p) \leq\ & \Phi(\lambda, x_1, x_2, y_1, y_2) + \Phi(\lambda, x_2, x_3, y_2, y_3) + \cdots \\
& + \Phi(\lambda, x_{p-2}, x_{p-1}, y_{p-2}, y_{p-1}) + \Phi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p) \\
\Phi(\mu, x_1, x_p, y_p, y_1) \leq\ & \Phi(\lambda, x_1, x_2, y_2, y_1) + \Phi(\lambda, x_2, x_3, y_3, y_2) + \cdots \\
& + \Phi(\lambda, x_{p-2}, x_{p-1}, y_{p-1}, y_{p-2}) + \Phi(\lambda, x_{p-1}, x_p, y_p, y_{p-1}) \\
\Phi(\mu, x_p, x_1, y_1, y_p) \leq\ & \Phi(\lambda, x_p, x_{p-1}, y_{p-1}, y_p) + \Phi(\lambda, x_{p-1}, x_{p-2}, y_{p-2}, y_{p-1}) \\
& + \cdots + \Phi(\lambda, x_3, x_2, y_2, y_3) + \Phi(\lambda, x_2, x_1, y_1, y_2) \\
\Phi(\mu, x_p, x_1, y_p, y_1) \leq\ & \Phi(\lambda, x_p, x_{p-1}, y_p, y_{p-1}) + \Phi(\lambda, x_{p-1}, x_{p-2}, y_{p-1}, y_{p-2}) \\
& + \cdots + \Phi(\lambda, x_3, x_2, y_3, y_2) + \Phi(\lambda, x_2, x_1, y_2, y_1).
\end{aligned}$$

(ii) *Suppose that M satisfies the $\triangleright$-triangle inequality. Given any fixed $x_1, x_2, \cdots, x_p, y_1, y_2, \cdots, y_p \in X$ and any fixed $\mu \in (0,1]$, there exists $\lambda \in (0,1)$ such that*

$$\begin{aligned}&\max\{\Phi(\mu, x_1, x_p, y_1, y_p), \Phi(\mu, x_1, x_p, y_p, y_1), \Phi(\mu, x_p, x_1, y_1, y_p), \Phi(\mu, x_p, x_1, y_p, y_1)\}\\&\quad\le \Phi(\lambda, x_1, x_2, y_1, y_2) + \Phi(\lambda, x_3, x_2, y_3, y_2) + \Phi(\lambda, x_4, x_3, y_4, y_3)\\&\qquad+ \cdots + \Phi(\lambda, x_p, x_{p-1}, y_p, y_{p-1})\end{aligned}$$

(iii) *Suppose that M satisfies the $\triangleleft$-triangle inequality. Given any fixed $x_1, x_2, \cdots, x_p, y_1, y_2, \cdots, y_p \in X$ and any fixed $\mu \in (0,1]$, there exists $\lambda \in (0,1)$, such that*

$$\begin{aligned}&\max\{\Phi(\mu, x_1, x_p, y_1, y_p), \Phi(\mu, x_1, x_p, y_p, y_1), \Phi(\mu, x_p, x_1, y_1, y_p), \Phi(\mu, x_p, x_1, y_p, y_1)\}\\&\quad\le \Phi(\lambda, x_2, x_1, y_2, y_1) + \Phi(\lambda, x_2, x_3, y_2, y_3) + \Phi(\lambda, x_3, x_4, y_3, y_4)\\&\qquad+ \cdots + \Phi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p)\end{aligned}$$

(iv) *Suppose that M satisfies the $\diamond$-triangle inequality. Given any fixed $x_1, x_2, \cdots, x_p, y_1, y_2, \cdots, y_p \in X$ and any fixed $\mu \in (0,1]$, there exists $\lambda \in (0,1)$, such that the following inequalities are satisfied.*

- *If p is even and $\Phi(\mu, x_1, x_p, y_1, y_p) < \infty$, then*

$$\begin{aligned}\Phi(\mu, x_1, x_p, y_1, y_p) &\le \Phi(\lambda, x_1, x_2, y_1, y_2) + \Phi(\lambda, x_2, x_3, y_2, y_3) + \Phi(\lambda, x_4, x_3, y_4, y_3)\\&\quad+ \Phi(\lambda, x_4, x_5, y_4, y_5) + \Phi(\lambda, x_6, x_5, y_6, y_5) + \Phi(\lambda, x_6, x_7, y_6, y_7)\\&\quad+ \cdots + \Phi(\lambda, x_p, x_{p-1}, y_p, y_{p-1}) \end{aligned} \tag{2}$$

- *If p is even and $\Phi(\mu, x_1, x_p, y_p, y_1) < \infty$, then*

$$\begin{aligned}\Phi(\mu, x_1, x_p, y_p, y_1) &\le \Phi(\lambda, x_1, x_2, y_2, y_1) + \Phi(\lambda, x_2, x_3, y_3, y_2) + \Phi(\lambda, x_4, x_3, y_3, y_4)\\&\quad+ \Phi(\lambda, x_4, x_5, y_5, y_4) + \Phi(\lambda, x_6, x_5, y_5, y_6) + \Phi(\lambda, x_6, x_7, y_7, y_6)\\&\quad+ \cdots + \Phi(\lambda, x_p, x_{p-1}, y_{p-1}, y_p) \end{aligned} \tag{3}$$

- *If p is even and $\Phi(\mu, x_p, x_1, y_1, y_p) < \infty$, then*

$$\begin{aligned}\Phi(\mu, x_p, x_1, y_1, y_p) &\le \Phi(\lambda, x_2, x_1, y_1, y_2) + \Phi(\lambda, x_3, x_2, y_2, y_3) + \Phi(\lambda, x_3, x_4, y_4, y_3)\\&\quad+ \Phi(\lambda, x_5, x_4, y_4, y_5) + \Phi(\lambda, x_5, x_6, y_6, y_5) + \Phi(\lambda, x_7, x_6, y_6, y_7)\\&\quad+ \cdots + \Phi(\lambda, x_{p-1}, x_p, y_p, y_{p-1}) \end{aligned} \tag{4}$$

- *If p is even and $\Phi(\mu, x_p, x_1, y_p, y_1) < \infty$, then*

$$\begin{aligned}\Phi(\mu, x_p, x_1, y_p, y_1) &\le \Phi(\lambda, x_2, x_1, y_2, y_1) + \Phi(\lambda, x_3, x_2, y_3, y_2) + \Phi(\lambda, x_3, x_4, y_3, y_4)\\&\quad+ \Phi(\lambda, x_5, x_4, y_5, y_4) + \Phi(\lambda, x_5, x_6, y_5, y_6) + \Phi(\lambda, x_7, x_6, y_7, y_6)\\&\quad+ \cdots + \Phi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p) \end{aligned} \tag{5}$$

- *If p is odd and $\Phi(\mu, x_1, x_p, y_1, y_p) < \infty$, then*

$$\begin{aligned}\Phi(\mu, x_1, x_p, y_1, y_p) &\le \Phi(\lambda, x_2, x_1, y_2, y_1) + \Phi(\lambda, x_3, x_2, y_3, y_2) + \Phi(\lambda, x_3, x_4, y_3, y_4)\\&\quad+ \Phi(\lambda, x_5, x_4, y_5, y_4) + \Phi(\lambda, x_5, x_6, y_5, y_6) + \Phi(\lambda, x_7, x_6, y_7, y_6)\\&\quad+ \cdots + \Phi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p) \end{aligned} \tag{6}$$

- *If p is odd and $\Phi(\mu, x_1, x_p, y_p, y_1) < \infty$, then*

$$\begin{aligned}\Phi(\mu, x_1, x_p, y_p, y_1) &\le \Phi(\lambda, x_2, x_1, y_1, y_2) + \Phi(\lambda, x_3, x_2, y_2, y_3) + \Phi(\lambda, x_3, x_4, y_4, y_3)\\&\quad+ \Phi(\lambda, x_5, x_4, y_4, y_5) + \Phi(\lambda, x_5, x_6, y_6, y_5) + \Phi(\lambda, x_7, x_6, y_6, y_7)\\&\quad+ \cdots + \Phi(\lambda, x_{p-1}, x_p, y_p, y_{p-1}) \end{aligned} \tag{7}$$

- *If $p$ is odd and $\Phi(\mu, x_p, x_1, y_1, y_p) < \infty$, then*

$$\Phi(\mu, x_p, x_1, y_1, y_p) \leq \Phi(\lambda, x_1, x_2, y_2, y_1) + \Phi(\lambda, x_2, x_3, y_3, y_2) + \Phi(\lambda, x_4, x_3, y_3, y_4) + \Phi(\lambda, x_4, x_5, y_5, y_4) + \Phi(\lambda, x_6, x_5, y_5, y_6) + \Phi(\lambda, x_6, x_7, y_7, y_6) + \cdots + \Phi(\lambda, x_p, x_{p-1}, y_{p-1}, y_p) \quad (8)$$

- *If $p$ is odd and $\Phi(\mu, x_p, x_1, y_p, y_1) < \infty$, then*

$$\Phi(\mu, x_p, x_1, y_p, y_1) \leq \Phi(\lambda, x_1, x_2, y_1, y_2) + \Phi(\lambda, x_2, x_3, y_2, y_3) + \Phi(\lambda, x_4, x_3, y_4, y_3) + \Phi(\lambda, x_4, x_5, y_4, y_5) + \Phi(\lambda, x_6, x_5, y_6, y_5) + \Phi(\lambda, x_6, x_7, y_6, y_7) + \cdots + \Phi(\lambda, x_p, x_{p-1}, y_p, y_{p-1}) \quad (9)$$

**Proof.** The proof is similar to the argument put foward in Wu [17] by considering $X^4$ instead of $X^2$. □

**Proposition 9.** *Let $(X, M)$ be a fuzzy semi-metric space, such that $M$ satisfies the rational condition, in which the t-norm $*$ is right-continuous at $0$ with respect to the first or second component. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences in $X$.*

(i) *Assume that $M$ satisfies the $\bowtie$-triangle inequality or the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality. Subsequently, we have the following results.*
- *$\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two $>$-Cauchy sequences if and only if, given any $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ satisfying $\Phi(\lambda, x_m, x_n, y_m, y_n) < \epsilon$ for $m > n \geq n_{\epsilon,\lambda}$.*
- *$\{x_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequences and $\{y_n\}_{n=1}^{\infty}$ is a $<$-Cauchy sequences if and only if, given any $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ satisfying $\Phi(\lambda, x_m, x_n, y_n, y_m) < \epsilon$ for $m > n \geq n_{\epsilon,\lambda}$.*
- *$\{x_n\}_{n=1}^{\infty}$ is a $<$-Cauchy sequences and $\{y_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequences if and only if, given any $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ satisfying $\Phi(\lambda, x_n, x_m, y_m, y_n) < \epsilon$ for $m > n \geq n_{\epsilon,\lambda}$.*
- *$\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two $<$-Cauchy sequences if and only if, given any $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ satisfying $\Phi(\lambda, x_n, x_m, y_n, y_m) < \epsilon$ for $m > n \geq n_{\epsilon,\lambda}$.*

(ii) *Assume that $M$ satisfies the $\diamond$-triangle inequality. Then, we have the following results.*
- *Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two $>$-Cauchy sequences. Suppose that $\Phi(\lambda, x_m, x_n, y_m, y_n) < \infty$ for all $\lambda \in (0, 1)$ and $m > n$. Subsequently, given any $\epsilon > 0$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ satisfying $\Phi(\lambda, x_m, x_n, y_m, y_n) < \epsilon$ for $m > n \geq n_{\epsilon,\lambda}$.*
- *Let $\{x_n\}_{n=1}^{\infty}$ be a $>$-Cauchy sequence and let $\{y_n\}_{n=1}^{\infty}$ be a $<$-Cauchy sequence. Suppose that $\Phi(\lambda, x_m, x_n, y_n, y_m) < \infty$ for any all $\lambda \in (0, 1)$ and $m > n$. Afterwards, given any $\epsilon > 0$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ satisfying $\Phi(\lambda, x_m, x_n, y_n, y_m) < \epsilon$ for $m > n \geq n_{\epsilon,\lambda}$.*
- *Let $\{x_n\}_{n=1}^{\infty}$ be a $<$-Cauchy sequence and let $\{y_n\}_{n=1}^{\infty}$ be a $>$-Cauchy sequence. Suppose that $\Phi(\lambda, x_n, x_m, y_m, y_n) < \infty$ for all $\lambda \in (0, 1)$ and $m > n$. Subsequently, given any $\epsilon > 0$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ satisfying $\Phi(\lambda, x_n, x_m, y_m, y_n) < \epsilon$ for $m > n \geq n_{\epsilon,\lambda}$.*
- *Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two $<$-Cauchy sequences. Suppose that $\Phi(\lambda, x_n, x_m, y_n, y_m) < \infty$ for all $\lambda \in (0, 1)$ and $m > n$. Subsequently, given any $\epsilon > 0$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ satisfying $\Phi(\lambda, x_n, x_m, y_n, y_m) < \epsilon$ for $m > n \geq n_{\epsilon,\lambda}$.*
- *Suppose that, given any $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ satisfying $\Phi(\lambda, x_m, x_n, y_m, y_n) < \epsilon$ for $m > n \geq n_{\epsilon,\lambda}$. Then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two $<$-Cauchy sequences.*
- *Suppose that, given any $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ satisfying $\Phi(\lambda, x_m, x_n, y_n, y_m) < \epsilon$ for $m > n \geq n_{\epsilon,\lambda}$. Then $\{x_n\}_{n=1}^{\infty}$ is a $<$-Cauchy sequences and $\{y_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequences.*
- *Suppose that, given any $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ satisfying $\Phi(\lambda, x_n, x_m, y_m, y_n) < \epsilon$ for $m > n \geq n_{\epsilon,\lambda}$. Subsequently, $\{x_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequences and $\{y_n\}_{n=1}^{\infty}$ is a $<$-Cauchy sequences.*

- *Suppose that, given any $\epsilon > 0$ and $\lambda \in (0,1)$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ satisfying $\Phi(\lambda, x_n, x_m, y_n, y_m) < \epsilon$ for $m > n \geq n_{\epsilon,\lambda}$. Afterwards, $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two $>$-Cauchy sequences.*

**Proof.** The proof is similar to the argument in Wu [17] by considering $X^4$ instead of $X^2$. □

## 4. Cauchy Sequences

Given any $a \in [0,1]$, for convenience, we write

$$(*a)^n = \overbrace{a * a * \cdots * a}^{n \text{ times}}$$

and

$$\left[*\eta\left(a,b,c,d,\frac{t}{k^n}\right)\right]^{2^n} = \overbrace{\eta\left(a,b,c,d,\frac{t}{k^n}\right) * \eta\left(a,b,c,d,\frac{t}{k^n}\right) * \cdots * \eta\left(a,b,c,d,\frac{t}{k^n}\right)}^{2^n \text{ times}}.$$

The following results will be used for further discussion.

**Proposition 10.** *Let $(X, M)$ be a fuzzy semi-metric space such that $M$ satisfies the rational condition in which the t-norm is right-continuous at 0 and left-continuous at 1 in the first or second component. Let $0 < k < 1$ be any fixed constant, and let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences in $X$.*

(i) *Suppose that $M$ satisfies the $\bowtie$-triangle inequality. Subsequently, we have the following results.*

- *Assume that there exist fixed elements $a_1, b_1, c_1, d_1 \in X$ satisfying*

$$\sup_{\lambda\in[0,1)} \Phi(\lambda, a_1, b_1, c_1, d_1) < \infty \tag{10}$$

*and*

$$\eta(x_n, x_{n+1}, y_n, y_{n+1}, t) \geq \left[*\eta\left(a_1, b_1, c_1, d_1, \frac{t}{k^n}\right)\right]^{2^n} \text{ for each } n \in \mathbb{N}. \tag{11}$$

*Afterwards, $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are $<$-Cauchy sequences.*

- *Assume that there exist fixed elements $a_2, b_2, c_2, d_2 \in X$ satisfying*

$$\sup_{\lambda\in[0,1)} \Phi(\lambda, a_2, b_2, c_2, d_2) < \infty \tag{12}$$

*and*

$$\eta(x_n, x_{n+1}, y_{n+1}, y_n, t) \geq \left[*\eta\left(a_2, b_2, c_2, d_2, \frac{t}{k^n}\right)\right]^{2^n} \text{ for each } n \in \mathbb{N}. \tag{13}$$

*Subsequently, $\{x_n\}_{n=1}^{\infty}$ is a $<$-Cauchy sequence and $\{y_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence.*

- *Assume that there exist fixed elements $a_3, b_3, c_3, d_3 \in X$ satisfying*

$$\sup_{\lambda\in[0,1)} \Phi(\lambda, a_3, b_3, c_3, d_3) < \infty \tag{14}$$

*and*

$$\eta(x_{n+1}, x_n, y_n, y_{n+1}, t) \geq \left[*\eta\left(a_3, b_3, c_3, d_3, \frac{t}{k^n}\right)\right]^{2^n} \text{ for each } n \in \mathbb{N}. \tag{15}$$

*Subsequently,* $\{x_n\}_{n=1}^{\infty}$ *is a* $>$*-Cauchy sequence and* $\{y_n\}_{n=1}^{\infty}$ *is a* $<$*-Cauchy sequence.*

- *Assume that there exist fixed elements* $a_4, b_4, c_4, d_4 \in X$ *satisfying*

$$\sup_{\lambda \in [0,1)} \Phi(\lambda, a_4, b_4, c_4, d_4) < \infty \tag{16}$$

*and*

$$\eta(x_{n+1}, x_n, y_{n+1}, y_n, t) \geq \left[ *\eta\left(a_4, b_4, c_4, d_4, \frac{t}{k^n}\right)\right]^{2^n} \text{ for each } n \in \mathbb{N}. \tag{17}$$

*Afterwards,* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are* $>$*-Cauchy sequences.*

(ii) *Suppose that the mapping* $M$ *satisfies the* $\triangleright$*-triangle inequality or the* $\triangleleft$*-triangle inequality, and that the conditions* (10)*,* (11)*,* (16) *and* (17) *are satisfied. Subsequently,* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are both* $>$*-Cauchy and* $<$*-Cauchy sequences. In other words,* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are Cauchy sequences.*

(iii) *Suppose that the mapping* $M$ *satisfies the* $\diamond$*-triangle inequality, and that any one of the following two conditions is satisfied:*

- *conditions* (10)*,* (11)*,* (16) *and* (17) *are satisfied;*
- *conditions* (12)*,* (13)*,* (14) *and* (15) *are satisfied.*

*Afterwards,* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are both* $>$*-Cauchy and* $<$*-Cauchy sequences.*

**Proof.** To prove part (i), if

$$\left[ *\eta\left(a_1, b_1, c_1, d_1, \frac{t}{k^n}\right)\right]^{2^n} \leq 1 - \lambda,$$

then, using Proposition 1, there exists $\bar{\lambda}(t) \in (0,1)$ satisfying

$$\eta\left(a_1, b_1, c_1, d_1, \frac{t}{k^n}\right) \leq 1 - \bar{\lambda}(t).$$

Let

$$\lambda_0 \equiv \inf_t \bar{\lambda}(t) \in [0,1).$$

Then $\lambda_0$ depends only on $\lambda$ and

$$\eta\left(a_1, b_1, c_1, d_1, \frac{t}{k^n}\right) \leq 1 - \inf_t \bar{\lambda}(t) \equiv 1 - \lambda_0.$$

It follows that

$$\left\{ t > 0 : \left[ *\eta\left(a_1, b_1, c_1, d_1, \frac{t}{k^n}\right)\right]^{2^n} \leq 1 - \lambda \right\} \subseteq \left\{ t > 0 : \eta\left(a_1, b_1, c_1, d_1, \frac{t}{k^n}\right) \leq 1 - \lambda_0 \right\}. \tag{18}$$

Therefore, we obtain

$$
\begin{aligned}
&\Phi(\lambda, x_n, x_{n+1}, y_n, y_{n+1}) \\
&\quad = \sup\{t > 0 : \eta(x_n, x_{n+1}, y_n, y_{n+1}, t) \leq 1 - \lambda\} \\
&\quad \leq \sup\left\{ t > 0 : \left[ *\eta\left(a_1, b_1, c_1, d_1, \frac{t}{k^n}\right)\right]^{2^n} \leq 1 - \lambda \right\} \text{ (by (11))} \\
&\quad \leq \sup\left\{ t > 0 : \eta\left(a_1, b_1, c_1, d_1, \frac{t}{k^n}\right) \leq 1 - \lambda_0 \right\} \text{ (by (18))} \\
&\quad = \sup\{k^n \cdot t > 0 : \eta(a_1, b_1, c_1, d_1, t) \leq 1 - \lambda_0\} \\
&\quad = k^n \cdot \sup\{t > 0 : \eta(a_1, b_1, c_1, d_1, t) \leq 1 - \lambda_0\} \\
&\quad = k^n \cdot \Phi(\lambda_0, a_1, b_1, c_1, d_1),
\end{aligned} \tag{19}
$$

where $\lambda_0$ only depends on $\lambda$. Now, we assume that $m, n \in \mathbb{N}$ with $m > n$. Given any $\mu \in (0, 1]$, by part (i) of Proposition 8, there exists $\lambda \in (0, 1)$, such that

$$
\begin{aligned}
&\Phi(\mu, x_n, x_m, y_n, y_m) \\
&\quad \leq \Phi(\lambda, x_n, x_{n+1}, y_n, y_{n+1}) + \Phi(\lambda, x_{n+1}, x_{n+2}, y_{n+1}, y_{n+2}) + \cdots + \Phi(\lambda, x_{m-1}, x_m, y_{m-1}, y_m) \\
&\quad \leq k^n \cdot \Phi(\lambda_0, a_1, b_1, c_1, d_1) + k^{n+1} \cdot \Phi(\lambda_0, a_1, b_1, c_1, d_1) + \cdots + k^{m-1} \cdot \Phi(\lambda_0, a_1, b_1, c_1, d_1) \text{ (by (19))} \\
&\quad = \Phi(\lambda_0, a_1, b_1, c_1, d_1) \cdot \frac{k^n \cdot (1 - k^{m-n})}{1 - k} \leq \Phi(\lambda_0, a_1, b_1, c_1, d_1) \cdot \frac{k^n}{1 - k} \\
&\quad \leq \left[ \sup_{\lambda \in [0,1)} \Phi(\lambda, a_1, b_1, c_1, d_1) \right] \cdot \frac{k^n}{1 - k} \to 0 \text{ as } n \to \infty,
\end{aligned} \tag{20}
$$

which also says that, given any $\epsilon \in (0, 1)$ and $\mu \in (0, 1)$, there exists $n_{\mu,\epsilon} \in \mathbb{N}$ such that $m > n \geq n_{\mu,\epsilon}$ implies $\Phi(\mu, x_n, x_m, y_n, y_m) < \epsilon$. By the fourth case of part (i) of Proposition 9, it follows that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are $<$-Cauchy sequences. The other results can be similarly obtained by using the corresponding cases of Proposition 9 and part (i) of Proposition 8.

To prove part (ii), we consider two cases below.

- Suppose that the mapping $M$ satisfies the $\triangleright$-triangle inequality. While using part (ii) of Proposition 8, we have

$$
\begin{aligned}
&\max\{\Phi(\mu, x_n, x_m, y_n, y_m), \Phi(\mu, x_m, x_n, y_n, y_m), \Phi(\mu, x_n, x_m, y_m, y_n), \Phi(\mu, x_m, x_n, y_m, y_n)\} \\
&\quad \leq \Phi(\lambda, x_m, x_{m-1}, y_m, y_{m-1}) + \Phi(\lambda, x_{m-1}, x_{m-2}, y_{m-1}, y_{m-2}) \\
&\qquad + \cdots + \Phi(\lambda, x_{n+2}, x_{n+1}, y_{n+2}, y_{n+1}) + \Phi(\lambda, x_n, x_{n+1}, y_n, y_{n+1}).
\end{aligned} \tag{21}
$$

  By referring to (19), we can similarly obtain

$$
\Phi(\lambda, x_{n+1}, x_n, y_{n+1}, y_n) \leq k^n \cdot \Phi(\lambda_0, a_4, b_4, c_4, d_4). \tag{22}
$$

  By using (19), (22), (21) and referring to (20), we have

$$
\begin{aligned}
&\max\{\Phi(\mu, x_n, x_m, y_n, y_m), \Phi(\mu, x_m, x_n, y_n, y_m), \Phi(\mu, x_n, x_m, y_m, y_n), \Phi(\mu, x_m, x_n, y_m, y_n)\} \\
&\quad \leq \max\left\{ \left[ \sup_{\lambda \in [0,1)} \Phi(\lambda, a_1, b_1, c_1, d_1) \right], \left[ \sup_{\lambda \in [0,1)} \Phi(\lambda, a_4, b_4, c_4, d_4) \right] \right\} \cdot \frac{k^n}{1 - k} \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

  Using the above argument, we can show that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are both $>$-Cauchy and $<$-Cauchy sequences in metric sense.

- Suppose that the mapping $M$ satisfies the $\triangleleft$-triangle inequality. While using part (iii) of Proposition 8, we have

$$\max\{\Phi(\mu, x_n, x_m, y_n, y_m), \Phi(\mu, x_m, x_n, y_n, y_m), \Phi(\mu, x_n, x_m, y_m, y_n), \Phi(\mu, x_m, x_n, y_m, y_n)\}$$
$$\leq \Phi(\lambda, x_{n+1}, x_n, y_{n+1}, y_n) + \Phi(\lambda, x_{n+1}, x_{n+2}, y_{n+1}, y_{n+2})$$
$$+ \cdots + \Phi(\lambda, x_{m-2}, x_{m-1}, y_{m-2}, y_{m-1}) + \Phi(\lambda, x_{m-1}, x_m, y_{m-1}, y_m).$$

Using the above argument, we can show that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are both $>$-Cauchy and $<$-Cauchy sequences in metric sense.

To prove part (iii), we consider two cases below.

- Assume that the conditions (10), (11), (16) and (17) are satisfied. If $p$ is even, then, using (2) and (5) in part (iv) of Proposition 8, we can similarly show that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are both $>$-Cauchy and $<$-Cauchy sequences in metric sense. If $p$ is odd, then, using (6) and (9) in Proposition 8, we can similarly obtain the desired results.
- Assume that the conditions (12), (13), (14) and (15) are satisfied. If $p$ is even, then, using (3) and (4) in part (iv) of Proposition 8, we can similarly show that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are both $>$-Cauchy and $<$-Cauchy sequences in the metric sense. If $p$ is odd, then, using (7) and (8) in Proposition 8, we can similarly obtain the desired results.

This completes the proof. □

## 5. Common Coupled Coincidence Points

In this section, we are going to investigate the common coupled coincidence points in fuzzy semi-metric space under some suitable conditions. We consider two mappings $T : X \times X \to X$ and $f : X \to X$.

- Recall that the mappings $T$ and $f$ commute when $f(T(x, y)) = T(f(x), f(y))$ for all $x, y \in X$.
- Recall that an element $(x^*, y^*) \in X \times X$ is called a coupled coincidence point of mappings $T$ and $f$ when $T(x^*, y^*) = f(x^*)$ and $T(y^*, x^*) = f(y^*)$. In particular, if $x^* = f(x^*) = T(x^*, y^*)$ and $y^* = f(y^*) = T(y^*, x^*)$, then $(x^*, y^*)$ is called a common coupled fixed point of $T$ and $f$.

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings from the product space $X \times X$ into $X$, and let $f$ be a mapping from $X$ into itself satisfying $T_n(X, X) \subseteq f(X)$ for all $n \in \mathbb{N}$. Given any two initial elements $x_0, y_0 \in X$, since $T_n(X, X) \subseteq f(X)$, there exist $x_1, y_1 \in X$ satisfying

$$f(x_1) = T_1(x_0, y_0) \text{ and } f(y_1) = T_1(y_0, x_0).$$

Similarly, there also exist $x_2, y_2 \in X$, satisfying

$$f(x_2) = T_2(x_1, y_1) \text{ and } f(y_2) = T_2(y_1, x_1).$$

Continuing this process, we can construct two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$, satisfying

$$f(x_n) = T_n(x_{n-1}, y_{n-1}) \text{ and } f(y_n) = T_n(y_{n-1}, x_{n-1}) \tag{23}$$

for $n \in \mathbb{N}$.

In the sequel, the common coupled coincidence points will be separately studied by considering the four different types of triangle inequalities.

**Theorem 1** (Satisfying the $\bowtie$-Triangle Inequality). *Let $(X, M)$ be a fuzzy semi-metric space, such that the mapping $M$ satisfies the rational condition and the $\bowtie$-triangle inequality. Suppose that the following conditions are satisfied.*

- *The t-norm $*$ is left-continuous with respect to the first or second component.*
- *Given any fixed $x, y \in X$, the mapping $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is left-continuous at each point $t \in (0, \infty)$.*
- *The mappings $T_n : X \times X \to X$ and $f : X \to X$ satisfy the inclusions $T_n(X, X) \subseteq f(X)$ for all $n \in \mathbb{N}$.*

- *The mappings $f$ and $T_n$ commute; that is, $f(T_n(x,y)) = T_n(f(x), f(y))$ for all $x, y \in X$ and all $n \in \mathbb{N}$.*
- *Given any $x, y, u, v \in X$, the following contractive inequality is satisfied:*

$$M(T_i(x,y), T_j(u,v), k_{ij} \cdot t) \geq M(f(x), f(u), t) * M(f(y), f(v), t), \tag{24}$$

*where $k_{ij}$ satisfies $0 < k_{ij} \leq k < 1$ for all $i, j \in \mathbb{N}$ and for some constant $k$.*

*Subsequently, we have the following results.*

(i) *Suppose that there exist $x^*, y^* \in X$ satisfying*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, f(x^*), T_1(x^*, y^*), f(y^*), T_1(y^*, x^*)) < \infty,$$

*and that any one of the following conditions is satisfied:*

(a) *$(X, M)$ is $(<, \triangleright)$-complete and $f$ is simultaneously $(\triangleright, \triangleright)$-continuous and $(\triangleright, \triangleleft)$-continuous with respect to $M$;*

(b) *$(X, M)$ is $(<, \triangleleft)$-complete and $f$ is simultaneously $(\triangleleft, \triangleright)$-continuous and $(\triangleleft, \triangleleft)$-continuous with respect to $M$.*

*Afterwards, the mappings $\{T_n\}_{n=1}^{\infty}$ and $f$ have a common coupled coincidence point $(x^\circ, y^\circ)$. We further assume that the following conditions are satisfied.*

- *The inequality (24) is replaced by the following inequality*

$$M(T_i(x,y), T_j(u,v), k_{ij} \cdot t) \geq M(f(x), f(u), t) \cdot M(f(y), f(v), t), \tag{25}$$

*where the t-norm $*$ is replaced by the product of real numbers.*

- *The mapping $M$ satisfies the distance condition in Definition 2.*
- *For any fixed $x, y \in X$ and $t > 0$, the following mapping*

$$\varrho(\alpha) = M\left(x, y, k^{\log_2 \alpha} \cdot t\right) \tag{26}$$

*is differentiable on $(0, \infty)$.*

*Afterwards, we have the following results.*

(A) *Suppose that $(\bar{x}, \bar{y})$ is another coupled coincidence point of mappings $f$ and $T_{n_0}$ for some $n_0 \in \mathbb{N}$. Subsequently, $f(x^\circ) = f(\bar{x})$ and $f(y^\circ) = f(\bar{y})$.*

(B) *There exists $(x^\circ, y^\circ) \in X \times X$ such that $(f(x^\circ), f(y^\circ)) \in X \times X$ is the common coupled fixed point of the mappings $\{T_n\}_{n=1}^{\infty}$.*

*Moreover, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained, as follows.*

- *Suppose that condition* (a) *is satisfied. Afterwards, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^{\triangleright}} x^\circ$ and $f(y_n) \xrightarrow{M^{\triangleright}} y^\circ$.*
- *Suppose that condition* (b) *is satisfied. Subsequently, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^{\triangleleft}} x^\circ$ and $f(y_n) \xrightarrow{M^{\triangleleft}} y^\circ$.*

*The sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are generated from the initial element $(x_0, y_0) = (x^*, y^*) \in X \times X$ according to (23).*

(ii) *Suppose that there exist $x^*, y^* \in X$ satisfying*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, T_1(x^*, y^*), f(x^*), T_1(y^*, x^*), f(y^*)) < \infty,$$

*and that any one of the following conditions is satisfied:*

(c) *$(X, M)$ is $(>, \triangleright)$-complete and $f$ is simultaneously $(\triangleright, \triangleright)$-continuous and $(\triangleright, \triangleleft)$-continuous with respect to $M$;*

(d) *$(X, M)$ is $(>, \triangleleft)$-complete and $f$ is simultaneously $(\triangleleft, \triangleright)$-continuous and $(\triangleleft, \triangleleft)$-continuous with respect to $M$.*

*Afterwards, we have the same result as part* (i)*.*

**Proof.** We can generate two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ from the initial element $x_0 = x^*$ and $y_0 = y^*$ according to (23). Then we have

$$f(x^*) = f(x_0) \text{ and } f(y^*) = f(y_0)$$

and

$$T_1(x^*, y^*) = T_1(x_0, y_0) = f(x_1) \text{ and } T_1(y^*, x^*) = T_1(y_0, x_0) = f(y_1).$$

To prove part (i), from (23) and (24), we obtain

$$\begin{aligned} M(f(x_1), f(x_2), t) &= M(T_1(x_0, y_0), T_2(x_1, y_1), t) \\ &\geq M\left(f(x_0), f(x_1), \frac{t}{k_{12}}\right) * M\left(f(y_0), f(y_1), \frac{t}{k_{12}}\right) \end{aligned}$$

and

$$\begin{aligned} M(f(y_1), f(y_2), t) &= M(T_1(y_0, x_0), T_2(y_1, x_1), t) \\ &\geq M\left(f(y_0), f(y_1), \frac{t}{k_{12}}\right) * M\left(f(x_0), f(x_1), \frac{t}{k_{12}}\right). \end{aligned}$$

By induction, we can obtain

$$\begin{aligned} M(f(x_n), f(x_{n+1}), t) &\geq \left[ *M\left(f(x_0), f(x_1), \frac{t}{\prod_{i=1}^{n} k_{i,i+1}}\right)\right]^{2^{n-1}} \\ &\quad * \left[ *M\left(f(y_0), f(y_1), \frac{t}{\prod_{i=1}^{n} k_{i,i+1}}\right)\right]^{2^{n-1}} \end{aligned} \tag{27}$$

and

$$\begin{aligned} M(f(y_n), f(y_{n+1}), t) &\geq \left[ *M\left(f(x_0), f(x_1), \frac{t}{\prod_{i=1}^{n} k_{i,i+1}}\right)\right]^{2^{n-1}} \\ &\quad * \left[ *M\left(f(y_0), f(y_1), \frac{t}{\prod_{i=1}^{n} k_{i,i+1}}\right)\right]^{2^{n-1}}. \end{aligned} \tag{28}$$

Part (i) of Proposition 2 says that the mapping $M(x, y, \cdot)$ is nondecreasing. Because $k_{i,i+1} \leq k$ for each $i \in \mathbb{N}$, using the increasing property of t-norm to (27) and (28), we also have

$$\begin{aligned} M(f(x_n), f(x_{n+1}), t) &\geq \left[ *M\left(f(x_0), f(x_1), \frac{t}{k^n}\right)\right]^{2^{n-1}} * \left[ *M\left(f(y_0), f(y_1), \frac{t}{k^n}\right)\right]^{2^{n-1}} \\ &= \left[ *\eta\left(f(x_0), f(x_1), f(y_0), f(y_1), \frac{t}{k^n}\right)\right]^{2^{n-1}} \end{aligned} \tag{29}$$

and

$$\begin{aligned} M(f(y_n), f(y_{n+1}), t) &\geq \left[ *M\left(f(x_0), f(x_1), \frac{t}{k^n}\right)\right]^{2^{n-1}} * \left[ *M\left(f(y_0), f(y_1), \frac{t}{k^n}\right)\right]^{2^{n-1}} \\ &= \left[ *\eta\left(f(x_0), f(x_1), f(y_0), f(y_1), \frac{t}{k^n}\right)\right]^{2^{n-1}}. \end{aligned} \tag{30}$$

Using the increasing property of t-norm to (29) and (30), we have

$$\eta(f(x_n), f(x_{n+1}), f(y_n), f(y_{n+1}), t) = M(f(x_n), f(x_{n+1}), t) * M(f(y_n), f(y_{n+1}), t)$$
$$\geq \left[ *\eta\left( f(x_0), f(x_1), f(y_0), f(y_1), \frac{t}{k^n} \right) \right]^{2^n}. \quad (31)$$

From part (i) of Proposition 10, it follows that $\{f(x_n)\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ are $<$-Cauchy sequences. We consider the following cases

- Suppose that condition (a) is satisfied. Because $(X, M)$ is $(<, \triangleright)$-complete, there exist $x^\circ, y^\circ \in X$, such that

$$f(x_n) \xrightarrow{M^\triangleright} x^\circ \text{ and } f(y_n) \xrightarrow{M^\triangleright} y^\circ \text{ as } n \to \infty. \quad (32)$$

  Since $f$ is simultaneously $(\triangleright, \triangleright)$-continuous and $(\triangleright, \triangleleft)$-continuous with respect to $M$, we have

$$f(f(x_n)) \xrightarrow{M^\triangleright} f(x^\circ) \text{ and } f(f(y_n)) \xrightarrow{M^\triangleright} f(y^\circ) \text{ as } n \to \infty$$

  and

$$f(f(x_n)) \xrightarrow{M^\triangleleft} f(x^\circ) \text{ and } f(f(y_n)) \xrightarrow{M^\triangleleft} f(y^\circ) \text{ as } n \to \infty,$$

  which say that, for all $t > 0$,

$$M(f(f(x_n)), f(x^\circ), t) \to 1- \text{ as } n \to \infty \quad (33)$$
$$M(f(f(y_n)), f(y^\circ), t) \to 1- \text{ as } n \to \infty \quad (34)$$
$$M(f(x^\circ), f(f(x_n)), t) \to 1- \text{ as } n \to \infty \quad (35)$$
$$M(f(y^\circ), f(f(y_n)), t) \to 1- \text{ as } n \to \infty. \quad (36)$$

- Suppose that condition (b) is satisfied. Since $(X, M)$ is $(<, \triangleleft)$-complete, there exist $x^\circ, y^\circ \in X$, such that

$$f(x_n) \xrightarrow{M^\triangleleft} x^\circ \text{ and } f(y_n) \xrightarrow{M^\triangleleft} y^\circ \text{ as } n \to \infty. \quad (37)$$

  Because $f$ is simultaneously $(\triangleleft, \triangleright)$-continuous and $(\triangleleft, \triangleleft)$-continuous with respect to $M$, we can similarly obtain (33)–(36).

  Using (23) and the commutativity of $T_n$ and $f$, we obtain

$$f(f(x_{n+1})) = f(T_{n+1}(x_n, y_n)) = T_{n+1}(f(x_n), f(y_n))) \quad (38)$$

and

$$f(f(y_{n+1})) = f(T_{n+1}(y_n, x_n)) = T_{n+1}(f(y_n), f(x_n))).$$

We shall show that $f(x^\circ) = T_n(x^\circ, y^\circ)$ and $f(y^\circ) = T_n(y^\circ, x^\circ)$ for all $n \in \mathbb{N}$. Now we have

$$M(f(f(x_{n+1})), T_n(x^\circ, y^\circ), kt) \geq M(f(f(x_{n+1})), T_n(x^\circ, y^\circ), k_{n+1,n} \cdot t)$$
$$= M(T_{n+1}(f(x_n), f(y_n))), T_n(x^\circ, y^\circ), k_{n+1,n} \cdot t) \text{ (by (38))}$$
$$\geq M(f(f(x_n)), f(x^\circ), t) * M(f(f(y_n)), f(y^\circ), t) \text{ (by (24))}. \quad (39)$$

Using Proposition 1 and applying (33) and (34) to (39), we obtain

$$\liminf_{n\to\infty} M(f(f(x_{n+1})), T_n(x^\circ, y^\circ), t)$$
$$\geq \lim_{n\to\infty} \left[ M\left( f(f(x_n)), f(x^\circ), \frac{t}{k} \right) * M\left( f(f(y_n)), f(y^\circ), \frac{t}{k} \right) \right] = 1 * 1 = 1,$$

which says that

$$1 \geq \limsup_{n\to\infty} M(f(f(x_{n+1})), T_n(x^\circ, y^\circ), t) \geq \liminf_{n\to\infty} M(f(f(x_{n+1})), T_n(x^\circ, y^\circ), t) \geq 1.$$

Therefore, we obtain

$$\lim_{n\to\infty} M(f(f(x_{n+1})), T_n(x^\circ, y^\circ), t) = 1, \text{ i.e., } M(f(f(x_{n+1})), T_n(x^\circ, y^\circ), t) \to 1-. \tag{40}$$

Using the $\bowtie$-triangle inequality, we see that

$$M(f(x^\circ), T_n(x^\circ, y^\circ), 2t) \geq M(f(x^\circ), f(f(x_{n+1})), t) * M(f(f(x_{n+1})), T_n(x^\circ, y^\circ), t).$$

While using the left-continuity of t-norm $*$ to (35) and (40), we obtain $M(f(x^\circ), T_n(x^\circ, y^\circ), 2t) = 1$ for all $t > 0$. Therefore we must have $f(x^\circ) = T_n(x^\circ, y^\circ)$ for all $n \in \mathbb{N}$. We can similarly show that $f(y^\circ) = T_n(y^\circ, x^\circ)$ for all $n \in \mathbb{N}$.

To prove property (A), let $(\bar{x}, \bar{y})$ be another coupled coincidence point of $f$ and $T_{n_0}$ for some $n_0 \in \mathbb{N}$, i.e., $f(\bar{x}) = T_{n_0}(\bar{x}, \bar{y})$ and $f(\bar{y}) = T_{n_0}(\bar{y}, \bar{x})$. Because the mapping $M(x, y, \cdot)$ is non-decreasing, by (25), we have

$$\begin{aligned} M(f(x^\circ), f(\bar{x}), t) &= M(T_{n_0}(x^\circ, y^\circ), T_{n_0}(\bar{x}, \bar{y}), t) \\ &\geq M\left(f(x^\circ), f(\bar{x}), \frac{t}{k_{n_0,n_0}}\right) \cdot M\left(f(y^\circ), f(\bar{y}), \frac{t}{k_{n_0,n_0}}\right) \\ &\geq M\left(f(x^\circ), f(\bar{x}), \frac{t}{k}\right) \cdot M\left(f(y^\circ), f(\bar{y}), \frac{t}{k}\right) \end{aligned} \tag{41}$$

and

$$\begin{aligned} M(f(y^\circ), f(\bar{y}), t) &= M(T_{n_0}(y^\circ, x^\circ), T_{n_0}(\bar{y}, \bar{x}), t) \\ &\geq M\left(f(y^\circ), f(\bar{y}), \frac{t}{k_{n_0,n_0}}\right) \cdot M\left(f(x^\circ), f(\bar{x}), \frac{t}{k_{n_0,n_0}}\right) \\ &\geq M\left(f(y^\circ), f(\bar{y}), \frac{t}{k}\right) \cdot M\left(f(x^\circ), f(\bar{x}), \frac{t}{k}\right). \end{aligned} \tag{42}$$

Therefore we obtain

$$\begin{aligned} M(f(x^\circ), f(\bar{x}), t) &\geq M\left(f(x^\circ), f(\bar{x}), \frac{t}{k}\right) \cdot M\left(f(y^\circ), f(\bar{y}), \frac{t}{k}\right) \text{ (by (41))} \\ &\geq \left[M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^2}\right) \cdot M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^2}\right)\right] \cdot \left[M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^2}\right) \cdot M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^2}\right)\right] \\ &\quad \text{(by (41) and (42))} \\ &= \left[M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^2}\right)\right]^2 \cdot \left[M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^2}\right)\right]^2 \\ &\geq \cdots \geq \left[M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^n}\right)\right]^{2^{n-1}} \cdot \left[M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^n}\right)\right]^{2^{n-1}} \\ &\quad \text{(by repeating to use (41) and (42))} \\ &\geq \left[M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^n}\right)\right]^{2^n} \cdot \left[M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^n}\right)\right]^{2^n} \\ &\quad \text{(since } M(x, y, t) \leq 1 \text{ for any } x, y \in X \text{ and } t > 0), \end{aligned} \tag{43}$$

Equivalently, we have

$$M(f(x^\circ), f(\bar{x}), t) \geq \left[M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^{\log_2 n}}\right)\right]^n \cdot \left[M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^{\log_2 n}}\right)\right]^n.$$

which can be rewritten as

$$M\left(f(x^\circ), f(\bar{x}), k^{\log_2 n} \cdot t\right) \geq [M(f(x^\circ), f(\bar{x}), t)]^n \cdot [M(f(y^\circ), f(\bar{y}), t)]^n. \tag{44}$$

We are going to claim that there exists $\bar{t} > 0$, such that $M(f(y^\circ), f(\bar{y}), t) \neq 0$ for all $t \geq \bar{t}$. We consider the following two cases.

- If $f(y^\circ) = f(\bar{y})$, then $M(f(y^\circ), f(\bar{y}), t) = 1$ for all $t > 0$.
- If $f(y^\circ) \neq f(\bar{y})$, then the distance condition says that there exits $\bar{t} > 0$ such that $M(f(y^\circ), f(\bar{y}), \bar{t}) \neq 0$. Part (i) of Proposition 2 says that the mapping $M(x, y, \cdot)$ is nondecreasing. It follows that $M(f(y^\circ), f(\bar{y}), t) \neq 0$ for all $t \geq \bar{t}$.

Therefore, from (44), for any fixed $t > 0$ with $t \geq \bar{t}$, we have

$$\left[M\left(f(x^\circ), f(\bar{x}), k^{\log_2 n} \cdot t\right)\right]^{1/n} \cdot \frac{1}{M(f(y^\circ), f(\bar{y}), t)} \geq M(f(x^\circ), f(\bar{x}), t). \tag{45}$$

Because $0 < k < 1$ and the mapping $M(x, y, \cdot)$ is non-decreasing, the function $\varrho$ defined in (26) is non-increasing, which says that $\varrho'(\alpha) \leq 0$ on $(0, \infty)$. Because $M$ satisfies the rational condition, we have

$$\lim_{t \to 0+} M(x, y, t) = 0 \tag{46}$$

for any fixed $x, y \in X$ with $x \neq y$. We consider

$$\varrho(\alpha) = M\left(f(x^\circ), f(\bar{x}), k^{\log_2 \alpha} \cdot t\right).$$

Suppose that $f(x^\circ) \neq f(\bar{x})$. Because $0 < k < 1$, it follows that $k^{\log_2 \alpha} \cdot t \to 0+$ as $\alpha \to \infty$. Therefore, (46) says that $\varrho(\alpha) \to 0+$ as $\alpha \to \infty$. Subsequently, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \left[M(f(x^\circ), f(\bar{x}), k^{\log_2 n} \cdot t)\right]^{1/n} \\
&= \lim_{n\to\infty} [\varrho(n)]^{1/n} = \lim_{n\to\infty} \exp\left[\frac{\ln \varrho(n)}{n}\right] = \exp\left[\lim_{n\to\infty} \frac{\varrho'(n)}{\varrho(n)}\right] \text{ (using the l'Hospital's rule)} \\
&= 0 \text{ (since } \varrho'(n) \leq 0 \text{ and } \varrho(n) \to 0+). 
\end{aligned} \tag{47}$$

By taking $n \to \infty$ in (45) and using (47), it follows that $M(f(x^\circ), f(\bar{x}), t) = 0$ for all $t \geq \bar{t}$. Because $f(x^\circ) \neq f(\bar{x})$, the distance condition says that there exits $t_0 > 0$, such that $M(f(x^\circ), f(\bar{x}), t_0) \neq 0$, i.e., $M(f(x^\circ), f(\bar{x}), t) \neq 0$ for all $t \geq t_0$ by the nondecreasing property of $M(x, y, \cdot)$, which contradicts $M(f(x^\circ), f(\bar{x}), t) = 0$ for all $t \geq \bar{t}$. Therefore, we must have $f(x^\circ) = f(\bar{x})$. We can similarly obtain $f(y^\circ) = f(\bar{y})$.

To prove property (B), using the commutativity of $T_n$ and $f$, we have

$$f(T_n(x^\circ, y^\circ)) = T_n(f(x^\circ), f(y^\circ)) = T_n(T_n(x^\circ, y^\circ), T_n(y^\circ, x^\circ)) \tag{48}$$

and

$$f(T_n(y^\circ, x^\circ)) = T_n(f(y^\circ), f(x^\circ)) = T_n(T_n(y^\circ, x^\circ), T_n(x^\circ, y^\circ)). \tag{49}$$

By regarding $\bar{x}$ as $T_n(x^\circ, y^\circ)$ and $\bar{y}$ as $T_n(y^\circ, x^\circ)$, the equalities (48) and (49) say that

$$f(\bar{x}) = T_n(\bar{x}, \bar{y}) \text{ and } f(\bar{y}) = T_n(\bar{y}, \bar{x}).$$

Therefore, using property (A), we must have

$$f(x^\circ) = f(\bar{x}) = f(T_n(x^\circ, y^\circ)) = T_n(f(x^\circ), f(y^\circ))$$

and

$$f(y^\circ) = f(\bar{y}) = f(T_n(y^\circ, x^\circ)) = T_n(f(y^\circ), f(x^\circ)),$$

which says that $(f(x^\circ), f(y^\circ)) \in X \times X$ is the common coupled fixed point of the mappings $\{T_n\}_{n=1}^{\infty}$.

To prove part (ii), we can similarly obtain

$$\eta(f(x_{n+1}),f(x_n),f(y_{n+1}),f(y_n),t) \geq \left[*\eta\left(f(x_1),f(x_0),f(y_1),f(y_0),\frac{t}{k^n}\right)\right]^{2^n}. \quad (50)$$

From part (i) of Proposition 10, it follows that $\{f(x_n)\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ are $>$-Cauchy sequences. We consider two cases below.

- Suppose that condition (c) is satisfied. Because $(X,M)$ is $(>,\triangleright)$-complete, there exist $x^\circ, y^\circ \in X$, such that

$$f(x_n) \stackrel{M^{\triangleright}}{\longrightarrow} x^\circ \text{ and } f(y_n) \stackrel{M^{\triangleright}}{\longrightarrow} y^\circ \text{ as } n \to \infty.$$

  Because $f$ is simultaneously $(\triangleright,\triangleright)$-continuous and $(\triangleright,\triangleleft)$-continuous with respect to $M$, we can similarly obtain (33)–(36).
- Suppose that condition (d) is satisfied. Because $(X,M)$ is $(>,\triangleleft)$-complete, there exist $x^\circ, y^\circ \in X$, such that

$$f(x_n) \stackrel{M^{\triangleleft}}{\longrightarrow} x^\circ \text{ and } f(y_n) \stackrel{M^{\triangleleft}}{\longrightarrow} y^\circ \text{ as } n \to \infty.$$

  Because $f$ is simultaneously $(\triangleleft,\triangleright)$-continuous and $(\triangleleft,\triangleleft)$-continuous with respect to $M$, we can similarly obtain (33)–(36).

The remaining proof follows from the similar argument of part (i). This completes the proof. □

In Theorem 1, since the fuzzy semi-metric $M$ is not necessarily symmetric, if the contractive inequalities (24) and (25) are not satisfied and, alternatively, the following converse-contractive inequalities

$$M(T_i(x,y),T_j(u,v),k_{ij}\cdot t) \geq M(f(u),f(x),t) * M(f(v),f(y),t)$$

and

$$M(T_i(x,y),T_j(u,v),k_{ij}\cdot t) \geq M(f(u),f(x),t) \cdot M(f(v),f(y),t)$$

are satisfied, then we can also obtain the desired results by assuming the different conditions.

**Theorem 2** (Satisfying the $\bowtie$-Triangle Inequality: Converse-Contractive Inequality)**.** *Let $(X,M)$ be a fuzzy semi-metric space, such that the mapping $M$ satisfies the rational condition and the $\bowtie$-triangle inequality. Suppose that the following conditions are satisfied.*

- *The first four conditions in Theorem 1 are satisfied.*
- *For any $x,y,u,v \in X$, the following converse-contractive inequality is satisfied:*

$$M(T_i(x,y),T_j(u,v),k_{ij}\cdot t) \geq M(f(u),f(x),t) * M(f(v),f(y),t), \quad (51)$$

  *where $k_{ij}$ satisfies $0 < k_{ij} \leq k < 1$ for all $i,j \in \mathbb{N}$ and for some constant $k$.*

  *Subsequently, we have the following results.*

(i) *Suppose that there exist $x^*, y^* \in X$ satisfying*

$$\sup_{\lambda\in(0,1)} \Phi(\lambda, f(x^*), T_1(x^*,y^*), f(y^*), T_1(y^*,x^*)) < \infty,$$

*and that any one of the following conditions is satisfied:*

  (a) *$(X,M)$ is $(<,\triangleright)$-complete and $f$ is $(\triangleright,\triangleright)$-continuous or $(\triangleright,\triangleleft)$-continuous with respect to $M$;*

  (b) *$(X,M)$ is $(<,\triangleleft)$-complete and $f$ is $(\triangleleft,\triangleright)$-continuous or $(\triangleleft,\triangleleft)$-continuous with respect to $M$.*

*Subsequently, the mappings* $\{T_n\}_{n=1}^{\infty}$ *and* $f$ *have a common coupled coincidence point* $(x^{\circ}, y^{\circ})$. *We further assume that the following conditions are satisfied.*

- *The inequality* (51) *is replaced by the following inequality*

$$M(T_i(x,y), T_j(u,v), k_{ij} \cdot t) \geq M(f(u), f(x), t) \cdot M(f(v), f(y), t),$$

*where the t-norm* $*$ *is replaced by the product of real numbers;*
- *The mapping* $M$ *satisfies the distance condition in Definition* 2.
- *For any fixed* $x, y \in X$ *and* $t > 0$, *the following mapping*

$$\varrho(\alpha) = M\left(x, y, k^{\log_2 \alpha} \cdot t\right)$$

*is differentiable on* $(0, \infty)$.

*Afterwards, we have the following results.*

(A) *Suppose that* $(\bar{x}, \bar{y})$ *is another coupled coincidence point of* $f$ *and* $T_{n_0}$ *for some* $n_0 \in \mathbb{N}$. *Then* $f(x^{\circ}) = f(\bar{x})$ *and* $f(y^{\circ}) = f(\bar{y})$.

(B) *There exists* $(x^{\circ}, y^{\circ}) \in X \times X$, *such that* $(f(x^{\circ}), f(y^{\circ})) \in X \times X$ *is the common coupled fixed point of the mappings* $\{T_n\}_{n=1}^{\infty}$.

*Moreover, the point* $(x^{\circ}, y^{\circ}) \in X \times X$ *can be obtained, as follows.*

- *Suppose that condition* (a) *is satisfied. Then the point* $(x^{\circ}, y^{\circ}) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^{\triangleright}} x^{\circ}$ *and* $f(y_n) \xrightarrow{M^{\triangleright}} y^{\circ}$.
- *Suppose that condition* (b) *is satisfied. Subsequently, the point* $(x^{\circ}, y^{\circ}) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^{\triangleleft}} x^{\circ}$ *and* $f(y_n) \xrightarrow{M^{\triangleleft}} y^{\circ}$.

*The sequences* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are generated from the initial element* $(x_0, y_0) = (x^*, y^*) \in X \times X$, *according to* (23).

(ii) *Suppose that there exist* $x^*, y^* \in X$ *satisfying*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, T_1(x^*, y^*), f(x^*), T_1(y^*, x^*), f(y^*)) < \infty,$$

*and that any one of the following conditions is satisfied:*

(c) $(X, M)$ *is* $(>, \triangleright)$*-complete and* $f$ *is* $(\triangleright, \triangleright)$*-continuous or* $(\triangleright, \triangleleft)$*-continuous with respect to* $M$;

(d) $(X, M)$ *is* $(>, \triangleleft)$*-complete and* $f$ *is* $(\triangleleft, \triangleright)$*-continuous or* $(\triangleleft, \triangleleft)$*-continuous with respect to* $M$;

*Subsequently, we have the same result as part* (i).

**Theorem 3** (Satisfying the $\triangleright$-Triangle Inequality). *Let* $(X, M)$ *be a fuzzy semi-metric space, such that the mapping* $M$ *satisfies the rational condition and the* $\triangleright$*-triangle inequality. Let* $(x_0, y_0) \in X \times X$ *be an initial element that generates the sequences* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *according to* (23). *Suppose that the following conditions are satisfied.*

- *The first four conditions in Theorem* 1 *are satisfied.*
- *The following contractive inequalities is satisfied*

$$M(T_i(x,y), T_j(u,v), k_{ij} \cdot t) \geq M(f(x), f(u), t) * M(f(y), f(v), t) \tag{52}$$

*or the following converse-contractive inequalities is satisfied*

$$M(T_i(x,y), T_j(u,v), k_{ij} \cdot t) \geq M(f(u), f(x), t) * M(f(v), f(y), t), \tag{53}$$

*where* $k_{ij}$ *satisfies* $0 < k_{ij} \leq k < 1$ *for all* $i, j \in \mathbb{N}$ *and for some constant* $k$.

- *There exist* $x^*, y^* \in X$ *satisfying*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, f(x^*), T_1(x^*, y^*), f(y^*), T_1(y^*, x^*)) < \infty$$

*and*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, T_1(x^*, y^*), f(x^*), T_1(y^*, x^*), f(y^*)) < \infty.$$

- *Any one of the following conditions is satisfied:*
  (a) $(X, M)$ *is* $(<, \triangleright)$*-complete or* $(>, \triangleright)$*-complete and* $f$ *is* $(\triangleright, \triangleleft)$*-continuous with respect to* $M$*;*
  (b) $(X, M)$ *is* $(<, \triangleleft)$*-complete or* $(>, \triangleleft)$*-complete and* $f$ *is* $(\triangleleft, \triangleleft)$*-continuous with respect to* $M$*.*

*Subsequently, the mappings* $\{T_n\}_{n=1}^{\infty}$ *and* $f$ *have a common coupled coincidence point* $(x^\circ, y^\circ)$*. We further assume that the following conditions are satisfied.*

- *The inequality* (52) *is replaced by the following inequality*

$$M(T_i(x, y), T_j(u, v), k_{ij} \cdot t) \geq M(f(x), f(u), t) \cdot M(f(y), f(v), t) \tag{54}$$

*and the inequality* (53) *is replaced by the following inequality*

$$M(T_i(x, y), T_j(u, v), k_{ij} \cdot t) \geq M(f(u), f(x), t) \cdot M(f(v), f(y), t), \tag{55}$$

*where the t-norm* $*$ *is replaced by the product of real numbers, such that any one of the inequalities* (54) *and* (55) *is satisfied.*

- *The mapping* $M$ *satisfies the distance condition in Definition* 2*.*
- *For any fixed* $x, y \in X$ *and* $t > 0$*, the following mapping*

$$\varrho(\alpha) = M\left(x, y, k^{\log_2 \alpha} \cdot t\right)$$

*is differentiable on* $(0, \infty)$*.*

*Afterwards, we have the following results.*

(A) *Suppose that* $(\bar{x}, \bar{y})$ *is another coupled coincidence point of* $f$ *and* $T_{n_0}$ *for some* $n_0 \in \mathbb{N}$*. Subsequently,* $f(x^\circ) = f(\bar{x})$ *and* $f(y^\circ) = f(\bar{y})$*.*

(B) *There exists* $(x^\circ, y^\circ) \in X \times X$*, such that* $(f(x^\circ), f(y^\circ)) \in X \times X$ *is the common coupled fixed point of the mappings* $\{T_n\}_{n=1}^{\infty}$*.*

*Moreover, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained as follows.*

- *Suppose that condition* (a) *is satisfied. Subsequently, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ *and* $f(y_n) \xrightarrow{M^\triangleright} y^\circ$*.*
- *Suppose that condition* (b) *is satisfied. Afterwards, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$ *and* $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$*.*

*The sequences* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are generated from the initial element* $(x_0, y_0) = (x^*, y^*) \in X \times X$ *according to* (23)*.*

**Theorem 4** (Satisfying the $\triangleleft$-Triangle Inequality)**.** *Let* $(X, M)$ *be a fuzzy semi-metric space such that the mapping* $M$ *satisfies the rational condition and the* $\triangleright$*-triangle inequality. Let* $(x_0, y_0) \in X \times X$ *be an initial element that generates the sequences* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *according to* (23)*. Suppose that the following conditions are satisfied.*

- *The first four conditions in Theorem* 1 *are satisfied.*
- *The following contractive inequalities is satisfied*

$$M(T_i(x, y), T_j(u, v), k_{ij} \cdot t) \geq M(f(x), f(u), t) * M(f(y), f(v), t) \tag{56}$$

*or the following converse-contractive inequalities is satisfied*

$$M(T_i(x,y),T_j(u,v),k_{ij}\cdot t)\geq M(f(u),f(x),t)*M(f(v),f(y),t), \tag{57}$$

*where $k_{ij}$ satisfies $0<k_{ij}\leq k<1$ for all $i,j\in\mathbb{N}$ and for some constant $k$.*

- *There exist $x^*,y^*\in X$ satisfying*

$$\sup_{\lambda\in(0,1)}\Phi(\lambda,f(x^*),T_1(x^*,y^*),f(y^*),T_1(y^*,x^*))<\infty$$

*and*

$$\sup_{\lambda\in(0,1)}\Phi(\lambda,T_1(x^*,y^*),f(x^*),T_1(y^*,x^*),f(y^*))<\infty;$$

- *Any one of the following conditions is satisfied:*
  (a) *$(X,M)$ is $(<,\triangleright)$-complete or $(>,\triangleright)$-complete and $f$ is $(\triangleright,\triangleright)$-continuous with respect to $M$;*
  (b) *$(X,M)$ is $(<,\triangleleft)$-complete or $(>,\triangleleft)$-complete and $f$ is $(\triangleleft,\triangleright)$-continuous with respect to $M$.*

*Subsequently, the mappings $\{T_n\}_{n=1}^{\infty}$ and $f$ have a common coupled coincidence point $(x^{\circ},y^{\circ})$. We further assume that the following conditions are satisfied.*

- *The inequality (56) is replaced by the following inequality*

$$M(T_i(x,y),T_j(u,v),k_{ij}\cdot t)\geq M(f(x),f(u),t)\cdot M(f(y),f(v),t) \tag{58}$$

*and the inequality (57) is replaced by the following inequality*

$$M(T_i(x,y),T_j(u,v),k_{ij}\cdot t)\geq M(f(u),f(x),t)\cdot M(f(v),f(y),t), \tag{59}$$

*where the t-norm $*$ is replaced by the product of real numbers, such that any one of the inequalities (58) and (59) is satisfied.*

- *The mapping $M$ satisfies the distance condition in Definition 2.*
- *For any fixed $x,y\in X$ and $t>0$, the following mapping*

$$\varrho(\alpha)=M\left(x,y,k^{\log_2\alpha}\cdot t\right)$$

*is differentiable on $(0,\infty)$.*

*Subsequently, we have the following results.*

(A) *Suppose that $(\bar{x},\bar{y})$ is another coupled coincidence point of $f$ and $T_{n_0}$ for some $n_0\in\mathbb{N}$. Subsequently, $f(x^{\circ})=f(\bar{x})$ and $f(y^{\circ})=f(\bar{y})$.*

(B) *There exists $(x^{\circ},y^{\circ})\in X\times X$ such that $(f(x^{\circ}),f(y^{\circ}))\in X\times X$ is the common coupled fixed point of the mappings $\{T_n\}_{n=1}^{\infty}$.*

*Moreover, the point $(x^{\circ},y^{\circ})\in X\times X$ can be obtained, as follows.*

- *Suppose that condition* (a) *is satisfied. Afterwards, the point $(x^{\circ},y^{\circ})\in X\times X$ can be obtained by taking the limit $f(x_n)\xrightarrow{M^{\triangleright}}x^{\circ}$ and $f(y_n)\xrightarrow{M^{\triangleright}}y^{\circ}$.*
- *Suppose that condition* (b) *is satisfied. Subsequently, the point $(x^{\circ},y^{\circ})\in X\times X$ can be obtained by taking the limit $f(x_n)\xrightarrow{M^{\triangleleft}}x^{\circ}$ and $f(y_n)\xrightarrow{M^{\triangleleft}}y^{\circ}$.*

*The sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are generated from the initial element $(x_0,y_0)=(x^*,y^*)\in X\times X$ according to (23).*

**Theorem 5** (Satisfying the $\diamond$-Triangle Inequality)**.** *Let $(X,M)$ be a fuzzy semi-metric space, such that the mapping $M$ satisfies the rational condition and the $\diamond$-triangle inequality. Suppose that the following conditions are satisfied.*

- *All five conditions in Theorem 1 are satisfied.*

- *There exist* $x^*, y^* \in X$ *satisfying*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, f(x^*), T_1(x^*, y^*), f(y^*), T_1(y^*, x^*)) < \infty$$

*and*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, T_1(x^*, y^*), f(x^*), T_1(y^*, x^*), f(y^*)) < \infty.$$

- *Any one of the following conditions is satisfied:*
  (a) $(X, M)$ *is* $(<, \triangleright)$*-complete or* $(>, \triangleright)$*-complete, and* $f$ *is* $(\triangleright, \triangleright)$*-continuous or* $(\triangleright, \triangleleft)$*-continuous with respect to* $M$*;*
  (b) $(X, M)$ *is* $(<, \triangleleft)$*-complete or* $(>, \triangleleft)$*-complete, and* $f$ *is* $(\triangleleft, \triangleright)$*-continuous or* $(\triangleleft, \triangleleft)$*-continuous with respect to* $M$*.*

*Subsequently, the mappings* $\{T_n\}_{n=1}^{\infty}$ *and* $f$ *have a common coupled coincidence point* $(x^\circ, y^\circ)$*. Moreover, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained, as follows.*

- *Suppose that condition* (a) *is satisfied. Afterwards, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ *and* $f(y_n) \xrightarrow{M^\triangleright} y^\circ$*.*
- *Suppose that condition* (b) *is satisfied. Subsequently, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$ *and* $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$*.*

*The sequences* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are generated from the initial element* $(x_0, y_0) = (x^*, y^*) \in X \times X$ *according to* (23)*.*

**Theorem 6** (Satisfying the $\diamond$-Triangle Inequality: Converse-Contractive Inequality)**.** *Let* $(X, M)$ *be a fuzzy semi-metric space, such that the mapping* $M$ *satisfies the rational condition and the* $\diamond$*-triangle inequality. Let* $(x_0, y_0) \in X \times X$ *be an initial element that generates the sequences* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *according to* (23)*. Suppose that the following conditions are satisfied.*

- *The first four conditions in Theorem 1 are satisfied.*
- *For any* $x, y, u, v \in X$*, the following converse-contractive inequality is satisfied:*

$$M(T_i(x,y), T_j(u,v), k_{ij} \cdot t) \geq M(f(u), f(x), t) * M(f(v), f(y), t),$$

*where* $k_{ij}$ *satisfies* $0 < k_{ij} \leq k < 1$ *for all* $i, j \in \mathbb{N}$ *and for some constant* $k$*.*

- *There exist* $x^*, y^* \in X$ *satisfying*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, f(x^*), T_1(x^*, y^*), f(y^*), T_1(y^*, x^*)) < \infty$$

*and*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, T_1(x^*, y^*), f(x^*), T_1(y^*, x^*), f(y^*)) < \infty.$$

- *Any one of the following conditions is satisfied:*
  (a) $(X, M)$ *is* $(<, \triangleright)$*-complete or* $(>, \triangleright)$*-complete and* $f$ *is simultaneously* $(\triangleright, \triangleright)$*-continuous and* $(\triangleright, \triangleleft)$*-continuous with respect to* $M$*;*
  (b) $(X, M)$ *is* $(<, \triangleleft)$*-complete or* $(>, \triangleleft)$*-complete and* $f$ *is simultaneously* $(\triangleleft, \triangleright)$*-continuous and* $(\triangleleft, \triangleleft)$*-continuous with respect to* $M$*.*

*Subsequently, the mappings* $\{T_n\}_{n=1}^{\infty}$ *and* $f$ *have a common coupled coincidence point* $(x^\circ, y^\circ)$*. Moreover, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained as follows.*

- *Suppose that condition* (a) *is satisfied. Afeterwards, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ *and* $f(y_n) \xrightarrow{M^\triangleright} y^\circ$*.*
- *Suppose that condition* (b) *is satisfied. Subsequently, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$ *and* $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$*.*

*The sequences* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are generated from the initial element* $(x_0, y_0) = (x^*, y^*) \in X \times X$ *according to* (23).

## 6. Common Coupled Fixed Points

Consider the mappings $T : X \times X \to X$ and $f : X \to X$. Recall that an element $(x^*, y^*) \in X \times X$ is called a common coupled fixed point when

$$x^* = f(x^*) = T(x^*, y^*) \text{ and } y^* = f(y^*) = T(y^*, x^*).$$

The common coupled fixed points are also the common coupled coincidence points. The uniqueness of common coupled coincidence points presented above was not guaranteed. In this section, we shall investigate the uniqueness of common coupled fixed points.

The contractive inequality and converse-contractive inequality should consider the product of real numbers instead of t-norm $*$ in order to obtain the unique common coupled fixed point.

**Theorem 7** (Satisfying the $\bowtie$-Triangle Inequality)**.** *Let* $(X, M)$ *be a fuzzy semi-metric space, such that the mapping M satisfies the rational condition and the* $\bowtie$*-triangle inequality. Suppose that the following conditions are satisfied.*

- *For any sequences* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *in* $[0, 1]$*, the following inequality is satisfied:*

$$\sup_n (a_n * b_n) \geq \left(\sup_n a_n\right) * \left(\sup_n b_n\right).$$

- *The t-norm* $*$ *is left-continuous with respect to the first or second component.*
- *Given any fixed* $x, y \in X$*, the mapping* $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ *is continuous on* $(0, \infty)$*.*
- *The mapping M satisfies the distance condition in Definition* 2*.*
- *Given any fixed* $x, y \in X$ *and* $t > 0$*, the following mapping*

$$\varrho(\alpha) = M\left(x, y, k^{\log_2 \alpha} \cdot t\right)$$

  *is differentiable on* $(0, \infty)$*.*
- *The mappings* $T_n : X \to X$ *and* $f : X \to X$ *satisfy the inclusion* $T_n(X, X) \subseteq f(X)$ *for all* $n \in \mathbb{N}$*.*
- *The mappings f and* $T_n$ *commute.*
- *Any one of the following conditions is satisfied:*
  - *the mapping f is simultaneously* $(\triangleright, \triangleright)$*-continuous and* $(\triangleright, \triangleleft)$*-continuous with respect to M;*
  - *the mapping f is simultaneously* $(\triangleleft, \triangleright)$*-continuous and* $(\triangleleft, \triangleleft)$*-continuous with respect to M.*
- *for any* $x, y, u, v \in X$*, the following contractive inequality is satisfied:*

$$M(T_i(x, y), T_j(u, v), k_{ij} \cdot t) \geq M(f(x), f(u), t) \cdot M(f(y), f(v), t), \tag{60}$$

  *where* $k_{ij}$ *satisfies* $0 < k_{ij} \leq k < 1$ *for all* $i, j \in \mathbb{N}$ *and for some constant k.*

  *Subsequently, we have the following results.*

(i) *Suppose that the space* $(X, M)$ *is simultaneously* $(<, \triangleright)$*-complete and* $(<, \triangleleft)$*-complete. We also assume that there exist* $x^*, y^* \in X$ *satisfying*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, f(x^*), T_1(x^*, y^*), f(y^*), T_1(y^*, x^*)) < \infty.$$

*Afterwards, the mappings* $\{T_n\}_{n=1}^{\infty}$ *and f have a unique common coupled fixed point* $(x^\circ, y^\circ)$*.*

(ii) *Suppose that the space $(X, M)$ is simultaneously $(>, \triangleright)$-complete and $(>, \triangleleft)$-complete. We also assume that there exist $x^*, y^* \in X$ satisfying*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, T_1(x^*, y^*), f(x^*), T_1(y^*, x^*), f(y^*)) < \infty.$$

*Then the mappings $\{T_n\}_{n=1}^{\infty}$ and $f$ have a unique common coupled fixed point $(x^\circ, y^\circ)$. Moreover, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained as follows.*

- *The point $x^\circ$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ or the limit $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$;*
- *The point $y^\circ$ can be obtained by taking the limit $f(y_n) \xrightarrow{M^\triangleright} y^\circ$ or the limit $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$.*

*The sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are generated from the initial element $(x_0, y_0) = (x^*, y^*) \in X \times X$ according to (23).*

**Proof.** According to (23), we can generate two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ from the initial element $x_0 = x^*$ and $y_0 = y^*$. To prove part (i), while using part (i) of Theorem 1, we have $f(x^\circ) = T_n(x^\circ, y^\circ)$ and $f(y^\circ) = T_n(y^\circ, x^\circ)$ for all $n \in \mathbb{N}$. According to the proof of part (i) of Theorem 1, we see that $\{f(x_n)\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ are $<$-Cauchy sequences. Since $(X, M)$ issimultaneously $(<, \triangleright)$-complete and $(<, \triangleleft)$-complete, using part (i) of Proposition 3, there exists $x^\circ, y^\circ \in X$ satisfying $f(x_n) \xrightarrow{M^\triangleright} x^\circ$, $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$, $f(y_n) \xrightarrow{M^\triangleright} y^\circ$ and $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$ as $n \to \infty$, which also says that $f(x_n) \xrightarrow{M} x^\circ$ and $f(y_n) \xrightarrow{M} y^\circ$ as $n \to \infty$.

Next, we are going to claim that $x^\circ$ is a fixed point of $f$. While using (60) and the nondecreasing property of $M(x, y, \cdot)$ by part (i) of Proposition 2, we have

$$\begin{aligned} M(f(x_{n+1}), f(x^\circ), t) &= M(T_{n+1}(x_n, y_n), T_n(x^\circ, y^\circ), t) \\ &\geq M\left(f(x_n), f(x^\circ), \frac{t}{k_{n+1,n}}\right) \cdot M\left(f(y_n), f(y^\circ), \frac{t}{k_{n+1,n}}\right) \\ &\geq M\left(f(x_n), f(x^\circ), \frac{t}{k}\right) \cdot M\left(f(y_n), f(y^\circ), \frac{t}{k}\right) \end{aligned} \tag{61}$$

and

$$\begin{aligned} M(f(y_{n+1}), f(y^\circ), t) &= M(T_{n+1}(y_n, x_n), T_n(y^\circ, x^\circ), t) \\ &\geq M\left(f(y_n), f(y^\circ), \frac{t}{k_{n+1,n}}\right) \cdot M\left(f(x_n), f(x^\circ), \frac{t}{k_{n+1,n}}\right) \\ &\geq M\left(f(y_n), f(y^\circ), \frac{t}{k}\right) \cdot M\left(f(x_n), f(x^\circ), \frac{t}{k}\right) \end{aligned} \tag{62}$$

Because $f(x_n) \xrightarrow{M} x^\circ$ and $f(y_n) \xrightarrow{M} y^\circ$ as $n \to \infty$, applying part (i) of Proposition 4 to (61) and (62), we obtain

$$M(x^\circ, f(x^\circ), t) \geq M\left(x^\circ, f(x^\circ), \frac{t}{k}\right) \cdot M\left(y^\circ, f(y^\circ), \frac{t}{k}\right).$$

and

$$M(y^\circ, f(y^\circ), t) \geq M\left(x^\circ, f(x^\circ), \frac{t}{k}\right) \cdot M\left(y^\circ, f(y^\circ), \frac{t}{k}\right).$$

By referring to (43), we can obtain

$$M(x^\circ, f(x^\circ), t) \geq \left[M\left(x^\circ, f(x^\circ), \frac{t}{k^n}\right)\right]^{2^n} \cdot \left[M\left(y^\circ, f(y^\circ), \frac{t}{k^n}\right)\right]^{2^n},$$

which is equivalent to

$$M(x^\circ, f(x^\circ), k^{\log_2 n} \cdot t) \geq [M(x^\circ, f(x^\circ), t)]^n \cdot [M(y^\circ, f(y^\circ), t)]^n. \tag{63}$$

We are going to claim that there exists $\bar{t} > 0$, such that $M(y^\circ, f(y^\circ), t) \neq 0$ for all $t \geq \bar{t}$. We consider the following cases.

- If $f(y^\circ) = y^\circ$, then $M(y^\circ, f(y^\circ), t) = 1$ for all $t > 0$.
- If $f(y^\circ) \neq y^\circ$, then the distance condition says that there exits $\bar{t} > 0$, such that $M(y^\circ, f(y^\circ), \bar{t}) \neq 0$. Part (i) of Proposition 2 says that the mapping $M(x, y, \cdot)$ is nondecreasing. Therefore, we have $M(y^\circ, f(y^\circ), t) \neq 0$ for all $t \geq \bar{t}$.

From (63), for any fixed $t > 0$ with $t \geq \bar{t}$, we have

$$\left[M\left(x^\circ, f(x^\circ), k^{\log_2 n} \cdot t\right)\right]^{1/n} \cdot \frac{1}{M(y^\circ, f(y^\circ), t)} \geq M(x^\circ, f(x^\circ), t). \tag{64}$$

Because $0 < k < 1$ and the mapping $M(x, y, \cdot)$ is nondecreasing, the function $\varrho$ that is defined in (26) is non-increasing, which says that $\varrho'(\alpha) \leq 0$ on $(0, \infty)$. Because $M$ satisfies the rational condition, we have

$$\lim_{t \to 0+} M(x, y, t) = 0 \tag{65}$$

for any fixed $x, y \in X$ with $x \neq y$. We consider

$$\varrho(\alpha) = M\left(x^\circ, f(x^\circ), k^{\log_2 \alpha} \cdot t\right).$$

Suppose that $f(x^\circ) \neq x^\circ$. Since $0 < k < 1$, it follows that $k^{\log_2 \alpha} \cdot t \to 0+$ as $\alpha \to \infty$. Therefore, (65) says that $\varrho(\alpha) \to 0+$ as $\alpha \to \infty$. Subsequently, we obtain

$$\begin{aligned} &\lim_{n\to\infty}\left[M(x^\circ, f(x^\circ), k^{\log_2 n} \cdot t)\right]^{1/n} \\ &= \lim_{n\to\infty}[\varrho(n)]^{1/n} = \lim_{n\to\infty}\exp\left[\frac{\ln \varrho(n)}{n}\right] = \exp\left[\lim_{n\to\infty}\frac{\varrho'(n)}{\varrho(n)}\right] \text{ (using the l'Hospital's rule)} \\ &= 0 \text{ (since } \varrho'(n) \leq 0 \text{ and } \varrho(n) \to 0+\text{).} \end{aligned} \tag{66}$$

Applying (66) to (64), we obtain $M(x^\circ, f(x^\circ), t) = 0$ for all $t \geq \bar{t}$. Because $f(x^\circ) \neq x^\circ$, the distance condition says that there exits $t_0 > 0$ such that $M(x^\circ, f(x^\circ), t_0) \neq 0$, i.e., $M(x^\circ, f(x^\circ), t) \neq 0$ for all $t \geq t_0$ by the nondecreasing property of $M(x, y, \cdot)$, which contradicts $M(x^\circ, f(x^\circ), t) = 0$ for all $t \geq \bar{t}$. Therefore we must have $f(x^\circ) = x^\circ$. We can similarly obtain $f(y^\circ) = y^\circ$; that is,

$$x^\circ = f(x^\circ) = T_n(x^\circ, y^\circ) \text{ and } y^\circ = f(y^\circ) = T_n(y^\circ, x^\circ)$$

for all $n \in \mathbb{N}$.

In order to prove the uniqueness, let $(\bar{x}, \bar{y})$ be another common coupled fixed point of $f$ and $\{T_n\}_{n=1}^\infty$, i.e., $\bar{x} = f(\bar{x}) = T_n(\bar{x}, \bar{y})$ and $\bar{y} = f(\bar{y}) = T_n(\bar{y}, \bar{x})$ for all $n \in \mathbb{N}$. The inequality (43) is equivalent to

$$M(x^\circ, \bar{x}, k^{\log_2 n} \cdot t) \geq [M(x^\circ, \bar{x}, t)]^n \cdot [M(y^\circ, \bar{y}, t)]^n. \tag{67}$$

We can similarly show that there exists $\hat{t} > 0$, such that $M(y^\circ, \bar{y}, t) \neq 0$ for all $t \geq \hat{t}$. Therefore, from (67), for any fixed $t > 0$ with $t \geq \hat{t}$, we have

$$\left[M\left(x^\circ, \bar{x}, k^{\log_2 n} \cdot t\right)\right]^{1/n} \cdot \frac{1}{M(y^\circ, \bar{y}, t)} \geq M(x^\circ, \bar{x}, t). \tag{68}$$

By referring to (68), it follows that $M(x^\circ, \bar{x}, t) = 0$ for all $t \geq \hat{t}$. Because $\bar{x} \neq x^\circ$, the distance condition says that there exits $t_0 > 0$, such that $M(x^\circ, \bar{x}, t_0) \neq 0$, i.e., $M(x^\circ, \bar{x}, t) \neq 0$ for all $t \geq t_0$ by the non-decreasing property of $M(x, y, \cdot)$, which contradicts $M(x^\circ, \bar{x}, t) = 0$ for all $t \geq \hat{t}$. Therefore, we must have $\bar{x} = x^\circ$. We can similarly obtain $\bar{y} = y^\circ$. This proves the uniqueness. Finally, part (ii) can be obtained by applying part (ii) of Theorem 1 to the above argument. This completes the proof. □

**Theorem 8** (Satisfying the $\bowtie$-Triangle Inequality: Converse-Contractive Inequality)**.** *Let $(X, M)$ be a fuzzy semi-metric space such that the mapping $M$ satisfies the rational condition and the $\bowtie$-triangle inequality. Suppose that the following conditions are satisfied.*

- *The first eight conditions of Theorem 7 are satisfied.*
- *For any $x, y, u, v \in X$, the following converse-contractive inequality is satisfied:*

$$M(T_i(x, y), T_j(u, v), k_{ij} \cdot t) \geq M(f(u), f(x), t) \cdot M(f(v), f(y), t), \tag{69}$$

*where $k_{ij}$ satisfies $0 < k_{ij} \leq k < 1$ for all $i, j \in \mathbb{N}$ and for some constant $k$.*

*Subsequently, we have the following results.*

(i) *Suppose that the space $(X, M)$ is simultaneously $(<, \triangleright)$-complete and $(<, \triangleleft)$-complete. We also assume that there exist $x^*, y^* \in X$ satisfying*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, f(x^*), T_1(x^*, y^*), f(y^*), T_1(y^*, x^*)) < \infty.$$

*Afterwards, the mappings $\{T_n\}_{n=1}^{\infty}$ and $f$ have a unique common coupled fixed point $(x^\circ, y^\circ)$.*

(ii) *Suppose that the space $(X, M)$ is simultaneously $(>, \triangleright)$-complete and $(>, \triangleleft)$-complete. We also assume that there exist $x^*, y^* \in X$ satisfying*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, T_1(x^*, y^*), f(x^*), T_1(y^*, x^*), f(y^*)) < \infty.$$

*Subsequently, the mappings $\{T_n\}_{n=1}^{\infty}$ and $f$ have a unique common coupled fixed point $(x^\circ, y^\circ)$.*

*Moreover, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained, as follows.*

- *The point $x^\circ$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ or the limit $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$.*
- *The point $y^\circ$ can be obtained by taking the limit $f(y_n) \xrightarrow{M^\triangleright} y^\circ$ or the limit $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$,*

*The sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are generated from the initial element $(x_0, y_0) = (x^*, y^*) \in X \times X$, according to (23).*

**Theorem 9** (Satisfying the $\triangleright$-Triangle Inequality)**.** *Let $(X, M)$ be a fuzzy semi-metric space, such that the mapping $M$ satisfies the rational condition and the $\triangleright$-triangle inequality. Suppose that the following conditions are satisfied.*

- *The first eight conditions of Theorem 7 are satisfied.*
- *The following contractive inequalities is satisfied*

$$M(T_i(x, y), T_j(u, v), k_{ij} \cdot t) \geq M(f(x), f(u), t) \cdot M(f(y), f(v), t) \tag{70}$$

*or the following converse-contractive inequalities is satisfied*

$$M(T_i(x, y), T_j(u, v), k_{ij} \cdot t) \geq M(f(u), f(x), t) \cdot M(f(v), f(y), t), \tag{71}$$

*where $k_{ij}$ satisfies $0 < k_{ij} \leq k < 1$ for all $i, j \in \mathbb{N}$ and for some constant $k$.*

- *There exist $x^*, y^* \in X$, satisfying*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, f(x^*), T_1(x^*, y^*), f(y^*), T_1(y^*, x^*)) < \infty$$

*and*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, T_1(x^*, y^*), f(x^*), T_1(y^*, x^*), f(y^*)) < \infty.$$

- *The mapping $f$ is $(\triangleright, \triangleleft)$-continuous or $(\triangleleft, \triangleleft)$-continuous with respect to $M$.*
- *Any one of the following conditions is satisfied:*
  - $(X, M)$ *is* $(<, \triangleright)$*-complete and* $(<, \triangleleft)$*-complete simultaneously;*
  - $(X, M)$ *is* $(>, \triangleright)$*-complete and* $(>, \triangleleft)$*-complete simultaneously.*

*Subsequently, the mappings $\{T_n\}_{n=1}^{\infty}$ and $f$ have a unique common coupled fixed point $(x^\circ, y^\circ)$. Moreover, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained, as follows.*

- *The point $x^\circ$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ or the limit $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$.*
- *The point $y^\circ$ can be obtained by taking the limit $f(y_n) \xrightarrow{M^\triangleright} y^\circ$ or the limit $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$.*

*The sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are generated from the initial element $(x_0, y_0) = (x^*, y^*) \in X \times X$ according to (23).*

**Theorem 10** (Satisfying the $\triangleleft$-Triangle Inequality)**.** *Let $(X, M)$ be a fuzzy semi-metric space, such that the mapping $M$ satisfies the rational condition and the $\triangleleft$-triangle inequality. Suppose that the following conditions are satisfied.*

- *The first eight conditions of Theorem 7 are satisfied.*
- *The following contractive inequalities is satisfied*

$$M(T_i(x, y), T_j(u, v), k_{ij} \cdot t) \geq M(f(x), f(u), t) \cdot M(f(y), f(v), t)$$

*or the following converse-contractive inequalities are satisfied*

$$M(T_i(x, y), T_j(u, v), k_{ij} \cdot t) \geq M(f(u), f(x), t) \cdot M(f(v), f(y), t),$$

*where $k_{ij}$ satisfies $0 < k_{ij} \leq k < 1$ for all $i, j \in \mathbb{N}$ and for some constant $k$.*

- *There exist $x^*, y^* \in X$ satisfying*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, f(x^*), T_1(x^*, y^*), f(y^*), T_1(y^*, x^*)) < \infty$$

*and*

$$\sup_{\lambda \in (0,1)} \Phi(\lambda, T_1(x^*, y^*), f(x^*), T_1(y^*, x^*), f(y^*)) < \infty.$$

- *The mapping $f$ is $(\triangleright, \triangleright)$-continuous or $(\triangleleft, \triangleright)$-continuous with respect to $M$.*
- *Any one of the following conditions is satisfied:*
  - $(X, M)$ *is* $(<, \triangleright)$*-complete and* $(<, \triangleleft)$*-complete simultaneously;*
  - $(X, M)$ *is* $(>, \triangleright)$*-complete and* $(>, \triangleleft)$*-complete simultaneously.*

*Subsequently, the mappings $\{T_n\}_{n=1}^{\infty}$ and $f$ have a unique common coupled fixed point $(x^\circ, y^\circ)$. Moreover, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained, as follows.*

- *The point $x^\circ$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ or the limit $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$.*
- *The point $y^\circ$ can be obtained by taking the limit $f(y_n) \xrightarrow{M^\triangleright} y^\circ$ or the limit $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$.*

*The sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are generated from the initial element $(x_0, y_0) = (x^*, y^*) \in X \times X$ according to (23).*

**Theorem 11** (Satisfying the $\diamond$-Triangle Inequality)**.** *Let $(X, M)$ be a fuzzy semi-metric space such that the mapping $M$ satisfies the rational condition and the $\diamond$-triangle inequality. Suppose that the following conditions are satisfied.*

- *All nine conditions of Theorem 7 are satisfied.*

- *There exist* $x^*, y^* \in X$ *satisfying*

$$\sup_{\lambda\in(0,1)} \Phi(\lambda, f(x^*), T_1(x^*,y^*), f(y^*), T_1(y^*,x^*)) < \infty$$

*and*

$$\sup_{\lambda\in(0,1)} \Phi(\lambda, T_1(x^*,y^*), f(x^*), T_1(y^*,x^*), f(y^*)) < \infty;$$

- *Any one of the following conditions is satisfied:*
  (a) $(X, M)$ *is* $(<, \triangleright)$*-complete or* $(>, \triangleright)$*-complete and* $f$ *is* $(\triangleright, \triangleright)$*-continuous or* $(\triangleright, \triangleleft)$*-continuous with respect to* $M$*;*
  (b) $(X, M)$ *is* $(<, \triangleleft)$*-complete or* $(>, \triangleleft)$*-complete and* $f$ *is* $(\triangleleft, \triangleright)$*-continuous or* $(\triangleleft, \triangleleft)$*-continuous with respect to* $M$*.*

*Afterwards, the mappings* $T$ *and* $f$ *have a unique common coupled fixed point* $(x^\circ, y^\circ)$ *Moreover, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained, as follows.*

- *Suppose that condition* (a) *is satisfied. Afterwards, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ *and* $f(y_n) \xrightarrow{M^\triangleright} y^\circ$*.*
- *Suppose that condition* (b) *is satisfied. Subsequently, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$ *and* $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$*.*

*The sequences* $\{x_n\}_{n=1}^\infty$ *and* $\{y_n\}_{n=1}^\infty$ *are generated from the initial element* $(x_0, y_0) = (x^*, y^*) \in X \times X$*, according to* (23)*.*

**Theorem 12** (Satisfying the $\diamond$-Triangle Inequality: Converse-Contractive Inequality)**.** *Let* $(X, M)$ *be a fuzzy semi-metric space, such that the mapping* $M$ *satisfies the rational condition and the* $\diamond$*-triangle inequality. Suppose that the following conditions are satisfied.*

- *The first eight conditions of Theorem* 7 *are satisfied.*
- *The following converse-contractive inequalities are satisfied*

$$M(T_i(x,y), T_j(u,v), k_{ij} \cdot t) \geq M(f(u), f(x), t) \cdot M(f(v), f(y), t),$$

*where* $k_{ij}$ *satisfies* $0 < k_{ij} \leq k < 1$ *for all* $i, j \in \mathbb{N}$ *and for some constant* $k$*.*

- *There exist* $x^*, y^* \in X$ *satisfying*

$$\sup_{\lambda\in(0,1)} \Phi(\lambda, f(x^*), T_1(x^*,y^*), f(y^*), T_1(y^*,x^*)) < \infty$$

*and*

$$\sup_{\lambda\in(0,1)} \Phi(\lambda, T_1(x^*,y^*), f(x^*), T_1(y^*,x^*), f(y^*)) < \infty.$$

- *Any one of the following conditions is satisfied:*
  (a) $(X, M)$ *is* $(<, \triangleright)$*-complete or* $(>, \triangleright)$*-complete and* $f$ *is* $(\triangleright, \triangleright)$*-continuous and* $(\triangleright, \triangleleft)$*-continuous with respect to* $M$*;*
  (b) $(X, M)$ *is* $(<, \triangleleft)$*-complete or* $(>, \triangleleft)$*-complete and* $f$ *is* $(\triangleleft, \triangleright)$*-continuous and* $(\triangleleft, \triangleleft)$*-continuous with respect to* $M$*.*

*Subsequently, the mappings* $T$ *and* $f$ *have a unique common coupled fixed point* $(x^\circ, y^\circ)$ *Moreover, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained, as follows.*

- *Suppose that condition* (a) *is satisfied. Subsequently, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ *and* $f(y_n) \xrightarrow{M^\triangleright} y^\circ$*.*
- *Suppose that condition* (b) *is satisfied. Afterwards, the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$ *and* $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$*.*

*The sequences* $\{x_n\}_{n=1}^\infty$ *and* $\{y_n\}_{n=1}^\infty$ *are generated from the initial element* $(x_0, y_0) = (x^*, y^*) \in X \times X$ *according to* (23)*.*

## 7. Conclusions

Four different kinds of triangle inequalities play the important role of studying the common coupled coincidence points and common coupled fixed points in fuzzy semi-metric spaces. We separately present the theorems of common coupled coincidence points that are based on the different kinds of triangle inequalities.

- Suppose that the fuzzy semi-metric space satisfies the $\bowtie$-triangle inequality. Theorem 1 studies the common coupled coincidence points. Because the symmetric condition is not satisfied. Theorem 2 also studies the common coupled coincidence points by considering the so-called converse-contractive inequality.
- Theorems 3 and 4 study the common coupled coincidence points when the fuzzy semi-metric space satisfies the $\triangleright$-triangle inequality and $\triangleleft$-triangle inequality, respectively.
- Suppose that the fuzzy semi-metric space satisfies the $\diamond$-triangle inequality. Theorem 5 studies the common coupled coincidence points, and Theorem 6 studies the common coupled coincidence points by considering the so-called converse-contractive inequality.

Because the common coupled fixed points are the common coupled coincidence points, Theorems 1–6 can also be used to present the common coupled fixed points. However, the uniqueness cannot be realized from Theorems 1–6. Section 6 studies the uniqueness of common coupled fixed points.

- Suppose that the fuzzy semi-metric space satisfies the $\bowtie$-triangle inequality. Theorem 7 studies the uniqueness of common coupled fixed points, and Theorem 8 also studies the uniqueness of common coupled fixed points by considering the so-called converse-contractive inequality.
- Theorems 9 and 10 study the uniqueness of common coupled fixed points when the fuzzy semi-metric space satisfies the $\triangleright$-triangle inequality and $\triangleleft$-triangle inequality, respectively.
- Suppose that the fuzzy semi-metric space satisfies the $\diamond$-triangle inequality. Theorem 11 studies the uniqueness of common coupled fixed points and Theorem 12 studies the uniqueness of common coupled fixed points by considering the so-called converse-contractive inequality.

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